The One-Round Voronoi Game*

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Abstract

In the one-round Voronoi game, the first player chooses an n-point set $\mathcal{W}$ in a square Q, and then the second player places another n-point set $\mathcal{B}$ into Q. The payoff for the second player is the fraction of the area of Q occupied by the regions of the points of $\mathcal{B}$ in the Voronoi diagram of $\mathcal{W} \cup \mathcal{B}$. We give a strategy for the second player that always guarantees him a payoff of at least $\frac{1}{2} + \alpha$, for a constant $\alpha > 0$ and every large enough n. This contrasts with the one-dimensional situation, with $Q = [0, 1]$, where the first player can always win more than 1/2.

1 Introduction

Competitive facility location studies the placement of sites by competing market players. Overviews of different models are the surveys by Friesz et al. [7], Eiselt and Laporte [3], and Eiselt et al. [4].

The *Voronoi game* is a simple geometric model for competitive facility location, where a site s "owns" the part of the playing arena that is closer to s than to any other site. We consider a two-player version with a square arena Q. The two players, White and Black, place points into Q. As in chess, White plays first. The goal of both players is to capture as much of the area of Q as possible, where the region captured by White is

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$R(\mathcal{W}, \mathcal{B}) = \{x \in Q : \text{dist}(x, \mathcal{W}) < \text{dist}(x, \mathcal{B})\}$ and the region captured by Black is $R(\mathcal{B}, \mathcal{W})$. Here $\mathcal{W}$ is the set of points of White, $\mathcal{B}$ is the set of points of Black, $\text{dist}(\cdot, \cdot)$ is the Euclidean distance and $\text{vol}(\cdot)$ is the Lebesgue measure. In other words, if we construct the Voronoi diagram of $\mathcal{W} \cup \mathcal{B}$, then each player captures the Voronoi regions (restricted to Q) of his point set and is rewarded proportionate to the measure of his captured set. The *payoff* for White is $\text{vol}(R(\mathcal{W}, \mathcal{B}))/\text{vol}(Q)$ and the payoff for Black is $\text{vol}(R(\mathcal{B}, \mathcal{W}))/\text{vol}(Q)$. (Of course, we can re-scale the board Q so that $\text{vol}(Q) = 1$, but in the subsequent considerations a different scaling seems more intuitive.)

Ahn et al. [1] studied a one-dimensional Voronoi game, where the arena Q is a line segment, and the game takes n rounds. In each round, White and Black place one point each. Ahn et al. showed that Black then has a winning strategy that guarantees a payoff of $1/2 + \varepsilon$, with $\varepsilon > 0$, but that White can force ε to be as small as he wishes. On the other hand, if only a single round is played, where White first places n points, followed by Black placing n points, then White has a winning strategy. In fact, if $Q = [0, 2n]$ and White plays on the odd integer points $\{1, 3, 5, \ldots, 2n - 1\}$, then Black's payoff is less than $1/2$.

In this paper we show that in the two-dimensional case Black, rather than White, has a winning strategy: For each set $\mathcal{W}$ of n points, there is a set $\mathcal{B}$ of n black points such that Black's payoff is at least $1/2 + \alpha$, for an absolute constant $\alpha > 0$ and n large enough.

From now on, let Q be the square $[0, \sqrt{n}]^2$, of area n, so that the average area per white point is 1. To win the game, Black needs to find n points such that their average area is at least $1/2 + \alpha$. We first show that it is very easy to find *one* such point—in fact, a *random* point in $Q \setminus \mathcal{W}$ has this property. Since this is the key idea of our proof, we first present it in a modified setting where the arena Q has the topology of a torus, eliminating boundary effects. We then proceed to prove this result for the square with its standard topology, showing how to handle the square boundary, and proceed to prove the result for n black points. Finally, we show that the result generalizes to higher dimensions as well.

2 The torus case

To present the (simple) main idea of our proofs in a setting free of technical complications due to effects near the boundary of Q, we assume in this section that the square Q has the topology of a torus. To be precise, we identify the left and right edges of Q, as well as the top and bottom edges, an we alter the Euclidean metric in Q accordingly.

Proposition 1 *There exist constants $\beta > 0$ and n_0 such that for every n-point set $\mathcal{W}$ in the square arena Q with torus topology, $n \geq n_0$, there is a point $x \in Q \setminus \mathcal{W}$ with $\text{vol}(R(x, \mathcal{W})) \geq \frac{1}{2} + \beta$. In fact, x can be selected uniformly at random: $\mathbf{E}[\text{vol}(R(x, \mathcal{W}))] \geq \frac{1}{2} + \beta$, where $\mathbf{E}[\cdot]$ denotes expectation with respect to uniform random selection of $x \in Q$.*

Proof: If there is a point $p \in Q$ such that $\text{dist}(p, \mathcal{W}) > \sqrt{n}/4$, then the proposition holds: With a constant probability the point x will grab an $\Omega(n)$ area. If n is large enough,[1] this is more than, say, 1. In the following we can therefore assume that no such point p exists.

[1] This is the only restriction on n in this proof, and in fact the lemma remains correct if the assumption $n \geq n_0$ is replaced by assuming that there is no empty disc of radius $\sqrt{n}/4$ in Q. When we start to take boundary effects into account, we will have to assume n to be larger by several orders of magnitude.

Let I_A denote the characteristic function of a set A. We have

$$\begin{aligned}\mathbf{E}[\mathrm{vol}(R(x,\mathcal{W}))] &= \frac{1}{\mathrm{vol}(Q)}\int_Q\int_Q I_{R(x,\mathcal{W})}(y)\,\mathrm{d}y\,\mathrm{d}x\\ &= \frac{1}{n}\int_Q \mathrm{vol}(\{x\in Q:\ y\in R(x,\mathcal{W})\})\,\mathrm{d}y\end{aligned}$$

by Fubini's theorem.

A point $y \in Q$ lies in $R(x,\mathcal{W})$ if and only if $\mathrm{dist}(y,x)\le r = \mathrm{dist}(y,\mathcal{W})$, and so

$$\{x\in Q:\ y\in R(x,\mathcal{W})\} = \{x\in Q:\ \mathrm{dist}(x,y)\le r\}.$$

Since $r\le \sqrt{n}/4$, this is a disc of radius r centered at y (possibly wrapping around the edges of Q).

Our integral thus becomes $\frac{\pi}{n}\int_Q \mathrm{dist}(y,\mathcal{W})^2\,\mathrm{d}y$, a quantity that we denote by $F_0(\mathcal{W})$. We split it into integrals over $\mathcal{W}$'s Voronoi cells:

$$F_0(\mathcal{W}) = \frac{\pi}{n}\sum_{w\in\mathcal{W}}\int_{\mathrm{cell}_{\mathcal{W}}(w)}\mathrm{dist}(y,w)^2\,\mathrm{d}y,$$

where $\mathrm{cell}_{\mathcal{W}}(w)$ is the region of w in the Voronoi diagram of $\mathcal{W}$ in Q.

Among all convex bodies $C\subset\mathbf{R}^2$ of area a, the integral $\int_C \mathrm{dist}(y,w)^2\,\mathrm{d}y$ is minimized by the disc C_0 of area a centered at w (somewhat informally, moving a piece of C closer to w decreases the integral, and such a move is possible for any C but that disc). Moreover, for later use we note that if C is a convex k-gon, then $\int_C \mathrm{dist}(y,w)^2\,\mathrm{d}y \ge (1+\varepsilon_k)\int_{C_0}\mathrm{dist}(y,w)^2\,\mathrm{d}y$ with a suitable small $\varepsilon_k>0$. (A detailed alternative argument will be given in Section 5.)

The value of that integral over C_0 is

$$\int_{C_0}\mathrm{dist}(y,w)^2\,\mathrm{d}y = \int_0^{\sqrt{a/\pi}} r^2\cdot 2\pi r\,\mathrm{d}r = \frac{a^2}{2\pi}.$$

Let us set $a_w := \mathrm{vol}(\mathrm{cell}_{\mathcal{W}}(w))$. Then

$$\begin{aligned}F_0(\mathcal{W}) &= \frac{\pi}{n}\sum_{w\in\mathcal{W}}\int_{\mathrm{cell}_{\mathcal{W}}(w)}\mathrm{dist}(y,w)^2\,\mathrm{d}y\\ &\ge \frac{1}{2n}\sum_{w\in\mathcal{W}}a_w^2 \ge \frac{1}{2n}\frac{\left(\sum_{w\in\mathcal{W}}a_w\right)^2}{n}\ge\frac{1}{2}\end{aligned}$$

by Cauchy-Schwarz.

So we see that for a random point x, the expected region size is at least $\frac{1}{2}$, but we want $\frac{1}{2}+\beta$. By the remark above, if $\mathrm{cell}_{\mathcal{W}}(w)$ has at most k sides, then $\int_{\mathrm{cell}_{\mathcal{W}}(w)}\mathrm{dist}(y,w)^2\,\mathrm{d}y \ge (1+\varepsilon_k)\cdot\frac{a_w^2}{2\pi}$. Let $\mathcal{W}_f\subseteq\mathcal{W}$ consist of the points whose regions in the Voronoi diagram of $\mathcal{W}$ have fewer than 12 sides. Since the average number of sides of a region in a planar Voronoi diagram is below 6 (using planarity[2] of the dual graph, the Delaunay triangulation), we have $|\mathcal{W}_f|\ge\frac{1}{2}n$.

[2] Strictly speaking, we have embedded it on a torus—the claim remains, however, true.

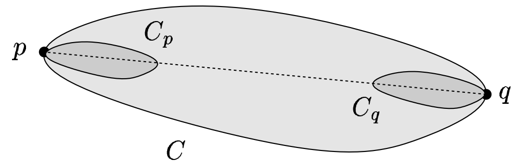

Figure 1: At least 1/16 of the area of C is not covered by $B(w, \frac{1}{4}D)$.

So we win the factor $1 + \varepsilon_{11}$ in at least half of the regions and lose nothing in the other regions. The only problem is that the regions of $\mathcal{W}_f$ could together occupy only a tiny fraction of the area of Q and then this win would not reach the threshold $\beta > 0$ that we seek. But if they occupy, say, less than $\frac{1}{4}$ of the total area then the average area of the remaining regions (of $\mathcal{W} \setminus \mathcal{W}_f$) is at least $\frac{3}{2}$ (at most $\frac{1}{2}n$ regions take up area at least $\frac{3}{4}n$). Then the Cauchy–Schwarz inequality used in the calculation above becomes strict and we win a constant factor in the regions of $\mathcal{W} \setminus \mathcal{W}_f$. □

3 The proof with boundary effects

The torus arena conveniently removed the need to consider the boundary effects. We now prove the same result for the square with boundary:

Proposition 2 *There exist constants $\beta > 0$ and n_0 such that for every n-point set $\mathcal{W} \subset Q$, $n \geq n_0$, we have*

$$\mathbf{E}[\mathrm{vol}(R(x, \mathcal{W}))] \geq \frac{1}{2} + \beta.$$

Proof: As in the proof of Proposition 1, we can rewrite the expected area as

$$\begin{aligned} F(\mathcal{W}) &= \frac{1}{n}\int_Q \mathrm{vol}(\{x \in Q :\ y \in R(x, \mathcal{W})\})\,\mathrm{d}y \\ &= \frac{1}{n}\int_Q \mathrm{vol}(B(y, \mathrm{dist}(y, \mathcal{W})) \cap Q)\,\mathrm{d}y \\ &= \frac{1}{n}\sum_{w \in \mathcal{W}} \int_{\mathrm{cell}_{\mathcal{W}}(w)} \mathrm{vol}(B(y, \mathrm{dist}(y, w)) \cap Q)\,\mathrm{d}y, \end{aligned}$$

where $B(x, r)$ is the disc of radius r centered at x. We want to bound $F(\mathcal{W})$ from below by $\frac{1}{2} + \beta$.

Let us choose a large constant D (the requirements on D will become apparent later). We call a region $\mathrm{cell}_{\mathcal{W}}(w)$ *long* if it has diameter at least D and *short* otherwise, and we denote by $\mathcal{W}_\ell$ and $\mathcal{W}_s$ the subsets of $\mathcal{W}$ corresponding to the long and short regions, respectively.

First we consider the long regions. We note that for any $w, y \in Q$,

$$\mathrm{vol}(B(y, \mathrm{dist}(y, w)) \cap Q) \geq \tfrac{1}{2} \cdot \mathrm{dist}(y, w)^2 \tag{1}$$

(the extreme case is w and y in opposite corners of Q).

Now let $w \in \mathcal{W}_\ell$ and write $C = \mathrm{cell}_\mathcal{W}(w)$. We claim that at least $\frac{1}{16}$ of the area of C lies at distance at least $\frac{1}{4}D$ from w; in other words, $\mathrm{vol}(C \setminus B(w, \frac{1}{4}D)) \geq \frac{1}{16}a_w$ (the constant can be improved). Let p, q be a diametrical pair of points of C, and place two copies C_p, C_q of $C/4$ inside C so that they share a common tangent to C at p and q, respectively, where $C/4$ is the shape resulting from shrinking C by a factor of 4. Clearly, the distance between C_p and C_q is $D/2$, and consequently, either C_p or C_q do not intersect $B = B(w, \frac{1}{4}D)$. Thus, the area of C not covered by $\mathcal{B}$ is at least $\mathrm{vol}(C_p) = \mathrm{vol}(C_q) = \mathrm{vol}(C)/16$. See Fig. 1.

It follows that

$$\int_{\mathrm{cell}_\mathcal{W}(w)} \mathrm{vol}(B(y, \mathrm{dist}(y,w)) \cap Q)\,\mathrm{d}y \geq \frac{1}{2} \cdot \frac{D^2}{16} \cdot \frac{1}{16}a_w > \frac{D^2}{2000}a_w$$

for every $w \in \mathcal{W}_\ell$, and so the contribution of the long regions to $F(\mathcal{W})$ is at least $\frac{D^2}{2000n}A_\ell$, where $A_\ell = \sum_{w \in \mathcal{W}_\ell} a_w$.

Next, we consider the short regions (of diameter at most D), and among those only the *inner* ones, whose distance to the boundary of Q is at least D. Let $\mathcal{W}_{si}$ be the corresponding subset of $\mathcal{W}$. We have $A_{si} = \sum_{w \in \mathcal{W}_{si}} a_w \geq n - 8D\sqrt{n} - A_\ell$. For the short inner regions, the disc $B(y, \mathrm{dist}(y,w))$ lies completely inside Q and so their contribution to $F(\mathcal{W})$ behaves as in the proof of Proposition 1; it equals

$$\frac{\pi}{n} \sum_{w \in \mathcal{W}_{si}} \int_{\mathrm{cell}_\mathcal{W}(w)} \mathrm{dist}(y,w)^2\,\mathrm{d}y.$$

As we saw above, this quantity is bounded below by

$$\frac{1}{2n} \sum_{w \in \mathcal{W}_{si}} a_w^2 \geq \frac{1}{2n}\frac{A_{si}^2}{|\mathcal{W}_{si}|}.$$

Now we distinguish several cases depending on the orders of magnitude of A_ℓ and $|\mathcal{W}_{si}|$. First suppose that $A_\ell \geq \frac{n}{2D}$; then the contribution of A_ℓ alone suffices: $F(\mathcal{W}) \geq \frac{D^2}{2000n}A_\ell \geq \frac{D}{4000} > \frac{1}{2} + \beta$ for D large enough. Next, let $A_\ell < \frac{n}{2D}$, which for large n implies $A_{si} \geq (1 - \frac{1}{D})n$. Now two cases are distinguished according to $|\mathcal{W}_{si}|$. For $|\mathcal{W}_{si}| \leq (1 - \frac{4}{D})n$, we obtain

$$F(\mathcal{W}) \geq \frac{1}{2n}\frac{A_{si}^2}{|\mathcal{W}_{si}|} \geq \frac{(1 - \frac{1}{D})^2}{2(1 - \frac{4}{D})} \geq \frac{1}{2} + \frac{1}{D}$$

which is the desired bound.

Finally it remains to deal with the case $A_{si} \geq (1 - \frac{1}{D})n$ and $|\mathcal{W}_{si}| \geq (1 - \frac{4}{D})n$. If D is very large, we are essentially in the situation analyzed in the proof of Proposition 1 and practically the same argument shows that $F(\mathcal{W}) \geq \frac{1}{2} + \beta$ in this case as well (using the fact that $\frac{1}{D}$ is much smaller than ε_{11}). □

4 The main result

A key ingredient in the proof of our main theorem is the following lemma, showing that if Black throws in δn points at random, instead of one as in Proposition 2, then his expected area gain still exceeds $\frac{1}{2}\delta n$ at least by a fixed fraction, provided that $\delta > 0$ is sufficiently small.

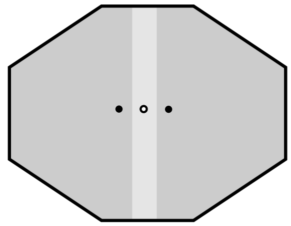

Figure 2: Two points of Black can take over almost a complete cell of White.

Lemma 3 *For every sufficiently large constant D, there exist constants $\beta_1 > 0$, $\delta > 0$, and n_0 such that for every n-point set $\mathcal{W} \subset Q$, $n \geq n_0$, if $\mathcal{B} \subset Q$ is obtained by δn independent random draws from the uniform distribution on Q, then $\mathbf{E}[\mathrm{vol}(R(\mathcal{B}, \mathcal{W}))] \geq (\frac{1}{2} + \beta_1)\delta n$. If the total area A_ℓ of the long regions (of diameter at least D) exceeds $\frac{n}{2D}$, then $\mathbf{E}[\mathrm{vol}(R(\mathcal{B}, \mathcal{W}))] \geq 2\delta n$.*

Proof: This is very similar to the proof of Proposition 2. Intuitively, for small δ, the δn independent random points are likely to interact very little and their expected area gain is likely to be nearly $(\delta - O(\delta^2))n$ times the expected area gain of a single point.

This time we have

$$\mathbf{E}[\mathrm{vol}(R(\mathcal{B}, \mathcal{W}))] \;\; = \;\; \int_Q \mathrm{Prob}\,[y \in R(\mathcal{B}, \mathcal{W})] \; \mathrm{d}y.$$

Here $P(y) = \mathrm{Prob}\,[y \in R(\mathcal{B}, \mathcal{W})]$ is the probability with respect to the random choice of the set $\mathcal{B}$. Namely,

$$\begin{aligned} P(y) &= \mathrm{Prob}\,[\mathcal{B} \cap B(y, \mathrm{dist}(y, \mathcal{W})) \neq \emptyset] \\ &= 1 - (\,\mathrm{Prob}\,[x \notin B(y, \mathrm{dist}(y, \mathcal{W}))])^{\delta n} \\ &= 1 - \Big(1 - \tfrac{1}{n} \cdot \mathrm{vol}(B(y, \mathrm{dist}(y, \mathcal{W})) \cap Q)\Big)^{\delta n}. \end{aligned}$$

Let us write $\rho(y) = \frac{1}{n} \cdot \mathrm{vol}(B(y, \mathrm{dist}(y, \mathcal{W})) \cap Q)$. If y lies in a short region of the Voronoi diagram of $\mathcal{W}$, then $\rho(y) \leq \frac{C_D}{n}$ with C_D depending only on D, and δC_D can be made as small as desired by choosing δ sufficiently small. Then we obtain $P(y) = 1 - (1 - \rho(y))^{\delta n} \geq \delta n \rho(y) + O((\delta n \rho(y))^2) \geq \delta n \rho(y) \cdot (1 - \gamma)$ with γ a small constant. Thus, the contribution of a short Voronoi region to $\mathbf{E}[\mathrm{vol}(R(\mathcal{B}, \mathcal{W}))]$ is at least $(1 - \gamma)\delta n$ times the contribution of that region to the expected area gained by a single random point as in Proposition 2. All the calculations involving short regions can be done in exactly the same way. It remains to show that if the total area A_ℓ of the long regions is at least $\frac{n}{2D}$, then these regions contribute at least $2\delta n$ to $\mathbf{E}[\mathrm{vol}(R(\mathcal{B}, \mathcal{W}))]$.

In the proof of Proposition 2, Eq. (1), we have shown $\rho(y) \geq \frac{1}{2n} \cdot \mathrm{dist}(y, w)^2$ for $y \in \mathrm{cell}_{\mathcal{W}}(w)$. We also know that $\mathrm{dist}(y, w) \geq \frac{1}{4}D$ for y in at least $\frac{1}{16}$ of the area of each long region. For these y, we have $P(y) \geq 1 - e^{-\rho(y)\delta n} \geq 1 - e^{-D^2\delta/200} \geq \frac{D^2\delta}{400}$ (assuming $\delta < D^{-2}$). The whole integral over all the long regions is then at least $\frac{1}{16}A_\ell \geq \frac{n}{32D}$ times this quantity and therefore larger than $2\delta n$ with ample room to spare. □

We can now prove our main theorem.

Theorem 4 *There exist constants $\alpha > 0$ and n_0 such that for every $n \geq n_0$, Black can always win at least $\frac{1}{2} + \alpha$ in the Voronoi game. That is, for every n-point set $\mathcal{W} \subset Q$ there exists an n-point set $\mathcal{B} \subset Q \setminus \mathcal{W}$ with* $\mathrm{vol}(R(\mathcal{B}, \mathcal{W})) \geq (\frac{1}{2} + \alpha)\mathrm{vol}(Q)$.

Proof: Let $w \in \mathcal{W}$. A *takeover* of w's region means that Black places two of his points very close to w with w as the center of symmetry. See Fig. 2. In this way, he captures almost all of $\mathrm{cell}_{\mathcal{W}}(w)$. This suggests the following strategy for Black: A takeover of the $\frac{1}{2}n$ largest White regions guarantees Black a payoff arbitrarily close to $\frac{1}{2}n$. This does not prove the theorem, in general, but it fails to do so only if almost all of White's regions have almost the same area. Thus, if more than εn White regions have area below $1 - \varepsilon$, for some constant $\varepsilon > 0$, then the takeover strategy implies the theorem. It therefore suffices to describe a strategy[3] for Black when all but εn of White's regions have area at least $1 - \varepsilon$.

First Black chooses a set $\mathcal{B}_0$ of δn points as in Lemma 3; that is, with $\mathrm{vol}(R(\mathcal{B}_0, \mathcal{W})) \geq (1 + \beta_1)\delta n$ and even with $\mathrm{vol}(R(\mathcal{B}_0, \mathcal{W})) \geq 2\delta n$ if $A_\ell \geq \frac{n}{2D}$.

If $A_\ell \geq \frac{n}{2D}$, then White now has n regions of total area $A_{\mathcal{W}} \leq (1 - 2\delta)n$ and Black still has $(1 - \delta)n$ points to play. He takes over the $\frac{1}{2}(1 - \delta)n$ largest among the current regions of White. In this way, Black has captured at least area arbitrarily close to

$$n - A_{\mathcal{W}} + \tfrac{1}{2}(1 - \delta)n \cdot \frac{A_{\mathcal{W}}}{n} = n - \tfrac{1}{2}(1 + \delta)A_{\mathcal{W}} > \tfrac{1}{2}(1 + \delta)n.$$

Next, we suppose that $A_\ell < \frac{n}{2D}$. Let us consider a point $w \in \mathcal{W}_s$ defining a short region and call w *contaminated* if Black has captured some point of $\mathrm{cell}_{\mathcal{W}}(w)$ by the set $\mathcal{B}_0$. A short region can be contaminated only by a point $b \in \mathcal{B}_0$ if $\mathrm{dist}(b, w) \leq 2D$. Therefore, the total area of contaminated short regions is $O(D^2\delta n) < \frac{n}{3}$, say, and so regions of total area at least $\frac{n}{2}$ remain uncontaminated. Now we use the assumption that all but εn of White's regions have area at least $1 - \varepsilon$. Black can now take over the $\frac{1}{2}(1 - \delta)n$ largest uncontaminated regions. This implies that the number of uncontaminated regions of size $\geq 1 - \varepsilon$ is at least $n/2 - \varepsilon n$. Thus, Black can now occupy at least $\min(n/2 - \varepsilon n, (1 - \delta)n/2) \geq \frac{1}{2}(1 - \delta)n - \varepsilon n$ cells, to gain total area at least

$$\begin{aligned} &(\tfrac{1}{2} + \beta_1)\delta n + \left(\tfrac{1}{2}(1 - \delta)n - \varepsilon n\right)(1 - \varepsilon) \\ = \;&(\tfrac{1}{2} + \beta_1)\delta n + \tfrac{1}{2}(1 - \delta - 2\varepsilon)(1 - \varepsilon)n. \end{aligned}$$

If ε is very small compared to δ and β_1, then this is at least $(\frac{1}{2} + \alpha)n$ with α close to $\beta_1\delta$. This concludes the proof of the theorem. □

5 The higher-dimensional case

The proof of Proposition 1 (and therefore of Lemma 3) exploited the fact that the Voronoi diagram is a planar graph, and therefore at least half of all Voronoi cells have at most 11 edges. In higher dimensions, though, the average number of facets of a Voronoi cell cannot be bounded by any constant, and so we must argue differently in order to show that the Voronoi cells cannot be all arbitrarily similar to a ball.

[3] A similar trick would also simplify the proof of Proposition 2 if we didn't want to prove the claim about a random point but only the existence of a point capturing at least $\frac{1}{2} + \beta$.

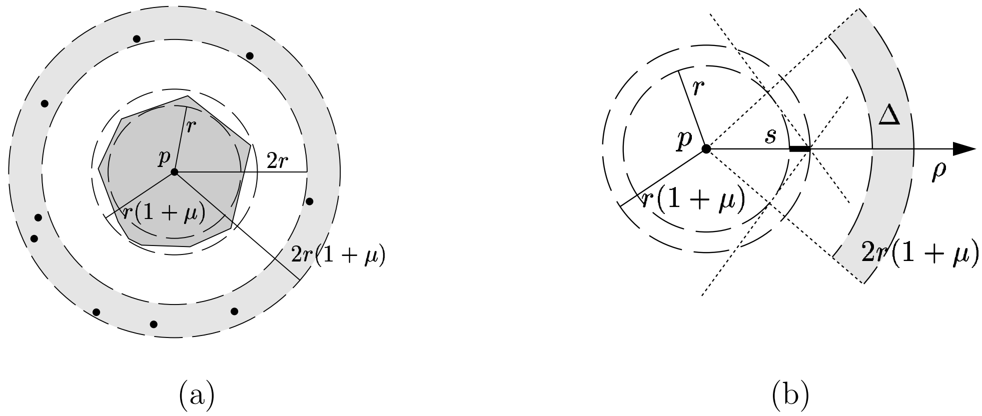

Figure 3: (a) If Voronoi cell C_p is $(1+\mu)$-spherical with center p, then the neighboring sites must be in a spherical shell around p. (b) The sites of $N(p)$ are densely spread in this spherical shell, as there must be a site inside the intersection Δ between the spherical shell and any cone of angular radius $4\sqrt{\mu}$.

Definition 5 *A convex body C is $(1+\mu)$-spherical with center p, for $\mu > 0$, if there exists a radius $r > 0$ such that $B(p, r) \subseteq C \subseteq B(p, r(1+\mu))$.*

Lemma 6 *If a convex body C in $\mathbb{R}^d$ is not $(1+\mu)$-spherical with center p, for some $p \in \mathbb{R}^d$ and $\mu > 0$, then*

$$\int_C c_d \cdot \mathrm{dist}(y,p)^d \,\mathrm{d}y \geq (1+\beta)L,$$

with

$$L = \int_\Delta c_d \cdot \mathrm{dist}(y,p)^d \,\mathrm{d}y = \frac{\mathrm{vol}(C)^2}{2}.$$

Here Δ is a ball of the same volume as C centered at p, $\beta > 0$ is a constant that depends only on μ and d, and c_d is the volume of the unit ball in $\mathbb{R}^d$.

Proof: Let $\Delta = B(p, R)$, where $R = (\mathrm{vol}(C)/c_d)^{1/d}$. Then

$$\begin{aligned} L &= \int_\Delta c_d \cdot \mathrm{dist}(y,p)^d \,\mathrm{d}y = \int_0^R (c_d r^d) \cdot (c_d d r^{d-1}) \,\mathrm{d}r \\ &= \frac{c_d^2}{2} R^{2d} = \frac{\mathrm{vol}(C)^2}{2}. \end{aligned}$$

As for the other claim, let r', R' be the largest (resp. smallest) radius so that $B(p, r') \subseteq C \subseteq B(p, R')$. Since C is not $(1+\mu)$-spherical with center p, it follows that there exists a positive constant β_1 such that $(1+\beta_1)R \leq R'$. In particular, this implies that there exists a constant β_2, such that $\mathrm{vol}(K) \geq \beta_2 \mathrm{vol}(C)$, where $K = C \setminus B(p, R(1+\beta_1/4))$. Namely,

$$\begin{aligned} \int_C c_d \cdot \mathrm{dist}(y,p)^d \,\mathrm{d}y &\geq \int_\Delta c_d \mathrm{dist}(y,p)^d \,\mathrm{d}y + \int_K c_d \left((R\,(1+\beta_1/4))^d - R^d \right) \mathrm{d}y \\ &\geq (1+\beta) \int_\Delta c_d \cdot \mathrm{dist}(y,p)^d \,\mathrm{d}y = (1+\beta)L, \end{aligned}$$

where $\beta > 0$ is an appropriate constant that depends only on d and μ. □

Lemma 7 *Let Q be a hypercube in $\mathbb{R}^d$, and let P be a set of points in Q. Let $V(P)$ denote the decomposition of Q into convex cells by the Voronoi diagram of P restricted to Q. Then there exists a constant $\mu > 0$, which depends only on d, such that the total volume of the cells that are $(1+\mu)$-spherical with respect to their defining site in P is bounded by* $\text{vol}(Q)/2$.

Proof: Consider a cell C_p of $V(P)$ that is $(1+\mu)$-spherical with center p. Let $B(p,r)$ be the largest ball with center p that is contained inside C_p. Let $N(p)$ be the set of points of P whose Voronoi cells have a common boundary with C_p.

Observe that the distance of any point of $N(p)$ to p is at least $2r$ and at most $2r(1+\mu)$. Furthermore, any angular cone of angular angle $4\sqrt{\mu}$ emanating from p must include a point of $N(p)$. Indeed, consider such a cone Z with a ray ρ as its rotational axis and angular radius $4\sqrt{\mu}$, where ρ emanates from p. Let s denote the intersection of ρ with the spherical shell $B(p,(1+\mu)r) \setminus B(p,r)$. Since one endpoint of s is outside C_p, and the other is inside C_p, it follows that there must be a point $q \in P$, so that the bisector of p and q intersects s. It is now straightforward to verify that q is inside Z. See Figure 3.

This implies that $N(p)$ is dense. Indeed, consider a point $q \in N(p)$. Its nearest point in $N(p)$ is at distance at most $2r(1+\mu)\cdot 2\cdot 4\sqrt{\mu} = O(r\sqrt{\mu})$. On the other hand, the Voronoi cell C_q of q has a point on its boundary of distance $\geq r$ from q (as it shares a boundary point u with C_p, $\text{dist}(u,p) \geq r$, and $\text{dist}(u,q) = \text{dist}(u,p)$). (This also implies that C_p is not adjacent to the boundary of Q.)

That is, C_q is not a γ-spherical, where $\gamma = \Omega(r/r\sqrt{\mu} - 1) = \Omega(1/\sqrt{\mu})$. By making μ small enough, we can ensure that C_q is not a $(1+\mu)$-spherical with center q.

We have shown that every $(1+\mu)$-spherical cell in $V(P)$ is surrounded by cells that are not $(1+\mu)$-spherical. We will charge the volume of such a μ-spherical cell to its surrounding cells as follows. For a point $p \in P$ whose Voronoi cell is a $(1+\mu)$-spherical with center p, let r_p be the radius of the largest ball contained inside C_p centered at p, and let $U_p = B(p, 1.8r_p)$ be the *region of influence* of p. Clearly, $U_p \cap P = \{p\}$ and $\text{vol}(U_p) \geq (1.8/(1+\mu))^d \text{vol}(C_p) \geq 2\text{vol}(C_p)$, for μ sufficiently small. By picking μ small enough, we can also guarantee that the regions of influence of the $(1+\mu)$-spherical cells of $V(P)$ are disjoint. We charge the volume of a $(1+\mu)$-spherical cell to its region of influence, establishing the claim. □

Plugging Lemmas 6 and 7 into the proof of Theorem 4 gives us the following result. The straightforward details are omitted.

Theorem 8 *There exist constants $\alpha > 0$ and n_0 depending only on the dimension d, such that for every $n \geq n_0$, Black can always win at least $\frac{1}{2} + \alpha$ in the Voronoi game played on arena Q, the d-dimensional hypercube. That is, for every n-point set $\mathcal{W} \subset Q$ there exists an n-point set $\mathcal{B} \subset Q \setminus \mathcal{W}$ with* $\text{vol}(R(\mathcal{B},\mathcal{W})) \geq (\frac{1}{2}+\alpha)\text{vol}(Q)$.

6 Conclusions and open problems

We considered the Voronoi game on a square or hypercube board Q, played in a single round: White starts by placing n points $\mathcal{W}$ in Q, then Black places another n points $\mathcal{B}$ disjoint from $\mathcal{W}$, and finally the winner is determined.

Our considerations appear to generalize without much change to sufficiently "fat" convex arenas in the plane. On the other hand, when the arena degenerates to a line segment, we have reached the one-dimensional case where White, not Black, has a winning strategy [1]. It would be interesting to understand the behavior of the game with a rectangular arena as a function of the aspect ratio of the rectangle.

What happens when the number of points played by White and Black are not identical? Specifically, let λ be a real number between 0 and 2. Consider the game where White plays n points and Black plays λn points. Let $f(\lambda, n)$ be the payoff to Black in this Voronoi game. It is not hard to show that $f(0, n) = 0$ and that $\lim_{n\to\infty} f(2, n) = 1$. We know that $f(\lambda, n) > (\frac{1}{2} + \varepsilon)\lambda$ for some positive ε and n large enough, as long as λ is bounded away from 0 and 2. It would be interesting to get a better idea of the behavior of f. Does $\lim_{n\to\infty} f(\lambda, n)$ exist for all λ?

We have shown that for any set of n white points, there is a black point that grabs a "large" Voronoi cell. It would be interesting to find configurations of the white points for which no black point can do too well. Obvious candidates are grid arrangements of the white points, such as the square grid or hexagonal grid.

In fact, if we ask for a configuration of the white points that minimizes the payoff of a *random* black point, it is known that the hexagonal grid is optimal if n is large enough. This follows from a result on the two-dimensional quantizer problem. In the quantizer problem, we want to quantize two-dimensional input values from a continuous domain (a ball $B \subset \mathbf{R}^2$, say) using $\log n$ bits. This is done by choosing a discrete quantizer set P of n points in B, and replacing the input value $x \in B$ by the closest point from P. Assuming uniform distribution of the input values, the *mean squared error* of a quantizer P is

$$\frac{1}{\mathrm{vol}(B)} \sum_{p \in P} \int_{\mathrm{cell}(p)} \mathrm{dist}(x, p)^2 \, \mathrm{d}x,$$

where cell(p) is the Voronoi cell of p in the Voronoi diagram of P (see Conway and Sloane [2]). Fejes Tóth [5] (see also [6]) showed that if n is sufficiently large, then the error is minimized by choosing P to be the hexagonal grid.

In the proof of Proposition 1, we showed that the expected payoff of a random black point is

$$\frac{\pi}{n} \sum_{p \in \mathcal{W}} \int_{\mathrm{cell}(w)} \mathrm{dist}(x, w)^2 \, \mathrm{d}x,$$

with the slight twist that here we assume torus topology. Assuming n is so large that we can ignore the difference in topology, this is proportional to the quantization error of $\mathcal{W}$, and so Fejes Tóth's result implies that the optimal choice of $\mathcal{W}$ is the hexagonal grid. An interesting open question is whether the hexagonal grid is also optimal if we consider the maximum possible area that a black point can grab.

The original version of the Voronoi game [1] is played in more than one round: White and Black alternate placing points on the board Q. The value of this game and the optimal strategies are still unknown for dimension higher than one. If the arena Q is symmetric, but the symmetry has no fixed point in Q, then Black can respond to each move of White with a point placed in the symmetric location. This guarantees a payoff of 1/2. Many obvious questions remain open: Can Black actually win the game for large n? What happens with asymmetric boards?

Finally, it seems that using a sliding grid argument one can derandomize the strategy of the Black player in the one-round Voronoi game investigated in this paper.

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