The Omega Test: a fast and practical integer programming algorithm for dependence analysis *

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Abstract

The Omega test is an integer programming algorithm that can determine whether a dependence exists between two array references, and if so, under what conditions. Conventional wisdom holds that integer programming techniques are far too expensive to be used for dependence analysis, except as a method of last resort for situations that cannot be decided by simpler methods. We present evidence that suggests this wisdom is wrong, and that the Omega test is competitive with approximate algorithms used in practice and suitable for use in production compilers.

The Omega test is based on an extension of Fourier-Motzkin variable elimination to integer programming, and has worst-case exponential time complexity. However, we show that for many situations in which other (polynomial) methods are accurate, the Omega test has low order polynomial time complexity.

The Omega test can be used to simplify integer programming problems, rather than just deciding them. This has many applications, including accurately and efficiently computing dependence direction and distance vectors.

1 Introduction

A fundamental analysis step in a parallelizing compiler (as well as many other software tools) is data dependence analysis for arrays: deciding if two references to an array can refer to the same element and if so, under what conditions. This information is used to determine allowable program transformations and optimizations. For example, we can decide that in the following program, no location of the array is both read and written. Therefore, the writes can be done in any order or in parallel.

```
for i = 1 to 100 do
  for j = i to 100 do
    A[i, j+1] = A[100,j]
```

There has been extensive study of methods for deciding array data dependences [All83, BC86, AK87, Ban88, Wol89, LYZ89, LY90, GKT91, MHL91]. Much of this work has focused on approximate methods that are guaranteed to be fast but only compute exact results in certain (commonly occurring) special cases. In other situations, approximate methods are conservative: they accurately report all actual dependences, but may report spurious dependences.

Data dependency problems are equivalent to deciding whether there exists an integer solution to a set of linear equalities and inequalities, a form of integer programming. The above problem would be formulated as an integer programming problem shown below. In this example, i and j refer to the values of the loop variables at the time the write is performed and i' and j' refer to the values of the loop variables at the time the read is performed.

$$1 \leq i \leq j \leq 100$$
$$1 \leq i' \leq j' \leq 100$$
$$i = 100$$
$$j + 1 = j'$$

Conventional wisdom holds that integer programming techniques are far too expensive to be used for dependence analysis, except as a method of last resort for situations that cannot be decided by simpler, special-case methods. We present evidence that suggests this wisdom is wrong. We describe the Omega test, which determines whether there is an integer solution to an arbitrary set of linear equalities and inequalities. We describe experiments that suggest that, for almost all programs, the average time required by the Omega test to determine the direction vectors for an array pair is less than 500 μsecs on a 12 MIPS workstation. We also found that the time required by the Omega test to analyze a problem is rarely more than twice the time required to scan the array subscripts and loop bounds. This would indicate that the Omega test is suitable for use in production compilers.

Conceptually, the Omega test combines new methods for eliminating equality constraints with an extension of Fourier-Motzkin variable elimination to integer programming. At a more detailed level, the Omega test incorporates a number of implementation details, such as computing unique constraint keys, that produce substantial speed improvements in practice.

Integer programming is a NP-Complete problem, and the Omega test has exponential worst-case time complexity. We show in Section 7 that in many situations in which other (polynomial) methods are accurate, the Omega test has low-order polynomial worst-case time complexity.

Dependence analysis is often structured as a decision problem: tests simply answer yes or no. Treated this way, determining dependence direction vectors [Wol82] may require a dependence test for each of an exponential number of direction vectors (dependence directions vectors are used to determine the validity of certain complex transformations such as loop interchange). To be competitive, a dependence analysis method must be able to short-cut this enumeration process (e.g., see [BC86, GKT91]). In Section 4, we show

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how the Omega test can be modified to *simplify* integer programming problems, rather than just *deciding* them. With this in hand, we can efficiently produce a set of constraints that precisely and concisely describes all possible dependency distance vectors. This information can be used directly in deciding the validity of program transformations, or standard direction and distance vectors can be quickly computed from it. These techniques are described in Section 5.1.

2 The Omega test

The Omega test determines whether there is an integer solution to an arbitrary set of linear equalities and inequalities, referred to as a problem. The input to the Omega test is a set of linear equalities (such as $\sum_{1\leq i\leq n} a_i x_i = c$) and inequalities (such as $\sum_{1\leq i\leq n} a_i x_i \geq c$). To simplify our presentation (and our algorithms), we define $x_0 = 1$ and use $\sum_{0\leq i\leq n} a_i x_i = 0$ and $\sum_{0\leq i\leq n} a_i x_i \geq 0$ as our standard representations, and we use V to denote the set of indices of the variables being manipulated (i.e., $V = \{i \mid 0 \leq i \leq n\}$).

2.1 Normalizing (and tightening) constraints

Throughout this paper, we assume that any constraint we are manipulating has been normalized. A normalized constraint in one in which all the coefficients are integers and the greatest common divisor of the coefficients (not including a_0) is 1.

If we are given a constraint with rational but not integer coefficients, we scale the constraint to produce integer coefficients (the algorithms described here do not produce any non-integer coefficients).

To normalize a constraint, we compute the greatest common divisor g of the coefficients $a_1, \ldots, a_n$. We then divide all the coefficients by g. If the constraint is an equality constraint and g does not evenly divide a_0, the constraint is unsatisfiable. If the constraint is an inequality constraint, we take the floor when dividing a_0 by g (i.e, we replace a_0 with $\lfloor a_0/g \rfloor$).

Taking floors in the constant term tightens the inequalities. If a problem P has rational but not integer solutions, tightening P may produce a problem without rational solutions, thus making it easier to determine that P has no integer solutions.

2.2 Equality constraints

Given a problem involving equality and inequality constraints, we first eliminate all the equality constraints, producing a new problem of inequality constraints that has integer solutions if and only if the original problem had integer solutions. Of course, in the process we might decide that the problem has no integer solutions regardless of the inequality constraints.

The Generalized GCD test [Banerjee88] can be used to eliminate integer constraints. However, we found the following approach better suited toward our needs, since it is somewhat simpler and more appropriate for situations in which additional equalities may be added later.

To eliminate the equality $\sum_{i\in V} a_i x_i = 0$, we first check if there exists a $j \neq 0$ such that $|a_j| = 1$. If so, we eliminate the constraint by solving for x_j and substitute the result into all other constraint.

Otherwise, let k be the index of the variable with the coefficient that has the smallest absolute value ($k \neq 0$) and let $m = |a_k| + 1$. We define $\widehat{\bmod}$ as follows:

$$a \mathbin{\widehat{\bmod}} b = \textbf{if } a \bmod b < b/2 \quad \textbf{then } a \bmod b \quad \textbf{else } (a \bmod b) - b \textbf{ fi}$$

We create a new variable α and produce the constraint:

$$m\alpha = \sum_{i\in V} (a_i \mathbin{\widehat{\bmod}} m) x_i$$

Note that $a_k \mathbin{\widehat{\bmod}} m = -\text{sign}(a_k)$. We then solve this constraint for x_k

$$x_k = -\text{sign}(a_k) m\alpha + \sum_{i\in V-\{k\}} \text{sign}(a_k)(a_i \mathbin{\widehat{\bmod}} m) x_i$$

and substitute the result in all constraints. In the original constraint, this substitution produces:

$$-|a_k| m\alpha + \sum_{i\in V-\{k\}} (a_i + |a_k|(a_i \mathbin{\widehat{\bmod}} m)) x_i = 0$$

Since $|a_k| = m - 1$, this is equal to

$$-|a_k| m\alpha + \sum_{i\in V-\{k\}} ((a_i - (a_i \mathbin{\widehat{\bmod}} m)) + m(a_i \mathbin{\widehat{\bmod}} m)) x_i = 0$$

Since all terms are now divisible by m, normalizing the constraint produces:

$$-|a_k|\alpha + \sum_{i\in V-\{k\}} ((\lfloor a_i/m + \frac{1}{2} \rfloor + (a_i \mathbin{\widehat{\bmod}} m)) x_i = 0$$

In the original constraint, the absolute value of the coefficient of α is the same as the absolute value of the original coefficient of x_k. For all other variables, this substitution changes the absolute value of each coefficient by a multiplicative factor of at most $1/m + 1/2$, Since $m \geq 3$, we need only perform this step $O(\log(\max_{i\in V-\{0\}} |a_i|))$ times before a unit coefficient appears and we can eliminate the constraint.

For example, consider the constraints:

$$\begin{aligned} 7x + 12y + 31z &= 17 \\ 3x + 5y + 14z &= 7 \end{aligned}$$

The methods above resolve these constraints as shown in Figure 1.

In this example, no contradictions arose during the solving of these constraints and there are no inequality constraints, so we know there exist integer solutions to the constraints. A contradiction arises if we encounter an equality constraint with zero coefficients and a non-zero constant term or an equality constraint with a constant term that is not divisible by the GCD of the coefficients of the variables.

substitution	resulting constraints
$x = -8\alpha - 4y - z - 1$	$-7\alpha - 2y + 3z = 3$ $-24\alpha - 7y + 11z = 10$
$y = \alpha + 3\beta$	$-3\alpha - 2\beta + z = 1$ $-31\alpha - 21\beta + 11z = 10$
$z = 3\alpha + 2\beta + 1$	$2\alpha + \beta = -1$
$\beta = -2\alpha - 1$	

Figure 1: Example of elimination of equatity constraints

2.3 Inequality constraints

The following process is used once all equality constraints have been eliminated. We first check to see if any two inequality constraints directly contradict one another (e.g., the constraints $3x + 5y \geq 2$ and $3x + 5y \leq 0$). If we find a contradiction, we report that the problem has no solutions. We can deal with equality constraints more efficiently than inequality constraints. Therefore, if we find a pair of tight inequalities (such as $6 \leq 3x + 2y$ and $3x + 2y \leq 6$), we replace them with the appropriate equality constraint and revert to our methods for dealing with equality constraints. While checking for contradictory pairs of constraints, we also eliminate constraints that are made redundant by another constraint (e.g., given $x + 2y \geq 0$ and $x + 2y \geq 5$, the first constraint is redundant).

If the problem involves at most one variable, we report that it has integer solutions. If the problem involves more than one variable, we apply Fourier-Motzkin variable elimination [DE73] in an attempt to prove that there are no integer solutions. In most cases arising in practice, this elimination is *exact*: it produces a new problem that has integer solutions if and only if the original problem has integer solutions. If this test cannot disprove the existence of integer solutions and the elimination is not exact, we apply an adaptation of the Fourier-Motzkin method we have devised that, if it reports true, guarantees that an integer solution exists. This test can confirm integer solutions in most of the cases that have integer solutions but not exact eliminations.

If neither of these tests are decisive, we then apply a test that makes use of the fact that the only integer solutions that can slip through the crack between the two tests must have very special forms.

2.3.1 The details

Consider two inequalities $\sum_{i \in V} a_i x_i \geq 0$ and $\sum_{i \in V} a'_i x_i \geq 0$, where $a_k > 0 > a'_k$. The first of these is a lower bound on x_k and the second is an upper bound. Resolving these in terms of x_k produces:

$$\begin{aligned} a_k x_k &\geq \sum_{i \in V - \{k\}} -a_i x_i \\ \sum_{i \in V - \{k\}} a'_i x_i &\geq -a'_k x_k \end{aligned}$$

Multiplying through by $-a'_k$ and a_k respectively gives:

$$\sum_{i \in V - \{k\}} a_k a'_i x_i \geq |a_k a'_k|\ x_k \geq \sum_{i \in V - \{k\}} a'_k a_i x_i$$

This is the key observation of Fourier-Motzkin variable elimination.

We first attempt to disprove the existence of integer solutions to P. We do this by checking if there is an integer solution to a new problem P' that contains all inequalities in P that do not involve x_k and all inequalities produced by combining (as shown above) each pair of upper bound and lower bound on x_k. For example, if $\sum_{i \in V} a_i x_i \geq 0$ is a lower bound on x_k and $\sum_{i \in V} a'_i x_i \geq 0$ is an upper bound on x_k in P, the problem P' contains:

$$\sum_{i \in V - \{k\}} a_k a'_i x_i \geq \sum_{i \in V - \{k\}} a'_k a_i x_i$$

There is a rational solution to P' if and only if there is a rational solution to P. This is standard Fourier-Motzkin variable elimination [DE73]. More directly related to the problem at hand, the lack of integer solutions to P' rules out the existence of integer solutions to P. Tightening the constraints of P' can be very important in verifying that P' does not have integer solutions.

There is an integer solution for x_k to

$$\sum_{i \in V - \{k\}} a_k a'_i x_i \geq |a_k a'_k|\ x_k \geq \sum_{i \in V - \{k\}} a'_k a_i x_i$$

if and only if there is a multiple of $|a_k a'_k|$ between the upper and lower bounds. Consider the case where the upper and lower bounds are as far apart as possible, and yet there is not a multiple of $|a_k a'_k|$ between them.

Since there doesn't exist a multiple of $|a_k a'_k|$ between the upper and lower bound, there must exist a j such that the upper bound is less than $(j+1)|a_k a'_k|$ and the lower bound is greater than $j|a_k a'_k|$. Since the upper bound is a multiple of a_k it is at most $(j+1)|a_k a'_k| - |a'_k|$ and since the lower bound is a multiple of a'_k it is at least $j|a_k a'_k| + a_k$. Therefore, the maximum distance between the upper and lower bound is $|a_k a'_k| - |a'_k| - a_k$. If the distance between the upper and lower bound is greater than this, a multiple of $a_k a'_k$ must exist between them.

Our adaptation of Fourier-Motzkin variable elimination is to produce a second problem P'', which contains every inequality in P that does not involve x_k and the result of combining each pair of upper and lower bounds on x_k in P to produce a new inequality

$$\sum_{i \in V - \{k\}} a_k a'_i x_i \geq (|a_k a'_k| - |a'_k| - a_k + 1) + \sum_{i \in V - \{k\}} a'_k a_i x_i$$

If there is an integer solution to P'', we know there is an integer value of x_k that extends that solution for P'' to be an integer solution to P.

When the set of integer points described by P' and P'' are the same, there will be an integer solution to P' if and only if there is an integer solution to P. This is referred to as an *exact elimination*. If all the upper bound coefficients of x_k are -1, or all the lower bound coefficients of x_k are 1, the elimination is guaranteed to be exact. The elimination is also guaranteed to be exact if the only constraints having non-unit coefficients for x_k produce inequalities of the form $a0 \geq 0$ (where a_0 is greater than zero in P''). In these cases, the Omega test takes advantage of the fact that we don't have to consider both P' and P''.

What if P' has integer solutions but P'' does not have integer solutions? Then we know that if there is an integer solution to P, it is tightly nestled between some pair of upper and lower bounds. Let m be the most negative coefficient of x_k in any inequality. We know that if an integer solution exists, there must exist some lower bound $\sum_{i\in V} a_i x_i \geq 0$ on x_k such that

$$|ma_k| - |m| - a_k + \sum_{i\in V-\{k\}} -|m|\, a_i x_i \geq |m|\, a_k x_k \geq \sum_{i\in V-\{k\}} -|m|\, a_i x_i$$

We consider each lower bound $\sum_{i\in V} a_i x_i \geq 0$ on x_k in turn. For j in the range 0 to $\lfloor(|ma_k|-|m|-a_k)/|m|\rfloor$, we produce a new problem P_j containing the constraints of P and the equality constraint:

$$a_k x_k = j + \sum_{i\in V-\{k\}} -a_i x_i$$

If we find an integer solution to some P_j, we report the existence of integer solutions to original problem.

If we don't find solutions for any P_j, we change in P the lower bound just considered to

$$a_k x_k \geq 1 + \lfloor\frac{|ma_k| - |m| - a_k}{|m|}\rfloor + \sum_{i\in V-\{k\}} -a_i x_i$$

and check if we can now disprove the existence of integer solutions to P. If so, we report that there are no integer solutions to P. Otherwise, we move on to the next lower bound on x_k. If we have exhausted the lower bounds on x_k, we report that no solution exists.

2.3.2 Choosing which variable to eliminate

We try to perform an exact elimination if possible, and choose the variable that minimizes the number of constraints resulting from the combination of upper and lower bounds. If we are forced to perform non-exact reductions, we choose a variable with coefficients as close to zero as possible. We expect that few, if any, inexact eliminations will arise in the analysis of typical programs.

2.3.3 An Omega test nightmare

To demonstrate (and show the limitations of) the techniques used, we illustrate the steps performed by the Omega test on an example designed to force the Omega test to work very hard for a small problem. Consider the inequalities

$$\begin{aligned} 3 \leq\ & 11x + 13y & \leq 21 \\ -8 \leq\ & 7x - 9y & \leq 6 \end{aligned}$$

There are no exact eliminations we can perform. We decide to eliminate x since the coefficients of x are (slightly) smaller. Rearranging the inequalities to be upper and lower bounds on x gives:

$$\begin{aligned} 3 - 13y \leq\ & 11x & \leq 21 - 13y \\ 9y - 8 \leq\ & 7x & \leq 9y + 6 \end{aligned}$$

P'

lower bound	upper bound	unnormalized combination
$33 - 143y \leq 121x$	$121x \leq 231 - 143y$	$198 \geq 0$
$21 - 91y \leq 77x$	$77x \leq 99y + 66$	$190y + 45 \geq 0$
$63y - 56 \leq 49x$	$49x \leq 63y + 42$	$98 \geq 0$
$99y - 88 \leq 77x$	$77x \leq 147 - 91y$	$235 \geq 190y$

P''

lower bound	upper bound	unnormalized combination
$(33 - 143y) + 100 \leq 121x$	$121x \leq 231 - 143y$	$98 \geq 0$
$(21 - 91y) + 60 \leq 77x$	$77x \leq 99y + 66$	$190y \geq 15$
$(63y - 56) + 36 \leq 49x$	$49x \leq 63y + 42$	$62 \geq 0$
$(99y - 88) + 60 \leq 77x$	$77x \leq 147 - 91y$	$175 \geq 190y$

Figure 2: Eliminations performed in example

We produce P' and P'' as shown in Figure 2. Since P' has integer solutions but P'' does not, we consider each lower bound in turn.

We first consider the lower bound $7x \geq 9y - 8$. The most negative coefficient of x is -11. This means we have to consider $P_j = P \wedge 7x = 9y - 8 + j$ for all j in the range $0 \leq j \leq \lfloor\frac{77-11-7}{11}\rfloor = 5$. None of these have solutions, so we add a restriction $7x \geq 9y - 8 + 6$ to P.

Since we are still unable to disprove the existence of integer solutions to P, we move on to the next lower bound, $11x \geq 3 - 13y$. We consider $P_j = P \wedge 11x \geq 3 - 13y + j$ for all j in the range $0 \leq j \leq \lfloor\frac{121-11-11}{11}\rfloor = 9$. We do not find integer solutions for any P_j and we have exhausted the lower bounds on x, so we report that P has no integer solutions.

The steps performed in this example appear complicated and expensive. However, this example was *designed* to be expensive to resolve. We do not expect situations this difficult to arise frequently in practice. Also, although many steps are performed in this process, our implementation of the Omega test takes only 4.5 milliseconds on a 12 MIPS workstation to perform them all.

Worse nightmares are possible: on problems with only 2 variables and 3 constraints, the Omega test can take time proportional to the absolute value of the coefficients. While this is a frightening possibility, we do not expect these situations to arise frequently in practice.

A decision on better methods for dealing with Omega test nightmares will have to wait until more experience is gained about the type of nightmares that occur in practice.

2.4 Implementation Details

In implementing the Omega test we used several algorithmic ideas and tricks that substantially improved our running time. We report some of those ideas here.

Equalities and inequalities are represented as vectors of coefficients. The Omega test is crafted so that the algorithms only need to deal with integers; no rational number representation scheme needs to be used.

Once we have eliminated all the equality constraints from a problem, we check for any variables that have no lower bounds or have no upper bounds. We refer to such variables as unbounded variables. Performing Fourier-Motzkin elimination on an unbounded variable simply deletes all the constraints involving it. We delete all constraints involving unbounded variables and then check if that has produced additional unbounded variables. We repeat this process until no unbounded variables remain.

Next, we normalize all the constraints and assign hash keys and constraint keys to them. We only do this to constraints that have been modified since the last time they were normalized. The constraint key of a constraint is a unique tag based on the coefficients of the variables in the constraint; two constraints have equal constraint keys if and only if they differ only in their constant term. Keys are both negative and positive, and the key of a constraint e_1 is the negation of the key of a constraint e_2 if and only if the coefficients of the variables in e_1 are the negation of the coefficients of the variables in e_2. We refer to this as opposing keys and opposing constraints. Constraint keys are assigned to constraints in constant expected time by recording in a hash table constraint keys previously assigned. We compute a hash key based on the coefficients of the constraint as an index into the hash table (hash keys are not guaranteed to be unique). Our method for computing hash keys is designed so that opposing constraints have opposing hash keys, which makes it easy to assign them opposing constraint keys. As constraints are normalized, we enter them into a table based on their constraint key. This allows us to check for redundant, contradictory or tight constraint pairs in constant time per constraint.

In the process of normalizing constraints, we check to see if any constraints involve more than one variable. After normalization, if we found no multi-variable constraints, we know the system must have solutions, and we return immediately.

Next, we examine the variables to decide which variable to eliminate. If we can perform an exact elimination, we perform the elimination in place (adding and deleting constraints from the current problem). Otherwise, we copy the constraints not involved in the elimination into two new problem data structures (for P' and P'') and then add the constraints produced by Fourier-Motzkin elimination. Since P' and P'' differ only in their constant term, we can share much of the work in creating these problems.

3 Nonlinear subscripts

Integer programming dependence analysis methods allows us to properly handle symbolic constants [LT88, HP90] and some types `min` and `max` functions in loop bounds [WT90] and and conditional assignments [LC90].

For example, even if we had no information about the value of `n`, we would like to be able to decide that there are no flow dependences in the following program:

```
for i = 1 to n do
   a[i+n] = a[i]
```

As previous authors have suggested, we can handle loop-invariant symbolic constants by adding them as additional variables to the integer programming problem. For example, the above problem would generate the following integer programming program (involving the variables i, i' and n):

$$1 \le i, i' \le n$$
$$i + n = i'$$

We also can accommodate integer division and integer remainder operations, something that does not appear to have been previously recognized. Assume an expression $\mathtt{e}$ appears in a program that can be expressed as

$$\mathtt{e} = \left(c + \sum_{i=1}^{n} a_i x_i\right) \;\mathtt{div}\; m$$

where m is a positive integer. To handle this, we define a new variable α and add the inequality constraints

$$0 \le -m\alpha + c + \sum_{i=1}^{n} a_i x_i \le m - 1$$

and use α as the value of $\mathtt{e}$. Similarly, if

$$\mathtt{e} = \left(c + \sum_{i=1}^{n} a_i x_i\right) \;\mathtt{mod}\; m$$

we would add the same inequality constraint but use

$$-m\alpha + c + \sum_{i=1}^{n} a_i x_i$$

as the value of $\mathtt{e}$.

4 Simplification of Integer Programming Problems

As described in Section 2, the Omega test simply *decides* if there is a solution to an integer programming problem. In this section, we describe how to adapt the Omega test to allow it to be used for *simplification*. When used this way, the Omega test is given as input an integer programming program P and a designation of a set $\widehat{V} \subset V$ as being protected variables. The Omega test simplifies P into one or more problems involving only variables in $\widehat{V}$ that describe all the possible values of the variables in $\widehat{V}$ such that there is an integer solution to P with those values. For example, simplifying the integer programming problem $\{0 \le a \le 5;\ b < a \le 5b\}$ while protecting a produces the problem $\{2 \le a \le 5\}$.

Actually, results of the simplification process can be a little more complicated than just described. The results may not be in terms of the variables in $\widehat{V}$. Instead, the results are given in terms of a set $\widehat{V}'$ of not more than $|\widehat{V}|$ variables (possibly including new variables), along with methods for calculating the appropriate values for the values of $\widehat{V}$ from the values of $\widehat{V}'$. For example, if asked to simplify the integer programming problem $\{a = 10b + 25c;\ a \ge 13\}$ while protecting a, the Omega test will produce $\{\alpha \ge 3;\ a = 5\alpha\}$.

Also, the simplification process may produce multiple simplified problems. For example, reducing the problem $\{5b \le a \le 6b\}$ while protecting a produces:

$$\{20 \le a\}$$
$$\{0 \le \alpha;\ a = 6\alpha\}$$
$$\{1 \le \alpha;\ a = 6\alpha - 1\}$$
$$\{2 \le \alpha;\ a = 6\alpha - 2\}$$
$$\{3 \le \alpha;\ a = 6\alpha - 3\}$$

4.1 Complicated results from simplification

All the Omega test does is simplify a problem to a system of constraints involving $|\widehat{V}|$ variables. If $|\widehat{V}|$ is greater than one, the simplified problem may involve redundant constraints, although it does not contain any contradictory ones (nor any redundant pairs).

4.2 Changes to the Omega test

Three of the changes required are simple, the other is not as simple. The quick changes are:

- If the current problem P involves only protected variables, check if there are integer solutions of P and if so, report P as one simplification.
- When performing an inexact Fourier-Motzkin elimination, simplify P'' and all the P_j for all lower bounds (not stopping when an integer solution is first verified). This could be expensive if simplifying a system involves many inexact eliminations. We do not believe this will occur in practice for the problems arising from dependency analysis.
- We never perform Fourier-Motzkin variable elimination on a protected variable. This could require us to perform a non-exact elimination in a situation where we could have performed an exact elimination if we were not protecting certain variables.

The not so simple change involves equalities. Given an equality constraint $\sum_{i\in V} a_i x_i = 0$, let g be the gcd of the coefficients of the non-protected variables. (we assume (as always) that the constraint is normalized).

- If $g = 0$, the constraint involves only protected variables. We use our standard methods to eliminate the constraint. This will result in the elimination of a protected variable. All substitutions performed in this process are recorded in a substitution log. These substitutions involve only protected variables.
- If $g = 1$, we use our standard techniques (Section 2.2) to find a substitution involving only unprotected variables that simplifies or eliminates the constraint.
- If $g > 1$, we create a new protected variable α, add the constraint:

  $$g\alpha = \sum_{i\in V}(a_i \ \widehat{\bmod}\ m)x_i$$

  Eliminating this new constraint will transform the original constraint so that the gcd of the non-protected variables is 1 (after normalization).

When we report a simplification, we also report all the substitutions involving protected variables made while solving the current problem.

4.3 Simplification with wildcards

As a modification of the approach described above, we could refuse to perform inexact reductions while performing simplification. The advantage of this is that we only report one simplified problem as our result. The disadvantage is that the simplified problem has additional variables (that should be treated as wildcards)

In the applications we have found for simplification, we have found simplification with wildcards to be more useful than producing multiple results.

5 Using simplification

This simplification technique can be used for several purposes. We describe some that have occurred to us.

5.1 Dependence direction and distance vectors

One problem with some dependence analysis methods is that they are only "yes/no" decision methods. In compilers and other program structuring tools, we need to know the data dependence direction vector [Wol82] and data dependence distance vector [KMC72, Mur71] describing the relation between the iterations in which the conflicting reads/writes occur. One way to determine dependence direction vectors is to make 3^L calls to the decision procedure (where L is the number of loops surrounding both references). In order to be competitive, a dependence analysis method must be able to short-cut this enumeration (for example, see [BC86, GKT91]).

In our method, we take the integer programming problem for determining if any dependence exists between two references, and introduce a new variable for the dependence distance in each shared loop (along with the appropriate equality constraints to define the value of the variable). We then simplify the problem down to the dependence distance variables. The simplified system may be a better way to describe dependence conditions than dependence directions and distances; it accurately describes more information than is typically contained in dependence direction vectors (such as when a dependence distance is always greater than 5).

Alternatively, we can use the simplified set of constraints to determine efficiently the dependence direction and distance vectors. We scan the dependences, and infer as much information as possible from constraints involving a single dependence distance variable. Any dependence distance variable that is uncoupled or who's sign is completely determined by uncoupled constraints is unprotected. If coupled variables were unprotected, we simplify the problem and repeat this process. Otherwise, we choose one protected variable and generate the subproblems for two or three possible signs for the variable (negative, zero or positive), and recursively explore those.

For example the dependence distances for the following array pair

```
for j = 0 to 20 do
  for i = max(-j,-10) to 0 do
    for k = max(-j,-10)-i to -1 do
      for l = 0 to 5 do
        a(l,i,j) = ...
              ... = a(l,k,i+j)
```

simplify to:

$$\begin{array}{ll} 1 \le \Delta i + \Delta j \le 10 & \\ 0 \le \Delta j \le 10 & \\ 1 \le \Delta j + \Delta i + \Delta k & \\ \Delta i + 2\Delta j \le 20 & \text{(Redundant)} \\ \Delta l = 0 & \end{array}$$

We first unprotect Δl, and the consider $\mathrm{sign}(\Delta j) = 0$ and $\mathrm{sign}(\Delta j) = 1$. Considering $\mathrm{sign}(\Delta j) = 0$ gives:

$$1 \le \Delta i \le 10; 1 \le \Delta i + \Delta k$$

We would then unprotect Δi (since we know sign(Δi) = 1) and simplify the problem, obtaining $-9 \leq \Delta k$, or a direction vector of $(=, <, *, =)$.

Returning to consideration of sign(Δj) = 1 produces:

$$-9 \leq \Delta i \leq 9$$
$$-9 \leq \Delta k$$
$$-9 \leq \Delta i + \Delta k$$
$$-18 \leq \Delta i + 2\Delta k \quad \text{(Redundant)}$$

Recursively analyzing the possibilities for the sign of Δi produces direction vectors of $(<, >, *, =)$, $(<, =, *, =)$ and $(<, <, *, =)$. This example is the most difficult example seen in our testing, requiring 1890 μsecs to analyze.

5.2 Summarizing Array References

In interprocedural analysis, we need to characterize the portions of an array that may be affected by a procedure call [Tri85, BK89, HK90, IJT91]. We can use the Omega test to obtain an accurate summary of the locations of an array that might be affected by a single assignment statement. We do this by setting up an integer programming problem involving variables for each array index and all loop variables and symbolic constants, and adding appropriate constraints for the loop bounds, subscript expressions, and so on. Simplifying this problem, while protecting the variables for the array indices and the symbolic constants, gives an accurate summary of the locations of the array affected by the assignment statement. The summary is not limited to convex polyhedron. The simplified problem will have solutions only for those locations that can actually be changed. Details such as strides are accurately represented.

The Omega test can easily be used to determine when two regions intersect. With more work, the Omega test can be used to check if one region is a subset of another. It is unclear how to use the Omega test to merge affected regions; however, the Omega test could be used to convert exact affected regions into approximate affect regions (such as described by [BK89, HK90]) and then those regions could be merged.

5.3 Determining Loop Bounds

The Omega test can be used to determine appropriate loop bounds when interchanging non-rectangular loops. Ths use of integer programming to perform this is described by [AI91].

6 Performance

We have implemented the Omega test in Wolfe's `tiny` tool [Wol91]. We handle `min` and `max` expressions in loop bounds and symbolic constants, and compute exact sets of direction vectors (as opposed to the compressed direction vectors normally generated by `tiny`). We applied this tool to the programs 1, 3, 4, 5 and 7 of the NASA NAS benchmark suite and to all the `tiny` source files distributed with `tiny`, (which include Cholesky decomposition, LU decomposition, several versions of wavefront algorithms, and several more contrived examples), as well as several of our own test programs. Programs 2 and 6 of the NAS benchmark make extensive use of index arrays. Since we do not provide special treatment for index arrays, we decided that it would be misleading to include them. The analysis of array pairs that have different constant subscripts (e.g., `a(4)` and `a(5)`) are

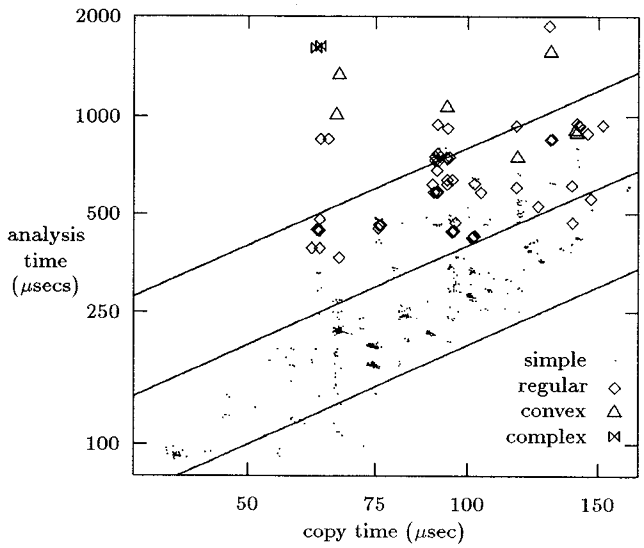

Figure 3: Omega Test Performance

not included in the figures reported here; those cases are detected while scanning the subscripts (thus both avoiding the analysis time and the time required to scan the loop bounds). Standard optimizations such as induction variable recognition and forward substitution were performed by hand. We did not compute input dependences (an input dependence is a dependence between two reads of the same location of an array) or dependences between array pairs that did not share at least one common loop.

We timed the Omega test on a Decstation 3100, a 12 MIPS workstation based on a MIPS R2000 CPU. Shown below are our results on the time per array pair required to analyze programs in the NASA NAS benchmark:

Program	average time	95%-tile time
#1: MXM	295 μsecs	329 μsecs
#3: CHOLSKY	469 μsecs	946 μsecs
#4:BTRIX	269 μsecs	420 μsecs
#5:GMTRY	209 μsecs	485 μsecs
#7:VPENTA	143 μsecs	220 μsecs

The third program of the NAS benchmark (CHOLSKY) is substantially more complicated that almost all real-world FORTRAN code, involving loops nested four deep, triplely subscripted arrays and groups of 3 coupled loop indices. We feel confident that it represents a good "worst-case example" for analyzing dusty deck FORTRAN code (excluding treatment of index arrays).

Our results on individual array pairs from all programs tested are shown in Figure 3. Each point is the timing result for a single array pair. To present the results in a somewhat machine independent fashion, the results are plotted on a log/log graph of analysis time vs. copying time (the time required just to copy the problem). All times were randomly perturbed by $\pm 1/2\mu$sec to spread out overlapping points. The diagonal lines are drawn at analysis time = 8 $\times$ copying time, 4 $\times$ copying time and 2 $\times$ copying time.

The analysis time is the total time required to analyze the array pair, calculate the appropriate direction vectors and add the dependences to dependence graph, excluding the time required to scan the array subscripts and loop bounds and build the constraints that describe the dependence between the array pairs.

Across a range of test programs, we found the following break-down for how time was spent by the Omega test: about 1/2 the time was spent dealing with inequality constraints, about 1/4 of the time was spent on dealing with equality constraints, and 1/4 of the time was spent examining simplified constraints to construct direction vectors. None of our test cases required inexact Fourier-Motzkin variable elimination.

To analyze our results, the set of constraints describing the dependence distances for each array pair were analyzed to remove any redundant constraints (this is not cost-effective normally). Based on the simplified constraints, each array pair was classified as follows:

simple Any case that does not involve coupled dependence distances.

regular A case where dependence distances are coupled, but all inequality constraints have unit coefficients (for example, $\{\Delta i \geq 0; \Delta i + \Delta j > 0\}$).

convex A case where the inequality constraints define a convex region but at least one constraint has a non-unit coefficient (for example, $\{0 \leq \Delta j \leq 10; 0 \leq \Delta i + \Delta j \leq 10; \Delta i + 2\Delta j \leq 10\}$ – the last constraint makes this non-regular).

complex A case where the inequality constraints define a non-convex region. We only encountered two such cases, one shown below and another one identical except that the lower bound of the i loop is 2.

```
for i = 1 to 10 do
  for j = 0 to 4 do
    a(i-j) = a(j)
  endfor
endfor
```

The flow/anti dependence distances for the example above are all the distances that satisfy $\{-4 \leq \Delta j \leq 4; -7 \leq \Delta i - \Delta j, \Delta i + \Delta j \leq 10; \Delta i \leq 9\}$ except for $\{\Delta i = 9; \Delta j = 0\}$.

Maydan, Hennessy and Lam [MHL91] use memoization to obtain better performance. Memoization could be added to the Omega test. However, the cost of computing a hash key and verifying a cache hit would be about 2-4 times the copying cost for a problem, and therefore adding caching to the Omega test would not produce significant savings for typical, simple cases and may produce little or no overall speed improvement.

We found that the cost of scanning array subscripts and loop bounds to build a dependence problem was typically 2-4 times the copying cost for the problem. Thus, for many array pairs the cost of building the dependence problem was nearly as large or even larger than the time spent analyzing the resulting problem. We have not spent much effort trying to improve the performance of the code that builds dependence problems. However, it is difficult to imagine building a dependence problem in much less than twice the time required to copy the problem. This suggests that for the majority of array pairs, using a dependence analysis algorithm significantly faster than the Omega test would not lead to significant overall speed improvements.

7 Polynomial time bounds

We first describe some general time bounds on parts of the Omega test, and then describe polynomial time bounds for cases where other polynomial time algorithms are accurate. In this section, we use m to denote the number of constraints and n to denote the denote the number of variables.

The time taken by the methods in Section 2.2 to eliminate one equality constraint is $O(mn \log |C|)$ worst-case time, where C is coefficient with the largest absolute value in the constraint. This cost arises from the fact that we might have to apply the perform $\log |C|$ substitutions before we can eliminate the constraint, and performing a substitution takes $O(nm)$ time.

Eliminating unbound variables takes $O(mnp)$ worst-case time, where p is the number of passes required to eliminate all the variables that become unbound. At least one variable is eliminated in each pass except the last.

Normalizing the constraints and checking for directly contradictory or redundant constraints requires $O(mn)$ expected time (the time bound is only expected, not worst-case, because hashing is used).

Producing the subproblems resulting from Fourier-Motzkin variable elimination takes time proportional to the size of the subproblems produced.

7.1 Special cases

During normalization, the Omega test checks to see if any variables are involved in constraints with other variables. If not (and checking for contradictory constraint pairs has not produced a contradiction), we know the problem has solutions and do not need to perform any additional computation. This applies iff the "Single Variable Per Constraint" (SVPC) test [MHL91] can be applied, which was found [MHL91] to be applicable in 1/3 of the unique cases found in the Perfect Club Benchmark (a higher percentage if duplicate cases were considered separately).

The "Acyclic Test" [MHL91] can be applied in exactly those cases that the Omega test can resolve just by eliminating unbound variables and performing exact eliminations that do not increase the number of constraints, a process that takes $O(mn^2)$ worst-case time. They found [MHL91] that this test could be applied in over 1/4 of the unique cases encountered.

The "Loop Residue" algorithm [Sho81] can be applied in just those cases where each constraint is of the form $x_i \geq x_j + c$, $x_i \geq c$, or $c \geq x_i$. In a set of constraints with this property, Fourier-Motzkin variable elimination is exact and preserves this property. On n variables, there can be at most $n^2 + n$ constraints of this form after eliminating redudant pairs. Thus, the Omega test will take $O(n^3)$ time to resolve a set of constraints that can be solved by the Loop Residue algorithm. Maydan, Hennessy and Lam [MHL91] found that the Loop Residue algorithm could be applied in 1/4 of the unique cases encounted in their study of the Perfect Club benchmark.

Maydan, Hennessy and Lam found that 91% of the cases they encountered could be determined by constant tests and Banerjee's Generalized GCD tests. Of the remaining 9% of the cases, they found that their SVPC, Acyclic or Loop Residue tests could be applied in 86% of the unique cases.

The Delta test [GKT91] works by searching for dependence distances that can be easily determined, and then propagating that information with the intent of making it possible to easily determine other dependence distances precisely. In the cases where their algorithm can determine a dependence distance without the use of MIV tests, the Omega test also will determine it efficiently (and in polynomial time) by a combination of solving equality constraints, tightening inequality constraints and converting tight inequality constraints into equality constraints. Since the Omega test treats the dependence analysis problem as a single integer programming problem, it automatically achieves the propagation effects of the Delta test. Therefore, any dependence analysis problem that can be solved by the Delta test without resorting to exponential algorithms or approximate methods (i.e., resorting to what they refer to as MIV tests) can be solved in polynomial time by the Omega test.

In their study of the RiCEPS, Perfect, SPEC benchmarks and LINPACK and EISPACK, they found that 97% percent of the cases could be solved without requiring the use of MIV tests.

Since the Omega test can solve effectively and in (effective) polynomial time any problem that be solved by any combination of the Single Variable Per Constraint test, the Acyclic test, the Loop Residue test and the Delta test, we expect that it should be able to solve more problems exactly and efficiently than any one of them alone.

8 Related work on Exact Dependence Analysis

The Constraint-Matrix test [Wal88] makes use of the simplex algorithm modified for integer programming. The Constraint-Matrix test can fail to terminate and not clear how efficiently it works in practice.

Lu and Chen describe [LC90] an integer programming algorithm for dependence analysis. However, their method appears prohibitively expensive for use in a production compiler.

Triolet [Tri85] used Fourier-Motzkin techniques for representing affected array regions in interprocedural analysis. Triolet found Fourier-Motzkin techniques to be expensive (22 to 28 times longer than using simpler methods for representing affected array regions).

Several implementations of Fourier-Motzkin variable elimination have been described for use in dependence analysis. The Power test described by Wolfe and Tseng [WT90] combines the Banerjee's Generalized GCD test, constraint tightening, and Fourier-Motzkin variable elimination. They take no special action when performing an inexact elimination except to flag the result as possibly being conservative. Fourier-Motzkin elimination is used by by Maydan, Hennessy and Lam [MHL91] if none of the other methods they use They use back substitution to determine a sample solution. If the sample solution is not integral, they suggest the use of branch and bound methods to verify or disprove the existence of integer solutions (they have not found the need to implement this thus far). Both Wolfe and Tseng [WT90] and Maydan, Hennessy and Lam [MHL91] suggest that due to the expense of Fourier-Motzkin variable elimination, simpler tests should be used instead in situations where they are known to be accurate.

Ancourt and Irigoin [AI91] describe the use of Fourier-Motzkin variable eliminate to simplify, or project, an integer programming problem (the concept described in Section 4) so as to determine loop bounds for iterating over an iteration space described by a set of linear inequalities. Their work has significant overlaps with ours. The key differences between our work and their work is as follows:

- They do not describe any special techniques for handling equality constraints, although they could easily use Banerjee's Generalized GCD test.
- They do not describe any performance data on their algorithm.
- They handle inexact elimination by introducing pseudo-linear constraints into the problem. There are useful useful for producing loops that iterate over a space described by a set of linear inequalities, but it is unclear how to use them when attempting to verify or disprove the existence of integer solutions.

When performing an inexact elimination, they produce a problem equal to our P'. They also consider a problem $\widehat{P}$ that is similar to our P'' except they use force the difference between the upper and lower bounds to be at least $|a_k a'_k| - |a'_k|$, as opposed to $|a_k a'_k| - |a'_k| - a_k + 1$ for P''. Since $\widehat{P}$ is more conservative than P'', using P'' gives better results.

They do not actually generate $\widehat{P}$ as a separate problem. Rather, they check the constraints in $\widehat{P}$ that differs from the constraints of P' and see if those constraints are redundant with respect to $\widehat{P}$. If so, then P' is an exact reduction. In other words, they check if $P' \subseteq \widehat{P}$. If so, since $\widehat{P} \subseteq P'$, we known that $P' = \widehat{P}$ and that the elimination is exact.

If some constraint in $\widehat{P}$ is not redundant with respect to the constraints of P', then they add *pseudo-linear* constraints to P' so that P' has integer solutions iff P has integer solutions. These pseudo-linear constraints appear useful and appropriate for determining loop bounds. However, they are difficult to use for determining the existence of integer solutions.

A recent report [IJT91] on the PIPS project mentions that Fourier-Motzkin variable elimination is used to analyze dependences (based on the work described in [AI91]). The methods used are not fully described, but the basic framework appears similar to that described in Section 5.1. It is not clear how the pseudo-linear constraints of [AI91] are handled. They point out that in many simply cases, Fourier-Motzkin variable elimination is fast and efficient. They state that using integer programming techniques for dependence analysis inccurs a very high cost (that is acceptable sinces PIPS is not a production system). They also state that in their implementation dependence testing does not take a noticeable amount of time compared with the whole parallelization process.

9 Source code availability

A `C` language implementation of the Omega test is freely available for anonymous `ftp` from `ftp.cs.umd.edu` in directory `pub/omega`.

10 Conclusions

Conservative dependence analysis methods may be efficacious for the demands of vectorizing compilers. Transforming programs so as to make efficient use of massively parallel SIMD computers is a much more demanding task. Also, programs that have undergone transformations such as look skewing and loop interchange present analysis problems substantially more difficult than encountered in typical dusty-deck FORTRAN.

Our studies have convinced us that the Omega test is a fast and practical method for performing data dependence analysis that is not only adequate for problems encountered in vectorizing FORTRAN code, but also for the demands of more sophisticated program transformation tools.

Performing simplification of integer programming problems is an exciting concept. We have discussed how it can be used to determine efficiently information about dependence direction and distance vectors, as well for several other uses. It much easier to describe and build program analysis and transformation tools. For example, it can be used for determining loop bounds after loop interchange [AI91], and we have made extensive use of it in work that considers loop transformations in a uniform manner [Pug91].

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