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Note

The Operators Governing Quantum Fluctuations of Yang-Mills Multi-Instantons on S^4 and Their Seeley Coefficients

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Abstract. We give explicit expressions for the Seeley coefficients of the fluctuation operator and the operator that appears in the Faddeev-Popov determinant, which arise in the calculation of quantum fluctuations around Yang-Mills multi-instantons.

In the calculation of quantum fluctuations around multi-instanton configurations it is of interest to know the Seeley coefficients for the fluctuation, and the gauge fixing operators [1]. In this note we shall give explicit expressions for these coefficients.

We work on S^4, the one-point compactification of $\mathbb{R}^4$. Let $\Box$ be a second order, self-adjoint, non-negative elliptic operator on S^4. Then it is well known [2] that the series

$$h_t(\Box) = \sum_\lambda e^{-t\lambda}$$

converges for any $t > 0$. The summation extends over all eigenvalues, λ, of $\Box$ with the appropriate multiplicities. Furthermore, $h_t(\Box)$ has an asymptotic expansion

$$h_t(\Box) \equiv \mathrm{Tr}\, e^{-t\Box} \sim t^{-2}\psi_2(\Box) + t^{-1}\psi_1(\Box) + \psi_0(\Box) + O(t^\delta), \delta > 0$$

for $t \downarrow 0$. The $\psi_k(\Box)$'s are known as the Seeley coefficients of $\Box$. Moreover each $\psi_k(\Box)$ can be expressed as an integral over S^4 of a certain measure $\psi_k(x|\Box)$dvol. $\psi_k(x|\Box)$ depends polynomially on the coefficients of $\Box$ and their derivatives. They can be expressed in terms of curvature invariants. In fact, the above asymptotic expansion is a consequence of a local expansion. Indeed, if $K_t(x, y)$ is the kernel of the operator $e^{-t\Box}$ then

$$K_t(x, x) \sim t^{-2}\psi_2(x|\Box) + t^{-1}\psi_1(x|\Box) + \psi_0(x|\Box) + O(t^\delta).$$

From this it also follows that

$$\hat{\psi}_k(x|\lambda\Box) = \lambda^{-k}\psi_k(x|\Box), \quad \lambda \in R^+, \quad k = 0, 1, 2.$$

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Let $P_k(S^4, G)$ be a principal bundle on S^4, characterized by its second Chern class, k. A multi-instanton with topological charge k is a connection on P_k. E is a bundle associated to P_k with a standard fibre the Lie algebra of G, on which G acts by the adjoint action.

Consider, now, the complex [3] which linearizes the self-duality equation $F(A) = *F(A)$.

$$0 \longrightarrow A^0 \xrightarrow{d_A} A^1 \xrightarrow{\sqrt{2}Pd_A} A^2_- \longrightarrow 0, \qquad (1)$$

where $A^p = \Gamma(\Lambda^p \otimes E) = p$-forms taking values in the Lie Algebra of G, and $P = 1/2(1-*)$, the projection operator into anti-self-dual 2-forms. (We have introduced $\sqrt{2}$ for convenience.) From Eq. (1) we construct the Laplacians

$$\Delta_0^A = d_A^* d_A, \qquad \Delta_1^A = 2d_A^* P d_A + d_A d_A^*.$$

It is well known [1] that Δ_1^A corresponds to the fluctuation operator, which governs quantum fluctuations around the self-dual connection A, whereas Δ_0^A is the operator which appears in the $F-P$ determinant. It is not surprising that the complex (1), which linearizes the self-duality equation, gives also the fluctuation operator, because the latter is obtained from the second variation of the action by retaining only quadratic terms.

Indeed, if we vary the Yang-Mills action, $\mathfrak{A}(A)$, along a straight line $A^t = A + t\eta$, then we get [4]

$$\frac{1}{2} \left. \frac{d^2\mathfrak{A}(A^t)}{dt^2} \right|_{t=0} = (\eta, 2d_A^* P d_A \eta) + O(\eta^3) \equiv (\eta, \tilde{\Delta}_1^A \eta) + O(\eta^3).$$

However, $\mathfrak{A}(A)$ is gauge invariant. So we must eliminate variations along gauge orbits. Thus, the correct fluctuation operator is given by a pair of equations

$$\tilde{\Delta}_1^A \eta = 0, \qquad d_A^* \eta = 0 \quad \text{(background gauge)}$$

or, equivalently by

$$\Delta_1^A = 2d_A^* P d_A + d_A d_A^*.$$

The operators $\Delta_p^A (p=0,1)$ are self-adjoint, second order and elliptic [1]. $h_t(\Delta_p^A)$ has, then, an asymptotic expansion. In what follows we shall calculate the Selley Coefficients functions $\psi(x|\Delta_p^A)$.

We shall use the conformally flat metric $g_{\mu\nu}(x) = \Omega(x)\delta_{\mu\nu}$, where $\Omega(x) = R^4/(x^2+R^2)^2$ and R is the radius of S^4. This is obtained from the stereographic projection on $\mathbb{R}^4$. (There is a factor of four missing in $g_{\mu\nu}$ so that $g_{\mu\nu} \xrightarrow{R\to\infty} \delta_{\mu\nu}$.) In this coordinate system

$$\begin{aligned}
\Delta_0^A &= -\Omega^{-1}\{\partial_\mu\partial_\mu + (2A_\mu + \Omega^{-1}\partial_\mu\Omega)\partial_\mu + (A_{\mu,\mu} + A_\mu A_\mu + \Omega^{-1}\partial_\mu \Omega A_\mu)\} \\
(\Delta_1^A)_{\mu\nu} &= -\Omega^{-1}\{\delta_{\mu\nu}\partial_\sigma\partial_\sigma + [\delta_{\mu\nu}2A_\sigma + \Omega\partial_\mu\Omega^{-1}\delta_{\sigma\nu} - \Omega\partial_\nu\Omega^{-1}\delta_{\sigma\mu}]\partial_\sigma \\
&\quad + [\delta_{\mu\nu}(A_{\sigma,\sigma} + A_\sigma A_\sigma) + \Omega\partial_\mu\Omega^{-1}A_\nu - \Omega\partial_\nu\Omega^{-1}A_\mu + \Omega\partial_\mu\partial_\nu\Omega^{-1} \\
&\quad + F_{\mu\nu} + {}^*F_{\mu\nu}]\}.
\end{aligned}$$

The Seeley Coefficient functions $\psi_k(x|\Delta_p^A)$ can be calculated by a cononical procedure applied to the coefficients of Δ_p^A [5]. $\psi_k(x|\Delta_p^A)$ are expressible in terms of curvature invariants which involve the curvature of the sphere and the bundle. $\psi_k(x|\Delta_p^A)$ are invariants of order $(4-2k)$ in the derivatives of the metric.

It turns out that the curvature invariants of S^4 (of order $\leqq 4$) are all constants.

Table 1

Order	Invariant
2	$K(g)=R^{\mu\nu}{}_{\nu\mu}=48/R^2$
4	$R^{\mu\nu}{}_{\nu\mu;\sigma\sigma}=0$
	$K(g)^2=2304/R^4$
	$\lvert R(g)\rvert^2=R^{\mu\nu\varrho\sigma}R_{\mu\nu\varrho\sigma}=384/R^4$
	$\lvert\mathrm{Ric}(g)\rvert^2=R^{\mu\sigma}{}_{\sigma}{}^{\nu}R_{\mu}{}^{\sigma}{}_{\sigma\nu}=576/R^4$

Thus the calculation of $\psi_k(x|\Delta_p^A)$ is simplified by choosing $x=0$. The results are tabulated below.

Table 2

k	$\psi_k(0\lvert\Delta_0^A)$
2	$1/(4\pi)^2 I$
1	$1/(4\pi)^2 8\cdot 1/R^2$
0	$1/(4\pi)^2\,[1/12 F_{\mu\nu}(0)F_{\mu\nu}(0)+464/15\;1/R^4]$
	$\psi_{k\lvert\mu\nu}(0\lvert\Delta_0^A)$
2	$1/(4\pi)^2 I\delta_{\mu\nu}$
1	$1/(4\pi)^2\,[F_{\mu\nu}(0)+{}^*F_{\mu\nu}(0)-4\delta_{\mu\nu}/R^2]$
0	$1/(4\pi)^2\,[\delta_{\mu\nu}1/12 F_{\varrho\sigma}(0)F_{\varrho\sigma}(0)+1/2(F_{\mu\kappa}+{}^*F_{\mu\kappa})(F_{\kappa\nu}+{}^*F_{\kappa\nu})$ $+1/6 D_\varrho D_\varrho(F_{\mu\nu}+{}^*F_{\mu\nu})-4/3\cdot 1/R^2\,{}^*F_{\mu\nu}-16/15\,\delta_{\mu\nu}/R^4]$

Where $D_\varrho=\partial_\varrho+A_\varrho$ is the covariant derivative in flat space.

It follows from Tables 1 and 2 that

$$\psi_1(x|\Delta_0^A)=1/(4\pi)^2\cdot K/6\,,$$

$$\psi_0(x|\Delta_0^A)=1/(4\pi)^2\,[1/12\,F_{\mu\nu}(x)F_{\mu\nu}(x)+aK^2+b|\mathrm{Ric}|^2+c|R|^2]\,,$$

where $2034a+576b+384c=464/15$. In fact, it is possible to show that $a=1/72$, and $c=-b=1/180$. Moreover,

$$\psi_{1|\mu\nu}(x|\Delta_1^A)=1/(4\pi)^2\,[F_{\mu\nu}(x)+{}^*F_{\mu\nu}(x)-1/12 K\delta_{\mu\nu}]\,,$$

$$\begin{aligned}\psi_{0|\mu\nu}(x|\Delta_1^A)=1/(4\pi)^2\,[&1/12\,\delta_{\mu\nu}F_{\varrho\sigma}(x)F_{\varrho\sigma}(x)\\ &+1/2(F_{\mu\kappa}(x)+{}^*F_{\mu\kappa}(x))(F_{\kappa\nu}(x)+{}^*F_{\kappa\nu}(x))\\ &+1/6 D_\varrho D_\varrho(F_{\mu\nu}(x)+{}^*F_{\mu\nu}(x))\\ &+(a'K^2+b'|\mathrm{Ric}|^2+c'|R|^2)\delta_{\mu\nu}\\ &-\tfrac{1}{36}{}^*F_{\mu\nu}(x)K]\,,\end{aligned}$$

where $2304a'+576b'+384c'=-16/15$.

A calculation of $\psi_k(0|\Delta_p^A)$ was also done by Lüscher [6] with identical results.

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