

# The operating diagram of a model of two competitors in a chemostat with an external inhibitor

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## Abstract

Understanding and exploiting the inhibition phenomenon, which promotes the stable coexistence of species, is a major challenge in the mathematical theory of the chemostat. Here, we study a model of two microbial species in a chemostat competing for a single resource in the presence of an external inhibitor. The model is a four-dimensional system of ordinary differential equations. Using general monotonic growth rate functions of the species and absorption rate of the inhibitor, we give a complete analysis for the existence and local stability of all steady states. We focus on the behavior of the system with respect of the three operating parameters represented by the dilution rate and the input concentrations of the substrate and the inhibitor. The operating diagram has the operating parameters as its coordinates and the various regions defined in it correspond to qualitatively different asymptotic behavior: washout, competitive exclusion of one species, coexistence of the species around a stable steady state and coexistence around a stable cycle. This bifurcation diagram which determines the effect of the operating parameters, is very useful to understand the model from both the mathematical and biological points of view, and is often constructed in the mathematical and biological literature.

*Keywords:* Chemostat, competition, inhibitor, stability, operating diagram

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## 1. Introduction

The chemostat is an important laboratory apparatus used for the continuous culture of micro-organisms. Competition for single and multiple resources, evolution of resource acquisition, and competition among micro-organisms have been investigated in ecology and biology using chemostats [1, 2, 3, 4]. A detailed mathematical description of competition in the chemostat may be found in [5, 6].

The basic chemostat model predicts that coexistence of two or more microbial populations competing for a single non-reproducing nutrient is not possible. Only the species with the lowest 'break-even' concentration survives, this is the species which consumes less substrate to attain its steady state [7]. This result, known

as the Competitive Exclusion Principle [8], was established under various hypotheses [9, 10, 11, 12]. The reader may consult [13, 14, 15] for a thorough account on the contributions of diverse authors. When the break-even concentrations are very close to each other an interesting phenomenon, known as practical coexistence, occurs [16].

Although this theoretical prediction has been corroborated by the experiences of Hansen and Hubell [17], the biodiversity found in nature as well as in wastewater treatment processes and biological reactors are exceptions to this principle. Several authors [18, 19, 20, 21, 22, 23, 24, 25, 26] studied the inhibition as a factor in the maintenance of the diversity of microbial ecosystems: Can the production of internal inhibitors or the introduction of external inhibitors induce the stable coexistence of competitors in a chemostat-like environment?

In this paper we consider the model introduced by Lenski and Hattingh [25]. In this model, two species compete for a single limiting resource in presence of

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an external inhibitor, like a pesticide or an antibiotic, to which one species is sensitive and the other is resistant. Moreover, the resistant species is able to detoxify the environment, that is to remove the inhibitor from the environment. For some values of the dilution rate, the sensitive species has the lowest break-even concentration and wins the competition in absence of the inhibitor. The presence of the inhibitor allows the coexistence of both species. The complete mathematical analysis of the model in [25] was provided by Hsu and Waltman [23]. Due to the importance of this phenomenon which promotes the stable coexistence of two species competing for a single resource, the results of [23] were discussed in the text book [6] and in the review paper [27]. The aim of this work is to revisit the results of [23, 27, 25, 6] and to discuss several important questions that were unanswered in these works.

The approach in [25] was to fix the biological parameters of the model, together with the dilution rate of the chemostat, and discuss the behavior of the model with respect to the input concentrations of the limiting nutrient and inhibitor, which are operating parameters of the model. Therefore these authors established the ‘operating diagram’ of the model: seven possible outcomes were shown, corresponding to seven regions of the operating diagram. Without detoxification, the competitive exclusion principle holds, see [25], Fig. 1(a). With detoxification two regions of stable coexistence of both species appear, see [25], Fig. 1(b). Using the Routh-Hurwitz theorem on local stability of the coexistence equilibrium, these authors emphasized on the fact that the coexistence equilibrium may be unstable. They gave conditions on the biological parameters for which the coexistence equilibrium becomes unstable. However they did not depicted the region of the operating parameters in which this behavior holds.

The approach in [23, 27, 6] was more mathematical. The authors rescaled the biological and operating parameters of the model, creating a ‘standard’ environment in which the operating parameters are fixed to the value 1. This rescaling is often used in the mathematical literature on the chemostat [6]. The authors established global results and shown that when the coexistence equilibrium is unstable then the model can have an attracting limit cycle. The theory developed in this standard environment potentially permits to present the operating diagrams of the model. However the operating diagram was not presented in [23, 27, 6]. Our main contribution is to present the operating diagram and to give its properties with respect of the biological parameters. The parameter space of the model is ten dimensional: seven biological parameters and three operating

parameters. Exploring all of it is not possible. Our approach to handle this question is to split the question in two intermediary questions. First we fix the biological parameters and present the operating diagram. Second we explore how the operating diagram varies when the biological parameters are changed. The problem is reduced to the determination of the sign of a set of five real valued functions of the dilution rate.

The operating diagram has the operating parameters as its coordinates and the various regions defined in it correspond to qualitatively different dynamics. This bifurcation diagram which determines the effect of the operating parameters, that are controlled by the operator and which are the dilution rate and the input concentrations, is very useful to understand the model from both the mathematical and biological points of view, and is often constructed in the mathematical and biological literature [28, 29, 30, 31, 32, 33, 34, 35, 36, 37].

In this paper we extend [23, 27, 6] by considering general growth functions and by describing the operating diagram. We extend also [25] by describing theoretically the various regions of the operating diagram. In particular we show that for the biological parameters in [25], for all values of the three operating parameters the coexistence equilibrium is stable whenever it exists and we clarify the question of the destabilisation of the coexistence equilibrium that was considered in [25] only through numerical exploration.

The organization of this paper is as follows. In Section 2, we present the model and some properties of its solutions. In Section 3, we discuss the existence and the local asymptotic stability of equilibria. In Section 4, we discuss global results. In Section 5, we present the operating diagrams. In Section 6, we consider examples and we give numerical simulations. A discussion follows in Section 7.

## 2. Mathematical model

The model of the chemostat with external inhibitor [23, 25, 27] we consider here is of the form

$$\begin{cases} S' &= (S^0 - S)D - f(p)f_1(S)\frac{x}{\gamma_1} - f_2(S)\frac{y}{\gamma_2}, \\ x' &= [f(p)f_1(S) - D]x, \\ y' &= [f_2(S) - D]y, \\ p' &= (p^0 - p)D - g(p)y, \end{cases} \quad (1)$$

with  $S(0) \geq 0$ ,  $x(0) > 0$ ,  $y(0) > 0$  and  $p(0) \geq 0$ .  $S(t)$  denotes the concentration of the substrate at time  $t$ ;  $x(t)$ ,  $y(t)$  are the concentrations of the competitors at time  $t$  and  $p(t)$  is the concentration of the external inhibitor.

Variables/ Parameters	Meanings of the Variables/Parameters	Units
$S, x, y, p$	Concentrations of substrate, species and inhibitor	mass/volume
$m_1, m_2$	Maximal growth rates of the competitors	1/time
$K_1, K_2$	Half saturation constants of the competitors	mass/volume
$\delta$	Maximal growth rates of detoxification	1/time
$K$	Half saturation constant of detoxification	mass/volume
$\mu$	Degree of sensitivity of $p$ on $x$	volume/mass
$\gamma_1, \gamma_2$	Growth yield coefficients	dimensionless

Table 1: Meanings and units of the variables and biological parameters.

$S^0 > 0$  is the input concentration of the nutrient,  $D > 0$  is the dilution rate of the chemostat and  $p^0 > 0$  is the input concentration of the inhibitor, all of which are assumed to be constant and are under the control of the experimenter. The parameters  $\gamma_i > 0, i = 1, 2$ , are the growth yield coefficients. The function  $f$  represents the degree of inhibition of  $p$  on the growth rate of  $x$ . The so-called functional responses  $f_i, i = 1, 2$ , represent the specific growth rates of the competitors and the function  $g$  represents the absorption rate of the external inhibitor relative to  $y$ . The global analysis of the model (1) was considered by Hsu and Waltman [23] when

$$\begin{aligned} f(p) &= e^{-\mu p}, & f_1(S) &= \frac{m_1 S}{K_1 + S}, \\ f_2(S) &= \frac{m_2 S}{K_2 + S}, & g(p) &= \frac{\delta p}{K + p}, \end{aligned} \quad (2)$$

where  $\mu, m_i, K_i, i = 1, 2, \delta$  and  $K$  are positive constant parameters whose meaning and units are given in Table 1. Here, except the three variable operating (or control) parameters, which are the input of the inhibitor  $p^0$ , the dilution rate  $D$  and the inflowing substrate  $S^0$ , all the other parameters are biological parameters which depend on the organisms, substrate and inhibitor considered.

In this paper, we consider the general model (1) without restricting ourselves to the special case (2). We suppose only that  $f, f_1, f_2$ , and  $g$  in system (1) are  $C^1$ -functions satisfying the following conditions, see Fig. 1(a):

- (H1)  $f(0) = 1, f(p) \geq 0$  and  $f'(p) < 0$  for all  $p > 0$ .
- (H2) For  $i = 1, 2, f_i(0) = 0$  and  $f'_i(S) > 0$  for all  $S \geq 0$ .
- (H3)  $g(0) = 0$  and  $g'(p) > 0$  for all  $p \geq 0$ .

Following [23, 27], without loss of generality, the operating parameters  $p^0, D$  and  $S^0$  together with the yields

$\gamma_1$  and  $\gamma_2$  can be fixed to 1. This is done by the following scaling of the dependent variables, and time:

$$\hat{S} = \frac{S}{S^0}, \quad \hat{x} = \frac{x}{S^0 \gamma_1}, \quad \hat{y} = \frac{y}{S^0 \gamma_2}, \quad \hat{p} = \frac{p}{p^0}, \quad \hat{t} = Dt, \quad (3)$$

and the following notations

$$\begin{aligned} \hat{f}(\hat{p}) &= f(p^0 \hat{p}), & \hat{g}(\hat{p}) &= \frac{S^0 \gamma_2}{p^0 D} g(p^0 \hat{p}), \\ \hat{f}_i(\hat{S}) &= \frac{1}{D} f_i(S^0 \hat{S}), & i &= 1, 2. \end{aligned} \quad (4)$$

Note that the functions  $\hat{f}, \hat{g}$  and  $\hat{f}_i, i = 1, 2$ , satisfy assumptions (H1), (H2) and (H3). Then, making the changes (3), (4) and dropping all the hats, model (1) is written in the simplified non-dimensional form

$$\begin{cases} S' &= 1 - S - f(p)f_1(S)x - f_2(S)y, \\ x' &= [f(p)f_1(S) - 1]x, \\ y' &= [f_2(S) - 1]y, \\ p' &= 1 - p - g(p)y, \end{cases} \quad (5)$$

where  $f, g$  and  $f_i, i = 1, 2$ , satisfy assumptions (H1), (H2) and (H3). This is the system we will analyze here.

The proof of the following result is standard and hence omitted.

**Proposition 1.** *For non-negative initial conditions, all solutions of system (5) are bounded and remain non-negative for all  $t > 0$ . Moreover, the compact set*

$$\Omega = \{(S, x, y, p) \in \mathbb{R}_+^4 : 0 \leq p \leq 1, S + x + y = 1\}$$

*is positively invariant and is a global attractor for system (5).*

### 3. Existence and stability of equilibria

As it was noticed in [23], the results obtained in [23, 25] for the specific functions (2) remain valid for the general monotone growth functions satisfying assumptions (H1), (H2) and (H3). In this section we give the results in this more general setting.

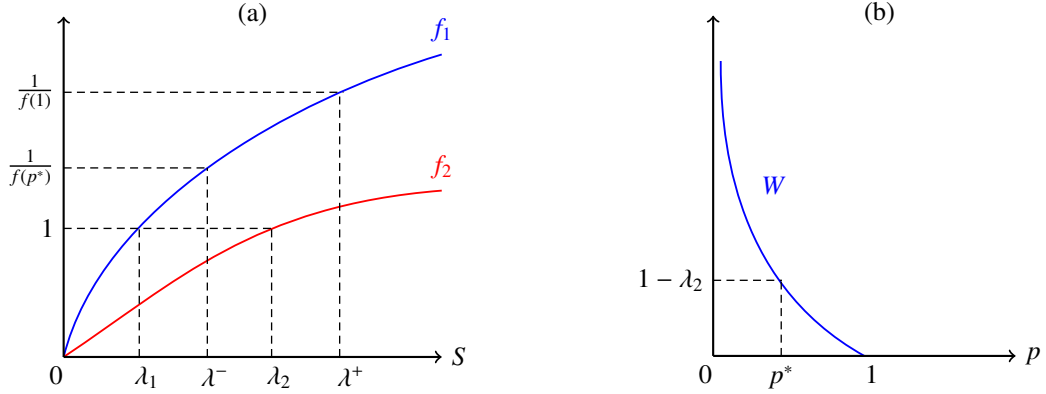


Figure 1: Illustrative graphs of functions (a):  $f_1$  and  $f_2$  and definitions of break-even concentrations  $\lambda_1$ ,  $\lambda_2$ ,  $\lambda^+$  and  $\lambda^-$  which are given by (6) and (8), respectively; (b):  $W$  with the unique positive solution  $p^*$  of equation  $W(p) = 1 - \lambda_2$ , when  $\lambda_2 < 1$ .

### 3.1. Existence of equilibria

Hereafter we use the following conditions and notations: for functions  $f$ ,  $f_i$ ,  $i = 1, 2$ , and  $g$  in (5), conditions (H1) to (H3) hold. When equations  $f_1(S) = 1$ ,  $f_2(S) = 1$  and  $f_1(S) = 1/f(1)$  have solutions, they are unique and then we define the break-even concentrations as:

$$\lambda_1 = f_1^{-1}(1), \quad \lambda_2 = f_2^{-1}(1), \quad \lambda^+ = f_1^{-1}\left(\frac{1}{f(1)}\right). \quad (6)$$

Otherwise, we put  $\lambda_1 = +\infty$ ,  $\lambda_2 = +\infty$  and  $\lambda^+ = +\infty$ . We define the function  $W$  by

$$W(p) = \frac{1-p}{g(p)}, \quad \text{for } p \in (0, 1].$$

Using (H3), for all  $p \in (0, 1)$  we have  $W(p) > 0$ ,  $W'(p) < 0$  and  $\lim_{p \rightarrow 0} W(p) = +\infty$ . Therefore, when  $\lambda_2 < 1$ , equation  $W(p) = 1 - \lambda_2$  admits a unique solution that we denote  $p^*$ :

$$W(p^*) = 1 - \lambda_2. \quad (7)$$

We have  $0 < p^* < 1$ , see Fig. 1(b).

If equation  $f_1(S) = 1/f(p^*)$  has a solution, it is unique and then we set

$$\lambda^- = f_1^{-1}\left(\frac{1}{f(p^*)}\right). \quad (8)$$

Otherwise, we let  $\lambda^- = +\infty$ . Since  $f$  is decreasing we have  $0 < f(1) < f(p^*) < 1$ . Therefore  $1/f(1) > 1/f(p^*)$  and, since  $f_1$  is increasing, the numbers  $\lambda_1$ ,  $\lambda^+$  and  $\lambda^-$  are related as follows, see Fig. 1(a):

$$\lambda_1 < \lambda^- < \lambda^+. \quad (9)$$

The existence of equilibria of system (5) is stated by the following result:

**Proposition 2.** Assume that (H1), (H2) and (H3) are satisfied. System (5) has the following equilibria:

- The washout equilibrium  $E_0 = (1, 0, 0, 1)$ , that always exists.
- The equilibrium  $E_1 = (\lambda^+, 1 - \lambda^+, 0, 1)$  of extinction of species  $y$ , where  $\lambda^+$  is given by (6). This equilibrium exists if and only if  $\lambda^+ < 1$ .
- The equilibrium  $E_2 = (\lambda_2, 0, 1 - \lambda_2, p^*)$  of extinction of species  $x$ , where  $\lambda_2$  and  $p^*$  are given by (6) and (7), respectively. This equilibrium exists if and only if  $\lambda_2 < 1$ .
- The coexistence equilibrium  $E_c = (\lambda_2, x_c, y_c, p_c)$ , where  $\lambda_2$  is given by (6) and  $p_c, y_c$  and  $x_c$  are given by

$$p_c = f^{-1}\left(\frac{1}{f_1(\lambda_2)}\right), \quad y_c = W(p_c), \quad x_c = 1 - \lambda_2 - y_c. \quad (10)$$

This equilibrium exists if and only if  $\lambda_2 < 1$  and  $\lambda^- < \lambda_2 < \lambda^+$ , where  $\lambda^+$  and  $\lambda^-$  are given by (6) and (8), respectively.

*Proof.* The proof is given in Appendix A.1.  $\square$

### 3.2. Local asymptotic stability of equilibria

For the study of the stability of equilibria it is convenient to use the change of variable

$$\Sigma = 1 - S - x - y$$

in system (5) that reveals the cascade structure of the system. Since  $\Sigma' = -\Sigma$ , the system (5) may then be

Equilibria	Existence	Local exponential stability
$E_0 = (1, 0, 0, 1)$	Always	$\lambda^+ > 1$ and $\lambda_2 > 1$
$E_1 = (\lambda^+, 1 - \lambda^+, 0, 1)$	$\lambda^+ < 1$	$\lambda^+ < \lambda_2$
$E_2 = (\lambda_2, 0, 1 - \lambda_2, p^*)$	$\lambda_2 < 1$	$\lambda_2 < \lambda^-$
$E_c = (\lambda_2, x_c, y_c, p_c)$	$\lambda^- < \lambda_2 < \min(\lambda^+, 1)$	$(A + B)(A + B + C)C > BEF$

Table 2: Existence and local asymptotic stability of equilibria of system (5) where  $A, B, C, E, F$  are defined by (14) and  $\lambda_1, \lambda_2, \lambda^+$  and  $\lambda^-$  are given by (6) and (8), respectively. Note that there is a typo in Table 2 of [27]: the condition  $\lambda_1 > 1$  of stability of  $E_0$  should be replaced by  $\lambda^+ > 1$ .

replaced by

$$\begin{cases} \Sigma' &= -\Sigma, \\ x' &= [f(p)f_1(1 - \Sigma - x - y) - 1]x, \\ y' &= [f_2(1 - \Sigma - x - y) - 1]y, \\ p' &= 1 - p - g(p)y. \end{cases} \quad (11)$$

The Jacobian matrix for the linearization of (11) at an equilibrium point  $E^* = (0, x^*, y^*, p^*)$  takes the triangular form

$$J = \begin{bmatrix} -1 & 0 \\ A & M \end{bmatrix},$$

where  $M$  is the square matrix

$$M = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & 0 \\ 0 & m_{32} & m_{33} \end{bmatrix}, \quad (12)$$

with

$$\begin{aligned} m_{11} &= f(p^*)f_1(1 - x^* - y^*) - 1 - x^*f'(p^*)f_1'(1 - x^* - y^*), \\ m_{12} &= -x^*f'(p^*)f_1'(1 - x^* - y^*), \\ m_{13} &= x^*f'(p^*)f_1(1 - x^* - y^*), \quad m_{21} = -y^*f_2'(1 - x^* - y^*), \\ m_{22} &= f_2(1 - x^* - y^*) - 1 - y^*f_2'(1 - x^* - y^*), \\ m_{32} &= -g(p^*), \quad m_{33} = -1 - y^*g'(p^*). \end{aligned}$$

Therefore, the eigenvalues of  $J$  are  $-1$ , together with the eigenvalues of matrix  $M$ . Hence the equilibrium point  $E^*$  is locally exponentially stable (LES) if and only if the eigenvalues of  $M$  are of negative real parts. The local stability of equilibria of system (5) is given by the following result.

**Proposition 3.** *Assume that (H1), (H2) and (H3) are satisfied. The stability of equilibria of (5) is as follows:*

- The equilibrium  $E_0$  is LES if and only if  $\lambda^+ > 1$  and  $\lambda_2 > 1$ .
- The equilibrium  $E_1$ , if it exists, has at least three dimensional stable manifolds and is LES if and only if  $\lambda^+ < \lambda_2$ .

- The equilibrium  $E_2$ , if it exists, has at least three dimensional stable manifolds and is LES if and only if  $\lambda_2 < \lambda^-$ .

- The equilibrium  $E_c$ , if it exists, is LES if and only if  $(A + B)(A + B + C)C > BEF$ , (13)

where  $A > 0, B > 0, C > 0, E > 0$  and  $F > 0$  and are defined by:

$$\begin{aligned} A &= f(p_c)f_1'(\lambda_2)x_c, & B &= f_2'(\lambda_2)y_c, \\ C &= 1 + g'(p_c)y_c, & E &= g(p_c), \\ F &= -f'(p_c)f_1(\lambda_2)x_c. \end{aligned} \quad (14)$$

*Proof.* The proof is given in Appendix A.2.  $\square$

We summarize the results on existence and local stability of equilibria of (5), given by Propositions 2 and 3, in Table 2. We observe that  $E_0$  is LES if and only if  $E_1$  and  $E_2$  do not exist, and  $E_c$  exists if and only if  $E_2$  exists and is unstable, and  $E_1$ , if it exists, is also unstable. One concludes that there is one and only one equilibrium which is stable.

#### 4. Global asymptotic stability of equilibria

The results of [23, 6] on the global asymptotic stability of equilibria of (5) obtained in the case where the growth functions are given by (2) can be extended without added difficulty to the general case where it is simply assumed that  $f, f_1, f_2$  and  $g$  satisfy hypotheses (H1), (H2) and (H3). For the convenience of the reader, we recall these results hereafter.

- If  $\lambda^+ > 1$  and  $\lambda_2 > 1$ , then the washout equilibrium  $E_0$  of system (5) exists and is globally asymptotically stable, see Proposition 3.1 in [23] or [6].
- If  $\lambda^+ < 1$  and  $\lambda^+ < \lambda_2$ , then the boundary equilibrium  $E_1$  of system (5) exists and is globally asymptotically stable with respect to solutions with  $x(0) > 0$ , see Theorem 5.5 in [23] or Theorem 5.1 in [6].

- If  $\lambda_2 < 1$  and  $\lambda_2 < \lambda^-$ , then the boundary equilibrium  $E_2$  of system (5) exists and is globally asymptotically stable with respect to solutions with  $y(0) > 0$ , see Theorem 5.4 in [23] or Theorem 5.2 in [6].
- Let  $\lambda^- < \lambda_2 < \lambda^+$ , that is, the positive equilibrium  $E_c$  exists, then the  $\omega$ -limit set of every solution of system (5) with positive initial conditions remains at a positive distance away from the boundary of  $\mathbb{R}_+^4$ . More precisely, for any solution  $(S(t), x(t), y(t), p(t))$  of (5) with  $x(0) > 0$  and  $y(0) > 0$ , one has:

$$\liminf_{t \rightarrow \infty} x(t) > 0 \quad \text{and} \quad \liminf_{t \rightarrow \infty} y(t) > 0 \quad (15)$$

See Theorem 6.1 in [23] or Theorem 7.1 in [6].

From a biological point of view, (15) guarantees the coexistence of both species  $x$  and  $y$ . However, it does not give the global asymptotic behavior. To study the global asymptotic behavior of system (5) when  $E_c$  exists, we need a supplementary condition on the following limiting system, obtained by putting  $\Sigma = 0$  in system (11):

$$\begin{cases} x' &= [f(p)f_1(1-x-y)-1]x, \\ y' &= [f_2(1-x-y)-1]y, \\ p' &= 1-p-g(p)y. \end{cases} \quad (16)$$

- Suppose that system (16) has no limit cycles. Then the positive equilibrium  $E_c$  is globally asymptotically stable with respect to solutions with positive initial conditions, see Theorem 6.2 in [23] or Theorem 7.2 in [6].

## 5. Operating diagrams

In this section we give our main result which is the discussion of the existence and stability of equilibria of (1) with respect of the operating parameters  $D$ ,  $p^0$  and  $S^0$ . We assume that  $f$ ,  $g$  and  $f_1$ ,  $f_2$  are fixed. The following change of dependent variables

$$\hat{x} = \frac{x}{\gamma_1}, \quad \hat{y} = \frac{y}{\gamma_2}, \quad (17)$$

which reduces (1) to

$$\begin{cases} S' &= (S^0 - S)D - f(p)f_1(S)\hat{x} - f_2(S)\hat{y}, \\ \hat{x}' &= [f(p)f_1(S) - D]\hat{x}, \\ \hat{y}' &= [f_2(S) - D]\hat{y}, \\ p' &= (p^0 - p)D - \hat{g}(p)\hat{y}, \end{cases} \quad (18)$$

where  $\hat{g}(p) = \gamma_2 g(p)$ . Note that  $\hat{g}$  satisfies (H3). Therefore, without loss of generality we can assume that the yields in (1) are equal to 1 ( $\gamma_1 = \gamma_2 = 1$ ). So, we consider the system

$$\begin{cases} S' &= (S^0 - S)D - f(p)f_1(S)x - f_2(S)y, \\ x' &= [f(p)f_1(S) - D]x, \\ y' &= [f_2(S) - D]y, \\ p' &= (p^0 - p)D - g(p)y. \end{cases} \quad (19)$$

### 5.1. Existence and stability of equilibria

To emphasize the dependence of the equilibria of (19) with respect to the operating parameters, we rewrite here the results of Section 3, obtained for the non dimensional system (5). Using the inverse functions  $f_1^{-1} : I_1 \rightarrow \mathbb{R}^+$  and  $f_2^{-1} : I_2 \rightarrow \mathbb{R}^+$  where

$$I_1 = [0, f_1(+\infty)), \quad I_2 = [0, f_2(+\infty)),$$

which are increasing, we define the break-even concentrations as

$$\begin{aligned} \lambda_1(D) &= f_1^{-1}(D), & \lambda_2(D) &= f_2^{-1}(D), \\ \lambda^+(D, p^0) &= f_1^{-1}\left(\frac{D}{f(p^0)}\right), \end{aligned} \quad (20)$$

which are the solutions of equations  $f_1(S) = D$ ,  $f_2(S) = D$  and  $f_1(S) = \frac{D}{f(p^0)}$ , respectively. Note that  $\lambda_1$  is defined on  $I_1$ ,  $\lambda_2$  is defined on  $I_2$  and  $\lambda^+$  is defined for  $(D, p^0)$  such that  $p^0 \geq 0$  and  $D/f(p^0) \in I_1$ .

We define the function  $W$  by

$$W(p, D, p^0) = \frac{(p^0 - p)D}{g(p)}, \quad \text{for } p \in (0, p^0].$$

Note that  $W$  is defined for  $(p, D, p^0)$  such that  $p \geq 0$ ,  $D \geq 0$  and  $0 < p \leq p^0$ . Note also that  $\frac{\partial W}{\partial p} < 0$ . Therefore, when  $\lambda_2 < S^0$ , equation  $W(p, D, p^0) = S^0 - \lambda_2$  has a unique solution denoted by  $p^* = p^*(D, p^0, S^0)$

$$W(p^*, D, p^0) = S^0 - \lambda_2. \quad (21)$$

If equation  $f_1(S) = D/f(p^*)$  has a solution, it is unique and then we set

$$\lambda^-(D, p^0, S^0) = f_1^{-1}\left(\frac{D}{f(p^*)}\right). \quad (22)$$

We define

$$p_c(D) = f^{-1}\left(\frac{D}{f_1(\lambda_2(D))}\right). \quad (23)$$

Note that  $p_c(D)$  is defined for  $D \in I_c$  where

$$I_c = \{D \in I_1 \cap I_2 : \lambda_1(D) < \lambda_2(D)\}. \quad (24)$$

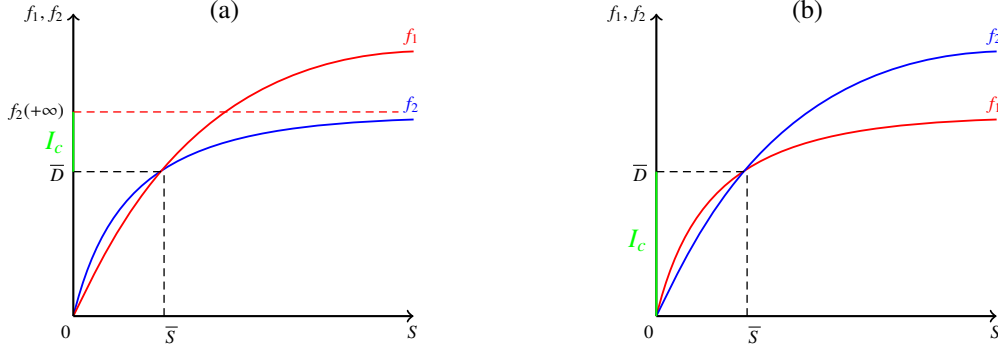


Figure 2: Graphs of  $f_1$  (in red) and  $f_2$  (in blue) when equation  $f_1(S) = f_2(S)$  has a positive solution  $S = \bar{S}$  and graphical depiction of  $I_c$ . (a):  $I_c = (\bar{D}, f_2(+\infty))$ . (b):  $I_c = (0, \bar{D})$  where  $\bar{D} = f_1(\bar{S}) = f_2(\bar{S})$ . The interval  $I_c$  is defined by (24)

Equilibria	Existence	Local exponential stability
$E_0 = (S^0, 0, 0, p^0)$	Always	$\lambda^+ > S^0$ & $\lambda_2 > S^0$
$E_1 = (\lambda^+, S^0 - \lambda^+, 0, p^0)$	$\lambda^+ < S^0$	$\lambda^+ < \lambda_2$
$E_2 = (\lambda_2, 0, S^0 - \lambda_2, p^*)$	$\lambda_2 < S^0$	$\lambda_2 < \lambda^-$
$E_c = (\lambda_2, x_c, y_c, p_c)$	$\lambda^- < \lambda_2 < \min(\lambda^+, S^0)$	$(A + B)(A + B + C)C > BEF$

Table 3: Existence and stability of equilibria of (19) where  $\lambda_1, \lambda_2, \lambda^+, \lambda^-, A, B, C, E$  and  $F$  are given by (20), (22) and (28).

For simplicity we assume that equation  $f_1(S) = f_2(S)$  has at most one positive solution  $S = \bar{S} > 0$ , see Fig. 2. This property holds when  $f_1(S)$  and  $f_2(S)$  are Monod functions. The case of multiple intersections can be treated similarly. We have

- $I_c = \emptyset$  if  $f_1(S) < f_2(S)$  for all  $S > 0$ .
- $I_c = (\bar{D}, f_2(+\infty))$ , if  $f_1(S) < f_2(S)$  for  $0 < S < \bar{S}$  and  $f_1(S) > f_2(S)$  for  $S > \bar{S}$ , see Fig. 2(a).
- $I_c = (0, \bar{D})$ , if  $f_1(S) > f_2(S)$  for  $0 < S < \bar{S}$  and  $f_1(S) < f_2(S)$  for  $S > \bar{S}$ , see Fig. 2(b).
- $I_c = (0, f_2(+\infty))$  if  $f_1(S) > f_2(S)$  for all  $S > 0$ .

We define

$$\begin{aligned} y_c(D, p^0) &= W(p_c(D), D, p^0), \\ x_c(D, p^0, S^0) &= S^0 - \lambda_2(D) - y_c(D, p^0). \end{aligned} \quad (25)$$

Note that  $y_c$  is defined for  $(D, p^0) \in J_c$ , where

$$J_c = \{(D, p^0) \in I_c \times \mathbb{R}^+ : 0 < p_c(D) < p^0\}, \quad (26)$$

with  $I_c$  defined by (24). Note that  $x_c$  is defined for  $(D, p^0, S^0) \in \mathcal{D}_c$  where

$$\mathcal{D}_c = \{(D, p^0, S^0) \in J_c \times \mathbb{R}^+ : S^0 > \lambda_2(D) + y_c(D, p^0)\}, \quad (27)$$

with  $J_c$  defined by (26).

To avoid cumbersome notations, and when there is no risk of confusion, we will omit to mention the operating parameters  $D, p^0$  and  $S^0$  in  $\lambda_2, \lambda^+, \lambda^-, p^*, p_c, y_c$  and  $x_c$ . Straightforward computations, similar to those used in the proofs of Propositions 2 and 3 show that the following result holds.

**Proposition 4.** *Assume that (H1), (H2) and (H3) are satisfied. Let  $\lambda_2, \lambda^+$  and  $\lambda^-$  be defined by (20) and (22), respectively. Let  $p^*$  be defined by (21). Let  $p_c, x_c$  and  $y_c$  be defined by (23) and (25), respectively. System (19) has the following equilibria:*

- The washout equilibrium  $E_0 = (S^0, 0, 0, p^0)$ .
- The boundary equilibrium  $E_1 = (\lambda^+, S^0 - \lambda^+, 0, p^0)$  of extinction of species  $y$ .
- The boundary equilibrium  $E_2 = (\lambda_2, 0, S^0 - \lambda_2, p^*)$  of extinction of species  $x$ .
- The positive equilibrium  $E_c = (\lambda_2, x_c, y_c, p_c)$  of co-existence of the species.

The conditions of existence and stability of these equilibria are given in Table 3 where  $A = A(D, p^0, S^0)$ ,  $B = B(D, p^0)$ ,  $C = C(D, p^0)$ ,  $E = E(D)$ ,  $F = F(D, p^0, S^0)$

Equilibria	Existence	Local exponential stability
$E_0 = (S^0, 0, 0, p^0)$	Always	$D > \max(f(p^0)f_1(S^0), f_2(S^0))$
$E_1 = (\lambda^+, S^0 - \lambda^+, 0, p^0)$	$D < f(p^0)f_1(S^0)$	$D < F_1(D, p^0)$
$E_2 = (\lambda_2, 0, S^0 - \lambda_2, p^*)$	$D < f_2(S^0)$	$S^0 < F_2(D, p^0)$
$E_c = (\lambda_2, x_c, y_c, p_c)$	$D > F_1(D, p^0) \& S^0 > F_2(D, p^0)$	$F_3(D, p^0, S^0) > 0$

Table 4: Existence and stability of equilibria of System (19) with respect of the operating parameters. The functions  $F_1, F_2, F_3$  are defined by (30), (31), (33), respectively, and  $\lambda_1, \lambda_2, \lambda^+$  and  $\lambda^-$  are given by 20 and 22, respectively.

are defined by

$$\begin{aligned}
A(D, p^0, S^0) &= \alpha(D)x_c(D, p^0, S^0), \\
B(D, p^0) &= \beta(D)y_c(D, p^0), \\
C(D, p^0) &= D + \gamma(D)y_c(D, p^0), \\
E(D) &= g(p_c(D)), \\
F(D, p^0, S^0) &= \phi(D)x_c(D, p^0, S^0).
\end{aligned} \tag{28}$$

Here  $\alpha(D), \beta(D), \gamma(D)$  and  $\phi(D)$  are given by

$$\begin{aligned}
\alpha &= f(p_c)f'_1(\lambda_2), \quad \beta = f'_2(\lambda_2), \quad \gamma = g'(p_c), \\
\phi &= -f'(p_c)f_1(\lambda_2).
\end{aligned} \tag{29}$$

### 5.2. Existence and stability of equilibria with respect of operating parameters

In what follows, our aim is to express the conditions of existence and stability of the equilibria in Proposition 4 with respect of the operating parameters  $D, p^0$  and  $S^0$ . For this purpose, we need the following definitions. We let

$$F_1(D, p^0) = f(p^0)f_1(f_2^{-1}(D)). \tag{30}$$

Note that  $F_1$  is defined on  $I_2 \times \mathbb{R}^+$ . We let

$$F_2(D, p^0) = f_2^{-1}(D) + W(p_c(D), D, p^0). \tag{31}$$

Note that  $F_2$  is defined for  $(D, p^0) \in J_c$ , where  $J_c$  is given by (26). We have the following result.

**Lemma 5.** *The following equivalences hold:*

$$p_c(D) < p^0 \iff D > F_1(D, p^0) \iff \lambda_2(D) < \lambda^+(D, p^0).$$

and

$$\begin{aligned}
p_c(D) > p^*(D, p^0, S^0) &\iff S^0 > F_2(D, p^0) \\
&\iff \lambda_2(D) > \lambda^-(D, p^0, S^0).
\end{aligned}$$

*Proof.* The proof is given in Appendix A.3.  $\square$

Using Lemma 5, the condition  $p_c(D) < p^0$  in the definition (26) of  $J_c$  can be written as follows

$$J_c = \{(D, p^0) : D \in I_c, D > F_1(D, p^0)\}.$$

If we assume that equation  $f(p^0)f_1(S) = f_2(S)$  has at most one positive solution  $S = \bar{S}(p^0) > 0$ , which is the case when  $f_1(S)$  and  $f_2(S)$  are Monod functions, then the function  $F_2$  is defined for all  $(D, p^0) \in I_c \times \mathbb{R}^+$  such that  $\bar{D}(p^0) < D < f_2(+\infty)$ , if  $f(p^0)f_1(S) > f_2(S)$  for  $0 < S < \bar{S}(p^0)$ , or  $0 < D < \bar{D}(p^0)$ , if  $f(p^0)f_1(S) < f_2(S)$  for  $0 < S < \bar{S}(p^0)$ . Here  $\bar{D}(p^0) = f_2(\bar{S}(p^0))$ .

We define also the function

$$F_3(D, p^0, S^0) = (A + B)(A + B + C)C - BEF, \tag{32}$$

where  $A, B, C, E$  and  $F$  are defined by (28). Note that  $F_3$  is defined for  $(D, p^0, S^0) \in \mathcal{D}_c$ , where  $\mathcal{D}_c$  is given by (27). Therefore, using Lemma 5, we have

$$\mathcal{D}_c = \{(D, p^0, S^0) : D \in I_c, D > F_1(D, p^0), S^0 > F_2(D, p^0)\}.$$

Using these notations, we have the following description of existence and stability of the equilibria of (19).

**Theorem 6.** *Assume that the hypotheses and notations of Proposition 4 hold. The conditions of existence and stability of equilibria of (19) can be expressed with respect to the operating parameters  $D, p^0$  and  $S^0$  as in Table 4. where  $F_1(D, p^0), F_2(D, p^0)$  and  $F_3(D, p^0, S^0)$  are defined by (30), (31) and (32), respectively.*

*Proof.* Using (20) and hypothesis (H2), the condition  $\lambda^+ > S^0$  and  $\lambda_2 > S^0$  of stability of  $E_0$  in Proposition 4 is equivalent to  $D > f(p^0)f_1(S^0)$  and  $D > f_2(S^0)$ . Similarly, the condition  $\lambda^+ < S^0$  [resp.  $\lambda_2 < S^0$ ] of existence of  $E_1$  [resp.  $E_2$ ] in Proposition 4 is equivalent to  $D < f(p^0)f_1(S^0)$  [resp.  $D < f_2(S^0)$ ].

We consider now the stability of  $E_1$  and  $E_2$ . Using Lemma 5, the condition  $\lambda^+ < \lambda_2$  of stability of  $E_1$  in Proposition 4 is equivalent to  $D < F_1(D, p^0)$ . On the other hand, using Lemma 5, the condition  $\lambda_2 < \lambda^-$  of stability of  $E_2$  in Proposition 4 is equivalent to  $S^0 < F_2(D, p^0)$ .

Let us consider now the existence and stability of  $E_c$ . Using Lemma 5, we see that the condition  $\lambda^- < \lambda_2 < \lambda^+$  of existence of  $E_c$  in Proposition 4 is equivalent to  $D > F_1(D, p^0)$  and  $S^0 > F_2(D, p^0)$ . Finally,



using the definition (32) of the function  $F_3$ , the condition of stability of  $E_c$  in Proposition 4 is equivalent to  $F_3(S^0, D, p^0) > 0$ .  $\square$

### 5.3. Instability of the positive equilibrium

We give now the necessary and sufficient conditions on the operating parameters  $D$ ,  $p^0$  and  $S^0$  such that the positive equilibrium  $E_c$  is unstable, that is we discuss the sign of  $F_3(D, p^0, S^0)$ . We have

$$F_3 = a_2x_c^2 + a_1x_c + a_0. \quad (33)$$

The coefficients  $a_2 = a_2(D, p^0)$ ,  $a_1 = a_1(D, p^0)$  and  $a_0 = a_0(D, p^0)$  of this polynomial are given by

$$a_2 = \alpha^2 C, \quad a_1 = \alpha(2B+C)C - \phi BE, \quad a_0 = B(B+C)C, \quad (34)$$

where  $B$ ,  $C$ ,  $E$  are defined by (28) and  $\alpha$ ,  $\phi$  are given by (29). Hence,  $F_3$  given by (33), appears as a second order polynomial in  $x_c$  whose coefficients are depending only on  $D$  and  $p^0$  and not on  $S^0$ . Let  $\Delta = \Delta(D, p^0)$  be the discriminant of  $F_3$ :

$$\Delta = a_1^2 - 4a_0a_2. \quad (35)$$

The roots of  $F_3$ ,

$$x_1(D, p^0) = \frac{-a_1 - \sqrt{\Delta}}{2a_2} \quad \text{and} \quad x_2(D, p^0) = \frac{-a_1 + \sqrt{\Delta}}{2a_2}, \quad (36)$$

exist and are positive if and only if

$$a_1(D, p^0) < 0 \quad \text{and} \quad \Delta(D, p^0) > 0. \quad (37)$$

We define the following functions

$$\begin{aligned} F_4(D, p^0) &= F_2(D, p^0) + x_1(D, p^0), \\ F_5(D, p^0) &= F_2(D, p^0) + x_2(D, p^0), \end{aligned} \quad (38)$$

where  $F_2(D, p^0)$  is given by (31) and  $x_1(D, p^0)$ ,  $x_2(D, p^0)$  are given by (36).

We give now necessary and sufficient conditions on the operating parameters  $D$  and  $p^0$  such that (37) hold. We have

$$a_1 = b_2y_c^2 + b_1y_c + b_0, \quad (39)$$

where the coefficients  $b_i = b_i(D)$  are given by

$$b_2 = \alpha\gamma(2\beta + \gamma), \quad b_1 = 2\alpha D(\beta + \gamma) - \beta E\phi, \quad b_0 = \alpha D^2, \quad (40)$$

with  $E$ ,  $\alpha$ ,  $\beta$ ,  $\gamma$  and  $\phi$  defined by (28) and (29), respectively. We have

$$\Delta = c_4y_c^4 + c_3y_c^3 + c_2y_c^2 + c_1y_c + c_0, \quad (41)$$

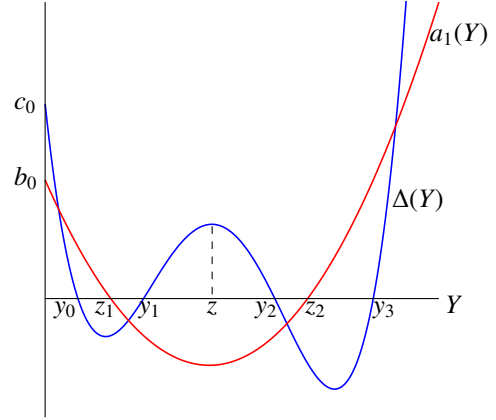


Figure 3: The graphs of  $a_1(Y)$  and  $\Delta(Y)$ , defined by (42), showing the relative positions of the roots  $y_i = y_i(D)$ ,  $i = 0 \dots 3$ , of the fourth order polynomial  $\Delta(Y)$  with respect to the roots  $z_1 = z_1(D)$  and  $z_2 = z_2(D)$  of the second order polynomial  $a_1(Y)$ , when  $a_1(z) < 0$  and  $\Delta(z) > 0$  where  $z = z(D)$  is the local maximum of  $\Delta(Y)$ .

where the coefficients  $c_i = c_i(D)$  are given by

$$c_4 = \alpha^2\gamma^4, \quad c_3 = 4\alpha^2D\gamma^3 - 2\alpha\beta\gamma E\phi(2\beta + \gamma),$$

$$c_2 = 6\alpha^2D^2\gamma^2 + \beta^2E^2\phi^2 - 4\alpha\beta(\beta + \gamma)DE\phi,$$

$$c_1 = 4\alpha^2\gamma D^3 - 2\alpha\beta D^2 E\phi, \quad c_0 = \alpha^2 D^4,$$

with  $E$ ,  $\alpha$ ,  $\beta$ ,  $\gamma$  and  $\phi$  defined by (28) and (29), respectively. Hence,  $a_1$  given by (39) and  $\Delta$  given by (41), appear as a second order polynomial and a fourth order polynomial in  $y_c$ , respectively, whose coefficients are depending only on  $D$  and not on  $p^0$  nor  $S^0$ . For the convenience of the notations we denote by

$$\begin{aligned} a_1(Y) &= b_2Y^2 + b_1Y + b_0, \\ \Delta(Y) &= c_4Y^4 + c_3Y^3 + c_2Y^2 + c_1Y + c_0, \end{aligned} \quad (42)$$

the polynomials (39) and (41). Notice first that, since  $b_2 > 0$ , we have  $a_1(Y) < 0$  if and only if  $z_1(D) < Y < z_2(D)$  where  $z_1(D)$  and  $z_2(D)$  are the positive real roots of  $a_1(Y)$ . This condition holds if and only if  $b_1(D) < 0$  and  $\Delta_1(D) > 0$  where  $\Delta = \Delta_1(D)$  is the discriminant of the polynomial  $a_1$ :

$$\Delta_1 = b_1^2 - 4b_0b_2. \quad (43)$$

If this discriminant is positive, the roots  $z_1(D)$  and  $z_2(D)$  are given by

$$z_1(D) = \frac{-b_1 - \sqrt{\Delta_1}}{2b_2} \quad \text{and} \quad z_2(D) = \frac{-b_1 + \sqrt{\Delta_1}}{2b_2}. \quad (44)$$

For the study of the sign of  $\Delta(Y)$  we use the following facts. Since  $a_1(z_1(D)) = a_1(z_2(D)) = 0$ , from  $\Delta = a_1^2 - 4a_0a_2$  we deduce that

$$\Delta(z_1(D)) < 0 \text{ and } \Delta(z_2(D)) < 0.$$

Therefore, from  $\Delta(0) = c_0 > 0$  and  $\Delta(+\infty) = +\infty$  it is deduced that the polynomial  $\Delta(Y)$  has at least two roots

$$y_0(D) \in (0, z_1(D)) \text{ and } y_3(D) \in (z_2(D), +\infty).$$

The condition (37) is satisfied if and only if  $\Delta(Y)$  takes positive values on the interval  $(z_1(D), z_2(D))$  on which  $a_1(Y) < 0$ , see Fig. 3. If this condition holds then  $\Delta(Y)$  has necessarily three extremal points, that is to say, its polynomial derivative,

$$\Delta'(Y) = 4c_4Y^3 + 3c_3Y^2 + 2c_2Y + c_1,$$

has three real roots. A necessarily and sufficient condition for this is that the discriminant of the polynomial  $\Delta'(Y)$  is positive. Let us denote by  $\Delta_2 = \Delta_2(D)$  this discriminant to emphasize its dependence on the sole operating parameter  $D$ :

$$\Delta_2 = -27d_0^2d_3^2 + 18d_0d_1d_2d_3 - 4d_0d_2^3 - 4d_1^3d_3 + d_1^2d_2^2, \quad (45)$$

where  $d_3 = 4c_4$ ,  $d_2 = 3c_3$ ,  $d_1 = 2c_2$  and  $d_0 = c_1$ . If  $\Delta_2(D) > 0$  then  $\Delta'(Y)$  has three real roots  $z_b(D)$ ,  $z(D)$  and  $z_\#(D)$  such that  $z_b(D) < z(D) < z_\#(D)$ . Thus

$$z(D) \text{ is the middle root of } \Delta'(Y). \quad (46)$$

If  $a_1(z(D)) < 0$  and  $\Delta(z(D)) > 0$  then the polynomial  $\Delta(Y)$  has two supplementary real roots  $y_1(D) \in (z_1(D), z(D))$  and  $y_2(D) \in (z(D), z_2(D))$ , see Fig. 3. Thus  $y_1(D)$  and  $y_2(D)$  are defined by

$$\begin{aligned} \Delta(y_1(D)) = \Delta(y_2(D)) = 0 \\ \text{and } z_1(D) < y_1(D) < y_2(D) < z_2(D). \end{aligned} \quad (47)$$

Therefore (37) holds only if  $D \in I_3$ , where  $I_3$  is the subset of  $I_c$  defined by

$$\begin{aligned} I_3 = \{D \in I_c : b_1(D) < 0, \Delta_1(D) > 0, \Delta_2(D) > 0, \\ a_1(z(D)) < 0, \Delta(z(D)) > 0\}, \end{aligned} \quad (48)$$

where  $I_c$  is defined by (24),  $b_1$  is given by (40),  $\Delta_1$  is given by (43),  $\Delta_2$  is given by (45),  $a_1$  is given by (34),  $\Delta$  is given by (35) and  $z(D)$  is given by (46). If  $D \in I_3$  then  $\Delta(Y) > 0$  and  $a_1(Y) < 0$  if and only if  $y_1(D) < Y < y_2(D)$ . We define the following functions

$$\begin{aligned} F_6(D) = p_c(D) + \frac{1}{D}y_1(D)g(p_c(D)), \\ F_7(D) = p_c(D) + \frac{1}{D}y_2(D)g(p_c(D)), \end{aligned} \quad (49)$$

where  $p_c(D)$  is defined by (23) and  $y_1(D)$ ,  $y_2(D)$  are given by (47). We can determine the sign of  $F_3$ , that is the stability of  $E_c$ , as stated in the following result.

**Theorem 7.** *The positive equilibrium  $E_c$  is unstable only if the subset  $I_3$  of  $I_c$  given by (48) is non empty. If this condition holds then  $E_c$  is unstable if and only if the three following conditions are satisfied by the operating parameters  $D$ ,  $p^0$  and  $S^0$ :*

1.  $D \in I_3$ ,
2.  $F_6(D) < p^0 < F_7(D)$  where  $F_6(D)$  and  $F_7(D)$  are given by (49),
3.  $F_4(D, p^0) < S^0 < F_5(D, p^0)$  where  $F_4(D, p^0)$  and  $F_5(D, p^0)$  are given by (38).

*Proof.* If  $\Delta > 0$ , the roots of  $F_3 = 0$  are  $x_1(D, p^0)$  and  $x_2(D, p^0)$ . Their product is equal to  $\frac{a_0}{a_2}$  which is positive. Therefore, the roots exist and are positive if and only if (37) holds. The roots  $z_1(D)$  and  $z_2(D)$  of  $a_1(Y)$ , given by (44), exist and are positive if and only if  $b_1(D) < 0$  and  $\Delta_1(D) > 0$  which are the two first conditions in the definition of  $I_3$  (48). Now,  $\Delta(Y)$  takes positive values between  $z_1(D)$  and  $z_2(D)$  if and only if the three last conditions in (48) hold. Let  $y_1(D)$  and  $y_2(D)$  be the roots of  $\Delta(Y)$  defined by (47), that is such that  $z_1(D) < y_1(D) < y_2(D) < z_2(D)$ . One has  $a_1 < 0$  and  $\Delta > 0$  if and only if

$$y_1(D) < y_c < y_2(D). \quad (50)$$

Using (25), we have

$$y_c(D, p^0) = D \frac{p^0 - p_c(D)}{g(p_c(D))}.$$

Therefore, (50) is equivalent to the condition 2 in the theorem. On the other hand  $F_3 < 0$  if and only if  $x_c$  is between the roots, that is,

$$x_1(D, p^0) < x_c < x_2(D, p^0), \quad (51)$$

where  $x_1(D, p^0)$  and  $x_2(D, p^0)$  are defined by (36). Using (25) and (31) we have  $x_c = S^0 - F_2(D, p^0)$ . Therefore, (51) is equivalent to the condition 3 in the theorem.  $\square$

#### 5.4. Construction of the operating diagram

The effect of the operating conditions on the asymptotic behavior of the system can be summarized with the aid of the operating diagram. The operating diagram has the operating parameters  $D$ ,  $S^0$  and  $p^0$  as its coordinates and the various regions defined in it correspond to qualitatively different asymptotic behaviors. It is not easy

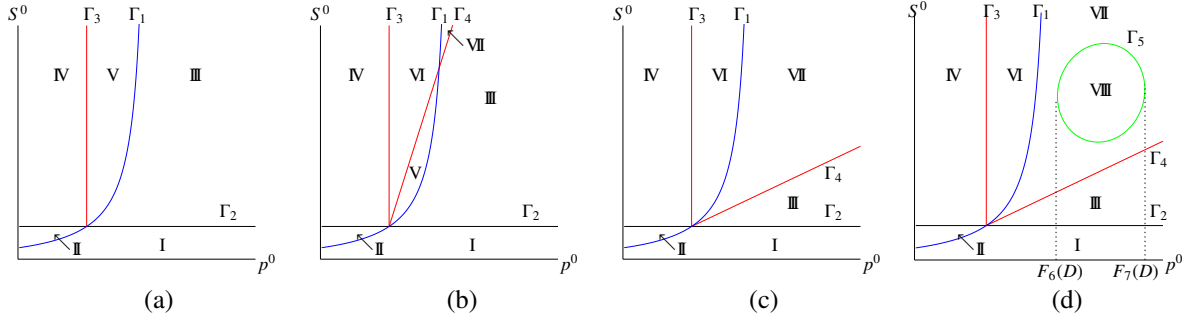


Figure 4: Illustrative operating diagrams for  $D$  fixed: The curves  $\Gamma_i$ ,  $i = 0 \dots 5$  defined in the Table 5 separate the operating plane  $(p^0, S^0)$  into eight regions labeled  $I, \dots, VIII$ . (a) corresponds to the case without detoxification; in case (b),  $E_c$  is stable whenever it exists; in case (c) the stability of  $E_c$  does not always occur and a region of instability can appear as shown in case (d). The existence and stability of equilibria  $E_0, E_1, E_2$  and  $E_c$  in the regions  $I, \dots, VIII$  of these diagrams are shown by the Table 6.

to represent the regions of existence and stability of the equilibria in the three dimensional space of the operating parameters  $D, p^0$  and  $S^0$ . For this reason we will fix the operating parameter  $D$  and show the regions of existence and stability in the operating plane  $(p^0, S^0)$ , see Fig. 4 and Fig. 5(b). The boundaries of the regions in the operating diagram are locations where bifurcations occur. In order to construct the operating diagram of the system one must compute these boundaries. These boundaries are defined by formulas (52), (53), (54), (55) and (56) below. The surface  $\Gamma_1$  defined by

$$\Gamma_1 := \{(D, p^0, S^0) : D = f(p^0)f_1(S^0)\} \quad (52)$$

is the border to which  $E_1$  exists. The surface  $\Gamma_2$  defined by

$$\Gamma_2 := \{(D, p^0, S^0) : D = f_2(S^0)\} \quad (53)$$

is the border to which  $E_2$  exists. The surface  $\Gamma_3$  defined by

$$\Gamma_3 := \{(D, p^0, S^0) : D = F_1(D, p^0), D < f(p^0)f_1(S^0)\} \quad (54)$$

is the border to which  $E_1$  is stable. The surface  $\Gamma_4$  defined by

$$\Gamma_4 := \{(D, p^0, S^0) : S^0 = F_2(D, p^0), D < f_2(S^0)\} \quad (55)$$

is the border to which  $E_2$  is stable. The surfaces  $\Gamma_3$  and  $\Gamma_4$  are the border to which  $E_c$  exists. The surface  $\Gamma_5$  defined by

$$\Gamma_5 := \{(D, p^0, S^0) : F_3(D, p^0, S^0) = 0\} \quad (56)$$

is the border to which  $E_c$  is unstable.

Table 5 gives the descriptions of these boundaries in the operating plane  $(p^0, S^0)$ , where  $D > 0$  is fixed. The

Boundary	Equation in $(p^0, S^0)$ , with $D$ fixed
$\Gamma_1$	Graph of $S^0 = f_1^{-1}\left(\frac{D}{f(p^0)}\right)$
$\Gamma_2$	Horizontal line $S^0 = \lambda_2(D)$
$\Gamma_3$	$\left\{ \begin{array}{l} \text{Vertical line } p^0 = p_c(D) \\ \text{and } S^0 > \lambda_2(D) \end{array} \right.$
$\Gamma_4$	$\left\{ \begin{array}{l} \text{Oblique line} \\ S^0 = W(p_c(D), D, p^0) + \lambda_2(D) \\ \text{and } p^0 > p_c(D) \end{array} \right.$
$\Gamma_5$	$\left\{ \begin{array}{l} \text{Graphs of } S^0 = F_4(D, p^0) \\ \text{or } S^0 = F_5(D, p^0) \end{array} \right.$

Table 5: Boundaries of the regions in the operating diagram.

curves  $\Gamma_i$ ,  $i = 1, 2, 3, 4$ , intersect at point  $(p^0, S^0)$  where  $p^0 = p_c(D)$  and  $S^0 = \lambda_2(D)$ , see Fig. 4 and Fig. 5(b). The curves  $\Gamma_i$ ,  $i = 1..4$ , separate the operating plane  $(p^0, S^0)$  into at most seven regions, as illustrated by Fig. 4, labeled I, II, III, IV, V, VI and VII. Some of these regions may be empty as shown on Fig. 4(a,c). Some of them may be not connected, as shown in Section 6.4, where region VII has two connected components, see Fig. 14(a). Fig. 4(a) corresponds to the case without detoxification, that is to say  $g(p) = 0$ , where regions VI and VII are empty. On the other hand, Fig. 4(c) corresponds to the case where the tangent of  $\Gamma_1$  at point  $(p_c, \lambda_2)$  is above  $\Gamma_4$ , where the region V has disappeared. Moreover subregions VIII  $\subset$  VII and IX  $\subset$  VI may occur, on which  $E_c$  is unstable, see Fig. 4(d) and Fig. 5(b). The construction of these regions will be explained in Section 5.6 below. The behavior of the system in each of the nine regions I, ..., IX is given by Theorem 6. This behavior is summarized in Table 6.

Before giving the construction of regions VIII and IX

	I	II	III	IV	V	VI	VII	VIII	IX
$E_0$	S	U	U	U	U	U	U	U	U
$E_1$		S		S	U	U			U
$E_2$			S	U	S	U	U	U	U
$E_c$						S	S	U	U

Table 6: Existence and stability of equilibria  $E_0$ ,  $E_1$ ,  $E_2$  and  $E_c$ , defined in Proposition 4, in the various the regions, labelled I to IX, of the operating diagrams in Fig. 4, Fig. 5(b), Fig. 7(b), Fig. 8(b), Fig. 9(b), Fig. 10(b), Fig. 11(b), Fig. 12(b), Fig. 13(b) and Fig. 14. The letter S (resp. U) means stable (resp. unstable) and no letter means that the equilibrium does not exist.

we state in the next section two sufficient conditions of stability of the positive equilibrium, for which the regions VIII and IX are necessarily empty.

### 5.5. Sufficient conditions of stability of the coexistence equilibrium

As a corollary of Theorem 7, it is easy to see that  $b_1(D) < 0$  is a necessary condition for the coexistence equilibrium  $E_c$  to be unstable. Therefore, one has the following result:

**Proposition 8.** *Let  $D$  be such that  $b_1(D) > 0$ . For all operating parameters  $S^0$  and  $p^0$ ,  $E_c$  is LES whenever it exists.*

As a consequence of this result, we have the following graphical condition which asserts that in the case depicted in Fig. 4(b), the positive equilibrium is stable for all values of the operating parameters.

**Proposition 9.** *Assume that  $D$  is such that the tangent of  $\Gamma_1$  at point  $(p_c, \lambda_2)$  is under  $\Gamma_4$ . Then  $b_1(D) > 0$ , so that for all operating parameters  $S^0$  and  $p^0$ ,  $E_c$  is LES whenever it exists.*

*Proof.* The proof is given in Appendix A.4.  $\square$

Assume that the condition in Proposition 9 holds. Since  $\Gamma_1$  has a vertical asymptote, it intersects the straight-line  $\Gamma_4$  at least at one second point, different from  $(p_c, \lambda_2)$ , see Fig. 4(b). For simplicity, we assume that there is only one such intersection point, which is the case for instance if  $\Gamma_1$  is convex. An example exhibiting a non convex curve  $\Gamma_1$  will be examined in Section 6.4. A sufficient condition for convexity of  $\Gamma_1$  ( $F'' \geq 0$ ) is given in the following result.

**Lemma 10.** *Assume that  $f_1'' \leq 0$  and  $(1/f)'' \geq 0$  then  $F'' \geq 0$ .*

*Proof.* From  $F(p) = f_1^{-1}\left(\frac{D}{f(p)}\right)$  we deduce that  $F'(p) = \frac{D}{f_1'(F(p))} \left(\frac{1}{f(p)}\right)'$ . Since  $\left(\frac{1}{f(p)}\right)' = \frac{-f'(p)}{f^2(p)} > 0$ , we have  $F'(p) > 0$ . Therefore

$$F''(p) = \frac{-Df_1''(F(p))F'(p)}{[f_1'(F(p))]^2} \left(\frac{1}{f(p)}\right)' + \frac{D}{f_1'(F(p))} \left(\frac{1}{f(p)}\right)''.$$

Since  $f_1'' \leq 0$ ,  $F' > 0$ ,  $\left(\frac{1}{f(p)}\right)' > 0$  and  $\left(\frac{1}{f(p)}\right)'' \geq 0$  we have  $F'' \geq 0$ .  $\square$

We give now a second sufficient condition on the biological parameters such that  $E_c$  is stable whenever it exists. For that we consider the model (19) with  $f$ ,  $f_1$ ,  $f_2$ , and  $g$  given by (2).

**Proposition 11.** *Let*

$$\mu_0 = \frac{2}{\sqrt{K^2 + 4K} - K} \ln\left(\frac{m_1 K_2}{m_2 K_1}\right), \quad \mu_1 = 2\frac{m_2 K_1}{m_1 K_2}.$$

*If*

$$K_1 < K_2 \text{ and } \mu_0 \leq \mu \leq \mu_1, \quad (57)$$

*then  $b_1(D) > 0$  for all  $D$ , so that for all operating parameters  $D$ ,  $S^0$  and  $p^0$ ,  $E_c$  is LES whenever it exists.*

*Proof.* The proof is given in Appendix A.5.  $\square$

### 5.6. Construction of the regions of instability

In the case depicted in Fig. 4(c),  $E_c$  is not necessarily stable as in the case depicted in Fig. 4(b). Therefore sub-regions VIII  $\subset$  VII and IX  $\subset$  VI may occur, on which  $E_c$  is unstable, see Fig. 4(d) and Fig. 5(b). We describe below how these regions VIII and IX are constructed. The necessary condition on  $D$  for which  $E_c$  is unstable is  $D \in I_3$ , as shown in Theorem 7. Assume that  $I_3$  defined by (48) is non empty. For simplicity, we assume that  $I_3$  is a sub-interval of  $I_c$ , as shown in Fig. 5(a). The curve  $H$  of equation

$$H = \{(D, p^0) : a_1(D, p^0) < 0, \Delta(D, p^0) = 0\} \quad (58)$$

separates the operating plane  $(D, p^0)$  in two regions: the bounded region  $\mathcal{U}$  in which  $a_1(D, p^0) < 0$  and  $\Delta(D, p^0) > 0$ , that is to say the condition (37) holds, and the region  $\mathcal{S}$  where this condition does not hold, see Fig. 5(a). Notice that  $H$  is simply the union of the graphs of functions  $p^0 = F_6(D)$  and  $p^0 = F_7(D)$ , given by (49), with  $D \in I_3$ . Let  $(D, p^0) \in \mathcal{U}$ . Therefore  $D \in I_3$  and  $F_6(D) < p^0 < F_7(D)$ . Hence, according to Theorem 7,  $E_c$  is unstable if and only if  $F_4(D, p^0) < S^0 < F_5(D, p^0)$ .

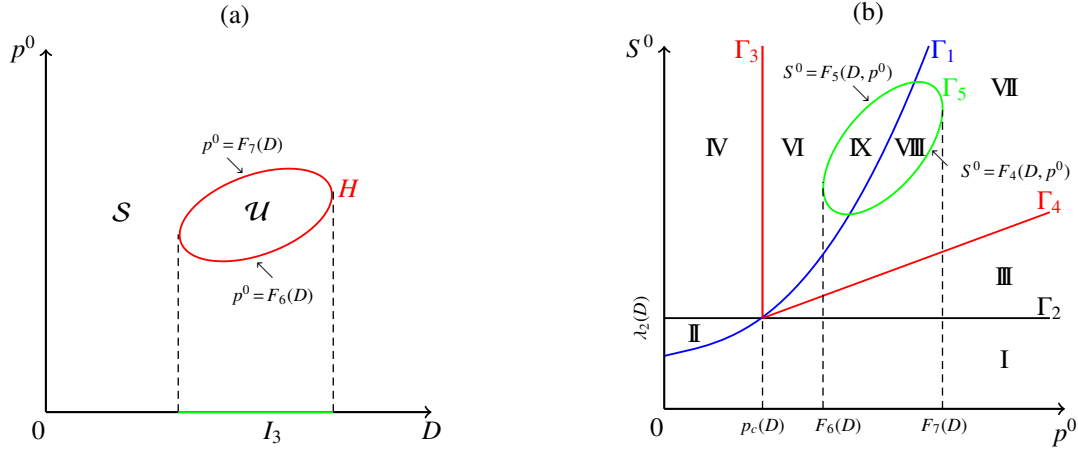


Figure 5: (a): The curve  $H$  defined by (58) separates the plane  $(D, p^0)$  in two regions  $\mathcal{U}$  and  $\mathcal{S}$  so that if  $(D, p^0) \in \mathcal{S}$ , the positive equilibrium  $E_c$  is stable for any  $S^0$ . The condition  $(D, p^0) \in \mathcal{U}$  is necessary of  $E_c$  to be unstable. (b): The construction of the curve  $\Gamma_5$  when  $D$  is fixed in the subset  $I_3$  and  $F_6(D) \leq p^0 \leq F_7(D)$ . The figure illustrate a case where  $\Gamma_5$  intersects  $\Gamma_1$ , so that both regions VIII and IX exist.

Case	Interval $I_3$	Figures
Left	$I_3 = (D_1, D_3)$ , where $D_1 < D_3 < D_2$	Figs. 7(a) and 10(a)
Center	$I_3 = (D_3, D_4)$ , where $D_1 < D_3 < D_4 < D_2$	Figs. 11(a) and 13(a)
Right	$I_3 = (D_3, D_2)$ , where $D_1 < D_3 < D_2$	Fig. 12(a)

Table 7: Three forms of interval  $I_3$ .

For  $D \in I_3$  fixed, equation  $F_3(D, p^0, S^0) = 0$  defines a closed curve  $\Gamma_5$  in the operating plane  $(p^0, S^0)$ , see Fig. 4(d) and Fig. 5(b). Notice that  $\Gamma_5$  is simply the union of the graphs of functions  $S^0 = F_4(D, p^0)$  and  $S^0 = F_5(D, p^0)$ , given by (38), with  $D \in I_3$  and  $F_6(D) \leq p^0 \leq F_7(D)$ . The curve  $\Gamma_5$  can have no intersection with the curve  $\Gamma_1$ , as in Fig. 4(d). It defines then a sub-region VIII of region VII. In this case we simply denote by the same letter VII, the complement of VIII in the region VII, see Fig. 4(d). On the other hand, the curve  $\Gamma_5$  can intersect the curve  $\Gamma_1$ , as in Fig. 5(b). It defines then two sub-regions VIII  $\subset$  VII and IX  $\subset$  VI. For simplicity we denote by the same letter VII, the complement of VIII in the region VII and by the same letter VI the complement of IX in the region VI, see Fig. 5(b). With these notations,  $E_c$  is unstable in the bounded sub-region VIII  $\cup$  IX of existence of  $E_c$ , and stable in its complementary region VI  $\cup$  VII.

In Fig. 5(a) we have assumed that  $I_3$  is a sub-interval of  $I_c$ . Actually, if we want to determine  $I_3$  we have to fix the biological parameters of the model, and then we have to plot the graphs of the five functions appearing in the definition (48) of  $I_3$ . Then, we consider the subset

on which these functions have the determined sign, see Fig. 6. Therefore the determination of stability of  $E_c$ , with respect to the biological and operating parameters, is reduced to the determination of the sign of functions depending only on the dilution rate  $D$ . It will be seen in the examples given in Section 6 that  $I_3$  is indeed a sub-interval of the interval  $I_c = (D_1, D_2)$ , defined by (24), of one of the three forms shown in Table 7

The operating diagrams shown in Fig. 4 and Fig. 5 are given only as illustrative examples, showing that our analysis gives a complete description of the behavior of the system for a large class of growth functions. Notice that for plotting operating diagrams we must choose functions  $f$ ,  $f_1$ ,  $f_2$ , and  $g$  in system (19) and fix the values of the biological parameters. We illustrate this in the Section 6 for various examples that have been considered in the literature.

### 5.7. The conjecture of Hsu and Waltman

Hsu and Waltman [23] conjectured that for the growth functions given by (2), if  $E_c$  and  $E_1$  exist and  $E_1$  is unstable, then the system has no limit cycle. A necessary condition for this conjecture to be true is that  $E_c$  is necessarily stable if  $E_1$  exists and is unstable.

Since  $E_1$  is unstable in region VI, if the Conjecture is true, then the sub-region IX of region VI on which  $E_c$  is unstable, cannot exist. We were not able to find a set of biological parameters such that this region is non empty and all the simulations given in Section 6 show that the region IX never exist for the Monod growth functions given by (2). However, we will show that this region can exist for Holling type 3 growth functions (59), as shown in Section 6.4. Therefore the Conjecture of Hsu and Waltman does not hold for general monotonic growth functions.

## 6. Examples

In this section, excepted for Section 6.4, we consider the model (19) with  $f$ ,  $f_1$ ,  $f_2$ , and  $g$  given by (2). Let us show the usefulness of our results on the construction of the operating diagram corresponding to various sets of biological parameters encountered in the literature, in particular those considered in [23, 25], see Table 8.

We consider the parameter values of Hsu and Waltman [23], see Case 1, in Table 8. The modification of one of these parameters given in Case 2 in Table 8, illustrates the dependence of the operating diagram with respect of the biological parameters, compare Figures 7(b) and 8(b).

We consider the parameter values of Lenski and Hattingh [25], see Case 3 in Table 8. The modifications of one of these biological parameters given in Case 4 or Case 5 in Table 8, illustrate again the dependence of the operating diagram with respect of the biological parameters, compare Figures 9(b), 10(b) and 11(b). These modifications clarify also the remarks of [25] on the destabilization of the coexistence equilibrium when the biological parameters are changed.

We consider also two set of biological parameters which are not taken from the existing literature, and are chosen due to their interesting properties, see Case 6 and 7, in Table 8.

Notice that the functions  $f_1$  and  $f$  in (2) satisfy the conditions in Lemma 10. Therefore  $\Gamma_1$  is convex and intersects  $\Gamma_4$  at most at one point different from  $(p_c(D), \lambda_2(D))$ .

### 6.1. Parameter values of Hsu and Waltman [23]

The parameter values used by [23] are given in Table 8, Case 1. It is shown in [23] that for  $D = S^0 = p^0 = 1$ ,  $E_c$  is unstable.

The plots of the functions  $b_1(D)$ ,  $\Delta_1(D)$ ,  $\Delta_2(D)$ ,  $a_1(z(D))$  and  $\Delta(z(D))$ , see Fig. 6, show that

$$I_3 = (0, D_3), \text{ with } D_3 \approx 2.0578.$$

The plot of the curve  $H$  defined by (58) is shown in Fig. 7(a). The curve  $H$  separates the plane  $(D, p^0)$  into two regions  $\mathcal{U}$  and  $\mathcal{S}$ . Now, we determine the operating diagram for  $D = 1 \in I_3$ . Notice that

$$F_6(1) \approx 0.4034, \quad F_7(1) \approx 2.108.$$

Therefore  $(D, p^0) = (1, 1) \in \mathcal{U}$ . The plot of the curve  $\Gamma_5$ , defined by (56), with  $D = 1$ , is shown in Fig. 7(b). Using Theorem 7, one has the following result:

**Claim 12.** *Let the biological parameters be given by Table 8, Case 1. The coexistence equilibrium  $E_c$  is unstable only if  $D \in I_3$  and  $(D, p^0) \in \mathcal{U}$  where  $\mathcal{U}$  is shown in Fig. 7(a). For  $D = 1$ ,  $E_c$  is unstable if and only if  $(S^0, p^0) \in \text{VIII}$ , where VIII is shown in Fig. 7(b).*

Notice that  $(S^0 = 1, p^0 = 1) \in \text{VIII}$ , which agrees with the result of [23]. The result of Claim 12 shows how extensive is the parameter region where the coexistence equilibrium is unstable.

We consider now the dependence of the operating diagram on the biological parameters. For this purpose we consider the parameter values given in Table 8, Case 2, where the parameters are the same as in Case 1, except that  $\delta$  is lowered from 50 to 0.02. The plot of the function  $b_1(D)$ , see Fig. 8(a), shows that  $b_1(D) > 0$  for all  $D \in [0, m_2)$ . Therefore, according to Proposition 8, we have the following result:

**Claim 13.** *Let the biological parameters be given by Table 8, Case 2. Since  $b_1(D) > 0$  for all  $D \in [0, m_2)$ , for all operating parameters  $D$ ,  $S^0$ ,  $p^0$ , the coexistence equilibrium  $E_c$  is LES whenever it exists*

The plot of the operating diagram, with  $D = 1$ , is shown in Fig. 8(b). The region VIII of instability of  $E_c$  does not exist as predicted by Claim 13. In this case the tangent of  $\Gamma_1$  at point  $(p_c, \lambda_2)$  is under  $\Gamma_4$ . Therefore all regions I, ..., VII exist and  $E_c$  is LES whenever it exists as predicted also by Proposition 9.

### 6.2. Parameter values of Lenski and Hattingh [25]

The parameter values used in [25] are given in Table 8, Case 3. It should be noticed that these authors considered yields  $\gamma_1 = \gamma_2 = 10^6$  in their model and used  $\delta = 10^{-6}$ . The rescaling (17) used to fix  $\gamma_1 = \gamma_2 = 1$  and the expression of  $\hat{g} = \gamma_2 g(p)$  in (18) show that we must rescale  $\delta$  to  $\delta = 1$ .

The plot of the function  $b_1(D)$ , see Fig. 9(a), shows that  $b_1(D) > 0$  for all  $D \in [0, m_2)$ . Therefore, according to Proposition 8, we have the following result:

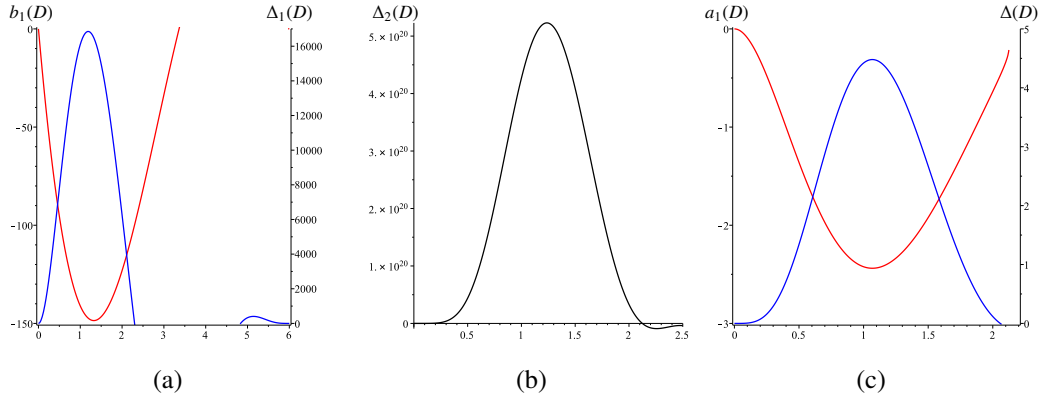


Figure 6: The numerical plots of (a):  $b_1(D)$  (in red) and  $\Delta_1(D)$  (in blue); (b):  $\Delta_2(D)$ ; (c):  $a_1(z(D))$  (in red) and  $\Delta(z(D))$  (in blue). These plots show that the conditions in the definition (48) of  $I_3$  hold for  $0 < D < D_3$  with  $D_3 \approx 2.0578$ . The biological parameters are given in Table 8, Case 1.

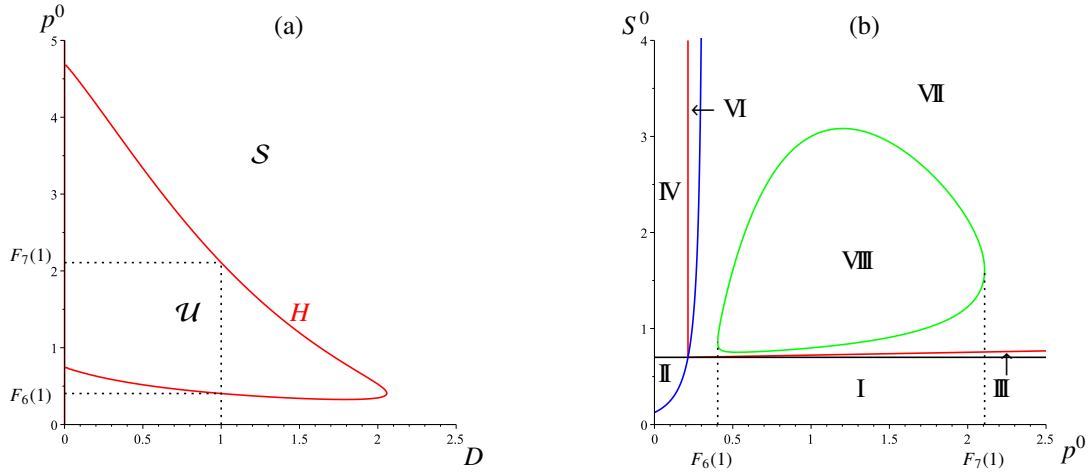


Figure 7: The biological parameters are given in Table 8, Case 1. (a): The subsets  $\mathcal{U}$  and  $\mathcal{S}$  defined in Figure 5(a). (b): The operating diagram  $(p^0, S^0)$  for  $D = 1$ . The existence and stability of equilibria in the regions I, II, ..., VIII are shown by Table 6. Regions V and IX are empty.

**Claim 14.** *Let the biological parameters be given by Table 8, Case 3. Since  $b_1(D) > 0$  for all  $D \in [0, m_2)$ , for all operating parameters  $D, S^0, p^0$ , the coexistence equilibrium  $E_c$  is LES whenever it exists*

The result of Claim 14 gives a justification of the claim of [25], that is, for  $D = 0.2$ , the coexistence equilibrium  $E_c$  is stable whenever it exists. The operating diagram in the plane  $(p^0, S^0)$  and  $D = 0.2$  is shown in Fig. 9(b). Our diagram is similar to the diagram in [25], Fig. 1(b).

In fact Claim 14 asserts that the stability occurs for all values of  $D$ , not only for  $D = 0.2$ , as it was stated in [25]. Let us give a proof of this result which is not based on a numerical plot as for the proof of Claim 13.

*Proof. (of Claim 14).* The biological parameters given

in Table 8, Case 3 satisfy the conditions (57) in Proposition 11. Indeed, we have  $\mu_0 \approx 0.76$  and  $\mu_1 = 1$ . Thus  $\mu_0 \leq \mu \leq \mu_1$ . According to Proposition 11, for all operating parameters  $D, S^0$  and  $p^0$ , the coexistence equilibrium  $E_c$  is stable whenever it exists.  $\square$

It should be noticed that for the biological parameters given in Table 8, Case 2, one has  $\mu_1 = 0.342$  and  $\mu = 5$ , so that the condition  $\mu \leq \mu_1$  is not satisfied. Hence we do not have Proposition 11 to our disposal. Therefore, in this case, we must use the plot of the curve  $b_1(D)$  to show that  $E_c$  is LES whenever it exists, see Figure 8.

We consider now the dependence of the operating diagram on the biological parameters. For this purpose, we notice that in [25] it was also stated that the instability of  $E_c$  can be accomplished in two different

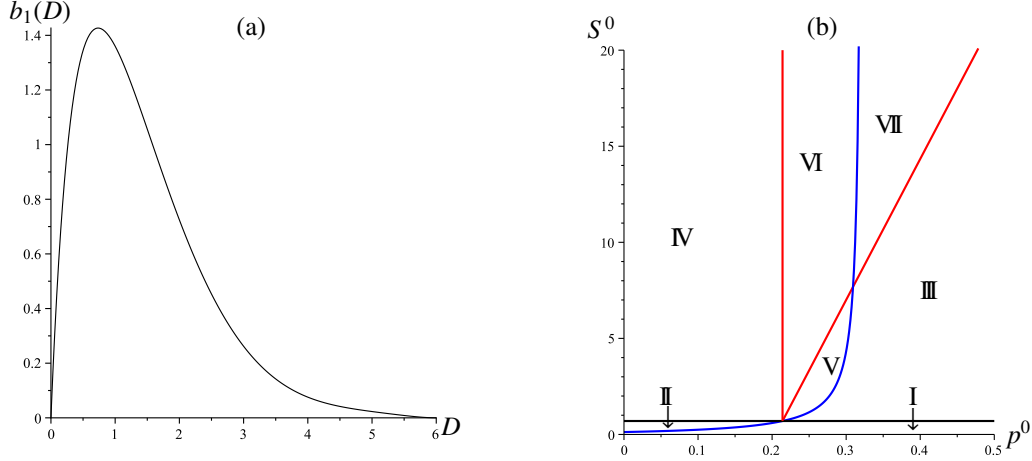


Figure 8: The biological parameters are given in Table 8, Case 2. (a): The plot of  $b_1(D)$  shows that  $b_1(D) > 0$  for all  $D$ , therefore  $E_c$  is LES whenever it exists. (b): The operating diagram  $(p^0, S^0)$  for  $D = 1$ . The existence and stability of equilibria in the regions I, II, ..., VII are shown by Table 6. Regions VIII and IX are empty.

ways. First, lowering  $K_1$  can destabilize  $E_c$ : the authors claimed that in the limiting case  $K_1 = 0$ ,  $E_c$  exists and is unstable when  $D = 0.2$ ,  $p^0 = 2.5$  and  $S^0 = 20$ . Second, lowering  $K$  while raising  $\mu$  may destabilize  $E_c$ : the authors claimed that in the limiting case  $K = 0$ , if  $\mu = 10$ , then  $E_c$  exists and is unstable when  $D = 0.2$ ,  $p^0 = 2.5$  and  $S^0 = 20$ . Let us show how our analysis can clarify this question.

It is seen from the inequalities  $\mu_0 \leq \mu \leq \mu_1$  in (57) that lowering  $K_1$  will raise  $\mu_0$  and diminish  $\mu_1$  so that the inequalities  $\mu_0 \leq \mu \leq \mu_1$  can be violated, giving hence the possibility of instability. On the other hand lowering  $K$  will raise  $\mu_0$ , so that the inequality  $\mu_0 \leq \mu$  can be violated, giving hence the possibility of instability.

Let us consider the parameter values given in Table 8, Case 4, where the parameters are the same as in Case 3, except that  $K$  is lowered from 10 to 0.035. For these parameter values  $\mu_0 \approx 4.06$  and  $\mu_1 = 1$ , thus the inequality  $\mu_0 \leq \mu$  is violated. Indeed, we can see that the instability of  $E_c$  can occur.

The plots of the functions  $b_1(D)$ ,  $\Delta_1(D)$ ,  $\Delta_2(D)$ ,  $a_1(z(D))$  and  $\Delta(z(D))$ , similar to those presented in Fig. 6, show that

$$I_3 = (0, D_3), \text{ with } D_3 \approx 0.2155.$$

The plot of the curve  $H$  defined by (58) is shown in Fig. 10(a). The curve  $H$  separates the plane  $(D, p^0)$  into two regions  $\mathcal{U}$  and  $\mathcal{S}$ . Now, we determine the operating diagram for  $D = 0.2 \in I_3$ . Notice that

$$F_6(0.2) \approx 3.602, \quad F_7(0.2) \approx 6.119.$$

The plot of the curve  $\Gamma_5$ , defined by (56), with  $D = 0.2$ ,

is shown in Fig. 10(b). Using Theorem 7, one has the following result:

**Claim 15.** *Let the biological parameters be given by Table 8, Case 4. The coexistence equilibrium  $E_c$  is unstable only if  $D \in I_3$  and  $(D, p^0) \in \mathcal{U}$ , where  $\mathcal{U}$  is shown in Fig. 10(a). For  $D = 0.2$ ,  $E_c$  is unstable if and only if  $(S^0, p^0) \in \text{VIII}$ , where VIII is shown in Fig. 10(b).*

Let us consider the parameter values given in Table 8, Case 5, where the parameters are the same as in Case 3, except that  $K_1$  is lowered from 5 to 0.4. For these parameter values  $\mu_0 \approx 3.51$  and  $\mu_1 = 0.08$ , thus the inequalities  $\mu_0 \leq \mu \leq \mu_1$  are violated. Indeed, we can see that the instability of  $E_c$  can occur.

The plots of the functions  $b_1(D)$ ,  $\Delta_1(D)$ ,  $\Delta_2(D)$ ,  $a_1(z(D))$  and  $\Delta(z(D))$ , similar to those presented in Fig. 6, show that

$$I_3 = (D_3, D_4), \text{ with } D_3 \approx 0.1691, D_4 \approx 0.5236.$$

The plot of the curve  $H$  defined by (58) is shown in Fig. 11(a). The curve  $H$  separates the plane  $(D, p^0)$  into two regions  $\mathcal{U}$  and  $\mathcal{S}$ . Now, we determine the operating diagram for  $D = 0.2 \in I_3$ . Notice that

$$F_6(0.2) \approx 1.608, \quad F_7(0.2) \approx 2.564.$$

The plot of the curve  $\Gamma_5$ , defined by (56), with  $D = 0.2$ , is shown in Fig. 11(b). Using Theorem 7, one has the following result:

**Claim 16.** *Let the biological parameters be given by Table 8, Case 5. The coexistence equilibrium  $E_c$  is unstable only if  $D \in I_3$  and  $(D, p^0) \in \mathcal{U}$ , where  $\mathcal{U}$  is shown in Fig. 11(a). For  $D = 0.2$ ,  $E_c$  is unstable if and only if  $(S^0, p^0) \in \text{VIII}$ , where VIII is shown in Fig. 11(b).*



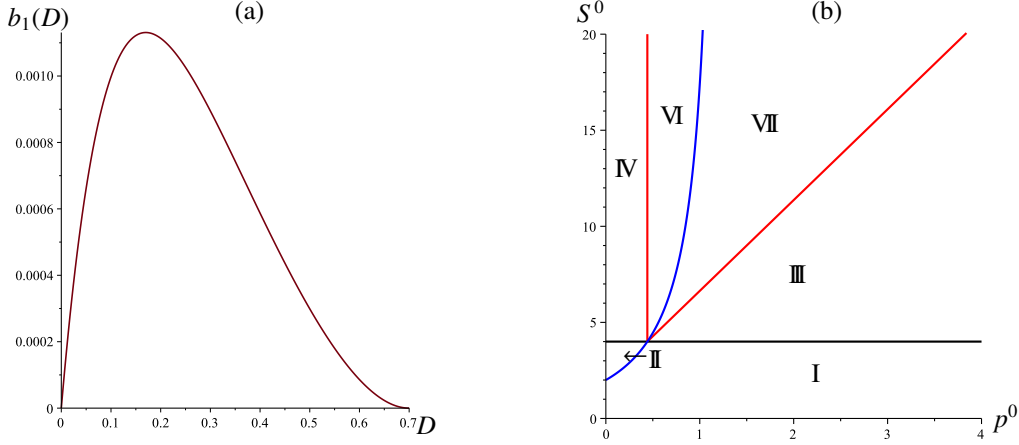


Figure 9: The biological parameters are given in Table 8, Case 3. (a): The plot of  $b_1(D)$  shows that  $b_1(D) > 0$  for all  $D$ , therefore  $E_c$  is LES whenever it exists. (b): The operating diagram  $(p^0, S^0)$  for  $D = 0.2$ . The existence and stability of equilibria in the regions I, II, ..., VII are shown by Table 6. Regions V, VIII and IX are empty.

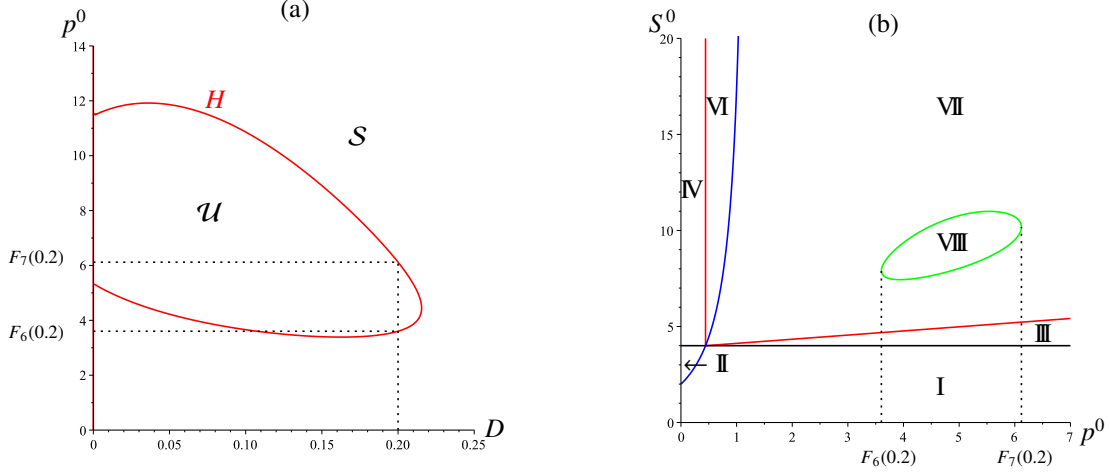


Figure 10: The biological parameters are given in Table 8, Case 4. (a): The subsets  $\mathcal{U}$  and  $\mathcal{S}$  defined in Figure 5(a). (b): The operating diagram  $(p^0, S^0)$  for  $D = 0.2$ . The existence and stability of equilibria in the regions I, II, ..., VIII are shown by Table 6. Regions V and IX are empty.

### 6.3. Parameter values with $I_3$ of type 'Right' of Table 7

All previous numerical plots of the operating diagram correspond to the case where  $I_3$  of type 'Left' or 'Center' of Table 7. The aim of this section is to provide a set of biological parameters for which  $I_3$  of type 'Right'.

Let us consider the parameter values given in Table 8, Case 6. The plots of the functions  $b_1(D)$ ,  $\Delta_1(D)$ ,  $\Delta_2(D)$ ,  $a_1(z(D))$  and  $\Delta(z(D))$ , similar to those presented in Fig. 6, show that

$$I_3 = (D_3, m_2), \text{ with } D_3 \approx 2.1219.$$

The plot of the curve  $H$  defined by (58) is shown in Fig. 12(a). The curve  $H$  separates the plane  $(D, p^0)$  into two regions  $\mathcal{U}$  and  $\mathcal{S}$ . Now, we determine the operating

diagram for  $D = 2.2 \in I_3$ . Notice that

$$F_6(2.2) \approx 2.0568, \quad F_7(2.2) \approx 6.6073.$$

The plot of the curve  $\Gamma_5$ , defined by (56), with  $D = 2.2$ , is shown in Fig. 12(b). Using Theorem 7, one has the following result:

**Claim 17.** *Let the biological parameters be given by Table 8, Case 6. The coexistence equilibrium  $E_c$  is unstable only if  $D \in I_3$  and  $(D, p^0) \in \mathcal{U}$ , where  $\mathcal{U}$  is shown in Fig. 12(a). For  $D = 2.2$ ,  $E_c$  is unstable if and only if  $(S^0, p^0) \in \text{VIII}$ , where VIII is shown in Fig. 12(b).*

### 6.4. An example with curve $\Gamma_5$ intersecting curve $\Gamma_1$

The aim of this section is to provide a set of biological parameters for which the curve  $\Gamma_5$  intersects the curve

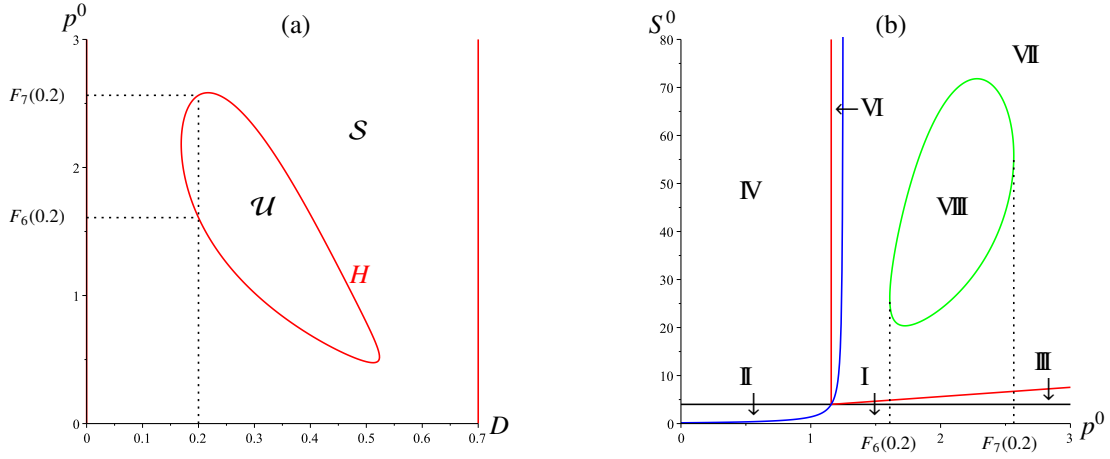


Figure 11: The biological parameters are given in Table 8, Case 5. (a): The subsets  $\mathcal{U}$  and  $\mathcal{S}$  defined in Figure 5(a). (b): The operating diagram  $(p^0, S^0)$  for  $D = 0.2$ . The existence and stability of equilibria in the regions I, II, ..., VIII are shown by Table 6. Regions V and IX are empty.

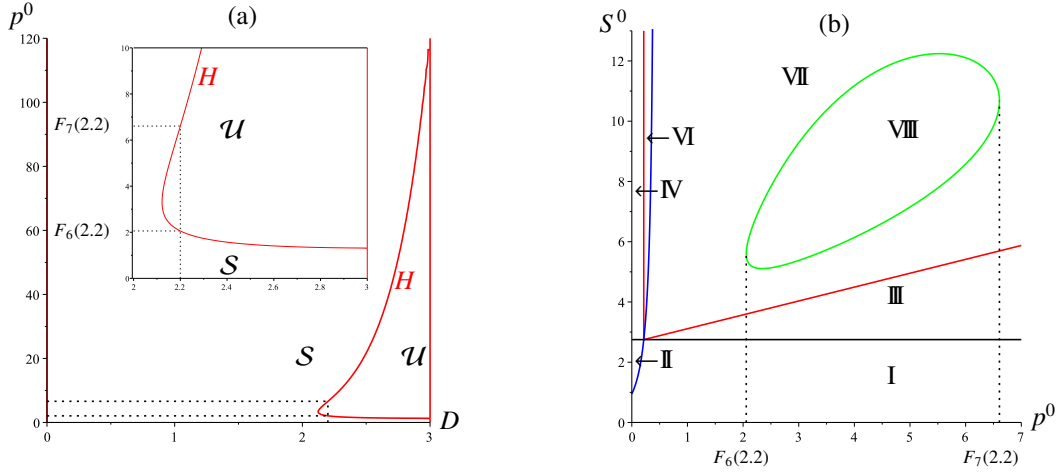


Figure 12: The biological parameters are given in Table 8, Case 6. (a): The subsets  $\mathcal{U}$  and  $\mathcal{S}$  defined in Figure 5(a). (b): The operating diagram  $(p^0, S^0)$  for  $D = 2.2$ . The existence and stability of equilibria in the regions I, II, ..., VIII are shown by Table 6. Regions V and IX are empty.

$\Gamma_1$ , so that both regions of instability VIII and IX exist. We were not able to find such a set of parameter values for the functions given in (2). The impossibility of finding such values for (2) is in agreement with the conjecture of Hsu and Waltman, as we mentioned in the Section 5.7.

We will consider another example of growth functions for which curve  $\Gamma_1$  is not convex and its intersection with  $\Gamma_5$  can occur. Let us consider the following growth functions

$$\begin{aligned} f(p) &= \frac{1}{1 + \mu p}, & f_1(S) &= \frac{m_1 S^2}{K_1^2 + S^2}, \\ f_2(S) &= \frac{m_2 S}{K_2 + S}, & g(p) &= \frac{\delta p}{K + p}. \end{aligned} \quad (59)$$

The meaning and the units of the variables and parameters are as for (2) and are given in Table 1. Notice that the function  $f_1$  in (59) does not satisfy the condition in Lemma 10. Therefore curve  $\Gamma_1$  is not necessarily convex and can intersect curve  $\Gamma_4$  at more than one point different from  $(p_c(D), \lambda_2(D))$ .

Assume that the parameter values are those given in Table 8, Case 7. The plots of the functions  $b_1(D)$ ,  $\Delta_1(D)$ ,  $\Delta_2(D)$ ,  $a_1(z(D))$  and  $\Delta(z(D))$ , similar to those presented in Fig. 6, show that

$$I_3 = (D_3, D_4), \text{ with } D_3 \approx 1.43610^{-4}, D_4 \approx 4.90110^{-4}.$$

The plot of the curve  $H$  defined by (58) is shown in Fig. 13(a). The curve  $H$  separates the plane  $(D, p^0)$  into two regions  $\mathcal{U}$  and  $\mathcal{S}$ . Now, we determine the operating

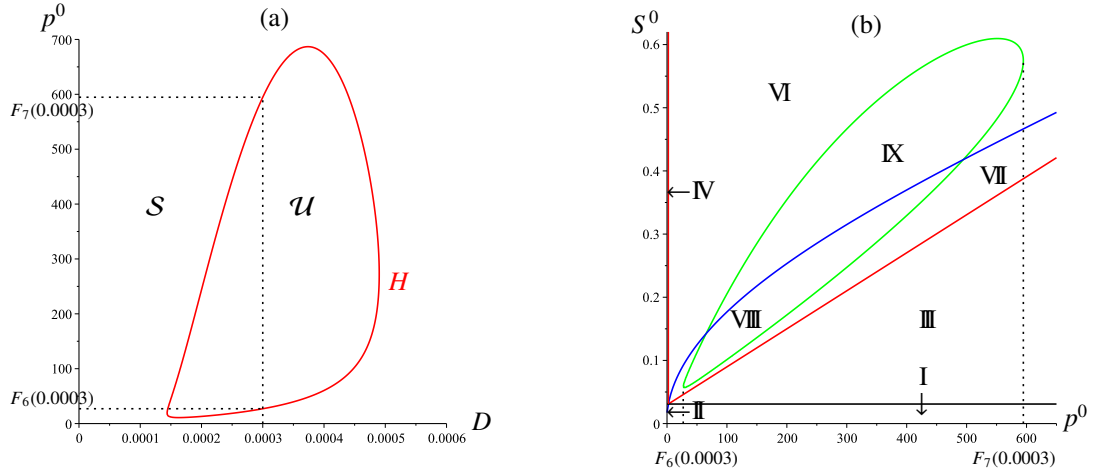


Figure 13: The biological parameters are given in Table 8, Case 7. (a): The subsets  $\mathcal{U}$  and  $\mathcal{S}$  defined in Figure 5(a). (b): The operating diagram  $(p^0, S^0)$  for  $D = 0.0003$ . The existence and stability of equilibria in the regions I, II, ..., IX are shown by Table 6.

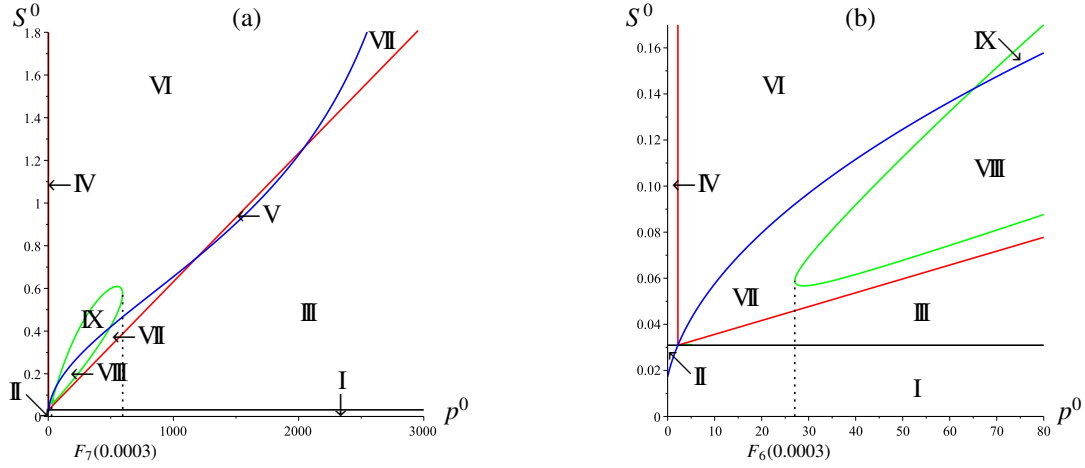


Figure 14: Supplementary information on the operating diagram shown in Fig. 13. (a): The full operating diagram. (b): A zoom near the origin. The nine regions I, ..., IX are non empty. The region VII is not connected. The existence and stability of equilibria are shown by Table 6.

diagram for  $D = 0.0003 \in I_3$ . Notice that

$$F_6(0.0003) \approx 27.05, \quad F_7(0.0003) \approx 594.7.$$

The plot of the curve  $\Gamma_5$ , defined by (56), with  $D = 0.0003$ , is shown in Fig. 13(b). Notice that in this example,  $\Gamma_5$  intersects  $\Gamma_1$ . Let us denote by VIII and IX the regions delimited by these curves as shown in Fig. 13(b). Using Theorem 7, one has the following result:

**Claim 18.** *Let the biological parameters be given by Table 8, Case 7. The coexistence equilibrium  $E_c$  is unstable only if  $D \in I_3$  and  $(D, p^0) \in \mathcal{U}$ , where  $\mathcal{U}$  is shown in Fig. 13(a). For  $D = 0.0003$ ,  $E_c$  is unstable if and only if  $(p^0, S^0) \in \text{VIII} \cup \text{IX}$ , where VIII and IX are shown in Fig. 13(b).*

This last example has many interesting properties:

- The region IX is non empty. Hence, the positive equilibrium  $E_c$  is unstable at the same time as the boundary equilibrium  $E_1$  exists and is unstable. Therefore it is a counter example of the Conjecture of Hsu and Waltman for the inhibition and growth function given in (59).
- The curve  $\Gamma_1$  is not convex. Even if its tangent at point  $(p_c, \lambda_2)$  is above the straight-line  $\Gamma_4$ , this curve intersect  $\Gamma_4$  at two points different from  $(p_c, \lambda_2)$ , so that the region V is not empty, and the region VII is not connected, as shown in Fig. 14(a).
- All nine regions I, ..., IX are non empty, as shown

Parameters	$m_1$	$m_2$	$K_1$	$K_2$	$\delta$	$K$	$\mu$	Figures
Case 1	5	6	0.5	3.5	50	0.1	5	Fig. 6 and Fig. 7
Case 2	5	6	0.5	3.5	0.02	0.1	5	Fig. 8
Case 3	0.7	0.7	5	10	1	10	1	Fig. 9
Case 4	0.7	0.7	5	10	1	0.035	1	Fig. 10
Case 5	0.7	0.7	0.4	10	1	10	1	Fig. 11
Case 6	8	3	2.5	1	5	0.01	3	Fig. 12
Case 7	1	0.01	1	1	0.5	0.007	1	Fig. 13 and Fig. 14

Table 8: Parameter values used in Section 6.

in Fig. 14 (a) and (b).

This example shows that for (59), according to the operating parameters, the model can exhibit nine different asymptotic behaviors. In contrast, for (2), according to the operating parameters, the model can exhibit only eight (and probably only seven, if the Conjecture of Hsu and Waltman is true) different asymptotic behaviors. Indeed, in this case region V can exist if and only if the tangent of  $\Gamma_1$  at point  $(p_c, \lambda_2)$  is under the straight-line  $\Gamma_4$ , and according to Proposition 9,  $E_c$  is necessarily stable. Therefore only the seven regions I, II, III, IV, V, VI and VII can exist. On the other hand, region V does not exist if and only if the tangent of  $\Gamma_1$  at point  $(p_c, \lambda_2)$  is above the straight-line  $\Gamma_4$ , and  $E_c$  can be unstable. Therefore only the eight regions I, II, III, IV, VI, VII, VIII and IX can exist. In fact, if the Conjecture of Hsu and Waltman is true, the region IX never exist for (2), so that the number of regions reduces to seven.

## 7. Discussion

In this work we have generalized the model of competition in the chemostat with an inhibitor [23, 25] by considering generic growth rate functions of competitors and absorption rate of external inhibitor. Our main contribution is the determination of the operating diagram and the analysis of its dependence with respect to the biological parameters. As it was noticed by Smith and Waltman [6], p. 252, the operating diagram is probably the most useful answer for the discussion of the behavior of the model with respect of the parameters. This diagram shows how robust or how extensive is the parameter region where coexistence occurs, where the coexistence equilibrium is stable and where it is unstable.

The operating diagram has the operating parameters  $D$ ,  $S^0$  and  $p^0$  as its coordinates and the various regions defined in it correspond to qualitatively different asymptotic behavior. As it is not easy to represent the regions

of existence and stability of the equilibria in a three dimensional space, we fixed the operating parameter  $D$  and showed the regions of existence and stability in the operating plane  $(p^0, S^0)$  in which the various outcomes occur. In this paper, we have fixed  $D$  since it is the choice made by Lenski and Hattingh [25]. These authors considered a set of biological parameters and they plotted the operating diagram for  $D = 0.2$ ,  $0 \leq p^0 \leq 5$  and  $0 \leq S^0 \leq 20$ . They claimed, without proof, that, when  $D = 0.2$ , the coexistence equilibrium is stable for all values of the operating parameters  $p^0$  and  $S^0$ . Since a numerical exploration of the whole range of parameters is not possible, it is necessary to develop a theory that support such claims. The theory developed in this paper completely solve this problem: we show that for the biological parameters in [25], for all values of the three operating parameters (not only for  $D = 0.2$  and  $p^0, S^0$  arbitrary) the coexistence equilibrium is stable whenever it exists. This theory permits also to clarify the question of the destabilisation of the coexistence equilibrium that was evocated in the Appendix of [25].

We applied our theory to the biological parameter values considered by Hsu and Waltman [23] and we gave the whole range of the operating parameters for which instability of the coexistence occurs. Figure 7 shows how extensive is the operating parameters region where coexistence occurs around a positive equilibrium and how extensive is the parameter region where this coexistence equilibrium is unstable. For instance the value  $D = 1$  considered [23] may be changed to any value  $0 < D < 2.0578$  and for such a value  $D$ , the set of  $(p^0, S^0)$  for which instability occurs can be theoretically predicted. For  $D = 1$  this set is shown in Figure 7(b), which shows that the particular choice  $p^0 = 1$ ,  $S^0 = 1$  for which the instability was shown in [23] can be changed in all the region VIII.

The sensitivity analysis of a mathematical model is a major problem in the literature. The parameter space is often multidimensional and exploring all of it is not possible. The theory developed in this paper gives a

complete answer to this problem for the model considered in [23, 25]. Decoupling the operating parameters and the biological parameters and fixing the biological parameters reduces the problem to a three dimensional parameter space that can be explored. Since the condition of instability is polynomial in the operating parameters  $S^0$  and  $p^0$ , as shown in Section 5.3, we reduced the problem of exploring all the three dimensional set of operating parameters to the determination of the signs of five real valued functions of only one variable (the dilution rate  $D$ ), see Theorem 7. This problem can be easily solved numerically. In Section 6 we have shown on seven well chosen examples that our theory is very efficient to address practical questions on the determination of the operating diagrams when the biological parameters are fixed. This section completes the theoretical results obtained in Sections 5. The choices of the biological parameters in Table 8 are motivated in the beginning of Section 6. For the biological parameters considered in [23, 25], we have changed some of them to illustrate the dependence of the operating parameters with respect of the biological parameters. Compare Figures 7(b) and 8(b). Compare Figures 9(b), 10(b) and 11(b).

From the biological viewpoint, the operating diagram indicates the possible choices of the operating parameters for a prescribed behaviour of the system. For instance, to maintain the coexistence of species in the chemostat, the ideally parameter values of  $D$ ,  $p^0$  and  $S^0$  should be chosen in the region of stable coexistence. If the model exhibit a region of destabilization, the operating diagram exhibits this region and gives to the experimenter useful informations on the experimental conditions that can demonstrate the oscillatory case. Therefore the operating diagram which determines the effect of the operating parameters, is very useful to understand the model from both the mathematical and biological points of view, and it is often constructed in the mathematical and biological literature, see the citations given in the introduction.

Our mathematical analysis of the model has revealed nine possible behaviors: washout, competitive exclusion of one species, coexistence of the species around a stable steady state and coexistence around a stable cycle. In the previous literature these nine possible behaviours were not all described and, even more, one of them was expected to never occur: it is the asymptotic behaviour corresponding to region the IX where the positive coexistence equilibrium exists and is unstable and at the same time the boundary equilibrium  $E_1$  exists and is unstable. Actually, a consequence of the Conjecture of Hsu and Waltman [23] is that when  $E_1$  exists and is unstable, the positive equilibrium  $E_c$  is necessarily

stable. This conjecture was formulated for the growth functions given in (2). We have shown in Section 6.4 that this conjecture does not hold for the growth functions given in (59).

Our study clarified what are the results in the existing literature that can be extended from the special case (2) to the general case of monotonic growth functions and what are the results that cannot be extended. Actually, it was noticed by Hsu and Waltman [23] that their results are probably valid for general monotone dynamics. Our findings show that the conditions on existence and local or global stability are easily extended to the general case, see Sections 3 and 4. However, some results are not valid in the general case: for instance, the Conjecture of Hsu and Waltman on the stability of  $E_c$ , when  $E_1$  exists and is unstable, does not hold in the general case of monotonic growth function, as shown in Section 6.4.

We were not able to find a set of biological parameters for (2) such that the region IX is not empty. Whether or not the region IX is always empty for (2) is the first step toward the study of the Conjecture of Hsu and Waltman and deserves further attention.

We can also fix the operating parameter  $p^0$  and show the regions of existence and stability in the operating plane  $(S^0, D)$  or fix the operating parameter  $S^0$  and show the regions of existence and stability in the operating plane  $(p^0, D)$ . These diagrams will be considered in a future work.

## Appendix A. Proofs

### Appendix A.1. Proof of Proposition 2

The steady states of (5) are the solutions of the set of equations

$$\begin{cases} 0 &= 1 - S - f(p)f_1(S)x - f_2(S)y, \\ 0 &= [f(p)f_1(S) - 1]x, \\ 0 &= [f_2(S) - 1]y, \\ 0 &= 1 - p - g(p)y. \end{cases} \quad (\text{A.1})$$

Therefore, besides the washout equilibrium  $E_0 = (1, 0, 0, 1)$  where both populations are extinct, that always exists, (5) has the following types of equilibria:

- $E_1 = (S_1, x_1, 0, p_1)$ , where second population is extinct and  $x_1 > 0$ .
- $E_2 = (S_2, 0, y_2, p_2)$ , where first population is extinct and  $y_2 > 0$ .
- $E_c = (S_c, x_c, y_c, p_c)$ , where both populations survive:  $x_c > 0, y_c > 0$ .

The components  $S = S_1$ ,  $x = x_1$  and  $p = p_1$  of the boundary equilibrium  $E_1$  are the solutions of (A.1) with  $x > 0$  and  $y = 0$ , that is  $p_1 = 1$  and

$$1 - S_1 = f(p_1)f_1(S_1)x_1, \quad (\text{A.2})$$

$$f(p_1)f_1(S_1) = 1. \quad (\text{A.3})$$

Therefore, from (A.3) we have  $f_1(S_1) = \frac{1}{f(1)}$ , that is  $S_1 = \lambda^+$ , where  $\lambda^+$  is given by (6). Then, using (A.2) we deduce that  $x_1 = 1 - \lambda^+$ . This equilibrium exists if and only if  $x_1 > 0$ , that is  $\lambda^+ < 1$ .

The components  $S = S_2$ ,  $y = y_2$  and  $p = p_2$  of the boundary equilibrium  $E_2$  are the solutions of (A.1) with  $x = 0$  and  $y > 0$ . Hence,  $f_2(S_2) = 1$ , that is  $S_2 = \lambda_2$ , where  $\lambda_2$  is given by (6) and

$$1 - S_2 = f_2(S_2)y_2, \quad (\text{A.4})$$

$$1 - p_2 = g(p_2)y_2. \quad (\text{A.5})$$

From (A.4) we have  $y_2 = 1 - \lambda_2$ . Then, using (A.5) we deduce that  $W(p_2) = 1 - \lambda_2$ , that is  $p_2 = p^*$ , where  $p^*$  is given by (7). This equilibrium exists if and only if  $y_2 > 0$  and  $p_2 > 0$ , that is  $\lambda_2 < 1$ .

The components of  $E_c = (S_c, x_c, y_c, p_c)$ , a positive equilibrium of (5), are the solutions of (A.1) with  $x > 0$  and  $y > 0$ . Hence,  $f_2(S_c) = 1$ , that is  $S_c = \lambda_2$ , where  $\lambda_2$  is given by (6) and

$$1 - S_c = f(p_c)f_1(S_c)x_c + f_2(S_c)y_c, \quad (\text{A.6})$$

$$f(p_c)f_1(S_c) = 1, \quad (\text{A.7})$$

$$1 - p_c = g(p_c)y_c. \quad (\text{A.8})$$

From (A.7) we have  $f(p_c) = \frac{1}{f_1(\lambda_2)}$ , from (A.8) we have  $y_c = W(p_c)$  and from (A.6) we have  $x_c = 1 - \lambda_2 - y_c$ . Therefore  $p_c, y_c$  and  $x_c$  are given by (10). Hence, a positive equilibrium  $E_c$  of system (5), if it exists, is unique. Let us study the condition of existence of  $E_c$ . We first note that

$$x_c + y_c = 1 - \lambda_2 > 0 \iff \lambda_2 < 1. \quad (\text{A.9})$$

Moreover, we have  $y_c > 0$  if and only if  $0 < p_c < 1$ . Using the fact that  $f$  is decreasing and  $f(0) = 1$  (hypothesis (H1)) we have

$$y_c > 0 \iff 0 < p_c < 1 \iff f(1) < f(p_c) < 1. \quad (\text{A.10})$$

Now, by (6) and (10) we have

$$1 = \frac{1}{f_1(\lambda_1)}, \quad f(1) = \frac{1}{f_1(\lambda^+)}, \quad f(p_c) = \frac{1}{f_1(\lambda_2)},$$

so that, by using the fact that  $f_1$  is increasing (hypothesis (H2)), we obtain from (A.10)

$$y_c > 0 \iff f_1(\lambda_1) < f_1(\lambda_2) < f_1(\lambda^+) \iff \lambda_1 < \lambda_2 < \lambda^+. \quad (\text{A.11})$$

On the other hand, by (7), (8) and (10) we have, respectively

$$W(p^*) = 1 - \lambda_2, \quad f(p^*) = \frac{1}{f_1(\lambda^-)}, \quad W(p_c) = y_c,$$

$$f(p_c) = \frac{1}{f_1(\lambda_2)}.$$

Hence, using the facts that  $W$  and  $f$  are decreasing, and  $f_1$  is increasing, we have

$$\begin{aligned} x_c > 0 &\iff y_c < 1 - \lambda_2 &\iff W(p_c) < W(p^*) \\ &\iff p_c > p^* &\iff f(p_c) < f(p^*) \\ &\iff f_1(\lambda_2) > f_1(\lambda^-) &\iff \lambda_2 > \lambda^-. \end{aligned} \quad (\text{A.12})$$

Taking into account (9), from (A.9), (A.11) and (A.12), we conclude finally that  $E_c$  exists if and only if  $\lambda^- < \lambda_2 < \min(\lambda^+, 1)$ .

#### Appendix A.2. Proof of Proposition 3

At washout equilibrium  $E_0$ , the matrix  $M$  defined by (12) is

$$M_0 = \begin{bmatrix} f(1)f_1(1) - 1 & 0 & 0 \\ 0 & f_2(1) - 1 & 0 \\ 0 & -g(1) & -1 \end{bmatrix}.$$

The eigenvalues of  $M_0$  are:  $-1$ ,  $f(1)f_1(1) - 1$  and  $f_2(1) - 1$ . Then, the equilibrium  $E_0$  is LES if and only if  $f(1)f_1(1) < 1$  and  $f_2(1) < 1$ , or equivalently,  $\lambda^+ > 1$  and  $\lambda_2 > 1$ .

Suppose that the equilibrium  $E_1$  exists, that is  $\lambda^+ < 1$ . At  $E_1$  the matrix  $M$  defined by (12) is

$$M_1 = \begin{bmatrix} -(1 - \lambda^+)f(1)f_1'(\lambda^+) & 0 & m_{13} \\ 0 & f_2(\lambda^+) - 1 & 0 \\ 0 & -g(1) & -1 \end{bmatrix},$$

where  $m_{13} = (1 - \lambda^+)f'(1)f_1(\lambda^+)$ . The eigenvalues of  $M_1$  are:  $-1$ ,  $-(1 - \lambda^+)f(1)f_1'(\lambda^+) < 0$  and  $f_2(\lambda^+) - 1$ . Since  $E_1$  has three negative eigenvalues, it has at least three dimensional stable manifolds. Moreover,  $E_1$  is LES if and only if  $f_2(\lambda^+) < 1$ , or equivalently,  $\lambda^+ < \lambda_2$ .

Suppose that the equilibrium  $E_2$  exists, that is  $\lambda_2 < 1$ . At  $E_2$  the matrix  $M$  defined by (12) is

$$M_2 = \begin{bmatrix} f(p^*)f_1(\lambda_2) - 1 & 0 & 0 \\ -(1 - \lambda_2)f_2'(\lambda_2) & -(1 - \lambda_2)f_2'(\lambda_2) & 0 \\ 0 & -g(p^*) & m_{33} \end{bmatrix},$$

where  $m_{33} = -1 - (1 - \lambda_2)g'(p^*)$ . The eigenvalues of  $M_2$  are:  $f(p^*)f_1(\lambda_2) - 1$ ,  $-(1 - \lambda_2)f_2'(\lambda_2) < 0$  and

$-1 - (1 - \lambda_2)g'(p^*) < 0$ . Since  $E_2$  has three negative eigenvalues, it has at least three dimensional stable manifolds. Moreover,  $E_2$  is LES if and only if  $f(p^*)f_1(\lambda_2) < 1$ , or equivalently,  $\lambda_2 < \lambda^-$ .

At  $E_c$ , the matrix  $M$  defined by (12) takes the form

$$M_c = \begin{bmatrix} -A & -A & -F \\ -B & -B & 0 \\ 0 & -E & -C \end{bmatrix},$$

where  $A, B, C, E$  and  $F$  are defined by (14). Obviously  $A > 0, B > 0, C > 0, E > 0$  and  $F > 0$ . The characteristic polynomial of  $M_c$  is given by

$$\lambda^3 + B_1\lambda^2 + B_2\lambda + B_3 = 0,$$

with  $B_1 = A + B + C, B_2 = C(A + B)$  and  $B_3 = BEF$ . Since  $B_1 > 0, B_2 > 0$  and  $B_3 > 0$ , by the Routh-Hurwitz criterion,  $E_c$  is LES if and only if  $B_1B_2 > B_3$  that is to say (13) holds.

#### Appendix A.3. Proof of Lemma 5

Using (20), hypothesis (H2) and the definition (30), we have

$$\begin{aligned} \lambda^+ < \lambda_2 &\iff f_1^{-1}\left(\frac{D}{f(p^0)}\right) < f_2^{-1}(D) \\ &\iff D < f(p^0)f_1\left(f_2^{-1}(D)\right) \\ &\iff D < F_1(D, p^0). \end{aligned}$$

On the other hand, using (20), (23) and hypotheses (H1) and (H2), we have

$$\begin{aligned} p_c(D) < p^0 &\iff f^{-1}\left(\frac{D}{f_1(\lambda_2)}\right) < f^{-1}\left(\frac{D}{f_1(\lambda^+)}\right) \\ &\iff \frac{D}{f_1(\lambda_2)} > \frac{D}{f_1(\lambda^+)} \\ &\iff f_1(\lambda_2) < f_1(\lambda^+) \iff \lambda_2 < \lambda^+. \end{aligned}$$

This completes the proof of first equivalences. Using (22) and (23), together with hypotheses (H1) and (H2), we have

$$\begin{aligned} \lambda_2 > \lambda^- &\iff f_1(\lambda_2) > f_1(\lambda^-) \iff \frac{D}{f(p_c)} > \frac{D}{f(p^*)} \\ &\iff f(p_c) < f(p^*) \iff p_c > p^*. \end{aligned} \quad (\text{A.13})$$

Using the fact that  $\frac{\partial W}{\partial p} < 0$  and (21), we have

$$\begin{aligned} p_c > p^* &\iff W(p_c, D, p^0) < W(p^*, D, p^0) \\ &\iff W(p_c(D), D, p^0) < S^0 - \lambda_2. \end{aligned} \quad (\text{A.14})$$

Therefore, from (A.14), together with  $\lambda_2 = f_2^{-1}(D)$  and the definition (31) of  $F_2$ , we deduce that  $p_c > p^*$  is equivalent to  $S^0 > F_2(D, p^0)$ . This completes the proof of second equivalences.

#### Appendix A.4. Proof of Proposition 9

Let  $D$  be fixed and let  $F(p^0) = f_1^{-1}\left(\frac{D}{f(p^0)}\right)$ . The curve  $\Gamma_1$  is the graph of the function  $S^0 = F(p^0)$ . Using  $f_1(\lambda_2)f(p_c) = D$ , we get that the slope  $F'(p_c)$  of the tangent of  $\Gamma_1$  at point  $(p_c, \lambda_2)$  is given by

$$F'(p_c) = \frac{1}{f_1'(F(p_c))} \frac{-Df'(p_c)}{f^2(p_c)} = \frac{1}{f_1'(\lambda_2)} \frac{-f'(p_c)f_1(\lambda_2)}{f(p_c)}.$$

Since the slope of the straight-line  $\Gamma_4$  is  $D/g(p_c)$ , the condition  $F'(p_c) < D/g(p_c)$  is equivalent to

$$Df_1'(\lambda_2)f(p_c) + f'(p_c)f_1(\lambda_2)g(p_c) > 0.$$

Straightforward computations show that

$$b_1(D) = f_2'(\lambda_2)M + Df(p_c)f_1'(\lambda_2)[f_2'(\lambda_2) + 2g'(p_c)],$$

where  $M = Df_1'(\lambda_2)f(p_c) + f'(p_c)f_1(\lambda_2)g(p_c) > 0$ . Hence  $b_1(D) > 0$ . Therefore, using Proposition 8, for all operating parameters  $S^0$  and  $p^0$  and whenever it exists,  $E_c$  is stable.

#### Appendix A.5. Proof of Proposition 11

Let us prove that  $b_1(D) > 0$  for all  $D \in I_c$  so that, using Proposition 8, for all operating parameters  $D, S^0$  and  $p^0$   $E_c$  is stable whenever it exists. Recall that  $b_1 = b_1(D)$  is given by (40)

$$b_1 = 2\alpha D(\beta + \gamma) - \beta E\phi,$$

where  $E$  is defined by (14) and  $\alpha, \beta, \gamma$  and  $\phi$  are defined by (29). From  $g(p) = \frac{\delta p}{K+p}$  and  $g'(p) = \frac{\delta K}{(K+p)^2}$  one has  $g(p) \leq g'(p)$  if and only if  $(K+p)p \leq K$ . Therefore  $p$  is between the roots of equation  $p^2 + Kp - K = 0$ . Since  $p \geq 0$  we have

$$g(p) \leq g'(p) \iff 0 \leq p \leq \frac{\sqrt{K^2 + 4K} - K}{2}. \quad (\text{A.15})$$

Straightforward calculations show that

$$p_c(D) = \frac{1}{\mu} \ln\left(\frac{m_1 K_2}{m_2 K_1 + D(K_2 - K_1)}\right).$$

Since  $K_1 < K_2$ ,  $p_c$  is decreasing. Thus, using  $\mu \geq \mu_0$ , one has

$$p_c(D) < p_c(0) = \frac{1}{\mu} \ln\left(\frac{m_1 K_2}{m_2 K_1}\right) \leq \frac{\sqrt{K^2 + 4K} - K}{2}.$$

Therefore, using (A.15), one has

$$g(p_c) \leq g'(p_c). \quad (\text{A.16})$$

Now using (A.16), (29) and (14) together with  $f'(p) = -\mu f(p)$ , one has

$$b_1(D) \geq f(p_c(D)) g(p_c(D)) M,$$

where  $M = 2Df'_1(\lambda_2(D)) - \mu f_1(\lambda_2(D))f'_2(\lambda_2(D))$ . Thus, if  $M > 0$  then  $b_1(D) > 0$ . Straightforward computations show that

$$f_1(\lambda_2(D)) = \frac{m_1 K_2 D}{m_2 K_1 + D(K_2 - K_1)}, \quad f'_2(\lambda_2(D)) = \frac{(m_2 - D)^2}{m_2 K_2},$$

$$f'_1(\lambda_2(D)) = \frac{m_1 K_1 (m_2 - D)^2}{(m_2 K_1 + D(K_2 - K_1))^2}.$$

Hence

$$M = \frac{m_1 D (D - m_2)^2}{m_2 (m_2 K_1 + D(K_2 - K_1))^2} N,$$

where  $N = K_1 m_2 (2 - \mu) + \mu D (K_1 - K_2)$ . Thus  $M$  is positive if and only if  $N > 0$ . Since  $p_c(D) > 0$  we have

$$D(K_1 - K_2) > m_2 K_1 - m_1 K_2,$$

from which we deduce that

$$N > K_1 m_2 (2 - \mu) + \mu (m_2 K_1 - m_1 K_2) = 2K_1 m_2 - \mu m_1 K_2.$$

Hence, from  $\mu \leq \mu_1$  one has  $N > 0$ . Therefore, if (57) holds then  $b_1(D) > 0$  for all  $D$ .

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