The Optimal Production of an Exhaustible Resource When Price is Exogenous and Stochastic*

by

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November, 1980

WP 1162-80

*Research leading to this paper was supported by the National Science Foundation under Grant No. SOC77-24573, and that support is gratefully acknowledged. The author also wishes to thank Robert Merton for helpful comments and suggestions.

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ABSTRACT

This paper examines the optimal production of a resource such as oil when its price is determined exogenously (e.g. by a cartel such as OPEC), and is subject to stochastic fluctuations away from an expected growth path. We first examine the dependence of production on extraction cost, and show that the conventional exponential decline curve is indeed optimal if marginal cost is constant with respect to the rate of extraction but is a hyperbolic function of the reserve level. We next show that uncertainty about future price affects the optimal production rate in two ways. First, if marginal cost is a convex (concave) function of the rate of production, stochastic fluctuations in price raise (lower) average cost over time, so that there is an incentive to speed up (slow down) production. Second, the "option" value of the reserve, i.e. the ability to withhold production indefinitely and never incur the cost of extraction, provides an incentive to slow down the rate of production.

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1. Introduction

Suppose you owned reserves of an exhaustible resource such as oil. How fast should you produce the resource if its price follows an exogenous growth path, and how should your rate of production be influenced by uncertainty over the future evolution of price?

Hotelling (1931) originally showed that with constant marginal extraction costs, you would produce at maximum capacity or else not at all, depending on whether price net of marginal cost (i.e. "rent") was expected to grow slower or faster than the rate of interest. Thus market clearing would ensure that rent grew at exactly the rate of interest, and producers would be indifferent about their rates of production. But clearly the producers of most resources in competitive markets today are far from indifferent over their production rates, and resource prices (whether in competitive or monopolistic markets) have usually not grown steadily over time as in the simple version of the Hotelling model.

One reason for this is that marginal production costs for most resources are usually not constant, but instead are likely to vary linearly or nonlinearly with the rate of production, and to depend (inversely) on the level of reserves as well. A second reason is that in most cases resource owners perceive the future price of the resource as uncertain, and even if those owners are risk-neutral, this can (as we will see) lead to a shift in behavior. In this paper we focus on these two issues, but recognizing of course that there are a number of other important factors that may also influence the rate of resource production and the behavior of resource markets.[1]

We will examine the optimal production of a (nondurable) exhaustible resource, e.g. oil, when its price follows an exogenous growth path, and may be subject to stochastic variation around that path. However we will not be concerned with the determination of the expected price trajectory, or the reasons for stochastic fluctuations around that trajectory. The reader might like to assume (and it would be reasonable to do so) that the resource price is controlled by a cartel (such as OPEC), and both the expected and realized price trajectories reflect a mixture of rational and (to economists) irrational behavior on the part of the cartel.[2]

We will see in this paper that uncertainty over the future price of the resource can affect the current production rate for two reasons. First, if marginal extraction cost is a nonlinear function of the production rate, stochastic fluctuations in price will lead (on average) to increases or decreases in cost over time, so that cost can be reduced by speeding up or slowing down the rate of depletion. Second, in-ground reserves of a resource can be thought of as an "option" on the future production of the resource; if the future price of the resource turns out to be much higher than the cost of extraction, it may well be desirable to "exercise" the option and produce the resource, but if instead price falls so that production would be unprofitable, the option need never be exercised, and the only loss is the cost of discovering or purchasing the reserve.[3] But this means that under future price uncertainty the current value of a unit of reserve is larger than the current price net of extraction cost, and as we will see, the greater the uncertainty the greater is the incentive to hold back production, and keep the option.

In the next section we set forth a simple deterministic model of optimal production in which price is assumed to grow exponentially at a rate less than the rate of interest, and production costs may be a general function of the rate of production and the level of reserves. The solution of that model is straightforward, but it is useful to examine the characteristics of the production trajectory under different assumptions about the cost function and the rate of growth of price. In Section 3 we introduce uncertainty by letting the price follow a stochastic process so that its future values are lognormally distributed around the expected growth path, and its variance grows linearly with the time horizon. We then solve the resulting stochastic optimization problem, but first ignoring the "option" value of the reserve, i.e. by calculating the expected value of the reserve based on the assumption that it is eventually extracted. This will enable is to examine the relationship between price uncertainty, the characteristics of extraction cost, and the rate of production. We will see that price uncertainty leads to faster (slower) production if marginal cost is a convex (concave) function of the rate of production. We then consider the value of an in-ground unit of reserves as an "option" on possible future production, and show that even if producers are risk neutral this implies a slowing down of production if there is price uncertainty. Finally we summarize our results and offer some concluding remarks in Section 4.

2. Optimal Production under Certainty

We assume that our resource producer begins with a known reserve level R_0, and has a total cost of extraction $C(q,R)$, with $C_q>0$, $C_{qq}\geq 0$, $C_R\leq 0$, $C_{qR}\leq 0$, $C_{qqR}\leq 0$, $C_{qRR}\geq 0$, and $C(0,R) = 0$. (For now we will assume that the inequalities hold for all of these partial derivatives; shortly we will consider special cases where some of the partials are equal to zero.) We also assume that the producer knows that the price of the resource p will grow at the rate α, where $0\leq\alpha<r$, and r is the rate of interest.[4] The producer's problem, then, is:

$$\max_{q(t)} \int_0^\infty [p(t)q(t)-C(q,R)]e^{-rt}dt \tag{1}$$

such that

$$\dot{R} = -q, \; R(0) = R_0 \tag{2}$$

$$\dot{p} = \alpha p, \; p(0) = p_0 \tag{3}$$

and $R(t), q(t) \geq 0$.

This is a straightforward optimal control problem. Define the Hamiltonian H as usual, and maximize with respect to q to get the discounted rent, or shadow price of a unit of reserves:

$$\lambda = (p-C_q)e^{-rt} \tag{4}$$

Now differentiate eqn. (4) with respect to time to get an expression for $\dot{\lambda}$, substitute (2) and (3) for $\dot{p}$ and $\dot{R}$, substitute $\dot{\lambda} = -\partial H/\partial R = C_R e^{-rt}$, and rearrange to obtain the equation that describes the dynamics of production:

$$\dot{q} = -\frac{1}{C_{qq}}[(r-\alpha)p - rC_q - C_{qR}q + C_R] \tag{5}$$

The optimal production and reserve trajectories are thus determined from the simultaneous solution of the three differential equations (2), (3), and (5), together with the two initial conditions for R(0) and p(0), and one terminal condition. Since at the terminal time T, H(T) = 0, the terminal condition will depend on the cost function C(q,R) and the rate of growth of price. If $C_R = 0$ or $C_R < 0$ but $C_{qR} \to a < \infty$ as $R \to 0$, the condition is $q(T) = 0, R(T) = (>) 0$ if $p(T) - C_q(T) > (=) 0$. This condition may also apply if $C_{qR} \to \infty$ as $R \to 0$, but if price grows fast enough, q(t), R(t) and $p(t)-C_q$ will all approach zero asymptotically. Finally, note that the assumption that $\alpha < r$ is essential; if $\alpha \geq r$ there is no incentive to produce at all.

The behavior of production is easiest to understand by examining some special cases. We begin with the case in which marginal cost is constant with respect to the rate of production, i.e. $C_{qq} = 0$, but C_R, $C_{qR} < 0$. This implies a singular solution, so that eqn. (5) no longer holds. Instead, maximization of the Hamiltonian gives:

$$q(t) = \begin{cases} \bar{q}_{max} & \text{if } (p-C_q) > \lambda e^{rt} \\ q^*(t) & \text{if } (p-C_q) = \lambda e^{rt} \\ 0 & \text{if } (p-C_q) < \lambda e^{rt} \end{cases} \tag{6}$$

If marginal cost did not depend on reserves, production would be set at either $\bar{q}_{max}$ or 0 depending on α, but with $C_{qR} < 0$, the interior solution $q^*(t)$ may apply for at least part of the time. We can determine the interior solution by differentiating the condition $(p-C_q) = \lambda e^{rt}$ with respect to time and rearranging:

$$q^*(t) = \frac{1}{C_{qR}} \left[(r-\alpha)p - rC_q + C_R \right] \tag{7}$$

Note that this implies that production is non-zero only if $r(p-C_q) > \alpha p - C_R$, i.e. the marginal profit from producing one unit must exceed the capitalized value of future gains plus future cost savings from all units extracted if the one unit were to be left in the ground.

Generally production will begin at either 0 or $\bar{q}_{max}$ and stay there until p and R are such that the interior condition holds, and then production will follow $q^*(t)$ for the remainder of the time. We can see this using as an example $C(q,R) = mq/R$. Now eqn. (7) implies that $p(t) = mr/(r-\alpha)R^3$. Differentiating this with respect to time and substituting αp for $\dot{p}$ gives an expression for $q^*(t)$:

$$q^*(t) = 3\alpha R = 3\alpha \left[\frac{mr}{(r-\alpha)p} \right]^{1/3} \tag{8}$$

so that when the interior solution applies, production is proportional to the reserve level. In fact this gives the familiar exponential decline curve for production, with the decline rate equal to 3α, i.e. $q^*(t) = 3\alpha R_0 e^{-3\alpha(t-t_0)}$. Note that if $\alpha = 0$ the interior solution never applies, and $q = \bar{q}_{max}$ throughout.

The optimal production and reserve trajectories are shown in Figure 1 for the situation where $q = \bar{q}_{max}$ initially, and in Figure 2 for the situation where $q = 0$ initially. In Figure 1, R_0 and p_0 are such that $R_0 > \left[mr/(r-\alpha)p_0\right]^{1/3}$, so that $q = \bar{q}_{max}$ until R falls and p rises to the point where $R = \left[mr/(r-\alpha)p\right]^{1/3}$, after which q and R follow the interior solution given by eqn. (8). Note from Figure 1 that as α becomes smaller, q remains at $\bar{q}_{max}$ longer, and the switch to the interior solution occurs at lower values of q and R. In Figure 2, R_0 is small relative to p_0 and α, so that $q = 0$ until p rises to the point where $R_0 = \left[mr/(r-\alpha)p\right]^{1/3}$, after which q and R again follow the interior solution. Note that if α is made larger, q remains at 0 longer, and the interior solution implies higher values of q and R. Letting T_2 be the time at which q switches from 0 to $q^*(t) > 0$, note that $T_2 \to \infty$ as $\alpha \to r$.

We observe, then, that even if marginal cost is constant with respect to the rate of output, dependence on the reserve level will imply an interior solution for the optimal production rate that will hold for at least part of the time. Furthermore, if marginal cost varies hyperbolically with the reserve level (which is at least approximately the case for many oil and gas reserves), the interior solution is the conventional exponential decline curve for production.

Now consider the alternative special case where cost is independent of reserves, i.e. $C_R = C_{qR} = 0$, but $C_{qq} > 0$. In this case an interior solution always applies, and eqn. (5) simplifies to:

$$\dot{q} = -\frac{1}{C_{qq}}\left[(r-\alpha)p - rC_q\right] \qquad (9)$$

The characteristics of the solution in this case will depend on both α and the shape of the marginal cost curve, and in particular the sign of C_{qqq}. Suppose $\alpha = 0$. Then clearly $\dot{q} < 0$ to exhaustion. Furthermore, if $C_{qqq} \geq 0$, $\ddot{q} < 0$ throughout as well. If $C_{qqq} < 0$ and is sufficiently large in magnitude, $\ddot{q}$ will be positive at first and later will turn and remain negative until exhaustion. These possibilities are shown as trajectories A and B in Figure 3.

Now suppose $\alpha > 0$. In this case the sign of $\dot{q}$ is not clear, at least during the initial period of production. With $C_R = 0$, discounted rent λ must be constant over time, so that if price is growing fast enough, production may be initially increasing. In such a case, $\dot{q}$ will change sign (q must fall to zero as $R \to 0$ since $H(T) = 0$), and can change sign only once. To see this, note that

$$\ddot{q} = -\frac{1}{C_{qq}}[r^2\lambda_0 e^{rt} - \alpha^2 p_0 e^{\alpha t}] \qquad (10)$$

so that $\ddot{q}$ is negative always, or else $\ddot{q}$ is positive initially, and turns negative later. Thus if α is large, optimal production can also follow trajectories C or D in Figure 3.

3. Production when Price is Stochastic

To introduce uncertainty over future values of price, we assume that price fluctuates from its expected growth path according to a stochastic process with independent increments. In particular, we replace eqn. (3) with

$$dp = \alpha p dt + \sigma p dz = \alpha p dt + \sigma p \varepsilon(t)\sqrt{dt} \qquad (11)$$

where $\varepsilon(t)$ is a serially uncorrelated normal random variable with zero mean and unit variance (i.e. $z(t)$ is a Wiener process).[5] Eqn. (11) implies that uncertainty about price grows with the time horizon, and that fluctuations in price occur continuously over time.

We assume that producers are risk neutral, so that the dynamic optimization problem is now:

$$\max_{q(t)} E_0 \int_0^\infty [p(t)q(t) - C(q,R)]\, e^{-rt} dt = E_0 \int_0^\infty \Pi_d(t) dt \qquad (12)$$

subject to the ordinary differential equation (2), the stochastic differential equation (11), and $R(t), q(t) \geq 0$.

Our approach is to first solve this problem under the assumption that p_0 and α are such that $q(t) > 0$, and that $q(t) > 0$ over the entire planning horizon (i.e. up to the point where exhaustion occurs or $C_q = p$). This can be done using stochastic dynamic programming, and although it ignores the possibility of withholding all production (perhaps indefinitely), it will allow us to determine how the effect of price uncertainty on current production depends on the characteristics of cost. Afterwards we will consider the ability of the producer to withhold production (of perhaps currently uneconomical reserves), while maintaining the option of producing in the future should price unexpectedly rise rapidly. As we will see, this leads to a quite different effect of price uncertainty.

We begin, then, by looking for an interior solution to the optimization problem set forth above. To do this, define the optimal value function:

$$J = J(R,p,t) = \max_{q(\tau)} E_t \int_t^{\infty} \Pi_d(\tau)\,d\tau \tag{13}$$

Since J is a function of the stochastic process p, the fundamental equation of optimality is:[6]

$$0 = \max_{q(t)} \{\Pi_d(t) + (1/dt)E_t dJ\}$$

$$= \max_{q(t)} \{\Pi_d(t) + J_t - qJ_R + \alpha p J_p + \tfrac{1}{2}\sigma^2 p^2 J_{pp}\} \tag{14}$$

Maximizing with respect to q gives:

$$\partial\Pi_d/\partial q = J_R \tag{15}$$

i.e. the usual result that the shadow price of the resource should always equal the incremental profit that could be obtained by selling an additional unit. Now differentiate eqn. (14) with respect to R:

$$\frac{\partial\Pi_d}{\partial R} + J_{Rt} - qJ_{RR} + \alpha p J_{Rp} + \tfrac{1}{2}\sigma^2 p^2 J_{Rpp} = 0, \tag{16}$$

and by Ito's Lemma note that this can be re-written as:

$$\partial \Pi_d/\partial R + (1/dt)E_t d(J_R) = 0 \tag{17}$$

To eliminate J from the problem, apply the operator $(1/dt)E_t d(\ )$ to both sides of (15), and combine the resulting equation with (17) to yield:

$$(1/dt)E_t d(\partial \Pi_d/\partial q) = - \partial \Pi_d/\partial R \tag{18}$$

Eqn. (18) is just a stochastic version of the well-known Euler equation from the calculus of variations. In its integral form it says that the marginal profit from selling 1 unit of reserves should just equal the expected sum of all discounted future increases in profit that would result from holding the unit in the ground.

Our objective is to obtain an equation analogous to eqn. (5) to explain the expected dynamics of production. To do this, substitute $\partial \Pi_d/\partial q = [p(t)-C_q]e^{-rt}$ and $\partial \Pi_d/\partial R = - C_R e^{-rt}$ into eqn. (18):

$$-r[p(t)-C_q] + (1/dt)E_t dp - (1/dt)E_t dC_q = C_R \tag{19}$$

Now note that $E_t dp = \alpha p dt$, and expand dC_q using Ito's Lemma:

$$dC_q = C_{qq}dq + C_{qR}dR + \frac{1}{2}C_{qqq}(dq)^2 + o(t) \tag{20}$$

where $o(t)$ represents terms that vanish as $dt \to 0$. Along an optimal trajectory $q = q^*(R,p)$, so that

$$E_t(dq)^2 = \sigma^2 p^2 q_p^2 dt + o(t) \tag{21}$$

where q is the (unknown) response of optimal production to a change in price. Now substituting eqns. (20) and (21) into (19) and rearranging, we obtain the equation, analogous to eqn. (5), that describes the expected dynamics of production:

$$\frac{1}{dt}E_t dq = -\frac{1}{C_{qq}}[(r-\alpha)p - rC_q - C_{qR}q + C_R + \frac{1}{2}\sigma^2 p^2 q_p^2 C_{qqq}] \quad (22)$$

Eqn. (22) tells us that the expected rate of change of production differs from the certainty case when marginal cost C_q is a nonlinear function of the rate of production. In particular, we see that when production is falling, price uncertainty causes it to fall faster (slower), so that production begins at a higher (lower) level, when marginal cost is a convex (concave) function of q. This deviation from the certainty case is easily understood by recognizing that stochastic variations in price imply changes in expected future marginal costs if $C_{qqq} \neq 0$. To see this, suppose $C_{qqq} > 0$, and random increases and decreases in price occur that balance out, leaving price unchanged on average. Clearly such fluctuations will have the net effect of raising marginal cost over time, since corresponding increases in optimal production will raise marginal cost more than corresponding decreases will lower it. This in turn implies an incentive to speed up production, and thereby reduce these expected increases in cost over time. If, on the other hand, $C_{qqq} < 0$, just the opposite holds, and there is an incentive to slow down production.

We see then that if marginal cost increases nonlinearly, price uncertainty will lead to changes in the expected rate of production. Does this mean that if marginal cost is constant or rises linearly with the rate of production, price uncertainty should have no effect on expected production? The answer is no, as we see if we remember that producers need not produce at all. This means that if current or <u>expected</u> price is below extraction cost for any marginal unit of the resource, the owner can keep the unit in the ground indefinitely but maintain the option of extracting it at some future time in the event that there is a sufficient (random) increase in price. Alternatively, suppose that production of a unit would be profitable, but just barely so. Then with future price sufficiently uncertain there is an incentive to keep the unit in the ground, since if price were to fall the only loss would be the small (unrealized) profit, whereas if price were to rise, extraction could then yield a relatively large profit.

This means that uncertainty over future price creates an incentive to slow down production. To see this, assume that $C_q(0,R) > 0$, and suppose for simplicity that $C_{qqq} = C_R = 0$. Now consider the present value of a marginal unit of reserves that can be extracted at some time t. Under certainty that value is

$$V = \max\{0, (p - C_q)e^{-rt}\} \tag{23}$$

i.e. extraction of the unit would occur only if $p(t) > C_q$. Further, it is easy to see from (23) that the value is constant over time, i.e. $dV/dt = 0$.

Now suppose future price is uncertain. The present value of the marginal unit is then just the expected value of the right-hand side of eqn. (23). Now consider the expected rate of change of that value. Since the right-hand side of (23) is a convex function of p, we have by Jensen's inequality:

$$\begin{aligned}(1/dt)E_0 dV &= (1/dt)E_0 \max\{0, d[(p-C_q)e^{-rt}]\} \\ &> (1/dt)\max\{0, E_0 d[(p-C_q)e^{-rt}]\} = 0\end{aligned} \tag{24}$$

Thus under uncertainty it is preferable not to extract the marginal unit that under certainty would have been extracted at time t, since that unit is now expected to rise in value over time. This means that given any particular current price (and given any expected rate of price increase α), production should be lower the greater the uncertainty over future price.

Uncertainty over future price therefore affects current production in two different ways. First, if marginal cost is a nonlinear function of the rate of production, future price uncertainty creates a cost-based incentive to alter the current rate of production -- speeding it up if the marginal cost function is convex, and slowing it down if it is concave. Second, whatever the characteristics of marginal cost, the fact that any in-ground unit has value as an option to extract -- or not extract -- in the future implies that future price uncertainty will cause a slowing down of production. Of course if the marginal cost function is convex, the net effect is ambiguous.

4. Concluding Remarks

Models of resource production often contain severe simplifying assumptions about the characteristics of cost and about knowledge of future price, and we have seen in this paper that such assumptions may lead to highly misleading results. For example, for most resources, and certainly for oil and gas, extraction cost is in fact usually not constant with respect to the reserve level and the rate of production as often assumed, and this means that resource owners should be far from indifferent about their rate of production. In particular, if marginal cost can be roughly characterized as a hyperbolic function of the reserve level, then the conventional exponential decline curve often used by petroleum engineers will apply.

It is also a fact that there is considerable uncertainty about the future prices of most resources. This is particularly true today for oil and other energy resources, where price determination by a politically unstable cartel makes market evolution highly unpredictable. We have seen that even if resource owners are risk neutral, uncertainty over future price will alter their current rates of production, in a way that depends on the characteristics of marginal cost.

In this paper price was viewed as exogenous, and we did not consider market equilibrium. This is reasonable in the case of oil, where price is now controlled (rationally or irrationally) by a cartel. The earlier paper by this author (1980) examined the effects of future demand and reserve uncertainty on the evolution of competitive market price, but made the assumption that marginal cost is constant with respect to the rate of production, and ignored the value of in-ground reserves as an option (that need not be exercised) on future production. In so doing it found that with constant marginal cost, under demand and/or reserve uncertainty the expected rate of change of rent, i.e. the competitive market price net of marginal cost, would still equal the rate of interest. It is easy to see from eqn. (24), however, that because of the option value of in-ground reserves, this is in fact not the case, and in expected value terms rent should rise at less than the rate of interest, since otherwise the expected present value of in-ground reserves would rise over time.

FOOTNOTES

1. For example, Pindyck (1978b) discusses the interrelationship between the rate of production and the rate of exploration and reserve accumulation, Levhari and Pindyck (1980) show how the durability of some resources affects their production and market price paths, and Newbery (1980) examines the implications of alternative market structures for production rates. Also, in an earlier paper Pindyck (1980) examined the effect of demand and/or reserve uncertainty on the expected evolution of the competitive market price, but that paper made simplifying assumptions about the characteristics of extraction cost, and also ignores the possibility that producers might withhold production that is currently uneconomical until (and if) price unexpectedly rises.

2. An earlier paper by this author (1978a) examined the optimal price behavior for an exhaustible resource cartel, taking non-cartel supply behavior as exogenous (and not dynamically optimal). For models in which the price and output behavior of both the cartel and the "competitive fringe" are dynamically consistent (i.e. Nash-Cournot models), see Salant (1976) for the case in which all producers face identical costs, and Newbery (1980) for the more general case in which there are differences in costs and/or discount rates.

3. The option value of an in-ground reserve is discussed in a recent paper by Tourinho (1970), who uses the standard Black-Scholes option pricing model to show how the value of the reserve grows as the degree of future price uncertainty grows.

(continued)

FOOTNOTES

(continued)

4. Whether price is determined in a quasi-competitive market or is set by a cartel, we would expect it to grow at a rate less than r as long as extraction costs are positive.

5. Eqn. (11) is the limiting form as $h \to 0$ of the discrete-time difference equation $p(t+h) - p(t) = \alpha p(t)h + \sigma p(t)\varepsilon(t)\sqrt{h}$ and $E[dp/p] = \alpha dt$, and $Var[dp/p] = \sigma^2 dt$. Note that p(t) is lognormally distributed, with $E_0[\log(p/t)/p(0))] = (\alpha - \frac{1}{2}\sigma^2)t$, and $Var[\log(p(t)/p(0))] = \sigma^2 t$. For an introduction to stochastic differential equations of the form of (11), see Cox and Miller (1965).

6. We use the notation $J_R = \partial J/\partial R$, etc. $(1/dt)E_t d(\ )$ is Ito's differential generator. For a discussion of the techniques used in this paper, and in particular the use of Ito's Lemma, see Kushner (1967), Merton (1971), or Chow (1979). Also, the approach here follows closely that used in this author's (1980) earlier paper on this subject.

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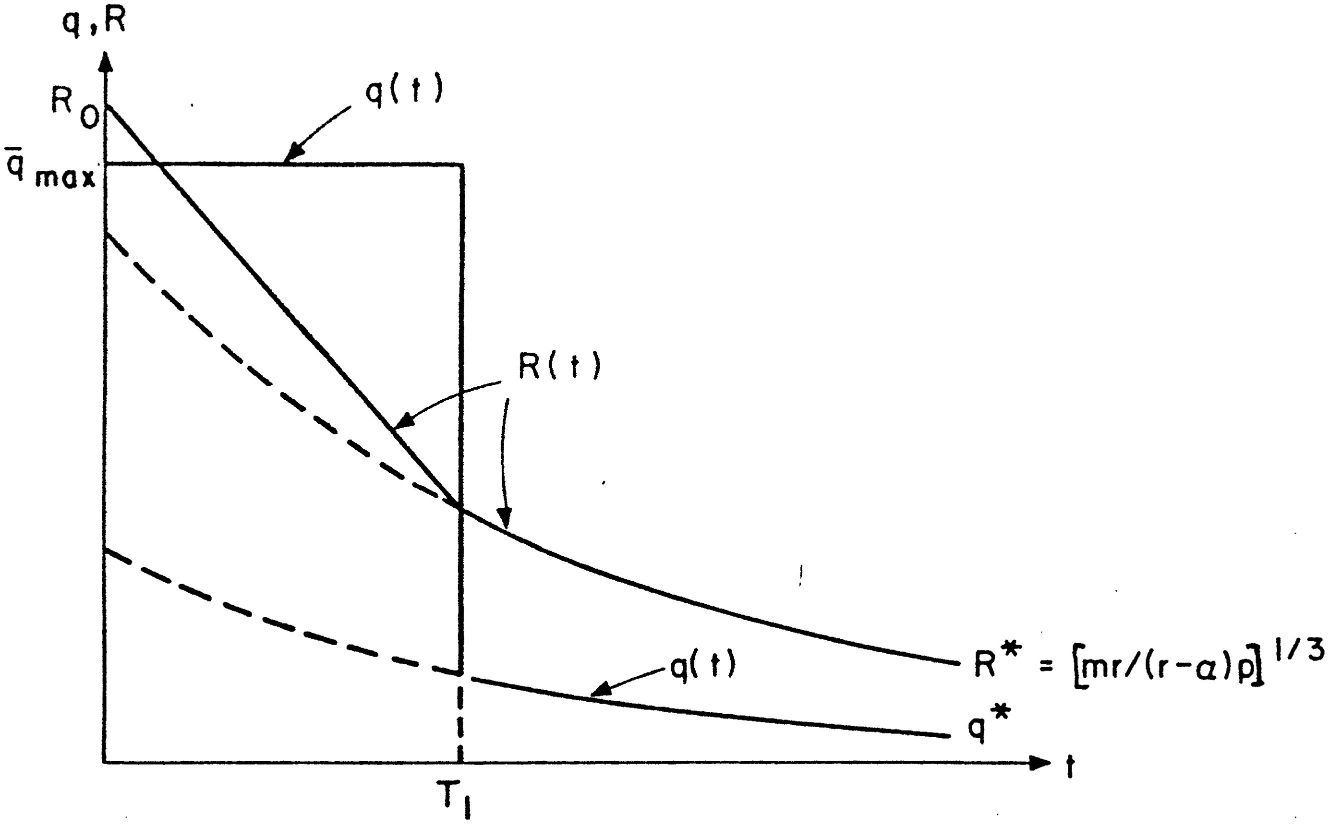

FIGURE 1: Production when $C_{qq} = 0$ and R_0 is Large Relative to P_0 and α

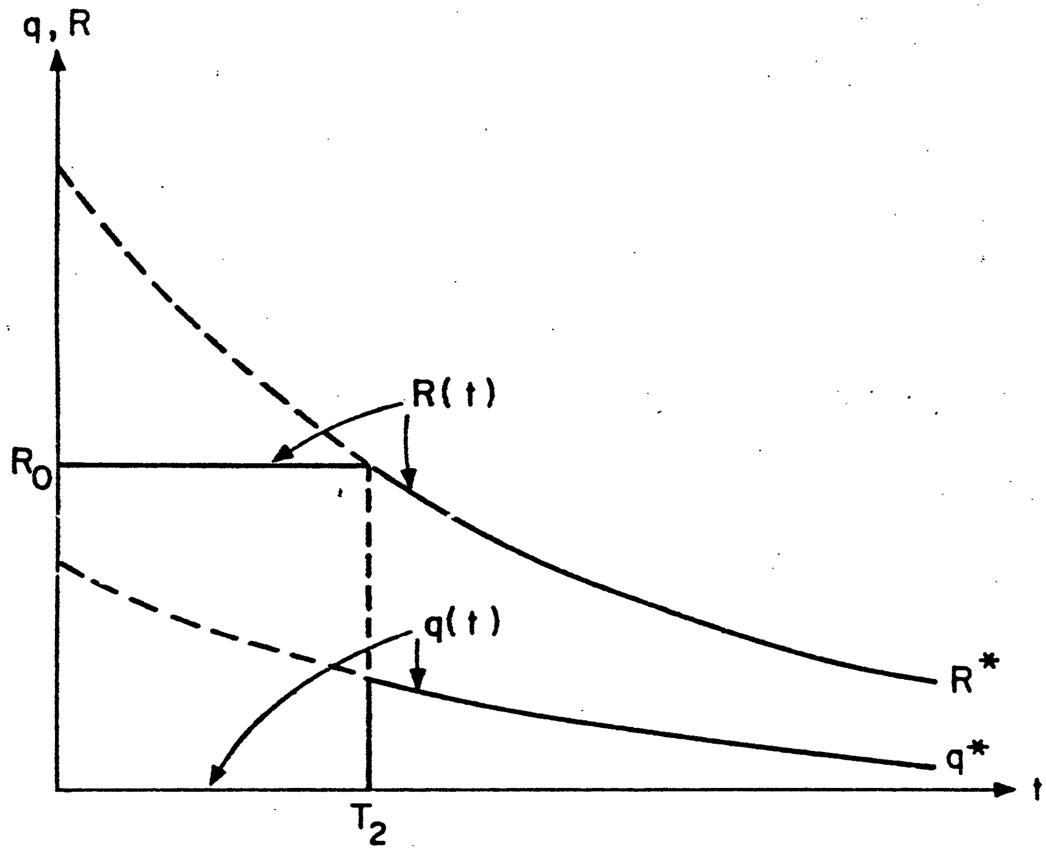

FIGURE 2: Production when $C_{qq} = 0$ and R_0 is Small Relative to P_0 and α

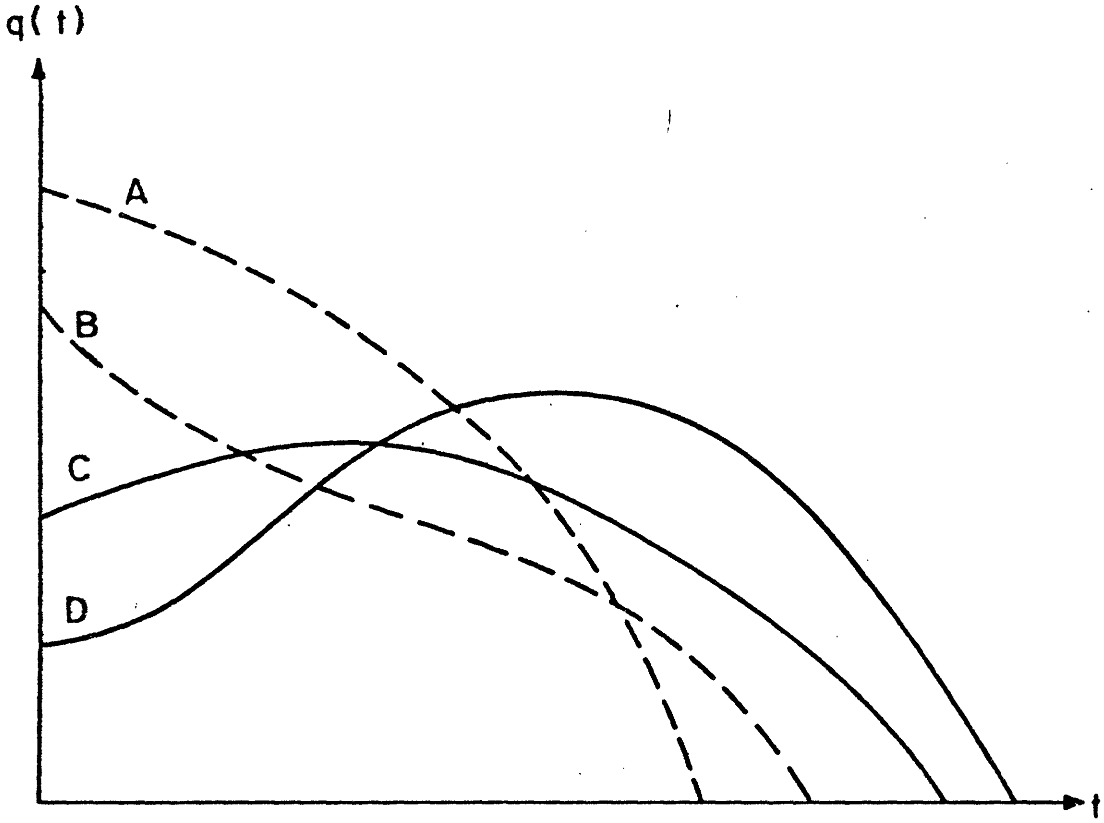

FIGURE 3: Production Trajectories when $C_R = 0$.