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THE OPERATOR EQUATION $THT = K$

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ABSTRACT. Let H and K be bounded positive operators on a Hilbert space, and assume that H is nonsingular. Then (i) there is at most one bounded positive operator T such that $THT=K$; (ii) a necessary and sufficient condition for the existence of such T is that $(H^{1/2}KH^{1/2})^{1/2} \leqq aH$ for some $a>0$, and then $\|T\| \leqq a$; (iii) this condition is satisfied if H is invertible or more generally if $K \leqq a^2 H$ for some $a>0$; (iv) an exact formula for T is given when H is invertible.

If H is a selfadjoint positive nuclear operator on a Hilbert space $\mathfrak{H}$, then the map $\varphi: A \to \mathrm{Tr}(AH)$ is a normal positive functional on the von Neumann algebra $B(\mathfrak{H})$. If $0 \leqq K \leqq H$ then the functional $\psi: A \to \mathrm{Tr}(AK)$ is majorized by φ. By S. Sakai's noncommutative Radon-Nikodym theorem [**3**] there is therefore a positive operator T with $\|T\| \leqq 1$ such that $\psi(A) = \varphi(TAT)$ for all A in $B(\mathfrak{H})$. Moreover, by [**4**, Lemma 15.4] the operator T is uniquely determined. Since the correspondence between normal positive functionals and positive nuclear operators is bijective this implies that $THT=K$. The purpose of this paper is to give a necessary and sufficient condition for the existence of a positive solution to the operator equation $THT=K$, with arbitrary H and K in $B(\mathfrak{H})_+$. Applications of the result to noncommutative integration theory can be found in [**2**].

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THEOREM. *Let H and K be selfadjoint positive operators in $B(\mathfrak{H})$, and assume that H is nonsingular. There is then at most one positive operator T in $B(\mathfrak{H})$ such that $THT=K$. A necessary and sufficient condition for the existence of such T is that $(H^{1/2}KH^{1/2})^{1/2} \leqq aH$ for some $a>0$; and then $\|T\| \leqq a$. This condition will be satisfied if H is invertible or, more generally, if $K \leqq a^2 H$ for some $a>0$.*

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PROOF. Suppose that S and T are positive operators in $B(\mathfrak{H})$ such that $SHS=THT$. Put $A=H^{1/2}S$ and $B=H^{1/2}T$. Then $A^*A=B^*B$ and from the polar decomposition $A=UB$, where U is a partial isometry such that U^*U is the range projection of B. Thus

$$H^{1/2}SH^{1/2} = AH^{1/2} = UBH^{1/2} = UH^{1/2}TH^{1/2}.$$

But $H^{1/2}SH^{1/2}$ and $H^{1/2}TH^{1/2}$ are both positive and since the polar decomposition (of $H^{1/2}SH^{1/2}$) is unique this implies that U is the range projection of $H^{1/2}T$. Thus $A=B$ and since H is assumed to be nonsingular this implies that $S=T$. It follows that the equation $THT=K$ can have at most one positive solution.

If $THT=K$ with T in $B(\mathfrak{H})_+$ then

$$(H^{1/2}KH^{1/2})^{1/2} = (H^{1/2}TH^{1/2}H^{1/2}TH^{1/2})^{1/2} = H^{1/2}TH^{1/2} \leqq \|T\|\ H.$$

Conversely, if $(H^{1/2}KH^{1/2})^{1/2} \leqq aH$ for some $a>0$ then $(H^{1/2}KH^{1/2})^{1/4}= a^{1/2}SH^{1/2}$ for some S in $B(\mathfrak{H})$ with $\|S\|\leqq 1$. This follows from a well-known variation of the polar decomposition theorem: If $A^*A\leqq B^*B$ define $S_0x= Ay$ for any x in $\mathfrak{H}$ such that $x=By$. Then S_0 extends uniquely to an operator S in $B(\mathfrak{H})$ with $\|S\|\leqq 1$ such that $A=SB$. Let $T=aS^*S$. Then $0\leqq T\leqq aI$ and

$$H^{1/2}THTH^{1/2} = (H^{1/2}TH^{1/2})^2 = (aH^{1/2}S^*SH^{1/2})^2 = H^{1/2}KH^{1/2}.$$

Since H is nonsingular this implies that $THT=K$.

If H is invertible then $I\leqq \|H^{-1}\|H$ so that each operator in $B(\mathfrak{H})_+$ is majorized by a suitable multiple of H. In this case the solution to the equation $THT=K$ is given by the formula $T = H^{-1/2}(H^{1/2}KH^{1/2})^{1/2}H^{-1/2}$.

Suppose now that $K\leqq a^2H$ for some $a>0$. Then $H^{1/2}KH^{1/2}\leqq a^2H^2$. Since the square root function is operator monotone (see [**1**]) this implies that $(H^{1/2}KH^{1/2})^{1/2}\leqq aH$ so that $THT=K$ from the above. This completes the proof of the theorem.

REFERENCES

1. G. K. Pedersen, *Some operator monotone functions*, Proc. Amer. Math. Soc. **36** (1972), 309–310.

2. G. K. Pedersen and M. Takesaki, *The Radon-Nikodym theorem for von Neumann algebras*, Acta Math. (to appear).

3. S. Sakai, *A Radon-Nikodym theorem in W*-algebras*, Bull. Amer. Math. Soc. **71** (1965), 149–151. MR **30** #5180.

4. M. Takesaki, *Tomita's theory of modular Hilbert algebras and its applications*, Lecture Notes in Math., vol. 128, Springer-Verlag, Berlin and New York, 1970. MR **42** #5061.

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