# 104. The 0-minimal Ideal of the Global of a Combinatorial Completely 0-simple Semigroup

By Takayuki TAMURA and Kenya YAMAOKA
Department of Mathematics, University of California, Davis

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**1. Introduction.** Let $S=\mathcal{M}^0(I, M, \{1\}; P)$ be a completely 0-simple combinatorial semigroup where $\{1\}$ is the trivial group and $P=(p_{ji})$ is an $M\times I$-matrix over $\{0,1\}$ (see [1], [2]). Let $\mathcal{P}(S)$ be the global i.e. the power semigroup of $S$. As proved in [3], $\mathcal{P}(S)$ has a unique 0-minimal ideal $C(S)$, and $C(S)\cong\mathcal{M}^0(\mathcal{P}(I), \mathcal{P}(M), \{1\}; \tilde{P})$ where $\tilde{P}=(\tilde{p}_{BA})$, $B\in\mathcal{P}(M)$, $A\in\mathcal{P}(I)$, $\tilde{p}_{BA}=1$ if $p_{ji}\neq 0$ for some $j\in B$, $i\in A$; $\tilde{p}_{BA}=0$ otherwise. According to [4], $\mathcal{P}(S)$ has a unique maximal regular zero-free $\mathcal{J}$-class $\mathcal{T}(S)$ and $\mathcal{T}^0(S)=\mathcal{T}(S)\cup\{0\}$ is completely 0-simple and $\mathcal{T}^0(S)\cong\mathcal{M}^0(\bar{Q}(I), \bar{Q}(M), \{1\}; \bar{P})$ where $\bar{Q}(I)$ is the set of all elements $A$ of $\mathcal{P}(I)$ satisfying; there is $j\in M$ such that $p_{ji}\neq 0$ for all $i\in A$; $\bar{Q}(M)$ is dually defined, $\bar{P}=(\bar{p}_{BA})$ where $\bar{p}_{BA}=1$ if $p_{ji}\neq 0$ for all $j\in B$, $i\in A$; $\bar{p}_{BA}=0$ otherwise. Let $S_1$ and $S_2$ be completely 0-simple semigroups, $S_1=\mathcal{M}^0(I_1, M_1, \{1\}; P_1)$, $S_2=\mathcal{M}^0(I_2, M_2, \{1\}; P_2)$. The first author has obtained the following:

**Lemma 1.** [4] *If* $\mathcal{T}^0(S_1)\cong\mathcal{T}^0(S_2)$, *then* $S_1\cong S_2$.

In this paper we will use this result to prove that $C(S_1)\cong C(S_2)$ implies $S_1\cong S_2$.

**2. Definitions.** Let $X=(x_{ji})$ be an $M\times I$-matrix (i.e. $j\in M, i\in I$) over $\{0,1\}$. Given $X$, define $X'=(x'_{ji})$ by $x'_{ji}=1$ if $x_{ji}=0$; $x'_{ji}=0$ if $x_{ji}=1$. Let $X=(x_{ji})$ and $Y=(y_{ji})$ be $M_1\times I_1$- and $M_2\times I_2$-matrices over $\{0,1\}$ respectively. We say $X$ is *equivalent* to $Y$, denoted by $X\sim Y$, if there are bijections $\sigma$: $M_1\to M_2$ and $\tau$: $I_1\to I_2$ such that $x_{ji}=y_{\sigma(j),\tau(i)}$ for all $j\in M$, $i\in I$. If every row and every column of $X$ contains 1, then $X$ is called a *sandwich*. Both $X$ and $X'$ are sandwiches if and only if every row and every column of $X$ contains both 0 and 1. Adjoin a new letter 1 to $M$ and $I$, namely, let $M^1=M\cup\{1\}$, $1\notin M$; $I^1=I\cup\{1\}$, $1\notin I$. Given an $M\times I$-matrix $X=(x_{ji})$ over $\{0,1\}$, we define an $M^1\times I^1$-matrix $X^1=(x^1_{ji})$ over $\{0,1\}$ as follows: For all $j\in M$, $i\in I$,

$$x^1_{ji}=x_{ji},\quad x^1_{j1}=x^1_{1i}=0,\quad x^1_{11}=1.$$

Then $(X^1)'$ is always a sandwich whether $X$ is so or not. Let $X=(x_{ji})$, $j\in M$, $i\in I$, and for each $B\in\mathcal{P}(M)$, $A\in\mathcal{P}(I)$, define a set $X_{BA}=\{x_{ji}: j\in B, i\in A\}$. Given an $M\times I$-matrix $X=(x_{ji})$ over $\{0,1\}$, we define a $\mathcal{P}(M)\times\mathcal{P}(I)$-matrices $\tilde{X}$ and $\bar{X}$ as follows:

$$\tilde{X}=(\tilde{x}_{BA}),\quad \tilde{x}_{BA}=\begin{cases}1 & \text{if } 1\in X_{BA}\\ 0 & \text{if } X_{BA}=\{0\},\end{cases}$$

$$\bar{X}=(\bar{x}_{BA}),\quad \bar{x}_{BA}=\begin{cases}1 & \text{if } X_{BA}=\{1\}\\ 0 & \text{if } 0\in X_{BA}.\end{cases}$$

**3. Preliminaries.** The following lemma is important for this paper.

**Lemma 2.** *Assume $X$ and $Y$ are $M\times I$-matrices over $\{0,1\}$. Then the following hold.*

(1) $(X')'=X$.

(2) *If $X\sim Y$, then $X'\sim Y'$.*

(3) $(\overline{X})'=\tilde{X}'$.

(4) *$\tilde{X}\sim\tilde{Y}$ implies $\tilde{X}^1\sim\tilde{Y}^1$.*

(5) *$X^1\sim Y^1$ implies $X\sim Y$.*

*Proof.* (1) and (2) are obvious.

(3) Immediate from the definition of $X'$, $\overline{X}$ and $\tilde{X}$.

(4) A $\mathcal{P}(M_1^1)\times\mathcal{P}(I_1^1)$-matrix $\tilde{X}^1$ can be obtained from a $\mathcal{P}(M_1)\times\mathcal{P}(I_1)$-matrix $\tilde{X}=(\tilde{x}_{BA})$ as follows:
Let $A^1=A\cup\{1\}$, $B^1=B\cup\{1\}$ where $A\in\mathcal{P}(I_1)$, $B\in\mathcal{P}(M_1)$. Entries of $\tilde{X}^1$ are defined by

$$\tilde{x}^1_{BA}=\tilde{x}^1_{BA}1=\tilde{x}^1_B1_A=\tilde{x}_{BA},\qquad \tilde{x}^1_B1_A1=\tilde{x}^1_{11}=1.$$

Similarly $\tilde{Y}^1$ is obtained from a $\mathcal{P}(M_2)\times\mathcal{P}(I_2)$-matrix $\tilde{Y}$. If $\tilde{X}\sim\tilde{Y}$ under bijections $\sigma:\mathcal{P}(M_1)\to\mathcal{P}(M_2)$ and $\tau:\mathcal{P}(I_1)\to\mathcal{P}(I_2)$, then we define bijections $\sigma^1:\mathcal{P}(M_1^1)\to\mathcal{P}(M_2^1)$ and $\tau^1:\mathcal{P}(I_1^1)\to\mathcal{P}(I_2^1)$ as follows: $\sigma^1B=\sigma B$ if $B\in\mathcal{P}(M_1)$; $\sigma^1B^1=(\sigma B)^1$ if $B\in\mathcal{P}(M_1)$; $\sigma^11=1$; $\tau^1A=\tau A$ if $A\in\mathcal{P}(I_1)$; $\tau^1A^1=(\tau A)^1$ if $A\in\mathcal{P}(I_1)$; $\tau^11=1$.

It follows that $\tilde{X}^1\sim\tilde{Y}^1$ under $\sigma^1$ and $\tau^1$.

(5) Assume $X^1=(x^1_{ji})$, $Y^1=(y^1_{ji})$ and $X^1\sim Y^1$ under bijections $\sigma^1: M_1^1\to M_2^1$ and $\tau^1: I_1^1\to I_2^1$. Let $D_1\times C_1$ be the set of all $(j,i)$ satisfying the following:

$$x^1_{ji}=1,\ x^1_{jk}=0 \text{ for all } k\in I_1,\ k\neq i;\ x^1_{li}=0 \text{ for all } l\in M,\ l\neq j.$$

Similarly $D_2\times C_2$ is defined from $Y^1$. Then $D_1$ is mapped to $D_2$ under $\sigma^1$, and $C_1$ is mapped to $C_2$ under $\tau^1$. However the submatrices $D_1\times C_1$-matrix $X_1=(x^1_{ji})$ and $D_2\times C_2$-matrix $Y_1=(y^1_{ji})$ are invertible matrices. For any bijection $\sigma_2: D_1\to D_2$ with $\sigma_2(1)=1$, there is a bijection $\tau_2: C_1\to C_2$ with $\tau_2(1)=1$ such that $X_1\sim Y_1$ under $\sigma_2$ and $\tau_2$. Now define $\bar{\sigma}_2: M_1^1\to M_2^1$ and $\bar{\tau}_2: I_1^1\to I_2^1$, respectively such that $\bar{\sigma}_2|D_1=\sigma_2$, $\bar{\sigma}_2|(M_1^1\backslash D_1)=\sigma^1|(M_1^1\backslash D_1)$ and $\bar{\tau}_2|C_2=\tau_2$, $\bar{\tau}_2|(I_1^1\backslash C_1)=\tau^1|(I_1^1\backslash C_1)$. Then $X^1\sim Y^1$ under $\bar{\sigma}_2$ and $\bar{\tau}_2$, but since $\bar{\sigma}_2(1)=1$ and $\bar{\tau}_2(1)=1$, we have $X\sim Y$ under $\bar{\sigma}_2|M_1$ and $\bar{\tau}_2|I_1$.

**4. Main theorem.** In the proof of the following, (1), (2), (3), (4), (5) denote (1), (2), (3), (4), (5) in Lemma 2 respectively.

**Theorem.** *If $S_1$ and $S_2$ are combinatorial, then*

$$\tilde{P}_1\sim\tilde{P}_2 \text{ implies } P_1\sim P_2$$

*in other words, $\mathcal{C}(S_1)\cong\mathcal{C}(S_2)$ implies $S_1\cong S_2$.*

*Proof.*

$$\tilde{P}_1\sim\tilde{P}_2 \underset{\text{by (4)}}{\Longrightarrow} \tilde{P}_1^1\sim\tilde{P}_2^1 \underset{\text{by (1), (3)}}{\Longrightarrow} \overline{((P_1^1)')'}\sim\overline{((P_2^1)')'}$$

$$\underset{\text{by (1), (2)}}{\Longrightarrow} \overline{(P_1^1)'}\sim\overline{(P_2^1)'} \underset{\text{by Lemma 1}}{\Longrightarrow} (P_1^1)'\sim(P_2^1)'$$

$$\underset{\text{by (1), (2)}}{\Longrightarrow} P_1^1\sim P_2^1 \underset{\text{by (5)}}{\Longrightarrow} P_1\sim P_2.$$

Speaking in terms of semigroups, the theorem says $\mathcal{C}(S_1)\cong\mathcal{C}(S_2)$ implies $S_1\cong S_2$. On the other hand, if $\mathcal{P}(S_1)\cong\mathcal{P}(S_2)$ then $\mathcal{C}(S_1)\cong\mathcal{C}(S_2)$ ([3]). Moreover the first author has obtained in [4] that if $\mathcal{P}(S_1)\cong\mathcal{P}(S_2)$ then $\mathcal{T}^0(S_1)\cong\mathcal{T}^0(S_2)$. Consequently

**Corollary.** *If $S_1$ and $S_2$ are combinatorial completely 0-simple semigroups, then the following are equivalent*

$$\mathcal{P}(S_1)\cong\mathcal{P}(S_2),\quad \mathcal{T}_1^0(S_1)\cong\mathcal{T}_2^0(S_2),\quad \mathcal{C}(S_1)\cong\mathcal{C}(S_2),\quad S_1\cong S_2.$$

5. **Remark.** If $S_1$ and $S_2$ are finite, the theorem can be directly proved by means of semilattice homomorphisms without using Lemma 1. This has been obtained by the second author and the proof will be reported in [5].

## References

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