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978-0-521-14394-3 - The One-Dimensional Hubbard Model

Fabian H. L. Essler, Holger Frahm, Frank Gohmann, Andreas Klumper and Vladimir E. Korepin
Frontmatter

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THE ONE-DIMENSIONAL HUBBARD MODEL

The description of a solid at a microscopic level is complex, involving the interaction of a huge number of its constituents, such as ions or electrons. It is impossible to solve the corresponding many-body problems analytically or numerically, although much insight can be gained from the analysis of simplified models. An important example is the Hubbard model, which describes interacting electrons in narrow energy bands, and which has been applied to problems as diverse as high- T_c superconductivity, band magnetism and the metal-insulator transition.

Remarkably, the one-dimensional Hubbard model can be solved exactly using the Bethe ansatz method. The resulting solution has become a laboratory for theoretical studies of non-perturbative effects in strongly correlated electron systems. Many methods devised to analyse such effects have been applied to this model, both to provide complementary insight into what is known from the exact solution and as an ultimate test of their quality.

This book presents a coherent, self-contained account of the exact solution of the Hubbard model in one dimension. The early chapters develop a self-contained introduction to Bethe's ansatz and its application to the one-dimensional Hubbard model, and will be accessible to beginning graduate students with a basic knowledge of quantum mechanics and statistical mechanics. The later chapters address more advanced topics, and are intended as a guide for researchers to some of the more recent scientific results in the field of integrable models.

The authors are distinguished researchers in the field of condensed matter physics and integrable systems, and have contributed significantly to the present understanding of the one-dimensional Hubbard model. FABIAN ESSLER is a University Lecturer in Condensed Matter Theory at Oxford University. HOLGER FRAHM is Professor of Theoretical Physics at the University of Hannover. FRANK GÖHMANN is a Lecturer at Wuppertal University, Germany. ANDREAS KLÜMPER is Professor of Theoretical Physics at Wuppertal University. VLADIMIR KOREPIN is Professor at the Yang Institute for Theoretical Physics, State University of New York at Stony Brook, and author of *Quantum Inverse Scattering Method and Correlation Functions* (Cambridge, 1993).

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Frontmatter[More information](#)

Contents

<i>Preface</i>	<i>page xi</i>
1 Introduction	1
1.1 On the origin of the Hubbard model	1
1.2 The Hubbard model – a paradigm in condensed matter physics	5
1.3 External fields	11
1.4 Conclusions	14
Appendices to Chapter 1	15
1.A Response to external fields	15
2 The Hubbard Hamiltonian and its symmetries	20
2.1 The Hamiltonian	20
2.2 Symmetries	25
2.3 Conclusions	35
Appendices to Chapter 2	36
2.A The strong coupling limit	36
2.B Continuum limits	45
3 The Bethe ansatz solution	50
3.1 The Hamiltonian in first quantization	51
3.2 Solution of the two-particle problem	54
3.3 Many-particle wave functions and Lieb-Wu equations	64
3.4 Symmetry properties of wave functions and states	67
3.5 The norm of the eigenfunctions	68
3.6 Conclusions	72
Appendices to Chapter 3	73
3.A Scalar products and projection operators	73
3.B Derivation of Bethe ansatz wave functions and Lieb-Wu equations	76
3.C Some technical details	94
3.D Highest weight property of the Bethe ansatz states with respect to total spin	96
3.E Explicit expressions for the amplitudes in the Bethe ansatz wave functions	101

Cambridge University Press

978-0-521-14394-3 - The One-Dimensional Hubbard Model

Fabian H. L. Essler, Holger Frahm, Frank Gohmann, Andreas Klumper and Vladimir E. Korepin
Frontmatter[More information](#)

vi

Contents

3.F	Lowest weight theorem for the η -pairing symmetry	105
3.G	Limiting cases of the Bethe ansatz solution	112
4	String hypothesis	120
4.1	String configurations	121
4.2	String solutions as bound states	125
4.3	Takahashi's equations	128
4.4	Completeness of the Bethe ansatz	131
4.5	Higher-level Bethe ansatz	133
	Appendices to Chapter 4	134
4.A	On deviations from the string hypothesis	134
4.B	Details about the enumeration of eigenstates	137
5	Thermodynamics in the Yang-Yang approach	149
5.1	A point of reference: noninteracting electrons	149
5.2	Thermodynamic Bethe Ansatz (TBA) equations	153
5.3	Thermodynamics	161
5.4	Infinite temperature limit	162
5.5	Zero temperature limit	163
	Appendices to Chapter 5	168
5.A	Zero temperature limit for $\varepsilon'_1(\Lambda)$	168
5.B	Properties of the integral equations at $T = 0$	168
6	Ground state properties in the thermodynamic limit	175
6.1	A point of reference: noninteracting electrons	175
6.2	Defining equations	177
6.3	Ground state phase diagram	178
6.4	Density and magnetization	184
6.5	Spin and charge velocities	187
6.6	Susceptibilities	188
6.7	Ground state energy	193
	Appendices to Chapter 6	195
6.A	Numerical solution of integral equations	195
6.B	Ground state properties in zero magnetic field	197
6.C	Small magnetic fields at half filling: application of the Wiener-Hopf method	202
7	Excited states at zero temperature	209
7.1	A point of reference: noninteracting electrons	210
7.2	Zero magnetic field and half-filled band	211
7.3	Root density formalism	225
7.4	Scattering matrix	236
7.5	'Physical' Bethe ansatz equations	242
7.6	Finite magnetic field and half-filled band	244
7.7	Zero magnetic field and less than half-filled band	253
7.8	Finite magnetic field and less than half-filled band	261
7.9	Empty band in the infinite volume	262

Cambridge University Press

978-0-521-14394-3 - The One-Dimensional Hubbard Model

Fabian H. L. Essler, Holger Frahm, Frank Gohmann, Andreas Klumper and Vladimir E. Korepin
Frontmatter[More information](#)*Contents*

vii

Appendices to Chapter 7	265
7.A Relating root-density and dressed-energy formalisms	265
7.B Lower bounds for $\varepsilon_n(0)$, $n \geq 2$ at half filling in a finite magnetic field	267
8 Finite size corrections at zero temperature	268
8.1 Generic case – the repulsive Hubbard model in a magnetic field	268
8.2 Special cases	276
8.3 Finite size spectrum of the open Hubbard chain	283
8.4 Relation of the dressed charge matrix to observables	290
Appendices to Chapter 8	294
8.A Wiener Hopf calculation of the dressed charge	294
9 Asymptotics of correlation functions	297
9.1 Low energy effective field theory at weak coupling	297
9.2 Conformal field theory and finite size scaling	303
9.3 Correlation functions of the one-dimensional Hubbard model	308
9.4 Correlation functions in momentum space	320
9.5 Correlation functions in the open boundary Hubbard chain	324
Appendices to Chapter 9	331
9.A Singular behaviour of momentum-space correlators	331
10 Scaling and continuum limits at half-filling	333
10.1 Construction of the scaling limit	333
10.2 The S-matrix in the scaling limit	335
10.3 Continuum limit	337
10.4 Correlation functions in the scaling limit	344
10.5 Correlation functions in the continuum limit	361
10.6 Finite temperatures	367
Appendices to Chapter 10	369
10.A Current algebra	369
10.B Two-particle form factors	371
10.C Correlation functions in the Gaussian model	372
11 Universal correlations at low density	376
11.1 The Hubbard model in the gas phase	377
11.2 Correlation functions of the impenetrable electron gas	383
11.3 Conclusions	392
12 The algebraic approach to the Hubbard model	393
12.1 Introduction to the quantum inverse scattering method	393
12.2 Shastry's R-matrix	411
12.3 Graded quantum inverse scattering method	425
12.4 The Hubbard model as a fundamental graded model	440
12.5 Solution of the quantum inverse problem	450
12.6 On the algebraic Bethe ansatz for the Hubbard model	452
12.7 Conclusions	470
Appendices to Chapter 12	472

Cambridge University Press

978-0-521-14394-3 - The One-Dimensional Hubbard Model

Fabian H. L. Essler, Holger Frahm, Frank Gohmann, Andreas Klumper and Vladimir E. Korepin
Frontmatter[More information](#)

viii

Contents

12.A	A proof that Shastry's R-matrix satisfies the Yang-Baxter equation	472
12.B	A proof of the inversion formula	479
12.C	A list of commutation relations	484
12.D	Some identities needed in the construction of the two-particle algebraic Bethe ansatz-states	484
12.E	An explicit expression for the fermionic R-operator of the Hubbard model	486
13	The path integral approach to thermodynamics	488
13.1	The quantum transfer matrix and integrability	489
13.2	The Heisenberg chain	496
13.3	Shastry's model as a classical analogue of the 1d Hubbard model	509
13.4	Diagonalization of the quantum transfer matrix	510
13.5	Associated auxiliary problem of difference type	514
13.6	Derivation of non-linear integral equations	519
13.7	Integral expression for the eigenvalue	525
13.8	Numerical results	536
13.9	Analytical solutions to the integral equations	547
13.10	Conclusions	555
	Appendices to Chapter 13	557
13.A	Derivation of TBA equations from fusion Hierarchy analysis	557
13.B	Derivation of single integral equation	560
14	The Yangian symmetry of the Hubbard model	563
14.1	Introduction	563
14.2	The variable-range-hopping Hamiltonian	564
14.3	Construction of the Yangian generators	566
14.4	Special cases	570
14.5	Conclusions	573
	Appendices to Chapter 14	575
14.A	Yangians	575
15	S-matrix and Yangian symmetry in the infinite interval limit	599
15.1	Preliminaries	599
15.2	Passage to the infinite interval	600
15.3	Yangian symmetry and commuting operators	605
15.4	Constructing N -particle states	607
15.5	Eigenvalues of quantum determinant and Hamiltonian	617
15.6	Conclusions	617
	Appendices to Chapter 15	618
15.A	Some useful formulae	618
16	Hubbard model in the attractive case	620
16.1	Half-filled case	622
16.2	The ground state and low lying excitations below half filling	625
16.3	Interaction with magnetic field	626

Cambridge University Press

978-0-521-14394-3 - The One-Dimensional Hubbard Model

Fabian H. L. Essler, Holger Frahm, Frank Gohmann, Andreas Klumper and Vladimir E. Korepin
Frontmatter[More information](#)

<i>Contents</i>		ix
16.4	Phase diagram	627
16.5	Critical behavior	628
16.6	Thermodynamics	630
	Appendices to Chapter 16	633
16.A	Appendix A	633
16.B	Appendix B	635
17	Mathematical appendices	638
17.1	Useful integrals	638
17.2	The Wiener-Hopf method	640
	<i>References</i>	643
	<i>Index</i>	669

Cambridge University Press

978-0-521-14394-3 - The One-Dimensional Hubbard Model

Fabian H. L. Essler, Holger Frahm, Frank Gohmann, Andreas Klumper and Vladimir E. Korepin
Frontmatter[More information](#)

Preface

On account of Lieb and Wu's 1968 Bethe ansatz solution, the one-dimensional Hubbard model has become a laboratory for theoretical studies of non-perturbative effects in strongly correlated electron systems. Many of the tools available for the analysis of such systems have been applied to this model, both to provide complementary insights to what is known from the exact solution or as an ultimate test of their quality. In parallel, due to the synthesis of new quasi one-dimensional materials and the refinement of experimental techniques, the one-dimensional Hubbard model has evolved from a toy model to a paradigm of experimental relevance for strongly correlated electron systems.

Due to the ongoing efforts to improve our understanding of one-dimensional correlated electron systems, there exists a large number of review articles and books covering various aspects of the general theory, as well as the Bethe ansatz and field theoretical methods. A collection of these works is listed in the General Bibliography below.

Still we felt – and many of our colleagues shared this view – that there would be a need for a coherent account of all of these aspects in a unified framework and from the perspective of the one-dimensional Hubbard model, which, moreover, would be accessible to beginners in the field. This motivated us to write this volume. It is intended to serve both as a textbook and as a monograph. The first chapters are supposed to provide a self-contained introduction to Bethe's ansatz and its application to the one-dimensional Hubbard model, accessible to beginning graduate students with only a basic knowledge of Quantum Mechanics and Statistical Mechanics. The later chapters address more advanced issues and are intended to guide the interested researcher to some of the more recent scientific developments in the field of integrable models.

Although this book concentrates on the one-dimensional Hubbard model, we would like to stress that the methods used in its solution are general in the sense that they apply equally well to other integrable models, some of which we actually deal with in passing. In fact, the application of Bethe's ansatz to the Hubbard model is more involved than in other cases. We expect the reader who has mastered the solution of the Hubbard model to be able to apply his/her knowledge readily to other integrable theories.

This volume does not pretend to cover its subject completely. Rather, we attempted to find a balance between being didactic and being comprehensive. Our selection of material

Cambridge University Press

978-0-521-14394-3 - The One-Dimensional Hubbard Model

Fabian H. L. Essler, Holger Frahm, Frank Gohmann, Andreas Klumper and Vladimir E. Korepin
Frontmatter[More information](#)

xii

Preface

was necessarily governed by our predispositions. We apologize if we have failed to cover important issues adequately.

Ultimately this book originates in the many collaborations between the authors over the last ten years, which are documented in the reference section at the end of the book. Although the material presented has matured in the discussions between us, it is not difficult to infer from our different styles which author bears primary responsibility for which chapter, namely FG for chapters 2, 3, 11, 12, 14, 15, FHLE for chapters 4–7, 10 and 17, HF for chapters 8 and 9, AK for chapter 13, VEK for chapter 16, and FG and FHLE jointly for chapter 1.

Throughout this project and in many fruitful collaborations before we have benefitted immeasurably from numerous discussions with our colleagues and friends A. M. Tselik, N. d'Ambrumenil, T. Deguchi, H. Fehske, F. Gebhard, F. D. M. Haldane, V. I. Inozemtsev, A. R. Its, E. Jeckelmann, G. Jüttner, N. Kawakami, R. M. Konik, E. H. Lieb, S. Lukyanov, M. J. Martins, S. Murakami, A. A. Nersisyan, K. Schoutens, H. Schulz, M. Shiroishi, F. Smirnov, J. Suzuki, M. Takahashi, M. Wadati, A. Weisse and J. Zittartz. Special thanks are due to Andreas Schadschneider for discussions and his constructive criticism after reading the entire manuscript. We are grateful to M. Bortz, A. Fledderjohann, M. Karbach, P. Boykens, A. Grage, M. Hartung, R. M. Konik and A. Seel for proofreading parts of the manuscript and helpful comments.

Despite the joint efforts of many dear friends we do not expect the first edition of such a thick volume to be free of misprints. We plan to keep a record of all misprints brought to our knowledge on our personal websites.

We thank the Physics Departments at Brookhaven National Laboratory and the Universities of Bayreuth, Dortmund, Hannover, Stony Brook, Warwick and Wuppertal for providing stimulating environments during the course of writing this book.

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Fabian H. L. Essler, Holger Frahm, Frank Gohmann, Andreas Klumper and Vladimir E. Korepin
Frontmatter[More information](#)

Preface

xiii

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978-0-521-14394-3 - The One-Dimensional Hubbard Model

Fabian H. L. Essler, Holger Frahm, Frank Gohmann, Andreas Klumper and Vladimir E. Korepin
Frontmatter[More information](#)

xiv

Preface

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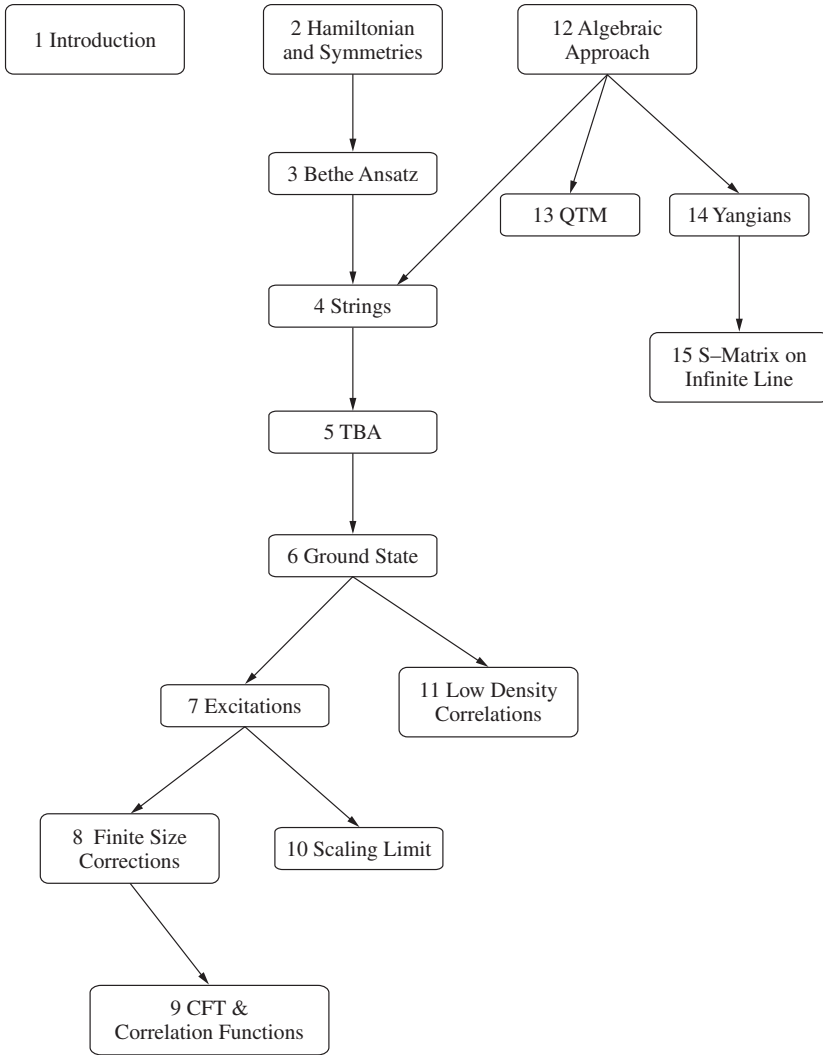
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978-0-521-14394-3 - The One-Dimensional Hubbard Model

Fabian H. L. Essler, Holger Frahm, Frank Gohmann, Andreas Klumper and Vladimir E. Korepin
Frontmatter[More information](#)*Preface*

xv

Instead of a reading guide

The figure shows the logical interdependence of the chapters and may serve the reader to find individual paths through this book. Chapters 16 and 17 have the character of appendices and are logically independent from the remaining part of the book.