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THE NUMERICAL STABILITY OF THE IMPROVED GARGANTINI METHOD*

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Abstract. The third order interval Gargantini method for the simultaneous inclusion of polynomial zeros was improved to the fourth order method by Carstensen and Petković [An improvement of Gargantini's simultaneous inclusion method for polynomial roots by Schroeder's correction, Appl. Numer. Math. 13 (1994), 453–468]. We investigate the numerical stability of this improved method in the presence of rounding errors. The dependence of the convergence rate of the considered method on the magnitude of rounding errors is studied.

1. Introduction

During the last forty years many techniques for a posteriori error estimates of approximations to polynomial zeros have been developed. One of the simplest and most efficient ways for the estimate of upper error bounds is the approach which uses circular interval arithmetic (see [4]). Iterative methods for the simultaneous inclusion of polynomial zeros produce resulting real or

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complex intervals containing the wanted zeros. In this manner information about upper error bounds of approximations to the zeros are automatically provided, see the books [1], [7], [9].

The aim of this paper is to study the numerical stability of the improved Gargantini simultaneous inclusion method, proposed in [2], in the presence of rounding errors that appear in the calculation on digital computers. In essence, we are interested in the following question: *What is the convergence behavior of the presented iterative method in the presence of round-off?* This problem was considered, for example, in [3], [8] and [10] for some specific inclusion methods.

For the reader's convenience we give some basic properties of circular complex arithmetic. More details can be found in the books [1], [7], [9].

A disk Z with center $\operatorname{mid} Z = c$ and radius $\operatorname{rad} Z = r$, that is $Z := \{z : |z - c| \le r\}$, will be denoted briefly by the parametric notation $Z = \{c; r\}$. The basic circular arithmetic operations are defined as follows:

$$\{c_1; r_1\} \pm \{c_2; r_2\} = \{c_1 \pm c_2; r_1 + r_2\},$$
$$\{c_1; r_1\} \cdot \{c_2; r_2\} = \{c_1 c_2; |c_1| r_2 + |c_2| r_1 + r_1 r_2\}.$$

Let us consider the inversion of a disk $Z = \{c; r\}$ which does not contain the origin, that is, $|c| > r$ holds. The inversion is introduced in two manners:

$$Z^{I_E} = \left\{ \frac{1}{z} : \ z \in \{c; r\} \right\} = \left\{ \frac{\bar{c}}{|c|^2 - r^2}; \frac{r}{|c|^2 - r^2} \right\} \quad \text{(exact inversion)}$$

and

$$Z^{I_C} = \left\{ \frac{1}{c}; \frac{r}{|c|(|c| - r)} \right\} \supseteq \left\{ \frac{1}{z} : \ z \in \{c; r\} \right\} \quad \text{(centered inversion).}$$

In the sequel, $\mathrm{INV}(Z)$ will denote one of the two inversions Z^{I_E}, Z^{I_C}. Following the introduced inversions, division is defined as

$$Z_1 : Z_2 = Z_1 \cdot Z_2^{I_E} \quad \text{or} \quad Z_1 : Z_2 = Z_1 \cdot Z_2^{I_C} \quad (0 \notin Z_2).$$

Two disks $Z_1 = \{c_1; r_1\}$ and $Z_2 = \{c_2; r_2\}$ are nonintersecting if and only if

$$|c_1 - c_2| > r_1 + r_2, \tag{1.1}$$

and $Z_2 \subseteq Z_1$ if and only if

$$|c_1 - c_2| \le r_1 - r_2. \tag{1.2}$$

It is easy to prove that, if $z \in Z$, then

$$\max\{0, |\operatorname{mid} Z| - \operatorname{rad} Z\} \le |z| \le |\operatorname{mid} Z| + |\operatorname{rad} Z|. \tag{1.3}$$

In what follows, disks in the complex plane will be denoted by capital letters.

2. The improved Gargantini interval method

Let

$$f(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$$

be a monic polynomial of degree $n \ge 3$ with simple zeros $\zeta_1, \ldots, \zeta_n$ and let $\mathcal{I}_n := \{1, \ldots, n\}$ be the index set. Let us assume that we have found nonintersecting disks $Z_1, \ldots, Z_n$ with the centers $z_i = \operatorname{mid} Z_i$ and the radii $r_i = \operatorname{rad} Z_i$ such that $\zeta_i \in Z_i$ $(i \in \mathcal{I}_n)$.

Applying the logarithmic derivatives we find

$$\frac{f'(z)}{f(z)} = \sum_{j=1}^{n} \frac{1}{z - \zeta_j},$$

wherefrom

$$\zeta_i = z - \frac{1}{\dfrac{1}{h_i} - \displaystyle\sum_{\substack{j=1 \\ j \ne i}}^{n} \frac{1}{z_i - \zeta_j}} \quad (i \in \mathcal{I}_n), \tag{2.1}$$

where $h_i = f(z_i)/f'(z_i)$ is Newton's correction. Taking inclusion disks Z_j instead the zeros ζ_j in (2.1), Gargantini [3] established the third order method for the simultaneous inclusion of all zeros of f,

$$\hat{Z}_i = z_i - \frac{1}{\dfrac{1}{h_i} - \displaystyle\sum_{\substack{j=1 \\ j \ne i}}^{n} \frac{1}{z_i - Z_j}} \quad (i \in \mathcal{I}_n;\ z_i = \operatorname{mid} Z_i). \tag{2.2}$$

Using Nourein's idea [6] Carstensen and Petković [2] improved the Gargantini method (2.2) by constructing the iterative formula

$$\hat{Z}_i = z_i - \operatorname{INV}_1\Big(\frac{1}{h_i} - \sum_{\substack{j=1 \\ j \ne i}}^{n} \operatorname{INV}_2(z_i - Z_j + h_j)\Big) \quad (i \in \mathcal{I}_n), \tag{2.3}$$

where $\mathrm{INV}_1,\ \mathrm{INV}_2 \in \{()^{I_E}, ()^{I_C}\}$.

The following theorems were proved in [2] for the inclusion method (2.3).

Theorem 2.1 *Let* $(Z_1, \ldots, Z_n) =: \big(Z_1^{(0)}, \ldots, Z_n^{(0)}\big)$ *be initial disks such that* $\zeta_i \in Z_i,\ i \in \mathcal{I}_n$ *and*

$$\rho \geq 4nr, \tag{2.4}$$

where

$$r := \max_{i=1,\ldots,n} \mathrm{rad}\,(Z_i), \quad \rho := \min_{\substack{i,j=1,\ldots,n \\ i\neq j}} |\mathrm{mid}\,(Z_i) - \mathrm{mid}\,(Z_j)|.$$

Then, the method (2.3) *is feasible, that is, it defines the sequences of disks* $\big(Z_i^{(m)} \,|\, i \in \mathcal{I}_n\big)_{m=0,1,\ldots}$*, for any* $i \in \mathcal{I}_n$ *and* $m = 0, 1, \ldots$ *there holds*

$$\zeta_i \in Z_i^{(m)}$$

and the sequences of radii $\big(\mathrm{rad}\,\big(Z_i^{(m)}\big)\big)_{m=0,1,\ldots}$ *tend towards zero.*

Theorem 2.2 *Let* $O((2.3))$ *denote the R-order of convergence of the radii of the inclusion method* (2.3)*. Then,*

$$O((2.3)) \geq \begin{cases} \dfrac{1}{2}(3+\sqrt{17}), & \text{if } \ \mathrm{INV}_2 = ()^{I_E}, \\ 4, & \text{if } \ \mathrm{INV}_2 = ()^{I_C}. \end{cases}$$

3. Iterative method in the presence of rounding errors

Any iterative procedure for finding the zeros of a function φ has to terminate in a finite number of iterations in the presence of round-off (see [3]). The termination shall occur as soon as the absolute value of the rounding error involved in the evaluation of φ in the proximity of a zero is of the same order of magnitude as $|\varphi|$.

Let f, f' denote the given polynomial and its derivative, and let $\Delta f, \Delta f'$ be upper bounds of the absolute value of the round-off occuring in the evaluation of f and f', respectively. These errors can be evaluated according to the rounding-error analysis introduced by Wilkinson [11] and depend on the number of significant digits of the mantissa for floating-point arithmetic operations.

The application of the iterative formula (2.3) requires that the single errors relative to the evaluation of f and f' are included in the determination of h_i because of the presence of rounding errors. Accordingly, we have to replace f by the disk $T_0 = \{f; \Delta f\}$ and the derivative f' by the disk $T_1 = \{f'; \Delta f'\}$ with the centers f and f' and the radii Δf and $\Delta f'$. Circular extension obtained by the above substitutions is

$$\frac{1}{h_i} = \frac{f'(z_i)}{f(z_i)} \in \{f'(z_i); \Delta f'(z_i)\} \cdot \{f(z_i); \Delta f(z_i)\}^{I_C}$$
$$= \left\{\frac{f'(z_i)}{f(z_i)}; \delta_i\right\} = \left\{\frac{1}{h_i}; \delta_i\right\} =: \frac{1}{H_i}.$$

In what follows, we shall operate with total rounding errors $\delta_i = \operatorname{rad}(1/H_i)$, which incorporates the single errors concerning the evaluation of f and f'.

In order that the inversion of the disk $T_0 = \{f; \Delta f\}$ (which appears in the evaluation of $1/H_i$) also produces a disk, we shall require $\Delta f < |f|$ in our analysis of the numerical stability of the scheme (2.3). Also, as it was already mentioned, the stopping criterion which determines the maximal number of iterations, is based on the comparison of $\Delta f_i^{(m)}$ with $|\Delta f(z_i^{(m)})|$ for each $i \in \mathcal{I}_n$.

In this paper we shall use the following notation. At the m-th iteration we denote the radius of $1/H_i^{(m)}$ by $\delta_i^{(m)}$. Also, we introduce the abbreviations

$$r^{(m)} = \max_{1\le i\le n} r_i^{(m)}, \quad \rho^{(m)} = \min_{\substack{1\le i,j\le n\\ i\ne j}} \left\{|z_i^{(m)} - z_j^{(m)}|\right\}.$$

In the new iterative formula only the scalars $1/h_i$ are replaced by the disks $1/H_i$, $i \in \mathcal{I}_n$. Thus, the algorithm (2.3) for the simultaneous inclusion of all simple zeros of a given polynomial in the presence of rounding errors has the form

$$\hat{Z}_i = z_i - \mathrm{INV}_1\Big(\frac{1}{H_i} - \sum_{\substack{j=1\\ j\ne i}}^{n} \mathrm{INV}_2(z_i - Z_j + H_j)\Big) \quad (i \in \mathcal{I}_n; m = 0, 1, \dots). \tag{3.1}$$

Since the method (2.3) has the best performance in the case when $\mathrm{INV}_1 = \mathrm{INV}_2 = ()^{I_C}$ (see Theorem 2.2 and [2]), in the continuation of our study we will use only the central inversion $()^{I_C}$ in (3.1).

4. Convergence analysis

In this section we shall investigate the dependence of the convergence order of the method (3.1) if the absolute value of the round-off δ_i by which the polynomial is evaluated in the proximity of a zero is of the order r/ρ^2. We note that a greater δ_i could cause the loss of the inclusion property $\zeta_i \in Z_i$, which makes the interval method pointless. On the other hand, dealing with δ_i significantly smaller than r/ρ^2, we approach the situation without rounding errors (iterative formula (2.3)). For this reason, the limit case $\delta_i = r/\rho^2$ can be regarded as the "worst case." To simplify calculating manipulations, we will introduce the coefficient of proportion $\alpha_i \in (\underline{\alpha}^{(i)}, 1]$ $(0 < \underline{\alpha}^{(i)} < 1)$ and write $\delta_i = \alpha_i r/\rho^2$, without loss of generality. Therefore, $\operatorname{rad}(1/H_i) = \alpha_i r/\rho^2$. From the analysis given in this section there follows that the value of the lower bound $\underline{\alpha}^{(i)}$ of α_i can be decreased arbitrarily. Besides, in our analysis we will assume that $\Delta f_i^{(m)} < \left|f(z_i^{(m)})\right|$.

For simplicity, we shall omit the iteration index always when there is no possibility of misunderstanding. Also, let

$$\varepsilon_i = z_i - \zeta_i, \quad \epsilon = \max_{i=1,\dots,n} |\varepsilon_i| \quad v_{ij} = z_i - z_j + h_j \quad a_i = \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{1}{z_i - \zeta_j}.$$

Before the convergence analysis, we will prove some necessary lemmas. In the sequel we will always assume that $n \geq 3$.

Lemma 4.1 *Let* $(Z_1, \dots, Z_n) =: \left(Z_1^{(0)}, \dots, Z_n^{(0)}\right)$ *be initial disks such that* $\zeta_i \in Z_i$, $i \in \mathcal{I}_n$, *and let* $\Delta f_i < |f(z_i)|$, $\delta_i \leq r/\rho^2$ *for all* $i \in \mathcal{I}_n$. *If (2.4) is satisfied, then the following implication holds for every* $i \in \mathcal{I}_n$

$$\zeta_i \in Z_i \Rightarrow \zeta_i \in Z_i - H_i.$$

Proof. Since $\zeta_i \in Z_i$, we conclude that $|\varepsilon_i| \leq r_i \leq r$ and

$$|a_i| \leq \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{1}{|z_i - z_j| - |z_j - \zeta_j|} \leq \frac{n-1}{\rho - r} = \frac{n-1}{r(\rho/r - 1)} \leq \frac{n-1}{r(4n-1)} < \frac{1}{4r},$$

which implies

$$|a_i \varepsilon_i| < \frac{1}{4}. \tag{4.1}$$

Besides, since

$$\frac{1}{h_i} = \frac{f'(z_i)}{f(z_i)} = \sum_{j=1}^{n} \frac{1}{z_i - \zeta_j} = \frac{1}{\varepsilon_i} + a_i,$$

by (4.1) we obtain

$$\frac{1}{|h_i|} = \frac{|1+\varepsilon_i a_i|}{|\varepsilon_i|} \geq \frac{1-|\varepsilon_i a_i|}{|\varepsilon_i|} \geq \frac{3}{4|\varepsilon_i|} \geq \frac{3}{4r}. \tag{4.2}$$

Using (2.4) we estimate

$$\frac{r}{\rho^2} \leq \frac{1}{144r} \leq \frac{1}{144r_i} \leq \frac{1}{144|\varepsilon_i|}. \tag{4.3}$$

From (4.2) and (4.3) it follows

$$\frac{1}{|h_i|} > \alpha_i \frac{r}{\rho^2}.$$

According to this we find

$$\frac{1}{H_i} = \left\{\frac{1}{h_i}; \delta_i\right\} = \left\{\frac{1}{h_i}; \frac{\alpha_i r}{\rho^2}\right\} \not\ni 0,$$

which means that the last inversion is well defined.

Applying the centered inversion, we obtain

$$H_i = \left\{\frac{1}{h_i}; \frac{\alpha_i r}{\rho^2}\right\}^{I_C} = \left\{h_i; \frac{\alpha_i |h_i|^2 r/\rho^2}{1-\alpha_i r|h_i|/\rho^2}\right\}. \tag{4.4}$$

We have to prove the implication

$$\zeta_i \in Z_i \;\Rightarrow\; \zeta_i \in Z_i - H_i,$$

that is, according to (1.2),

$$|z_i - \zeta_i| \leq r_i \;\Rightarrow\; |z_i - \operatorname{mid} H_i - \zeta_i| < r_i - \operatorname{rad} H_i. \tag{4.5}$$

Using (4.1) we find

$$\begin{aligned} |z_i - \operatorname{mid} H_i - \zeta_i| = |\varepsilon_i - h_i| &= \left|\varepsilon_i - \frac{\varepsilon_i}{1+\varepsilon_i a_i}\right| \leq \frac{|\varepsilon_i|^2 |a_i|}{1-|\varepsilon_i a_i|} \\ &\leq \frac{4}{3}|\varepsilon_i|^2 \cdot |a_i| < \frac{1}{3}|\varepsilon_i| \leq \frac{1}{3} r_i. \end{aligned} \tag{4.6}$$

From (4.2) we have $|h_i| \leq \frac{4}{3}r$ and by (2.4) we find

$$x_i := \frac{\alpha_i |h_i| r}{\rho^2} \leq \frac{\frac{4}{3}r^2\alpha_i}{\rho^2} \leq \frac{4\alpha_i}{3}\left(\frac{1}{4n}\right)^2 \leq \frac{1}{108}.$$

According to the last inequality and (4.4) we obtain

$$\operatorname{rad} H_i = \frac{|h_i| x_i}{1 - x_i} < \frac{\frac{4}{3} r_i \cdot \frac{1}{108}}{1 - \frac{1}{108}} < \frac{r_i}{80}. \tag{4.7}$$

Combining (4.6) and (4.7) we conclude that the inequality (4.5) holds, whence $\zeta_i \in Z_i - H_i$.

Therefore, if the zero ζ_i belongs to the disk Z_i, under the condition (2.4) this zero will also belong to the "shifted disk" $Z_i - H_i$. This property is of the crucial importance in preserving the inclusion property of the iterative formula (3.1). □

Lemma 4.2 *Let* $(Z_1, \ldots, Z_n) =: \big(Z_1^{(0)}, \ldots, Z_n^{(0)}\big)$ *be initial disks such that* $\zeta_i \in Z_i$, $i \in \mathcal{I}_n$ *and let* $\Delta f_i < |f(z_i)|$, $\delta_i \le r/\rho^2$ *for all* $i \in \mathcal{I}_n$. *If (2.4) is satisfied, there holds*

(i) $\zeta_i \in \hat{Z}_i$;

(ii) $\hat{r} < \dfrac{7}{18} r$;

(iii) $\hat{\rho} > \rho - \dfrac{25}{9} r$;

(iv) $\dfrac{\hat{r}}{\hat{\rho}} < \dfrac{1}{4n}$;

(v) $\hat{Z}_1, \ldots, \hat{Z}_n$ *are pairwise disjoint.*

Proof. Of (i): Since $\zeta_i \in Z_i - H_i$ for any $i \in \mathcal{I}_n$ (see Lemma 4.1) by the inclusion property from (2.1) we obtain $\zeta_i \in \hat{Z}_i$.

Of (ii): According to (4.4) we obtain

$$z_i - Z_j + H_j \subset \Big\{ v_{ij}; r + 2\frac{r^3}{\rho^2} \Big\} \subset \Big\{ v_{ij}; 2r \Big\}.$$

Using (2.4) we find

$$|v_{ij}| \ge |z_i - z_j| - |h_j| \ge \rho - \frac{4}{3}|\varepsilon_i| \ge \rho - \frac{4}{3} r \ge 4nr - \frac{4}{3} r > 10r. \tag{4.8}$$

Since $|v_{ij}| > 10r > 2r$, we conclude that $\{v_{ij}; 2r\} \not\ni 0$, i.e. the inversions INV_2 in (3.1) exist.

Since

$$\operatorname{rad}\Big(\frac{1}{z_i - Z_j + H_j}\Big) < \frac{2r}{\Big(\rho - \dfrac{4}{3} r\Big)\Big(\rho - \dfrac{10}{3} r\Big)},$$

by (4.8) and (2.4) we estimate

$$\operatorname{rad}\Big(\frac{1}{H_i}-\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{1}{z_i-Z_j+H_j}\Big)<\frac{r}{\rho^2}+\frac{2(n-1)r}{\Big(\rho-\frac{4}{3}r\Big)\Big(\rho-\frac{10}{3}r\Big)}<4(n-1)\frac{r}{\rho^2}$$

and

$$\Big|\operatorname{mid}\Big(\frac{1}{H_i}-\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{1}{z_i-Z_j+H_j}\Big)\Big|>\frac{1}{|h_i|}-\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{1}{|v_{ij}|}\geq\frac{3}{4|\varepsilon_i|}-\frac{n-1}{\rho-\frac{4}{3}|\varepsilon_i|}>\frac{2}{5r}.$$

Using the last two inequalities we conclude that

$$\frac{2}{5r}-4(n-1)\frac{r}{\rho^2}>\frac{3}{10r}>0,$$

which furnishes

$$\Big|\operatorname{mid}\Big(\frac{1}{H_i}-\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{1}{z_i-Z_j+H_j}\Big)\Big|>\operatorname{rad}\Big(\frac{1}{H_i}-\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{1}{z_i-Z_j+H_j}\Big).$$

According to the last inequality we infer that INV_1 exists, thus the method (3.1) is feasible.

We further estimate

$$\operatorname{rad}\Bigg(\mathrm{INV}_1\Big(\frac{1}{H_i}-\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{1}{z_i-Z_j+H_j}\Big)\Bigg)<\frac{4(n-1)\frac{r}{\rho^2}}{\frac{2}{5|\varepsilon_i|}\Big(\frac{2}{5|\varepsilon_i|}-\frac{1}{32|\varepsilon_i|}\Big)}$$
$$<28(n-1)\frac{r|\varepsilon_i|^2}{\rho^2}\leq 28(n-1)\frac{r^3}{\rho^2} \tag{4.9}$$

and

$$\hat{r}<\frac{7}{18}r. \tag{4.10}$$

Of (iii): Using (4.10) and

$$|\hat{z}_i-z_i|\leq|\hat{z}_i-\zeta_i|+|z_i-\zeta_i|\leq\hat{r}_i+r_i<\frac{25}{18}r,$$

we conclude that

$$\hat{\rho}=|\hat{z}_i-\hat{z}_j|\geq|z_i-z_j|-|z_i-\hat{z}_i|-|z_j-\hat{z}_j|>\rho-\frac{25}{9}r. \tag{4.11}$$

Of (iv): According to (4.10) and (4.11) we find

$$\frac{\hat{r}}{\hat{\rho}} < \frac{\frac{7}{18}r}{\rho\Big(1-\frac{25}{9}\frac{r}{\rho}\Big)} < \frac{3}{5}\frac{r}{\rho} \leq \frac{3}{5}\cdot\frac{1}{4n} < \frac{1}{4n}.$$

Of (v): The inequality $\rho/r \geq 4n$ implies that $\hat{Z}_1, \ldots, \hat{Z}_n$ are pairwise disjoint which follows from (4.11), (1.1) and

$$\hat{\rho} = |\hat{z}_i - \hat{z}_j| > 4nr - \frac{25}{9}r > 9r > \hat{r}_i + \hat{r}_j. \quad \Box$$

Theorem 4.1 *Let* $(Z_1, \ldots, Z_n) =: \big(Z_1^{(0)}, \ldots, Z_n^{(0)}\big)$ *be initial disks such that* $\zeta_i \in Z_i$, $i \in \mathcal{I}_n$ *and let* $\Delta f_i < |f(z_i)|$, $\delta_i \leq r/\rho^2$ *for all* $i \in \mathcal{I}_n$. *If* $\rho^{(0)} \geq 4nr^{(0)}$ *holds then the method* (3.1) *is feasible, that is, it defines the sequences of disks* $\big(Z_i^{(m)} \,|\, i \in \mathcal{I}_n\big)_{m=0,1,\ldots}$, $\zeta_i \in Z_i^{(m)}$ *for any* $i \in \mathcal{I}_n$ *and* $m = 0, 1, \ldots,$ *and the sequences of radii* $\big(\mathrm{rad}\,\big(Z_i^{(m)}\big)\big)_{m=0,1,\ldots}$ *tend towards zero.*

Proof. From Lemma 4.2 we conclude by induction on $m = 0, 1, \ldots$ that the inclusion method (3.1) defines disks $Z_i^{(m)}$ including ζ_i for any $m = 0, 1, \ldots,$ $i \in \mathcal{I}_n$ such that

$$\frac{\max\limits_{i=1,\ldots,n} \mathrm{rad}\,\big(Z_i^{(m)}\big)}{\min\limits_{i,j=1,\ldots,n,\, i\neq j}\big|\mathrm{mid}\,\big(Z_i^{(m)}\big) - \mathrm{mid}\,\big(Z_i^{(m)}\big)\big|} \leq \frac{1}{4n}.$$

Therefore the conclusions of Lemma 4.2 hold in any iteration step. In particular, (ii) shows that the inclusion method (3.1) converges. $\Box$

Lemma 4.3 *If* $\Delta f_i < |f(z_i)|$ *and* $\delta_i \leq r/\rho^2$ *for all* $i \in \mathcal{I}_n$, *then the following is valid for the inclusion method* (3.1)*:*

(i) $r^{(m+1)} = O\big(r^{(m)}\big(\epsilon^{(m)}\big)^2\big)$;

(ii) $\epsilon^{(m+1)} = O\big(\big(\epsilon^{(m)}\big)^4\big)$.

Proof. Of (i): Immediately follows from (4.9).

Of (ii): Since

$$\frac{1}{h_i} = \sum_{j=1}^{n} \frac{1}{z_i - \zeta_j},$$

we find

$$\frac{1}{h_i} - \sum_{\substack{j=1\\j\neq i}}^{n} \frac{1}{z_i - z_j + h_j} = \frac{1}{z_i - \zeta_i} + \sum_{\substack{j=1\\j\neq i}}^{n} \Big(\frac{1}{z_i - \zeta_j} - \frac{1}{z_i - z_j + h_j} \Big)$$
$$= \frac{1}{\varepsilon_i} - \sum_{\substack{j=1\\j\neq i}}^{n} \frac{z_j - h_j - \zeta_j}{(z_i - \zeta_j)(z_i - z_j + h_j)}.$$

From (4.6) we see that $|z_i - \zeta_i - h_i| = O(\varepsilon_i^2)$ and from (3.1) we obtain for all $i \in \mathcal{I}_n$

$$\varepsilon_i^{(m+1)} = \varepsilon_i^{(m)} - \frac{1}{\dfrac{1}{\varepsilon_i^{(m)}} + O\big(\big(\varepsilon_i^{(m)}\big)^2\big)} = \frac{O\big(\big(\varepsilon_i^{(m)}\big)^4\big)}{1 + O\big(\big(\varepsilon_i^{(m)}\big)^3\big)}.$$

Since the denominator is bounded and tends to 1 when $\varepsilon_i^{(m)} \to 0$, we conclude that

$$\epsilon^{(m+1)} = O\big(\big(\epsilon^{(m)}\big)^4\big). \qquad \square$$

In Lemma 4.3 we have the case of two mutually dependent sequences. In order to determine their order of convergence, we will use the following special case of Theorem 4 presented in [5]:

Theorem 4.2 *Given the error-recursion*

$$u_i^{(m+1)} \leq \gamma_i \prod_{j=1}^{k} \big(u_j^{(m)}\big)^{t_{ij}}, \quad (i \in \mathcal{I}_k;\ m \geq 0),$$

where $t_{ij} \geq 0$, $\gamma_i > 0$, $i, j \in \mathcal{I}_k$. Denote the matrix of exponents with T_k, that is $T_k = [t_{ij}]_{k\times k}$. If the nonnegative matrix T_k has the spectral radius $\rho(T_k) > 1$ and a corresponding eigenvector $x_\rho > 0$, then the R-order of all sequences $\big(u_i^{(m)}\big)$, $i \in \mathcal{I}_k$, is bounded below by the spectral radius $\rho(T_k)$.

The matrix $T_k = [t_{ij}]$ will be called the R-matrix.

Let $O_R(IM)$ denote the R-order of convergence of an iterative method IM. Now we can give the convergence theorem for the interval method (3.1).

Theorem 4.3 *Let $O_R(3.1)$ denote the R-order of the iterative interval methods (3.1). If $\Delta f_i < |f(z_i)|$ and $\delta_i \leq r/\rho^2$ for all $i \in \mathcal{I}_n$, then $O_R(3.1) = 4$.*

Proof. Let us assume that $\epsilon^{(0)} \approx r^{(0)}$ and $\epsilon^{(m)} \leq r^{(m)}$ for $m \geq 1$. From Lemma 4.3 we have

$$\epsilon^{(m+1)} \sim \left(\epsilon^{(m)}\right)^4, \qquad r^{(m+1)} \sim \left(\epsilon^{(m)}\right)^2 r^{(m)}, \tag{4.12}$$

where $a \sim b$ $(a, b > 0)$ means $a = O(b)$. From the relations (4.12) we form the R-matrix

$$T_2 = \begin{bmatrix} 4 & 0 \\ 2 & 1 \end{bmatrix}$$

with the spectral radius $\rho(T_2) = 4$ and the corresponding eigenvector $x_\rho = (3, 2) > 0$. According to Theorem 4.2, we obtain

$$O_R(3.1) \geq \rho(T_2) = 4. \qquad \square$$

From the Theorem 4.3 we can conclude that the method (3.1) preserves the convergence rate (equal to four) even in the case when the value of the incorporating roundoff is of the same order of magnitude as the disk size. This means that the improved Gargantini interval method (2.3) possesses very good numerical stability in the presence of rounding errors.

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