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The Odd-Distance Plane Graph

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Abstract The vertices of the odd-distance graph are the points of the plane $\mathbb{R}^2$. Two points are connected by an edge if their Euclidean distance is an odd integer. We prove that the chromatic number of this graph is at least five. We also prove that the odd-distance graph in $\mathbb{R}^2$ is countably choosable, while such a graph in $\mathbb{R}^3$ is not.

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1 Introduction

In 1950 Edward Nelson, a student at the University of Chicago, formulated the *alternative four-color problem*: What is the minimum number of colors for coloring the points of the plane so that points at unit distance apart receive distinct colors. Nelson himself showed that at least four colors are needed. Soon after learning about the problem from Ed Nelson, John Isbell proved that the plane can be colored with seven colors. Fifty seven years later, and numerous efforts by many researchers, these are still the best known bounds. Some authors call it a disappointment, a disaster, others call it frustrating. We would like to call it a great opportunity as evidenced by its high popularity and interesting history. After all, at least five mathematicians were credited with it. Without a doubt, its popularity can be traced to its simplicity, its elusive solution and Paul Erdős who repeatedly publicized it in his presentations and papers. It is commonly referred to as the *unit-distance problem*. It produced many variations, numerous papers, dissertations and head-aches, but no improvements on the original bounds. Its history was traced by A. Soifer [10]. A nice account of the problem and its derivatives can be found in the book *Research Problems in Discrete Geometry* [2].

In this paper we present yet another variation of the unit-distance graph: the *odd-distance graph*. It is a simple, natural generalization of the unit-distance graph. We wish to color the points of the plane so that points at odd distance apart receive distinct colors. Since four points in the plane cannot have pairwise odd integral distances this graph does not contain K_4 as a subgraph [6, 9]. It is thus natural to ask whether the plane can be colored with a finite number of colors such that points at odd integral distance receive distinct colors. In 1994, M. Rosenfeld asked Paul Erdős this question during the *25th South-Eastern International Conference*. Erdős [4] presented this problem in his talk. Erdős also asked to determine the maximum number of odd distances among n points in $\mathbb{R}^2$. It is curious to note that an identical question for the unit distance was investigated by Erdős already in 1946 [3] but not in the context of graphs. Since the odd-distance graph spanned by n points does not contain K_4 as a subgraph, this number is bounded by Turán's function. L. Piepmeyer [8] showed that the complete tri-partite graph $K_{m,m,m}$ can be embedded in the plane (actually on a circle) in such a way that two vertices connected by an edge will be at odd distance apart. Note that this is a faithful embedding (no other vertices are at odd distance apart). As an aside, answering a question of Erdős, Maehara, Ota and Tokushige [7] proved that every finite graph admits a faithful representation in $\mathbb{R}^2$ in which only vertices connected by an edge will be at an integral distance apart.

It is interesting to compare the odd-distance graph and the unit-distance graph. Clearly, the unit-distance graph is a subgraph of the odd-distance graph, its diameter is 2 while the diameter of the unit-distance graph is not bounded, any two vertices have countably many common neighbors while in the unit-distance graph any two points have at most two common neighbors. As a result of these differences, some problems that are easy for one graph are difficult for the other and vice versa. For instance, the upper bound for the chromatic number of the unit-distance graph is rather easy to obtain while the same bound for the odd-distance graph seems more difficult. Since any two distinct circles have at most two points in common it is easy to find many graphs which are not subgraphs of the unit-distance graph. For instance, $K_{2,3}$ is such a graph. On the other hand, the only forbidden subgraph of the odd-distance graph known to us is K_4. Finding the maximum number of edges among all subgraphs of order n is unknown for the unit-distance graph but follows a very predictive upper bound for the odd-distance graph as proved in [8]. It is interesting to note that P. Erdős formulated this question for the unit-distance graph 4 years before Nelson, but not in the context of graphs, see [3].

In this paper we use the unit triangular grid to construct a graph of order 21 with chromatic number five which is a subgraph of the odd-distance graph. We could not find any finite upper bound other than the trivial bound $\aleph_0$ (we suspect that this graph cannot be colored with a finite number of colors). We also show that the rational odd-distance graph is 2-colorable. Furthermore, we study the choosability of the odd-distance graph. Since the unit-distance graph has finite regular subgraphs of arbitrarily large degree (such as the d-dimensional cube), it follows from a result by N. Alon [1] that the unit-distance graph and hence the odd-distance graph are not finitely choosable. By assuming the axiom of choice we prove that the odd-distance graph in $\mathbb{R}^2$ is $\aleph_0$-choosable. In contrast, we prove that the odd-distance graph in $\mathbb{R}^3$ is not $\aleph_0$-choosable.

2 The Odd-Distance Graph

Let $D \subset \mathbb{N}$ be a set of numbers and let V be a metric space with metric distance d. We denote by $G^D(V)$ the graph whose vertices are the points in V and whose edges are between points in V whose distance is in D. Odd will denote the set of all odd natural numbers.

2.1 The Triangular Grid

In this section we use the triangular grid to prove that $\chi(G^{\mathrm{Odd}}(\mathbb{R}^2)) \geq 5$.

Definition 1 The triangular grid is the metric space on the set $T = \{(n + \frac{m}{2}, \frac{m}{2}\sqrt{3}) : n, m \in \mathbb{Z}\} \subset \mathbb{R}^2$ with the Euclidean distance.

In $G^{\mathrm{Odd}}(T)$, aside from points at odd distance along the grid lines, there are other points on the grid at odd distance. In particular, consider the points B and D in Fig. 1. The triangle ΔBCD has sides 3 and 5 along the grid, $\angle BCD = 120°$. Hence,

$$\begin{aligned} d(B,D)^2 &= d(B,C)^2 + d(C,D)^2 - 2\cos(120°)d(B,C)d(C,D) \\ &= 5^2 + 3^2 + 3 \cdot 5 = 49 \end{aligned}$$

or $d(B,D) = 7$.

Lemma 1 *Let f be a proper 4-coloring of $G^{\mathrm{Odd}}(T)$. For any two points X and Y on the grid line of T at distance 8, we have $f(X) = f(Y)$.*

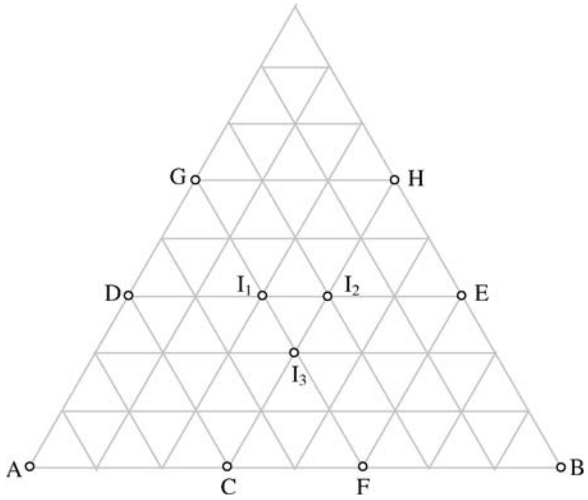

Fig. 1 A part of the triangular grid in which each *side* of each *small equilateral triangle* is 1 and hence, the distance from A to B is 8

Proof Let c_1, c_2, c_3, c_4 denote the four colors. Let A and B be two points at distance 8 on the same grid line of T such that $f(A) \neq f(B)$, say $f(A) = c_1$ and $f(B) = c_2$, cf. Fig. 1. In the following, we will consider other points whose relative position to A and B is depicted in Fig. 1. Then, since $d(A, C) = 3$ and $d(C, B) = 5$, C must have a different color than c_1 and c_2, say c_3. Since $d(A, D) = d(C, D) = 3$, and $d(B, D) = 7$, $f(D) = c_4$. Similarly, it follows that $f(E) = c_3$, $f(F) = c_4$, $f(G) = c_3$, and $f(H) = c_4$. Since each of I_1, I_2, I_3 is at distance 3 from one point colored c_3 and one colored c_4, we have $f(I_1), f(I_2), f(I_3) \in \{c_1, c_2\}$. This is a contradiction since they form a triangle with side 1. □

Corollary 1 $\chi(G^{\mathrm{Odd}}(\mathbb{R}^2)) \geq 5$.

Proof Take two copies of the points depicted in Fig. 1. Rotate the second copy around the point A so that the image of B in the second copy will be at distance 1 from B. By Lemma 1, this configuration is not 4-colorable. □

The graph used in the proof of the lower bound uses only four odd distances $1, 3, 5, 7$. It is easy to observe that for any finite set of distances D, $\chi(G^D(\mathbb{R}^2))$ is finite. Indeed, for any $d \in D$, it is well-known that $\chi(G^{\{d\}}(\mathbb{R}^2)) \leq 7$, and therefore to color $G^D(\mathbb{R}^2)$ we can use $|D|$-tuples of 7 colors. Hence, $\chi(G^D(\mathbb{R}^2)) \leq 7^{|D|}$.

The following theorem shows that the triangular grid by itself is 4-colorable.

Theorem 1 *Let T be the triangular grid. Then* $\chi^{\mathrm{Odd}}(T) = 4$.

Proof It is easy to show that $\chi^{\mathrm{Odd}}(T) > 3$.

The coloring function f is defined as follows:

$$f\left(n + \frac{m}{2}, \frac{m}{2}\sqrt{3}\right) = \begin{cases} 1, & \text{if } n \text{ and } m \text{ are both even,} \\ 2, & \text{if } n \text{ is odd and } m \text{ is even,} \\ 3, & \text{if } n \text{ is even and } m \text{ is odd,} \\ 4, & \text{if } n \text{ and } m \text{ are both odd.} \end{cases}$$

Note that all the color classes of this coloring are translations of the color class of the color 1. Hence, it is enough to show that any two points colored with color 1, do not have an odd distance.

Let $(n_1 + \frac{m_1}{2}, \frac{m_1}{2}\sqrt{3})$ and $(n_2 + \frac{m_2}{2}, \frac{m_2}{2}\sqrt{3})$ be two points colored with color 1. Then

$$\left\| \left(n_1 + \frac{m_1}{2}, \frac{m_1}{2}\sqrt{3}\right) - \left(n_2 + \frac{m_2}{2}, \frac{m_2}{2}\sqrt{3}\right) \right\|^2$$
$$= \left(n_1 - n_2 + \frac{m_1 - m_2}{2}\right)^2 + \left(\frac{m_1 - m_2}{2}\sqrt{3}\right)^2$$
$$= (n_1 - n_2)^2 + (n_1 - n_2)(m_1 - m_2) + (m_1 - m_2)^2$$

which is an even integer. Therefore, their distance is either even or non-integral. Hence, f is a proper 4-coloring of T and $\chi^{\mathrm{Odd}}(T) = 4$. □

2.2 The Chromatic Number of $G^{\mathrm{Odd}}(\mathbb{Q}^2)$

It is known that the chromatic number of $G^{\{1\}}(\mathbb{Q}^2)$ is two [5, 11]. In Theorem 2, we conclude the same for $G^{\mathrm{Odd}}(\mathbb{Q}^2)$. We begin with two simple observations about Pythagorean triples.

Lemma 2 *Let $x, y, z \in \mathbb{Z}$ be a Pythagorean triple, i.e., $x^2 + y^2 = z^2$. Then the following statements are both true.*

1. *If $z \equiv 0 \pmod 2$ then $x \equiv y \equiv 0 \pmod 2$.*
2. *If $z \equiv 1 \pmod 2$ then one of x and y is even and the other is odd.*

Corollary 2 *Let $x, y, z \in \mathbb{Z}$ be a Pythagorean triple. If 2^k divides z, then 2^k divides both x and y.*

We need the following to prove that there are no odd cycles in $G^{\mathrm{Odd}}(\mathbb{Q}^2)$.

Lemma 3 *Let $X = (x, y) = (2^k m, 2^l n)$ and $U = (u, v) = (2^p r, 2^q s)$ for some $k, l, p, q, m, n, r, s \in \mathbb{Z}$, where m, n, r, s are all odd. If $d(X, U)$ is an odd integer, then $x - u$ and $y - v$ are both integers with different parity modulo 2.*

Proof Assume $d(X, U) = o$ for some odd integer o:

$$\|(x, y) - (u, v)\|^2 = \left(2^k m - 2^p r\right)^2 + \left(2^l n - 2^q s\right)^2 = o^2. \tag{1}$$

It is enough to show that $x - u \in \mathbb{Z}$, since then, by symmetry, $y - v \in \mathbb{Z}$, and the rest of the lemma follows by Lemma 2.

Case 1 Assume $k, p \geq 0$ (similarly for $l, q \geq 0$). Then $x - u = 2^k m - 2^p r$ is an integer.

Case 2 Assume $p < 0 \leq k$ (similarly for $k < 0 \leq p$, $q < 0 \leq l$ and $l < 0 \leq q$). Then

$$\left(2^{k-p} m - r\right)^2 + \left(2^{l-p} n - 2^{q-p} s\right)^2 = \left(2^{-p} o\right)^2.$$

Since both $2^{-p}o$ and $2^{k-p}m-r$ are integers, $2^{l-p}n-2^{q-p}s$ is also an integer. Therefore, by Corollary 2, 2^{-p} divides $2^{k-p}m-r$. But since $k-p \geq -p$, 2^{-p} divides $2^{k-p}m$. Hence 2^{-p} divides r. But this is impossible since r is odd.

Case 3 Assume $p \leq k < 0$ and $q \leq l < 0$. Let $a=-k$, $b=-p$, $c=-l$ and $d=-q$. Then $0 < a \leq b$ and $0 < c \leq d$ and (1) becomes

$$\left(\frac{m}{2^a}-\frac{r}{2^b}\right)^2+\left(\frac{n}{2^c}-\frac{s}{2^d}\right)^2=o^2.$$

Assume without loss of generality that $b \geq d$. Then

$$\left(2^{b-a}m-r\right)^2+\left(2^{b-c}n-2^{b-d}s\right)^2=\left(2^b o\right)^2.$$

Since all the terms in the above equality are integers and $2^b|(2^b o)$, by Corollary 2, $2^b|(2^{b-a}m-r)$. But since m and r are both odd, this is possible only if $b=a$, and in this case $2^b|m-r$, i.e.,

$$x-u=\frac{m}{2^a}-\frac{r}{2^b}=\frac{m-r}{2^b}\in\mathbb{Z}.$$

□

Theorem 2 *The graph $G^{\mathrm{Odd}}(\mathbb{Q}^2)$ is bipartite.*

Proof Assume by contradiction that it contains an odd cycle, say (x_1, y_1), (x_2, y_2), $\ldots, (x_{2t+1}, y_{2t+1})$ for some $t \in \mathbb{N}$. Let $x_i=\frac{2^{k_i}n_i}{m}$ and $y_i=\frac{2^{l_i}p_i}{m}$ for some $k_i, l_i, n_i, p_i \in \mathbb{Z}$ for $1 \leq i \leq 2t+1$ and $m \in \mathbb{Z}$ and n_i, p_i and m are odd. Therefore, $(2^{k_1}n_1, 2^{l_1}p_1)$, $(2^{k_2}n_2, 2^{l_2}p_2), \ldots, (2^{k_{2t+1}}n_{2t+1}, 2^{l_{2t+1}}p_{2t+1})$ is also an odd cycle in $G^{\mathrm{Odd}}(\mathbb{Q}^2)$. By Lemma 3, $(2^{k_i}n_i-2^{k_{i+1}}n_{i+1})$, $(2^{l_i}p_i-2^{l_{i+1}}p_{i+1}) \in \mathbb{Z}$ and they have different parity for all $1 \leq i \leq 2t+1$ where indices are taken mod $(2t+1)$. Thus we have

$$\sum_{i=1}^{2t+1}\left(2^{k_i}n_i-2^{k_{i+1}}n_{i+1}\right)+\left(2^{l_i}p_i-2^{l_{i+1}}p_{i+1}\right)\equiv 1 \bmod 2,$$

since there are $2t+1$ terms on the left hand side and each of them is odd. On the other hand,

$$\begin{aligned}&\sum_{i=1}^{2t+1}\left(2^{k_i}n_i-2^{k_{i+1}}n_{i+1}\right)+\left(2^{l_i}p_i-2^{l_{i+1}}p_{i+1}\right)\\&\quad=\sum_{i=1}^{2t+1}2^{k_i}n_i-\sum_{i=1}^{2t+1}2^{k_{i+1}}n_{i+1}+\sum_{i=1}^{2t+1}2^{l_i}p_i-\sum_{i=1}^{2t+1}2^{l_{i+1}}p_{i+1}=0,\end{aligned}$$

a contradiction. Therefore, $G^{\mathrm{Odd}}(\mathbb{Q}^2)$ is a bipartite graph. □

3 Choosability

A graph G is κ-choosable if for every assignment of an arbitrary set of colors of cardinality κ to each vertex of G it is possible to properly color G in such a way that each vertex receives a color from its assigned set. As noted in the introduction the odd-distance graph contains finite regular subgraphs of arbitrary large minimum degree, and hence it is not finitely choosable.

In this section we first prove that the odd-distance graph $G^{\mathrm{Odd}}(\mathbb{R}^2)$ is $\aleph_0$-choosable, while the odd-distance graph $G^{\mathrm{Odd}}(\mathbb{R}^3)$ is not $\aleph_0$-choosable.

3.1 Choosability of $G^{\mathrm{Odd}}(\mathbb{R}^2)$

Let S be a set. For a cardinal κ, we denote by $S^{\leq\kappa}$ the set of all subsets of S of cardinality at most κ. In particular, $S^{\leq\aleph_0}$ denotes the set of all countable subsets of S. Similarly, $S^{<\aleph_0}$ denotes the set of all finite subsets of S.

Definition 2 Let G be an infinite (not necessarily countable) graph and κ be any cardinal. We say that G has the *κ-property* if there is a function $F: V(G)^{<\aleph_0} \to V(G)^{\leq\kappa}$ such that for any infinite subset $X \subset V(G)$ of cardinality at most κ and for any y which is connected to all vertices in X, there exists a finite subset $X' \subset X$ such that $y \in F(X')$.

We will prove that if an infinite graph G has the κ-property then it is κ-choosable. We then show that the odd-distance graph $G^{\mathrm{Odd}}(\mathbb{R}^2)$ has the $\aleph_0$-property and hence is countably choosable.

Theorem 3 *Let $\kappa \geq \aleph_0$ be a cardinal number. If G has the κ-property then it is κ-choosable.*

Proof Let F be the function from the definition of the κ-property. Assuming the well ordering property of sets, let $\{x_\alpha\}_\alpha$ be a well ordering of the vertices of G where α are all ordinals such that $\alpha < \lambda$ and λ is the smallest ordinal of all ordinal types of cardinality $|V(G)|$. Let L_x be the list of colors of size κ assigned to the vertex x. For each ordinal α such that $\alpha < \lambda$ define the set V_α inductively as follows:

1. $V_{\alpha,0} = \{x_\beta, \beta < \alpha\}$.
2. $V_{\alpha,n+1} = V_{\alpha,n} \bigcup_{Z \in V_{\alpha,n}^{<\aleph_0}} F(Z)$.
3. $V_\alpha = \bigcup_{n=0}^{\infty} V_{\alpha,n}$.

Observation 1 *For any integer $n \in \mathbb{N}$, $|V_{\alpha,n}| \leq |\alpha| + \kappa$.*

Proof The claim can be easily proved by induction.

For the base case $n = 0$, the claim is true, since $|V_{\alpha,0}| = |\{x_\beta,\ \beta < \alpha\}| = |\alpha| \leq |\alpha| + \kappa$.

For the inductive step assume that $|V_{\alpha,n}| \leq |\alpha| + \kappa$. Since $|V_{\alpha,n}^{<\aleph_0}| = |V_{\alpha,n}| \leq |\alpha| + \kappa$, and since for every finite Z, $|F(Z)| \leq \kappa$, $|V_{\alpha,n+1}| \leq |\alpha| + \kappa + (|\alpha| + \kappa)\kappa = |\alpha| + \kappa$. □

Observation 2

1. *The sets $\{V_\alpha\}_{\alpha<\lambda}$ form a nested sequence of subsets of $V(G)$.*

2. $\bigcup_{\alpha<\lambda} V_\alpha = V(G)$.
3. *For every* α *such that* $\alpha < \lambda$, V_α *is closed under* F, *i.e., for every finite set* $X \subset V_\alpha$, $F(X) \subseteq V_\alpha$.

We now use transfinite recursion on α to define a family of colorings ϕ_α of the vertices in V_α such that:

(C1) For every $x \in V_\alpha$, $\phi_\alpha(x) \in L_x$.
(C2) If $\alpha > \beta$ then ϕ_α extends ϕ_β, i.e., ϕ_α agrees with ϕ_β on V_β. This is well defined since $V_\beta \subset V_\alpha$ by Observation 2.

If $\alpha \le \kappa$ then a simple greedy coloring algorithm can be used to assign the valid color for each vertex of V_α. If α is a limit ordinal then define $\phi_\alpha = \bigcup_{\gamma<\alpha} \phi_\gamma$. It is easy to see that both (C1) and (C2) hold for ϕ_α. Finally, assume that $\alpha = \beta + 1$ for some ordinal β. By Observation 2, $V_\alpha = V_\beta \uplus (V_\alpha - V_\beta)$. Let $\phi_\alpha(y) = \phi_\beta(y)$, for all $y \in V_\beta$.

Let G' be the induced subgraph of G on $V_\alpha - V_\beta$. For every $y \in V(G')$, we will construct a list $L'_y \subseteq L_y$ such that $|L'_y| = \kappa$. Note that any list-coloring ϕ of G' from these lists is compatible with ϕ_β, i.e., ϕ and ϕ_β together form a coloring of V_α. We achieve this by removing all colors from the list of $y \in V(G')$ that are assigned to its neighbors in V_β. Let $X_y = \{x \in V_\beta;\ xy \in E(G)\}$. For every $y \in V(G')$, let $L'_y = L_y - \{\phi_\beta(x);\ x \in X_y\}$. It remains to show that $|L'_y| = \kappa$. This is obviously true if $|X_y| < \kappa$. We now show that $|X_y| \ge \kappa$ cannot occur. Since G has the κ-property, there is a finite subset $X'_y \subset X_y$ such that $y \in F(X'_y)$. But $X'_y \subset X_y \subset V_\beta$. Since V_β is closed under F, $F(X'_y) \subseteq V_\beta$. Hence, $y \in V_\beta$ but this contradicts the choice of y.

We now use transfinite induction to construct the desired coloring of G. Let G be an infinite graph. Assume that G has the κ property and $V(G)$ is well ordered then G is κ-*choosable*. The proof proceeds by transfinite induction on λ, the ordinal of the well ordered set $V(G)$. For the base case, as noted before, if $\lambda \le \kappa$ then G is κ-choosable.

G' is a subgraph of G and $V(G')$ is a well ordered subset of $V(G)$. The lists L'_y have cardinality κ. We wish to use the induction hypothesis to prove that G' can be properly colored using colors from the lists L'_y. For that we need to show that G' has the κ-property and the ordinal of $V(G')$ is less than λ.

Let the well ordered set $V(G')$ have ordinal λ'. First, we have $|V(G')| = |V_\alpha - V_\beta| \le |V_\alpha| \le |\alpha| + \kappa$. Since $|\alpha| < |\lambda|$ and $\kappa < |\lambda|$, it follows that $|V(G')| < |\lambda|$, and therefore $\lambda' < \lambda$. Second, define a function F' as follows: for every $S \in V(G')^{<\aleph_0}$, let $F'(S) = F(S) \cap V(G')$. To verify the κ-property for G', consider an infinite subset $X \subset V(G')$ of cardinality at most κ, and a vertex $y \in V(G')$ which is connected to all vertices in X. Since G has the κ-property, there exists a finite set $X' \subset X$ such that $y \in F(X')$. By definition, $F'(X') = F(X') \cap V(G')$, and hence $y \in F'(X')$. This proves the κ-property for G'.

Therefore, by induction hypothesis on G', G' is κ-*choosable*. Let ψ be a coloring of G' from the lists L'_y. We define $\phi_\alpha(y) = \psi(y)$, for all $y \in V_\alpha - V_\beta$, which completes the definition of ϕ_α on V_α. Now, it is trivial to see that both properties (C1) and (C2) hold for ϕ_α.

To complete the proof define the coloring ϕ of G from lists L_x as $\phi = \bigcup_{|\alpha|<\lambda} \phi_\alpha$. □

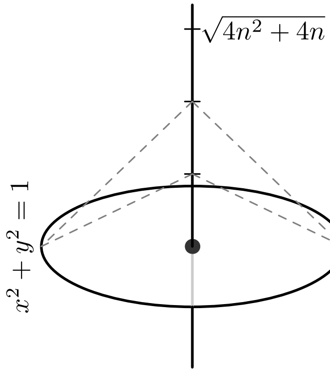

Fig. 2 A subgraph of $G^{\text{Odd}}(\mathbb{R}^3)$. *The points* on *a circle* belong to set A and *the vertical line* contains the points in B

Theorem 4 *The integer-distance graph $G^{\mathbb{N}}(\mathbb{R}^2)$ has the $\aleph_0$-property.*

Proof For every finite set $X = \{x_1, \ldots, x_k\}$ let

$$F(X) = \begin{cases} X, & \text{if } k = 1, \\ \{y; \exists x_i, x_j \in X, d(y, x_i), d(y, x_j) \in \mathbb{N}\}, & \text{otherwise.} \end{cases}$$

Clearly, $|F(x)| \le \aleph_0$ and if X is an infinite countable set and there exists y such that $yx \in E(G)$ for every $x \in X$ then by definition of F, for any finite subset $X' \subset X$ with at least two elements, $y \in F(X')$. □

Corollary 3 *The integer-distance graph $G^{\mathbb{N}}(\mathbb{R}^2)$ is $\aleph_0$-choosable.*

Corollary 4 *The odd-distance graph $G^{\text{Odd}}(\mathbb{R}^2)$ is $\aleph_0$-choosable.*

Note that this is best possible since arbitrarily large complete bipartite graphs are subgraphs of $G^{\text{Odd}}(\mathbb{R}^2)$ and their choosability is growing with their size.

Corollary 5 *The unit-distance graph $G^{\{1\}}(\mathbb{R}^2)$ is $\aleph_0$-choosable.*

3.2 Choosability of $G^{\text{Odd}}(\mathbb{R}^3)$

We will show that the odd-distance graph $G^{\text{Odd}}(\mathbb{R}^3)$ is not $\aleph_0$-choosable.

Theorem 5 *The odd-distance graph $G^{\text{Odd}}(\mathbb{R}^3)$ is not $\aleph_0$-choosable.*

Proof Consider the set of points

$$A = \{(x, y, 0) : x^2 + y^2 = 1\}$$

and

$$B = \{b_n = (0, 0, \sqrt{4n^2 + 4n}) : n \in \mathbb{N}\},$$

cf. Fig. 2. Note that for any $a \in A$ and any $b_n \in B$, $d(a, b_n) = 2n + 1$, an odd integer.

Since $|A| = 2^{\aleph_0}$, there is a one-to-one correspondence ψ between A and the set of all infinite subsets of $\mathbb{N}$. For each vertex of $G^{\mathrm{Odd}}(\mathbb{R}^3)$, we define the list of available colors as follows. For every $a \in A$, let $L_a = \psi(a)$. For each $n \in \mathbb{N}$, let $L_{b_n} = \{n, n + 1, n + 2, \dots\}$. For all other points of $G^{\mathrm{Odd}}(\mathbb{R}^3)$ let the corresponding lists be $\mathbb{N}$. Let f be an $\aleph_0$-coloring of $G^{\mathrm{Odd}}(\mathbb{R}^3)$ with respect to these lists of colors. It follows from the definition of L_{b_n} with $b_n \in B$ that $f(B)$ is infinite so that there is a point $a \in A$ such that $L_a = f(B)$. But since the distance from a to any point in B is odd, f is not a proper coloring, a contradiction. □

Corollary 6 *The odd-distance graph $G^{\mathrm{Odd}}(\mathbb{R}^3)$ does not have the $\aleph_0$-property.*

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References

1. Alon, N.: Restricted colorings of graphs. In: Proc. 14th British Combinatorial Conference. Surveys in Combinatorics, London Mathematical Society Lecture Notes Series, vol. 187, pp. 1–33. Cambridge University Press, Cambridge (1993)
2. Brass, P., Moser, W., Pach, J.: Research Problems in Discrete Geometry, pp. 234–244. Springer, Berlin (2005)
3. Erdős, P.: On sets of distances of n points. Am. Math. Mon. **53**, 248–250 (1946)
4. Erdős, P.: Twenty five years of questions and answers. In: Proc. 25th South-Eastern International Conference on Combinatorics, Graph Theory and Computing (1994)
5. Guldan, F.: On a problem of colouring the real plane. Math. Bohem. **116**, 309–318 (1991)
6. Graham, R., Rothschild, B., Strauss, E.: Are there $n + 2$ points in E^n with odd integral distances? Am. Math. Mon. **81**, 21–25 (1974)
7. Maehara, H., Ota, K., Tokushige, N.: Every graph is an integral distance graph in the plane. J. Comb. Theory Ser. A **80**, 290–294 (1997)
8. Piepmeyer, L.: The maximum number of odd integral distances between points in the plane. Discrete Comput. Geom. **16**, 113–115 (1996)
9. Rosenfeld, M.: Odd integral distances among point in the plane. Geombinatorics **5**, 156–159 (1996)
10. Soifer, A.: Chromatic number of the plane & its relatives. Part I: the problem & its history. Geombinatorics **12**, 131–148 (2003)
11. Woodall, D.R.: Distance realized by sets covering the plane. J. Comb. Theory Ser. A **14**, 187–200 (1973)