THE OPENING OF A GRIFFITH CRACK UNDER INTERNAL PRESSURE*

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1. The determination of the distribution of stress in the neighbourhood of a crack in an elastic body is of importance in the discussion of certain properties of the solid state. The theory of cracks in a two-dimensional elastic medium was first developed by Griffith[1] who succeeded in solving the equations of elastic equilibrium in two dimensions for a space bounded by two concentric coaxial ellipses; by considering the inner ellipse to be of zero eccentricity and by assuming that the major axis of the outer ellipse was very large Griffith then derived the solution corresponding to a very thin crack in the interor of an infinite elastic solid. Because of the nature of the coordinate system employed by Griffith the expressions he derives for the components of stress in the vicinity of the crack do not lend themselves easily to computation. An alternative method of determining the distribution of stress in the neighbourhood of a Griffith crack was given recently by one of us[2] making use of a complex stress-function stated by Westergaard.[3] This method suffers from the disadvantage that the Westergaard stress-function refers only to the case in which the Griffith crack is opened under the action of a *uniform* internal pressure; the stress-function corresponding to a variable internal pressure does not appear to be known.

In the present note we discuss the distribution of stress in the neighbourhood of a Griffith crack which is subject to an internal pressure, which may vary along the length of the crack, by considering the corresponding boundary value problem for a semi-infinite two-dimensional medium. The analysis is the exact analogue of that for the three-dimensional "circular" cracks developed in the previous paper[2] except that now we employ a Fourier cosine transform method in place of the Hankel transform method used there. A method is given for determining the shape of the crack resulting from the application of a variable internal pressure to a very thin crevice in the interior of an elastic solid, and for determining the distribution of stress throughout the solid. The converse problem of determining the distribution of pressure necessary to open a crevice to a crack of prescribed shape is also considered. As an example of the use of the method the expressions for the components of stress, due to the opening of a crack under a uniform pressure, are derived and are found to be in agreement with those found in the earlier paper.[2]

2. We consider the distribution of stress in the interior of an infinite two-dimensional elastic medium when a very thin internal crack $-c \leqq y \leqq c$, $x=0$ is opened under the action of a pressure which may be considered to vary in magnitude along the length of the crack. For simplicity we shall consider the symmetrical case in which the applied pressure is a function of $|y|$ but the analysis may easily be extended to the

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[1] A. A. Griffith, Phil. Trans. (A) **221**, 163 (1921).

[2] I. N. Sneddon, Proc. Roy. Soc. (A) (in the press).

[3] H. M. Westergaard, J. Appl. Mech. **6**, A49 (1939).

more general case in which there is no such symmetry. The stress in such a medium may be described by three components of stress σ_x, σ_y and τ_{xy}; the corresponding components of the displacement vector will be denoted by u_x and u_y. The differential equations determining the stress-components are[4]

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0, \qquad (1) \qquad\qquad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0. \qquad (2)$$

The boundary conditions to be satisfied are that all the components of stress and of the displacement vector must tend to zero as x^2+y^2 tends to infinity, and that

$$\tau_{xy} = 0, \qquad \sigma_x = -p(y), \tag{3}$$

when $x=0$ and $-c \leqq y \leqq c$.

It is obvious from the symmetry about the axis $x=0$ that the problem of determining the distribution of stress in the neighbourhood of the crevice is equivalent to that of determining the stress in the semi-infinite elastic medium $x \geqq 0$ when the boundary $x=0$ is subjected to the following conditions:

(i) $\tau_{xy}=0$, for all values of y,

(ii) $\sigma_x = -p(y)$, $|y| \leqq c$,

$u_x = 0$ $\quad |y| \geqq c$.

From the symmetry about the second axis $y=0$ we may take as solutions of the elastic equations (1) and (2) the expressions:[5]

$$\sigma_x = \frac{2}{\pi}\int_0^\infty \phi(\rho)(1+\rho x)e^{-\rho x}\cos \rho y d\rho, \tag{4}$$

$$\sigma_y = \frac{2}{\pi}\int_0^\infty \phi(\rho)(1-\rho x)e^{-\rho x}\cos \rho y d\rho, \tag{5}$$

$$\tau_{xy} = \frac{2x}{\pi}\int_0^\infty \rho\phi(\rho)e^{-\rho x}\sin \rho y d\rho. \tag{6}$$

These expressions satisfy the equations of equilibrium and the boundary condition (i) above; the function $\phi(\rho)$ is determined from the set of conditions (ii). The components of the displacement vector are similarly found to be

$$u_x = -\frac{2(1+\sigma)}{\pi E}\int_0^\infty \phi(\rho)e^{-\rho x}\{2(1-\sigma)+\rho x\}\frac{\cos \rho y}{\rho}d\rho, \tag{7}$$

$$u_y = \frac{2(1+\sigma)}{\pi E}\int_0^\infty \phi(\rho)e^{-\rho x}\{(1-2\sigma)-\rho x\}\frac{\sin \rho y}{\rho}d\rho. \tag{8}$$

When $x=0$, equations (4) and (7) reduce to

$$\sigma_x = \frac{2}{\pi}\int_0^\infty \phi(\rho)\cos \rho y d\rho, \tag{9}$$

$$u_x = -\frac{4(1-\sigma^2)}{\pi E}\int_0^\infty \phi(\rho)\frac{\cos \rho y}{\rho}d\rho. \tag{10}$$

[4] A. E. H. Love, *The mathematical theory of elasticity*, 4th ed., Cambridge, 1934, p. 208.

[5] I. N. Sneddon, Proc. Cambridge Phil. Soc. **40**, 229 (1944).

If we insert the boundary conditions (ii) into Eqs. (9) and (10) and make the substitutions

$$\rho = \xi/c, \qquad y = \eta c, \qquad g(\eta) = -c\left(\frac{\pi}{2\eta}\right)^{1/2} p(\eta c), \qquad \bar{\phi}\left(\frac{\xi}{c}\right) = \xi^{1/2}F(\xi), \tag{11}$$

we obtain a pair of "dual" integral equations

$$\left.\begin{aligned} \int_0^\infty \xi F(\xi) J_{-1/2}(\xi\eta)\,d\xi &= g(\eta), \qquad 0 < \eta < 1 \\ \int_0^\infty F(\xi) J_{-1/2}(\xi\eta)\,d\xi &= 0, \qquad \eta > 1 \end{aligned}\right\} \tag{12}$$

for the determination of the function $F(\xi)$. Once $F(\xi)$ has been found, $\bar{\phi}(\rho)$ can be written down and the components of stress calculated by means of Eqs. (4), (5) and (6).

3. The dual integral equations (12) are a special case of a pair of equations considered by Busbridge;[6] the solution may be obtained by substituting $\alpha=1$, $\nu=-1/2$ in the general solution given in the paper.[6] In this we obtain

$$F(\xi) = \sqrt{\frac{2}{\pi}}\,\xi^{1/2}\Big[J_0(\xi)\int_0^1 y^{1/2}(1-y^2)^{1/2}g(y)\,dy + \xi\int_0^1 u^{1/2}(1-u^2)^{1/2}du\int_0^1 g(yu)\,y^{5/2}J_1(\xi y)\,dy\Big]. \tag{13}$$

Thus if the pressure $p(y)$ is given by a Taylor series of the form

$$p(y) = p_0\sum_{n=0}^{\infty} a_n\left(\frac{y}{c}\right)^n, \tag{14}$$

convergent for $-c \leqq y \leqq c$, then the corresponding expression for $\bar{\phi}(\rho)$ is readily found to be

$$\bar{\phi}(\rho) = -\tfrac{1}{2}p_0 c^2 \pi^{1/2}\rho\sum_{n=0}^{\infty}\frac{\Gamma(\frac{1}{2}n+\frac{1}{2})}{\Gamma(\frac{1}{2}n+2)}\,a_n\left\{J_0(c\rho) + c\rho\int_0^1 y^{n+2}J_1(c\rho y)\,dy\right\}. \tag{15}$$

Substituting for $\bar{\phi}(\rho)$ from Eq. (15) into Eq. (10) and making use of the results[7]

$$\int_0^\infty J_0(c\rho)\cos\rho y\,d\rho = \frac{1}{\sqrt{c^2-y^2}}, \qquad 0 < y < c$$

$$\int_0^\infty \rho J_1(c\rho)\cos\rho y\,d\rho = \frac{c}{(c^2-y^2)^{3/2}}, \qquad 0 < y < c$$

we find that the normal component of the displacement along the crack is given by w, where

$$w = \frac{2(1-\sigma^2)p_0 c}{\sqrt{\pi}\cdot E}\sum_{n=0}^{\infty}\frac{\Gamma(\frac{1}{2}n+\frac{1}{2})}{\Gamma(\frac{1}{2}n+2)}\,a_n\left\{\frac{c}{\sqrt{c^2-y^2}} + \left(\frac{y}{c}\right)^{n+1}\int_1^{c/y}\frac{u^{n+3}du}{(u^2-1)^{3/2}}\right\}. \tag{16}$$

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[6] I. W. Busbridge, Proc. London Math. Soc. (2) **44,** 115 (1938).

[7] G. N. Watson, *The theory of Bessel functions,* 2nd ed., Cambridge, 1944, p. 405.

For the case of a uniform pressure p_0 we take $a_0=1$, $a_n=0$, $n\geqq 1$ and find

$$w = \frac{2(1-\sigma^2)p_0}{E}\sqrt{c^2-y^2}. \tag{17}$$

If we write

$$b = 2(1-\sigma^2)p_0c/E,$$

Eq. (17) reduces to the form

$$\frac{y^2}{c^2}+\frac{w^2}{b^2} = 1$$

which shows that the effect of the uniform pressure is to widen the crevice into an elliptic crack.

4. It is also of interest to determine what distribution of pressure will produce a crack of prescribed shape. In this case we assume that the value of the normal displacement u_x is known *all* along the y-axis; we have

$$u_x = \begin{cases} w(y), & y \leqq |c|, \quad x = 0, \\ 0, & y \geqq |c|, \quad x = 0. \end{cases}$$

Inverting Eq. (10) by the Fourier cosine rule and substituting this value for u_x we have

$$\bar{\phi}(\rho) = -\frac{E}{2(1-\sigma^2)}\rho\int_0^c w(y)\cos\rho y dy. \tag{18}$$

With this value of $\bar{\phi}(\rho)$ in Eqs. (4), (5) and (6) we obtain expressions for the components of stress in the interior of the elastic solid.

For example if we take

$$w(y) = \epsilon\left(1-\frac{y^2}{c^2}\right),$$

then, from Eq. (18)

$$\bar{\phi}(\rho) = -\frac{E\epsilon}{(1-\sigma^2)c\rho}\left(\frac{\sin c\rho}{c\rho}-\cos c\rho\right). \tag{19}$$

Substituting from (19) into Eq. (9) we obtain for the normal component of the stress along $x=0$,

$$\sigma_x = -\frac{2E\epsilon}{\pi(1-\sigma^2)c}\left[1-\frac{y}{c}\int_0^\infty \frac{\sin u \sin\dfrac{yu}{c}}{u}du\right]. \tag{20}$$

Now

$$\int_0^\infty \frac{\cos qx-\cos px}{x}dx = \tfrac{1}{2}\log\frac{p^2}{q^2},$$

so that Eq. (20) reduces to

$$\sigma_x = -\frac{2E\epsilon}{\pi(1-\sigma^2)c}\left[1-\frac{y}{2c}\log\frac{c+y}{c-y}\right], \qquad 0 < y < c, \tag{21}$$

giving the normal component of stress along the crack. This stress is negative when $y=0$ but becomes positive for a value of y between 0 and c, so that if a crack of this shape is to be maintained the applied stress must be tensile (and very large) near the edges $y=\pm c$ of the crack.

5. Expressions for the potential functions $\omega(z)$, $\Omega(z)$ of Stevenson corresponding to this problem can easily be deduced from the analysis of Section 3. It was shown by Stevenson,[8] that if we write

$$\Theta = \sigma_x + \sigma_y; \qquad \Phi = \sigma_x - \sigma_y + 2i\tau_{xy}, \qquad D = u_x + iu_y$$

then the components of the stress and the displacement can be expressed in terms of two "potential" functions $\omega(z)$, $\Omega(z)$ by means of the equations

$$\left.\begin{aligned} D &= \frac{1+\sigma}{4} E\{(3-4\sigma)\Omega(z) - z\bar{\Omega}'(\bar{z}) - \bar{\omega}'(\bar{z})\} \\ 2\Theta &= \Omega'(z) + \bar{\Omega}'(\bar{z}) \\ -2\Phi &= z\bar{\Omega}''(\bar{z}) + \bar{\omega}''(\bar{z}) \end{aligned}\right\} \tag{22}$$

in the absence of body forces.

It follows from Eqs. (4) to (8) that the stresses and the components of the displacement vector may be derived from the potential functions

$$\Omega(z) = -\frac{4}{\pi}\int_0^\infty \frac{\bar{\phi}(\rho)}{\rho} e^{-\rho z} d\rho, \qquad \omega'(z) = \frac{4}{\pi}\int_0 \frac{\bar{\phi}(\rho)}{\rho}(1+\rho z)e^{-\rho z}d\rho, \tag{23}$$

where $\bar{\phi}(\rho)$ is given by Eq. (15) in the case where the applied internal pressure is given by Eq. (14).

6. We now consider the distribution of stress in the solid when the crevice $-c \leqq y \leqq c$, $x=0$ is opened up by the action of a uniform pressure p_0. Taking $a_0=1$, $a_n=0$, $n>0$, in Eq. (15) we obtain for $\bar{\phi}(\rho)$ the expression

$$\bar{\phi}(\rho) = -\tfrac{1}{4}\pi p_0 c^2 \rho \left\{ J_0(c\rho) + \frac{1}{c^2\rho^2}\int_0^{c\rho} z^2 J_1(z)dz \right\}.$$

Now,

$$\int_0^{c\rho} z^2 J_1(z)dz = c^2\rho^2 J_2(c\rho)$$

and, by a well-known recurrence relation,

$$J_0(c\rho) + J_2(c\rho) = \frac{2}{c\rho} J_1(c\rho)$$

so that

$$\bar{\phi}(\rho) = -\tfrac{1}{2}\pi p_0 c J_1(c\rho). \tag{24}$$

Substituting from Eq. (24) into (4), (5) and (6) we obtain the equations

$$\tfrac{1}{2}(\sigma_x + \sigma_y) = -p_0 c \int_0^\infty e^{-\rho x} \cos \rho y J_1(c\rho) d\rho, \tag{25}$$

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[8] A. C. Stevenson, Phil. Mag., (7) **34**, 766 (1943).

$$\tfrac{1}{2}(\sigma_y - \sigma_x) = p_0 c x \int_0^\infty \rho e^{-\rho x} \cos \rho y J_1(c\rho) d\rho, \tag{26}$$

$$\tau_{xy} = - p_0 c x \int_0^\infty \rho e^{-\rho x} \sin \rho y J_1(c\rho) d\rho \tag{27}$$

for the determination of the components of stress.

Now,

$$\int_0^\infty \rho e^{-\rho z} J_1(c\rho) d\rho = c(c^2 + z^2)^{-3/2},$$

so that writing

$$z = x + iy = re^{i\theta}, \qquad z - ic = r_1 e^{i\theta_1}, \qquad z + ic = r_2 e^{i\theta_2} \tag{28}$$

we obtain the formula

$$\int_0^\infty J_1(c\rho)\rho e^{-\rho x}(\cos \rho y - i \sin \rho y) d\rho = \frac{c}{(r_1 r_2)^{3/2}} e^{-i3/2(\theta_1+\theta_2)}. \tag{29}$$

In a similar way we can establish that

$$\int_0^\infty J_1(c\rho) e^{-\rho x}(\cos \rho y - i \sin \rho y) d\rho = \frac{1}{c}\left\{1 - \frac{r}{(r_1 r_2)^{1/2}} e^{i(\theta - \frac{1}{2}\theta_1 - \frac{1}{2}\theta_2)}\right\}. \tag{30}$$

Equating real and imaginary parts in Eqs. (29) and (30) and substituting into (25), (26), and (27) we obtain the expressions

$$\left.\begin{aligned} \tfrac{1}{2}(\sigma_x + \sigma_y) &= p_0\left\{\frac{r}{(r_1 r_2)^{1/2}} \cos\left(\theta - \tfrac{1}{2}\theta_1 - \tfrac{1}{2}\theta_2\right) - 1\right\}, \\ \tfrac{1}{2}(\sigma_y - \sigma_x) &= p_0 \frac{r \cos\theta}{c}\left(\frac{c^2}{r_1 r_2}\right)^{3/2} \cos\frac{3}{2}(\theta_1 + \theta_2), \\ \tau_{xy} &= - p_0 \frac{r \cos\theta}{c}\left(\frac{c^2}{r_1 r_2}\right)^{3/2} \sin\frac{3}{2}(\theta_1 + \theta_2) \end{aligned}\right\} \tag{31}$$

for the components of stress. Equations (31) are agreement with those derived in the previous paper;[2] in making the comparison it should be noted that the angles θ, θ_1, θ_2 of this note are the complements of the angles denoted by these symbols in the paper quoted.

It follows from Eqs. (23) that these equations are a consequence of the Stevenson equations (22) if we write

$$\Omega(z) = 2p_0[\sqrt{c^2 + z^2} - z], \qquad \omega'(z) = - 2p_0 c^2 (c^2 + z^2)^{-\frac{1}{2}}.$$