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THE ORBIFOLD COHOMOLOGY RING OF SIMPLICIAL TORIC STACK BUNDLES

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Abstract. We introduce a new quotient construction of toric Deligne–Mumford stacks. We use this new construction to define toric stack bundles which generalize the construction of toric bundles by Sankaran and Uma [*Comment. Math. Helv.* **78** (2003) 540–554]. The orbifold Chow ring of such toric stack bundles is computed. We show that the orbifold Chow ring of the toric stack bundle and the Chow ring of its crepant resolution are fibres of a flat family, generalizing a result of Borisov–Chen–Smith.

1. Introduction

For a complex algebraic orbifold (or equivalently a smooth Deligne–Mumford stack), the orbifold cohomology was constructed using the genus zero and degree zero orbifold Gromov–Witten invariants of Deligne–Mumford stacks, see [AGV], [CR1], [CR2]. In this paper, we explicitly compute the orbifold cohomology ring of toric stack bundles. These are bundles over a smooth base variety B with fibers toric Deligne–Mumford stacks of [BCS].

From [Cox], to a simplicial fan Σ with n rays, one can associate a simplicial toric variety $X(\Sigma)$ expressed as a quotient Z/G, where Z is an open subset of $\mathbb{C}^n$ and G is a subgroup of $(\mathbb{C}^\times)^n$. Let $T:=(\mathbb{C}^\times)^n/G$ be the torus acting on $X(\Sigma)$. Given a principle T-bundle $E\to B$, one can form a fibre bundle ${}^EX(\Sigma):=E\times_T X(\Sigma)\to B$ over B with fibers isomorphic to $X(\Sigma)$. The cohomology ring of ${}^EX(\Sigma)$ was computed in [SU].

Generalizing Cox's construction, Borisov, Chen, and Smith [BCS] constructed toric Deligne–Mumford stacks. A toric Deligne–Mumford stack is defined in terms of a stacky fan $\mathbf{\Sigma}=(N,\Sigma,\beta)$, where N is a finitely generated Abelian group, $\Sigma\subset\overline{N}=N\otimes_{\mathbb{Z}}\mathbb{Q}$ is a simplicial fan and $\beta:\mathbb{Z}^n\longrightarrow N$ is a map determined by the elements $\{b_1,\ldots,b_n\}$ in N. They require that β has finite

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cokernel and $\{\overline{b}_1,\ldots,\overline{b}_n\}$ generate the simplicial fan Σ, where $\overline{b}_i$ is the image of b_i under the natural map $N \longrightarrow \overline{N}$. (Note that n equals the number of rays in Σ.) The toric Deligne–Mumford stack $\mathcal{X}(\mathbf{\Sigma})$ associated to $\mathbf{\Sigma}$ is defined to be the quotient stack $[Z/G]$, where Z is the same as in the quotient construction of toric varieties, G is the product of an algebraic torus and a finite Abelian group, and the action is through a group homomorphism $\alpha: G \longrightarrow (\mathbb{C}^\times)^n$ determined by the stacky fan. Let $P \longrightarrow B$ be a principal $(\mathbb{C}^\times)^n$-bundle over a smooth variety B. The group G acts on the fibre product $P \times_{(\mathbb{C}^\times)^n} Z$ via the map α. Define ${}^P\mathcal{X}(\mathbf{\Sigma})$ to be the quotient stack $[(P \times_{(\mathbb{C}^\times)^n} Z)/G]$ which we write as $P \times_{(\mathbb{C}^\times)^n} \mathcal{X}(\mathbf{\Sigma})$. The fibre bundle ${}^P\mathcal{X}(\mathbf{\Sigma}) \longrightarrow B$ is called a toric stack bundle over B whose fibre is the toric Deligne–Mumford stack $\mathcal{X}(\mathbf{\Sigma})$.

Both Cox and Borisov–Chen–Smith's construction used the minimal presentation of a toric variety (stack) as a quotient. One may expect that toric Deligne–Mumford stacks can be represented as a quotient stack of a larger space Z by a larger group G. For example, the classifying stack $\mathcal{B}\mu_3 = [pt/\mu_3]$ is a toric Deligne–Mumford stack in the sense of Borisov, Chen, and Smith, where the corresponding stacky fan is $(\mathbb{Z}_3, 0, 0)$. The stack $\mathcal{B}\mu_3$ is isomorphic to the stack $[\mathbb{C}^\times/\mathbb{C}^\times]$, where $\mathbb{C}^\times$ acts on $\mathbb{C}^\times$ by $\lambda x \mapsto \lambda^3 x$. Given a line, bundle $L \longrightarrow B$ over B. Applying the construction above yields, a μ_3-gerbe $[(L^\times \times_{\mathbb{C}^\times} \mathbb{C}^\times)/\mathbb{C}^\times]$ over B which is nontrivial if L is. The presentation $[pt/\mu_3]$ only produces trivial gerbes.

Motivated by the study of gerbes, the above discussion suggests that it is desirable to work with other presentations of toric Deligne–Mumford stacks. For this purpose, we introduce the notion of *extended stacky fans*. An extended stacky fan is a triple $\mathbf{\Sigma^e} := (N, \Sigma, \beta^e)$, where N and Σ are the same as in the stacky fan $\mathbf{\Sigma}$, but $\beta^e: \mathbb{Z}^m \longrightarrow N$ is determined by $\{b_1,\ldots,b_n\}$ and additional elements $\{b_{n+1},\ldots,b_m\}$ in N. An extended stacky fan $\mathbf{\Sigma^e}$ has an underlying stacky fan $\mathbf{\Sigma}$. Using $\mathbf{\Sigma^e}$, we define a quotient stack $\mathcal{X}(\mathbf{\Sigma^e}) := [Z^e/G^e]$, where $Z^e = Z \times (\mathbb{C}^\times)^{m-n}$ and G^e acts on Z^e through the homomorphism $\alpha^e: G^e \longrightarrow (\mathbb{C}^\times)^m$ determined by the extended stacky fan. We prove that $\mathcal{X}(\mathbf{\Sigma^e})$ is isomorphic to the toric Deligne–Mumford stack $\mathcal{X}(\mathbf{\Sigma})$. So, enlarging the presentation from the minimal ones of Cox and Borisov–Chen–Smith is encoded in the extended stacky fan. For example, let $N = \mathbb{Z}_3$, let $\beta^e: \mathbb{Z} \longrightarrow N$ be the map defined by $b_1 = 1 \in \mathbb{Z}_3$, then $\mathbf{\Sigma^e} = (N, \Sigma, \beta^e)$ is an extended stacky fan. (Note that this is not a stacky fan.) The toric Deligne–Mumford stack is $\mathcal{X}(\mathbf{\Sigma^e}) = [\mathbb{C}^\times/\mathbb{C}^\times]$ which is isomorphic to $[pt/\mu_3]$.

Let $P \longrightarrow B$ be a principal $(\mathbb{C}^\times)^m$-bundle. The group G^e acts on the fibre product $P \times_{(\mathbb{C}^\times)^m} Z^e$ via the map α^e. The quotient stack ${}^P\mathcal{X}(\mathbf{\Sigma^e}) := [(P \times_{(\mathbb{C}^\times)^m} Z^e)/G^e]$ is called a toric stack bundle over B whose fibre is isomorphic to the toric Deligne–Mumford stack $\mathcal{X}(\mathbf{\Sigma^e})$.

In [BCS], Borisov, Chen, and Smith computed the orbifold Chow ring of toric Deligne–Mumford stacks. The computation in the special case of weighted projective stack was pursued in [Jiang1]. In this paper, we compute the orbifold cohomology ring of ${}^P\mathcal{X}(\mathbf{\Sigma^e})$. To describe the result, we introduce line bundles ξ_θ for $\theta \in M = N^*$. For $\theta \in M$, let $\chi^\theta : (\mathbb{C}^\times)^m \longrightarrow \mathbb{C}^\times$ be the map induced by $\theta \circ \beta^e : \mathbb{Z}^m \longrightarrow \mathbb{Z}$. The bundle $\xi_\theta \longrightarrow B$ is the line bundle $P \times_{\chi^\theta} \mathbb{C}$. We introduce the deformed ring $A^*(B)[N]^{\mathbf{\Sigma^e}} = A^*(B) \otimes \mathbb{Q}[N]^{\mathbf{\Sigma^e}}$, where $\mathbb{Q}[N]^{\mathbf{\Sigma^e}} := \bigoplus_{c \in N} \mathbb{Q}y^c$, y is a formal variable and $A^*(B)$ is the Chow ring of B. The multiplication of $\mathbb{Q}[N]^{\mathbf{\Sigma^e}}$ is given by:

$$(1) \quad y^{c_1} \cdot y^{c_2} := \begin{cases} y^{c_1+c_2} & \text{if there is a cone } \sigma \in \Sigma \text{ such that } \overline{c}_1 \in \sigma, \overline{c}_2 \in \sigma, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathcal{I}({}^P\mathbf{\Sigma^e})$ be the ideal in $A^*(B)[N]^{\mathbf{\Sigma^e}}$ generated by the elements:

$$(2) \qquad \left(c_1(\xi_\theta) + \sum_{i=1}^{n} \theta(b_i) y^{b_i} \right)_{\theta \in M},$$

and $A^*_{orb}({}^P\mathcal{X}(\mathbf{\Sigma^e}))$ the orbifold Chow ring of the toric stack bundle ${}^P\mathcal{X}(\mathbf{\Sigma^e})$.

THEOREM 1.1. *If ${}^P\mathcal{X}(\mathbf{\Sigma^e}) \longrightarrow B$ is a toric stack bundle over a smooth variety B whose fibre is the toric Deligne–Mumford stack $\mathcal{X}(\mathbf{\Sigma^e})$ associated to an extended stacky fan $\mathbf{\Sigma^e}$, then we have an isomorphism of $\mathbb{Q}$-graded rings:*

$$A^*_{orb}({}^P\mathcal{X}(\mathbf{\Sigma^e})) \cong \frac{A^*(B)[N]^{\mathbf{\Sigma^e}}}{\mathcal{I}({}^P\mathbf{\Sigma^e})}.$$

The extra data $\{b_{n+1}, \ldots, b_m\}$ in the extended stacky fan $\mathbf{\Sigma^e}$ does affect the structure of ${}^P\mathcal{X}(\mathbf{\Sigma^e})$, but does not affect its orbifold cohomology ring.

To prove this theorem, we show that twisting by the $(\mathbb{C}^\times)^m$-bundle P does not "twist" the components of the inertia stack of the toric Deligne–Mumford stack $\mathcal{X}(\mathbf{\Sigma^e})$. Thus, we can describe the components of the inertia stack $\mathcal{I}({}^P\mathcal{X}(\mathbf{\Sigma^e}))$ of ${}^P\mathcal{X}(\mathbf{\Sigma^e})$ using $Box(\mathbf{\Sigma^e})$, in a manner analogous to [BCS]. This makes it possible to use the similar methods as in [BCS] to determine 3-twisted sectors, obstruction bundles and compute the orbifold Chow ring of ${}^P\mathcal{X}(\mathbf{\Sigma^e})$.

As an example, let N be a finite Abelian group and $\beta^e : \mathbb{Z} \longrightarrow N$ any homomorphism, then $\mathbf{\Sigma^e} = (N, 0, \beta^e)$ is an extended stacky fan. The toric Deligne–Mumford stack is $\mathcal{X}(\mathbf{\Sigma^e}) = \mathcal{B}\mu$, where $\mu = Hom(N, \mathbb{C}^\times)$. Twisting this toric Deligne–Mumford stack by a line bundle L over a smooth variety B gives a μ-gerbe $\mathcal{X}$ over B. So, no matter if the gerbe is trivial or not, our result gives that $H^*_{orb}(\mathcal{X}, \mathbb{Q}) = H^*(B, \mathbb{Q}) \otimes H^*_{orb}(\mathcal{B}\mu, \mathbb{Q})$.

The paper is organized as follows. In Section 2, we introduce extended stacky fans and their associated toric Deligne–Mumford stacks. In Section 3, we define toric stack bundles and discuss their properties. In Section 4, we describe the orbifold Chow ring of toric stack bundles. In Section 5, we discuss the μ-gerbe $\mathcal{X}$ mentioned above. Finally, in Section 6, we give some applications to crepant resolutions. We generalize a result of Borisov, Chen, and Smith showing that the orbifold Chow ring of a toric stack bundle and the Chow ring of its crepant resolution can be put into a flat family.

Conventions. In this paper, we work entirely algebraically over the field of complex numbers. Chow rings and orbifold Chow rings are taken with rational coefficients. By an orbifold, we mean a smooth Deligne–Mumford stack with trivial generic stabilizer. We refer to [BCS] for the construction of Gale dual $(\beta^e)^\vee : \mathbb{Z}^m \to DG(\beta)$ from $\beta^e : \mathbb{Z}^m \to N$. We denote by N^* the dual of N and $N \to \overline{N}$ the natural map modulo torsion.

2. A new quotient representation of toric Deligne–Mumford stacks

In this section, we introduce extended stacky fans and construct a new representation of toric Deligne–Mumford stacks.

We refer to [BCS] the construction and notation of toric Deligne–Mumford stacks. Let N be a finitely generated Abelian group of *rank* d and $\overline{N}$ the lattice generated by N in the d-dimensional vector space $N_{\mathbb{Q}} := N \otimes_{\mathbb{Z}} \mathbb{Q}$. Write $\overline{b}$ for the image of b under the natural map $N \longrightarrow \overline{N}$. Let Σ be a rational *simplicial* fan in $N_{\mathbb{Q}}$. Suppose $\rho_1, \ldots, \rho_n$ are the rays in Σ. We fix $b_i \in N$ for $1 \leq i \leq n$ such that $\overline{b}_i$ generates the cone ρ_i. Let $\{b_{n+1}, \ldots, b_m\} \subset N$. We consider the homomorphism $\beta^e : \mathbb{Z}^m \longrightarrow N$ determined by the elements $\{b_1, \ldots, b_m\}$. We require that β^e has finite cokernel.

Definition 2.1. The triple $\mathbf{\Sigma^e} := (N, \Sigma, \beta^e)$ is called an extended stacky fan.

It is easy to see that any extended stacky fan $\mathbf{\Sigma^e} = (N, \Sigma, \beta^e)$ naturally determines a stacky fan $\mathbf{\Sigma} := (N, \Sigma, \beta)$, where $\beta : \mathbb{Z}^n \longrightarrow N$ is given by $\{b_1, \ldots, b_n\}$. Now since β^e has finite cokernel, by Proposition 2.2 in [BCS], we have exact sequences:

$$0 \longrightarrow DG(\beta^e)^* \longrightarrow \mathbb{Z}^m \xrightarrow{\beta^e} N \longrightarrow Coker(\beta^e) \longrightarrow 0,$$
$$0 \longrightarrow N^* \longrightarrow \mathbb{Z}^m \xrightarrow{(\beta^e)^\vee} DG(\beta^e) \longrightarrow Coker((\beta^e)^\vee) \longrightarrow 0,$$

where $(\beta^e)^\vee$ is the Gale dual of β^e. As a $\mathbb{Z}$-module, $\mathbb{C}^\times$ is divisible, so it is an injective $\mathbb{Z}$-module, and hence from [Lang], the functor $Hom_{\mathbb{Z}}(-, \mathbb{C}^\times)$ is exact. We get the exact sequence:

$$1 \longrightarrow Hom_{\mathbb{Z}}(Coker((\beta^e)^\vee), \mathbb{C}^\times) \longrightarrow Hom_{\mathbb{Z}}(DG(\beta^e), \mathbb{C}^\times) \longrightarrow Hom_{\mathbb{Z}}(\mathbb{Z}^m, \mathbb{C}^\times) \longrightarrow Hom_{\mathbb{Z}}(N^*, \mathbb{C}^\times) \longrightarrow 1.$$

Let $\mu := Hom_{\mathbb{Z}}(Coker((\beta^e)^\vee), \mathbb{C}^\times)$, we have the exact sequence:

$$1 \longrightarrow \mu \longrightarrow G^e \xrightarrow{\alpha^e} (\mathbb{C}^\times)^m \longrightarrow T \longrightarrow 1. \tag{3}$$

From [BCS], the toric Deligne–Mumford stack $\mathcal{X}(\mathbf{\Sigma}) = [Z/G]$ is a quotient stack, where they use the method of quotient construction of toric varieties [Cox]. Define $Z^e := Z \times (\mathbb{C}^\times)^{m-n}$, then there exists a natural action of $(\mathbb{C}^\times)^m$ on Z^e. The group G^e acts on Z^e through the map α^e in (3). The quotient stack $[Z^e/G^e]$ is associated to the groupoid $Z^e \times G^e \rightrightarrows Z^e$. Define the morphism $\varphi : Z^e \times G^e \longrightarrow Z^e \times Z^e$ to be $\varphi(x,g) = (x, g \cdot x)$. Since $Z^e = Z \times (\mathbb{C}^\times)^{m-n}$, we can mimic the proof the Lemma 3.1 in [BCS] to show that φ is finite which means that the stack $[Z^e/G^e]$ is a Deligne–Mumford stack.

Lemma 2.2. *The morphism $\varphi : Z^e \times G^e \longrightarrow Z^e \times Z^e$ is a finite morphism.*

Definition 2.3. For an extended stacky fan $\mathbf{\Sigma^e} = (N, \Sigma, \beta^e)$, we denote the quotient stack $[Z^e/G^e]$ by $\mathcal{X}(\mathbf{\Sigma^e})$.

Proposition 2.4. *For an extended stacky fan $\mathbf{\Sigma^e} = (N, \Sigma, \beta^e)$, the stack $\mathcal{X}(\mathbf{\Sigma^e})$ is isomorphic to the toric Deligne–Mumford stack $\mathcal{X}(\mathbf{\Sigma})$ associated to the underlying stacky fan $\mathbf{\Sigma}$.*

Proof. From the definitions of extended stacky fan $\mathbf{\Sigma^e}$ and stacky fan $\mathbf{\Sigma}$, we have the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}^n & \longrightarrow & \mathbb{Z}^m & \longrightarrow & \mathbb{Z}^{m-n} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \beta^e} & & \downarrow{\scriptstyle \tilde{\beta}} & & \\ 0 & \longrightarrow & N & \xrightarrow{id} & N & \longrightarrow & 0 & \longrightarrow & 0. \end{array}$$

From the definition of Gale dual, we compute that $DG(\tilde{\beta}) = \mathbb{Z}^{m-n}$ and $\tilde{\beta}^\vee$ is an isomorphism. So, from Lemma 2.3 in [BCS], applying the Gale dual and the $Hom_{\mathbb{Z}}(-, \mathbb{C}^\times)$ functor to the above diagram we get:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & G & \xrightarrow{\varphi_1} & G^e & \longrightarrow & (\mathbb{C}^\times)^{m-n} & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha^e} & & \downarrow{\scriptstyle \tilde{\alpha}} & & \\ 1 & \longrightarrow & (\mathbb{C}^\times)^n & \longrightarrow & (\mathbb{C}^\times)^m & \longrightarrow & (\mathbb{C}^\times)^{m-n} & \longrightarrow & 1. \end{array} \tag{4}$$

We define the morphism $\varphi_0 : Z \longrightarrow Z^e = Z \times (\mathbb{C}^\times)^{m-n}$ to be the inclusion defined by $z \longmapsto (z, 1)$. So, $(\varphi_0 \times \varphi_1, \varphi_0) : (Z \times G \rightrightarrows Z) \longrightarrow (Z^e \times G^e \rightrightarrows Z^e)$ defines a morphism between groupoids. Let $\varphi : [Z/G] \longrightarrow [Z^e/G^e]$ be the morphism of stacks induced from $(\varphi_0 \times \varphi_1, \varphi_0)$. From the above commutative diagram we have the following commutative diagram:

$$\begin{array}{ccc} Z \times G & \xrightarrow{\varphi_0 \times \varphi_1} & Z^e \times G^e \\ \downarrow{\scriptstyle (s,t)} & & \downarrow{\scriptstyle (s,t)} \\ Z \times Z & \xrightarrow{\varphi_0 \times \varphi_0} & Z^e \times Z^e. \end{array}$$

In (4), $\tilde{\alpha}$ is an isomorphism, which implies that the left square in (4) is Cartesian. So, the above commutative diagram is Cartesian and $\varphi\colon [Z/G] \longrightarrow [Z^e/G^e]$ is injective. Given an element $(z_1,\ldots,z_n,z_{n+1},\ldots,z_m)\in Z^e$, there exists an element $g^e\in(\mathbb{C}^\times)^{m-n}$ such that $g^e\cdot(z_1,\ldots,z_n,z_{n+1},\ldots,z_m)=(z_1,\ldots,z_n,1,\ldots,1)$. From (4), g^e determines an element in G^e, so φ is surjective. We conclude that the stacks $\mathcal{X}(\mathbf{\Sigma^e})$ and $\mathcal{X}(\mathbf{\Sigma})$ are isomorphic. $\square$

Remark. In view of Proposition 2.4, we call $\mathcal{X}(\mathbf{\Sigma^e})$ the toric Deligne–Mumford stack associated to the extended stacky fan $\mathbf{\Sigma^e}$.

Let $X(\Sigma)$ be the simplicial toric variety associated to the simplicial fan Σ in the extended stacky fan $\mathbf{\Sigma^e}$. We have the following corollary.

Corollary 2.5. *Given an extended stacky fan $\mathbf{\Sigma^e}$, the coarse moduli space of the toric Deligne–Mumford stack $\mathcal{X}(\mathbf{\Sigma^e})$ is also the simplicial toric variety $X(\Sigma)$.*

As in [BCS], for each top dimensional cone σ in Σ, denote by $Box(\sigma)$ to be the set of elements $v\in N$ such that $\overline{v}=\sum_{\rho_i\subseteq\sigma}a_i\overline{b}_i$ for some $0\leq a_i<1$. The elements in $Box(\sigma)$ are in one-to-one correspondence with the elements in the finite group $N(\sigma)=N/N_\sigma$, where $N(\sigma)$ is a local group of the stack $\mathcal{X}(\mathbf{\Sigma^e})$. If $\tau\subseteq\sigma$ is a low dimensional cone, we define $Box(\tau)$ to be the set of elements in $v\in N$ such that $\overline{v}=\sum_{\rho_i\subseteq\tau}a_i\overline{b}_i$, where $0\leq a_i<1$. It is easy to see that $Box(\tau)\subset Box(\sigma)$. In fact, the elements in $Box(\tau)$ generate a subgroup of the local group $N(\sigma)$. Let $Box(\mathbf{\Sigma^e})$ be the union of $Box(\sigma)$ for all d-dimensional cones $\sigma\in\Sigma$. For $v_1,\ldots,v_n\in N$, let $\sigma(\overline{v}_1,\ldots,\overline{v}_n)$ be the unique minimal cone in Σ containing $\overline{v}_1,\ldots,\overline{v}_n$.

3. The toric stack bundle ${}^P\mathcal{X}(\mathbf{\Sigma^e})$

In this section, we introduce the toric stack bundle ${}^P\mathcal{X}(\mathbf{\Sigma^e})$ and determine its twisted sectors. Let $P\longrightarrow B$ be a principal $(\mathbb{C}^\times)^m$-bundle over a smooth variety B. Through the map α^e in (3), G^e acts on the fibre product $P\times_{(\mathbb{C}^\times)^m} Z^e$.

Definition 3.1. We define the toric stack bundle ${}^P\mathcal{X}(\mathbf{\Sigma^e})\longrightarrow B$ to be the quotient stack

$$ {}^P\mathcal{X}(\mathbf{\Sigma^e}) := \big[\big(P\times_{(\mathbb{C}^\times)^m} Z^e\big)/G^e\big]. \tag{5}$$

Let $\phi:\mathbb{Z}^m\longrightarrow\mathbb{Z}^m$ be the map given by $e_i\longmapsto e_i$ for $1\leq i\leq n$, and $e_j\longmapsto e_j+\sum_{i=1}^n a_i^j e_i$ for $n+1\leq j\leq m$, where $a_i^j\in\mathbb{Z}$. Then from the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z}^m & \xrightarrow{\phi} & \mathbb{Z}^m & \longrightarrow & 0 & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle \tilde{\beta}^e} & & \downarrow{\scriptstyle \beta^e} & & \downarrow{\scriptstyle id} & & \\
0 & \longrightarrow & N & \xrightarrow{id} & N & \longrightarrow & 0 & \longrightarrow & 0,
\end{array}$$

we obtain a new extended stacky fan $\tilde{\mathbf{\Sigma}}^{\mathbf{e}} = (N, \Sigma, \tilde{\beta}^e)$, where the extra data in $\tilde{\mathbf{\Sigma}}^{\mathbf{e}}$ are $b'_{n+1} = b_{n+1} + \sum_{i=1}^n a_i^{n+1} b_i, \ldots, b'_m = b_m + \sum_{i=1}^n a_i^m b_i$. The map ϕ gives a map $\mathbb{C}^n \times (\mathbb{C}^\times)^{m-n} \longrightarrow \mathbb{C}^n \times (\mathbb{C}^\times)^{m-n}$ which is the identity on the first factor and given by ϕ on the second factor. Since the map in the above diagram doesn't change the fan in the extended stacky fans, we have a map $\varphi_0 : P \times_{(\mathbb{C}^\times)^m} Z^e \longrightarrow P \times_{(\mathbb{C}^\times)^m} Z^e$. We may use the same argument as that in Proposition 2.4 to prove that ${}^P\mathcal{X}(\mathbf{\Sigma}^{\mathbf{e}}) \cong {}^P\mathcal{X}(\tilde{\mathbf{\Sigma}}^{\mathbf{e}})$. This means that we can always choose the extra data $\{b_{n+1}, \ldots, b_m\}$, so that $b_j = \sum_{i=1}^n a_i b_i$ for $j = n+1, \ldots, m$ and $0 \le a_i < 1$. These extra data are actually in the $Box(\mathbf{\Sigma}^{\mathbf{e}})$.

EXAMPLE. By above, the extra data can be chosen to lie in $Box(\mathbf{\Sigma}^{\mathbf{e}})$. In this example, we prove that they cannot be put into the torsion subgroup of N. Let $N = \mathbb{Z}$ and $b_1 = 2, b_2 = -2$. Then $\Sigma = \{b_1, b_2\}$ is a simplicial fan in $N_{\mathbb{Q}}$. Let $\mathbf{\Sigma}^{\mathbf{e}} = (N, \Sigma, \beta^e)$, where $\beta^e : \mathbb{Z}^3 \longrightarrow \mathbb{Z}$ is determined by $\{b_1, b_2, b_3 = 1\}$. We compute that $DG(\beta^e) = \mathbb{Z}^2$ and the Gale dual $(\beta^e)^\vee : \mathbb{Z}^3 \longrightarrow \mathbb{Z}^2$ is given by the matrix

$$\begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \end{bmatrix}.$$

The toric Deligne–Mumford stack is $\mathcal{X}(\mathbf{\Sigma}^{\mathbf{e}}) = [(\mathbb{C}^2 - \{0\}) \times \mathbb{C}^\times / (\mathbb{C}^\times)^2]$, where the action is given by $(\lambda_1, \lambda_2)(x, y, z) = (\lambda_1 \lambda_2^{-1} \cdot x, \lambda_1 \cdot y, \lambda_2^2 \cdot z)$. We get $\mathcal{X}(\mathbf{\Sigma}^{\mathbf{e}}) = \mathbb{P}^1 \times [\mathbb{C}^\times / \mathbb{C}^\times] = \mathbb{P}^1 \times \mathcal{B}\mu_2$. Now, let $\tilde{\mathbf{\Sigma}}^{\mathbf{e}} = (N, \Sigma, \tilde{\beta}^e)$, where $\tilde{\beta}^e : \mathbb{Z}^3 \longrightarrow \mathbb{Z}$ is determined by $\{b_1, b_2, \tilde{b}_3 = 0\}$, then we compute that $DG(\beta^e) = \mathbb{Z}^2 \oplus \mathbb{Z}_2$ and the Gale dual $(\tilde{\beta}^e)^\vee : \mathbb{Z}^3 \longrightarrow \mathbb{Z}^2 \oplus \mathbb{Z}_2$ is given by the matrix

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

The toric Deligne–Mumford stack is $\mathcal{X}(\tilde{\mathbf{\Sigma}}^{\mathbf{e}}) = [(\mathbb{C}^2 - \{0\}) \times \mathbb{C}^\times / (\mathbb{C}^\times)^2 \times \mu_2]$, where the action is given by $(\lambda_1, \lambda_2, \lambda_3)(x, y, z) = (\lambda_1 \cdot x, \lambda_1 \cdot y, \lambda_2 \cdot z)$. We get $\mathcal{X}(\tilde{\mathbf{\Sigma}}^{\mathbf{e}}) = [\mathbb{P}^1 / \mu_2] = \mathbb{P}^1 \times \mathcal{B}\mu_2$. Let $B = \mathbb{P}^1$ and $P = \mathbb{C}^\times \oplus \mathbb{C}^\times \oplus \mathcal{O}(-1)^\times$, then ${}^P\mathcal{X}(\mathbf{\Sigma}^{\mathbf{e}})$ is a nontrivial μ_2-gerbe over $\mathbb{P}^1 \times \mathbb{P}^1$ coming from the line bundle $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(0, -1)$. Let $Q = \mathcal{O}(n_1)^\times \oplus \mathcal{O}(n_2)^\times \oplus \mathcal{O}(n_3)^\times$, then ${}^Q\mathcal{X}(\tilde{\mathbf{\Sigma}}^{\mathbf{e}})$ is the trivial μ_2-gerbe over the $\mathbb{P}^1$-bundle E over $\mathbb{P}^1$. So ${}^P\mathcal{X}(\mathbf{\Sigma}^{\mathbf{e}})$ is not isomorphic to ${}^Q\mathcal{X}(\tilde{\mathbf{\Sigma}}^{\mathbf{e}})$ for any Q.

From Corollary 2.5, $\mathcal{X}(\mathbf{\Sigma}^{\mathbf{e}})$ has the coarse moduli space $X(\Sigma)$ which is the simplicial toric variety associated to the simplicial fan Σ. From the exact sequence in (3), a $(\mathbb{C}^\times)^m$-bundle over B naturally determines a T-bundle over B. Let $E \longrightarrow B$ be the principal T-bundle induced by P, then we have the twists ${}^P\mathcal{X}_{red}(\mathbf{\Sigma}^{\mathbf{e}}) \longrightarrow B$ with fibre the toric orbifold $\mathcal{X}_{red}(\mathbf{\Sigma}^{\mathbf{e}})$ and ${}^EX(\Sigma) \longrightarrow B$ with fibre the simplicial toric variety $X(\Sigma)$, where ${}^P\mathcal{X}_{red}(\mathbf{\Sigma}^{\mathbf{e}}) := [(P \times_{(\mathbb{C}^\times)^m} Z^e)/\overline{G}^e]$, ${}^EX(\Sigma) := E \times_T X(\Sigma)$, and $\overline{G}^e = Im(\alpha^e)$ in (3). We obtain the exact sequence:

$$1 \longrightarrow \mu \longrightarrow G^e \xrightarrow{\alpha^e} \overline{G}^e \longrightarrow 1. \tag{6}$$

From [DP], we have the following proposition.

PROPOSITION 3.2. *${}^P\mathcal{X}(\mathbf{\Sigma^e})$ is a μ-gerbe over ${}^P\mathcal{X}_{red}(\mathbf{\Sigma^e})$ for a finite Abelian group μ.*

REMARK. In fact, any toric Deligne–Mumford stack is a μ-gerbe over the underlying toric orbifold for a finite Abelian group μ and some kind of μ-gerbes over toric Deligne–Mumford stacks are again toric Deligne–Mumford stacks, see [Jiang2].

Because any toric stack bundle is a μ-gerbe over the corresponding toric orbifold bundle and can be represented as a quotient stack, we have the following propositions.

PROPOSITION 3.3. *The simplicial toric bundle ${}^EX(\Sigma)$ is the coarse moduli space of the toric stack bundle ${}^P\mathcal{X}(\mathbf{\Sigma^e})$ and the toric orbifold bundle ${}^P\mathcal{X}_{red}(\mathbf{\Sigma^e})$.*

Proof. The toric stack bundle ${}^P\mathcal{X}(\mathbf{\Sigma^e})$ is a μ-gerbe over the simplicial toric orbifold bundle ${}^P\mathcal{X}_{red}(\mathbf{\Sigma^e})$ for a finite Abelian group μ. The stacks ${}^P\mathcal{X}(\mathbf{\Sigma^e}) = [(P \times_{(\mathbb{C}^\times)^m} Z^e)/G^e]$ and ${}^P\mathcal{X}_{red}(\mathbf{\Sigma^e}) = [(P \times_{(\mathbb{C}^\times)^m} Z^e)/\overline{G}^e]$ are quotient stacks. Taking geometric quotient, we have the coarse moduli space $(P \times_{(\mathbb{C}^\times)^m} Z^e)//\overline{G}^e = (P \times Z^e)//(\mathbb{C}^\times)^m \times \overline{G}^e$. From Corollary 2.5, $X(\Sigma) = Z//\overline{G} = Z^e//\overline{G}^e$ so,

$$E \times_T (Z^e//\overline{G}^e) = \big(P \times_{(\mathbb{C}^\times)^m} T\big) \times_T (Z^e//\overline{G}^e) = (P \times Z^e)//(\mathbb{C}^\times)^m \times \overline{G}^e.$$

From the universal geometric quotients in [KM], ${}^EX(\Sigma)$ is the coarse moduli space of ${}^P\mathcal{X}(\mathbf{\Sigma^e})$ and ${}^P\mathcal{X}_{red}(\mathbf{\Sigma^e})$. □

PROPOSITION 3.4. *The toric stack bundle ${}^P\mathcal{X}(\mathbf{\Sigma^e})$ is a Deligne–Mumford stack.*

Proof. From (5), ${}^P\mathcal{X}(\mathbf{\Sigma^e}) = [(P \times_{(\mathbb{C}^\times)^m} Z^e)/G^e]$ is a quotient stack, where G^e acts trivially on P. The action of G^e on Z^e has finite, reduced stabilizers because the stack $[Z^e/G^e]$ is a Deligne–Mumford stack, so the action of G^e on $P \times_{(\mathbb{C}^\times)^m} Z^e$ also has finite, reduced stabilizers. From Corollary 2.2 of [Ed], ${}^P\mathcal{X}(\mathbf{\Sigma^e})$ is a Deligne–Mumford stack. □

For an extended stacky fan $\mathbf{\Sigma^e}$, let $\sigma \in \Sigma$ be a cone, we define

$$link(\sigma) := \{\tau : \sigma + \tau \in \Sigma, \sigma \cap \tau = 0\}.$$

Let $\{\tilde{\rho}_1, \ldots, \tilde{\rho}_l\}$ be the rays in $link(\sigma)$. Then $\mathbf{\Sigma^e}/\sigma = (N(\sigma) = N/N_\sigma, \Sigma/\sigma, \beta^e(\sigma))$ is an extended stacky fan, where $\beta^e(\sigma) : \mathbb{Z}^{l+m-n} \longrightarrow N(\sigma)$ is given by the images of $b_1, \dots, b_l, b_{n+1}, \dots, b_m$ under $N \longrightarrow N(\sigma)$. From the construction of toric Deligne–Mumford stacks, we have $\mathcal{X}(\mathbf{\Sigma^e}/\sigma) := [Z^e(\sigma)/G^e(\sigma)]$, where $Z^e(\sigma) = (\mathbb{A}^l - \mathbb{V}(J_{\Sigma/\sigma})) \times (\mathbb{C}^\times)^{m-n} = Z(\sigma) \times (\mathbb{C}^\times)^{m-n}$, $G^e(\sigma) = Hom_{\mathbb{Z}}(DG(\beta^e(\sigma)), \mathbb{C}^\times)$. We have an action of $(\mathbb{C}^\times)^m$ on $Z^e(\sigma)$ induced by the natural action of $(\mathbb{C}^\times)^{l+m-n}$ on $Z^e(\sigma)$ and the projection $(\mathbb{C}^\times)^m \longrightarrow (\mathbb{C}^\times)^{l+m-n}$. We consider

$$\begin{aligned}{}^P\mathcal{X}(\mathbf{\Sigma^e}/\sigma) &= \big[\big(P \times_{(\mathbb{C}^\times)^m} (\mathbb{C}^\times)^{l+m-n} \times_{(\mathbb{C}^\times)^{l+m-n}} Z^e(\sigma)\big)/G^e(\sigma)\big] \\ &= \big[\big(P \times_{(\mathbb{C}^\times)^m} Z^e(\sigma)\big)/G^e(\sigma)\big].\end{aligned}$$

Then we have the following proposition.

Proposition 3.5. *Let σ be a cone in the extended stacky fan $\mathbf{\Sigma^e}$, then ${}^P\mathcal{X}(\mathbf{\Sigma^e}/\sigma)$ defines a closed substack of ${}^P\mathcal{X}(\mathbf{\Sigma^e})$.*

Proof. Let $W^e(\sigma)$ be the closed subvariety of Z^e defined by $J(\sigma) := \langle z_i : \rho_i \subseteq \sigma \rangle$ in $\mathbb{C}[z_1, \dots, z_n, z_{n+1}^{\pm 1}, \dots, z_m^{\pm 1}]$, then we see that $W^e(\sigma) = W(\sigma) \times (\mathbb{C}^\times)^{m-n}$, where $W(\sigma)$ is the closed subvariety of Z defined by $J(\sigma) := \langle z_i : \rho_i \subseteq \sigma \rangle$ in $\mathbb{C}[z_1, \dots, z_n]$. From [BCS], there is a map $\varphi_0 : W(\sigma) \longrightarrow Z(\sigma)$ which is $(\mathbb{C}^\times)^n$-equivariant. We define the map $W^e(\sigma) \longrightarrow Z^e(\sigma)$ by $\varphi_0 \times 1$. Twisting it by the bundle P, we have a map $\varphi_0 : P \times_{(\mathbb{C}^\times)^m} W^e(\sigma) \longrightarrow P \times_{(\mathbb{C}^\times)^m} Z^e(\sigma)$. From the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}^{n-l} & \longrightarrow & \mathbb{Z}^m & \longrightarrow & \mathbb{Z}^{l+m-n} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle \overline{\beta}^e} & & \downarrow{\scriptstyle \overline{\beta}^e(\sigma)} & & \\ 0 & \longrightarrow & N_\sigma & \longrightarrow & N & \longrightarrow & N(\sigma) & \longrightarrow & 0, \end{array}$$

applying Gale dual and $Hom_{\mathbb{Z}}(-, \mathbb{C}^\times)$ functor we get:

$$\begin{array}{ccc} G^e & \xrightarrow{\varphi_1} & G^e(\sigma) \\ \downarrow{\scriptstyle \alpha^e} & & \downarrow{\scriptstyle \alpha^e(\sigma)} \\ (\mathbb{C}^\times)^m & \longrightarrow & (\mathbb{C}^\times)^{l+m-n}. \end{array} \tag{7}$$

So, we get a map of groupoids: $\varphi_0 \times \varphi_1 : P \times_{(\mathbb{C}^\times)^m} W^e(\sigma) \times G^e \longrightarrow P \times_{(\mathbb{C}^\times)^m} Z^e(\sigma) \times G^e(\sigma)$ which is Morita equivalent. So, we have an isomorphism of stacks $[(P \times_{(\mathbb{C}^\times)^m} W(\sigma))/G^e] \cong [(P \times_{(\mathbb{C}^\times)^m} Z^e(\sigma))/G^e(\sigma)]$. Since, $W^e(\sigma)$ is a subvariety of Z^e, and $P \times_{(\mathbb{C}^\times)^m} W^e(\sigma)$ is a subvariety of $P \times_{(\mathbb{C}^\times)^m} Z^e$, so, $[(P \times_{(\mathbb{C}^\times)^m} W^e(\sigma))/G^e]$ is a substack of $[(P \times_{(\mathbb{C}^\times)^m} Z^e)/G^e] = {}^P\mathcal{X}(\mathbf{\Sigma^e})$. So, ${}^P\mathcal{X}(\mathbf{\Sigma^e}/\sigma)$ is a closed substack of ${}^P\mathcal{X}(\mathbf{\Sigma^e})$. □

Remark. From [BCS], $W(\sigma) = Z^{\langle g_1, \dots, g_r \rangle}$ for some group elements $g_1, \dots, g_r \in G$. From Proposition 2.4, the toric Deligne–Mumford stack $[Z^e(\sigma)/G^e(\sigma)]$ is isomorphic to the stack $[Z(\sigma)/G(\sigma)]$. Let $g_1, \dots, g_r$ still represent the elements in G^e through the map φ_1 in (4). Then $W^e(\sigma) = (Z^e)^{\langle g_1, \dots, g_r \rangle}$.

Proposition 3.6. *Let ${}^P\mathcal{X}(\mathbf{\Sigma^e}) \longrightarrow B$ be a toric stack bundle over a smooth variety B whose fibre $\mathcal{X}(\mathbf{\Sigma^e})$ is the toric Deligne–Mumford stack associated to the extended stacky fan $\mathbf{\Sigma^e}$, then rth inertia stack of this toric stack bundle is*

$$\mathcal{I}_r({}^P\mathcal{X}(\mathbf{\Sigma^e})) = \coprod_{(v_1,\ldots,v_r)\in Box(\mathbf{\Sigma^e})^r} {}^P\mathcal{X}\big(\mathbf{\Sigma^e}/\sigma(\overline{v}_1,\ldots,\overline{v}_r)\big).$$

Proof. From (5), ${}^P\mathcal{X}(\mathbf{\Sigma^e}) = [(P\times_{(\mathbb{C}^\times)^m} Z^e)/G^e]$ is a quotient stack. Because G^e is an Abelian group and the action has finite, reduced stabilizers, we have the rth inertia stack:

$$\mathcal{I}_r({}^P\mathcal{X}(\mathbf{\Sigma^e})) = \bigg[\bigg(\coprod_{(g_1,\ldots,g_r)\in(G^e)^r} \big(P\times_{(\mathbb{C}^\times)^m} Z^e\big)^H\bigg)/G^e\bigg],$$

where H is the subgroup in G^e generated by the elements $g_1,\ldots,g_r$. From Lemma 4.6 in [BCS], there is a map from $Box(\mathbf{\Sigma^e})$ to G. So, from the map φ_1 in (4), we have a map $\rho\colon Box(\mathbf{\Sigma^e}) \longrightarrow G^e$ such that $\rho(v) = g(v)$. For a r-tuple $(v_1,\ldots,v_r)$ in the $Box(\mathbf{\Sigma^e})$, from Proposition 3.5 and the above Remark, we have: ${}^P\mathcal{X}(\mathbf{\Sigma^e}/\sigma(\overline{v}_1,\ldots,\overline{v}_r)) \cong [P\times_{(\mathbb{C}^\times)^m}(Z^e)^H/G^e]$. Taking the disjoint union over all r-tuples in $Box(\mathbf{\Sigma^e})$ we get a map:

$$\psi: \coprod_{(v_1,\ldots,v_r)\in Box(\mathbf{\Sigma^e})^r} {}^P\mathcal{X}\big(\mathbf{\Sigma^e}/\sigma(\overline{v}_1,\ldots,\overline{v}_r)\big) \longrightarrow \mathcal{I}_r({}^P\mathcal{X}(\mathbf{\Sigma^e})).$$

The toric stack bundle ${}^P\mathcal{X}(\mathbf{\Sigma^e})$ locally is the product of a smooth variety with the toric Deligne–Mumford stack $\mathcal{X}(\mathbf{\Sigma^e})$. From [BCS], the map ψ is an isomorphism locally in the Zariski topology of the base B, so ψ is an isomorphism globally. We complete the proof of the proposition. □

Remark. For any pair $(v_1,v_2)\in Box(\mathbf{\Sigma^e})^2$, there exists a unique element $v_3\in Box(\mathbf{\Sigma^e})$ such that $v_1+v_2+v_3\in N$. This means that in the local group $N/N_{\sigma(\overline{v}_1,\overline{v}_2)}$, the corresponding group elements g_1,g_2,g_3 satisfy $g_1g_2g_3=1$. So, this implies that $\sigma(\overline{v}_1,\overline{v}_2,\overline{v}_3) = \sigma(\overline{v}_1,\overline{v}_2)$. In fact, Proposition 3.6 determines all 3-twisted sectors of the toric stack bundle ${}^P\mathcal{X}(\mathbf{\Sigma^e})$. See also [Po], [Jiang1] for the case of toric varieties.

4. The orbifold cohomology ring

In this section, we describe the ring structure of the orbifold cohomology of toric stack bundles.

4.1. The module structure on $A^*_{orb}({}^P\mathcal{X}(\mathbf{\Sigma^e}))$. Let $\mathbf{\Sigma^e}$ be an extended stacky fan, $P\to B$ a $(\mathbb{C}^\times)^m$-bundle and ${}^P\mathcal{X}(\mathbf{\Sigma^e}) \longrightarrow B$ the associated toric stack bundle. Let $M=N^*$ be the dual of N. For $\theta\in M$, let $\chi^\theta : (\mathbb{C}^\times)^m \longrightarrow \mathbb{C}^\times$ be the map induced by $\theta\circ\beta^e:\mathbb{Z}^m\longrightarrow\mathbb{Z}$. Let $\xi_\theta\longrightarrow B$ be the line bundle $P\times_{\chi^\theta}\mathbb{C}$. We give several definitions.

Definition 4.1. Let $A^*(B)$ denote the Chow ring over $\mathbb{Q}$ of the smooth variety B. Define the deformed ring $A^*(B)[N]^{\mathbf{\Sigma^e}}$ as follows: $A^*(B)[N]^{\mathbf{\Sigma^e}} = A^*(B)\otimes_{\mathbb{Q}}\mathbb{Q}[N]^{\mathbf{\Sigma^e}}$, $\mathbb{Q}[N]^{\mathbf{\Sigma^e}} = \bigoplus_{c\in N}\mathbb{Q}y^c$, where y is a formal variable. Multiplication is given by (1).

The deformed ring $A^*(B)[N]^{\mathbf{\Sigma^e}}$ has a $\mathbb{Q}$-grading defined as follows: if $\overline{c} = \sum_{\rho_i\subseteq\sigma(\overline{c})} a_i\overline{b}_i$, $deg(y^c) = \sum a_i \in \mathbb{Q}$. If $\gamma\in A^*(B)$, then $deg(\gamma\cdot y^c) = deg(\gamma) + deg(y^c)$.

Definition 4.2. Let $\mathbf{\Sigma^e} = (N,\Sigma,\beta^e)$ be an extended stacky fan. Let S_Σ be the ring $A^*(B)[x_1,\ldots,x_n]/I_\Sigma$, where the ideal I_Σ is generated by the square-free monomials $\{x_{i_1}\cdots x_{i_s} : \rho_{i_1}+\cdots+\rho_{i_s}\notin\Sigma\}$.

Note that S_Σ is a subring of $A^*(B)[N]^{\mathbf{\Sigma^e}}$ given by the map $x_i\longmapsto y^{b_i}$ for $1\leq i\leq n$. Let $\{\rho_1,\ldots,\rho_n\}$ be the rays of $\mathbf{\Sigma^e}$, then each ρ_i corresponds to a line bundle L_i over the toric Deligne–Mumford stack $\mathcal{X}(\mathbf{\Sigma^e})$. This line bundle can be defined as follows. The line bundle L_i on the toric Deligne–Mumford stack $\mathcal{X}(\mathbf{\Sigma})$ is given by the trivial line bundle $\mathbb{C}\times Z$ over Z with the G action on $\mathbb{C}$ given by the ith component α_i of $\alpha: G\longrightarrow(\mathbb{C}^\times)^n$ in (3) when $\mathbf{\Sigma^e}=\mathbf{\Sigma}$. From (4), we have:

$$\begin{array}{ccc} G & \xrightarrow{\varphi_1} & G^e \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\alpha^e} \\ (\mathbb{C}^\times)^n & \xrightarrow{i} & (\mathbb{C}^\times)^m. \end{array} \tag{8}$$

Definition 4.3. For each ρ_i, define the line bundle L_i over $\mathcal{X}(\mathbf{\Sigma^e})$ to be the quotient of the trivial line bundle $Z^e\times\mathbb{C}$ over Z^e under the action of G^e on $\mathbb{C}$ through the component of α^e such that the pullback component in α through (8) is α_i. Twisting it by the principal $(\mathbb{C}^\times)^m$-bundle P, we get the line bundle $\mathcal{L}_i$ over the toric stack bundle ${}^P\mathcal{X}(\mathbf{\Sigma^e})$.

Let $\mathcal{I}({}^P\mathbf{\Sigma^e})$ be the ideal in (2). We first describe the ordinary Chow ring of the toric stack bundle ${}^P\mathcal{X}(\mathbf{\Sigma^e})$.

Lemma 4.4. *Let ${}^P\mathcal{X}(\mathbf{\Sigma^e})\longrightarrow B$ be a toric stack bundle over a smooth variety B whose fibre $\mathcal{X}(\mathbf{\Sigma^e})$ is the toric Deligne–Mumford stack associated to the extended stacky fan $\mathbf{\Sigma^e}$, then there is an isomorphism of $\mathbb{Q}$-graded rings:*

$$\frac{S_\Sigma}{\mathcal{I}({}^P\mathbf{\Sigma^e})}\cong A^*({}^P\mathcal{X}(\mathbf{\Sigma^e}))$$

given by $x_i\longmapsto c_1(\mathcal{L}_i)$.

Proof. From Corollary 2.5, let $X(\Sigma)$ be the coarse moduli space of the toric Deligne–Mumford stack $\mathcal{X}(\mathbf{\Sigma^e})$. Let $E\longrightarrow B$ be the principal T-bundle induced from the $(\mathbb{C}^\times)^m$-bundle P. Then from Proposition 3.3, ${}^EX(\Sigma)$ is the coarse moduli space of the toric stack bundle ${}^P\mathcal{X}(\mathbf{\Sigma^e})$. Let a_i be the first lattice vector in the ray generated by $\overline{b}_i$, then $\overline{b}_i = l_i a_i$ for some positive integer l_i. The ideal $\mathcal{I}({}^P\mathbf{\Sigma^e})$ in (2) also defines an ideal in S_Σ. From [SU], we have

$$\frac{S_\Sigma}{\mathcal{I}({}^P\mathbf{\Sigma^e})} \cong A^*({}^EX(\Sigma)),$$

which is given by $x_i \longmapsto E(V(\rho_i))$, where $E(V(\rho_i))$ is the associated bundle over B corresponding to the T-invariant divisor $V(\rho_i)$. From [V], the Chow ring of the stack ${}^P\mathcal{X}(\mathbf{\Sigma^e})$ is isomorphic to the Chow ring of its coarse moduli space ${}^EX(\Sigma)$ given by $c_1(\mathcal{L}_i) \longmapsto l_i^{-1} \cdot E(V(\rho_i))$. Then we conclude by $c_1(\xi_\theta) + \sum_{i=1}^n \theta(a_i) l_i y^{b_i} = c_1(\xi_\theta) + \sum_{i=1}^n \theta(b_i) y^{b_i}$. □

Now, we discuss the module structure of $A^*_{orb}({}^P\mathcal{X}(\mathbf{\Sigma^e}))$. Because Σ is a simplicial fan, we have the following lemma.

LEMMA 4.5. *For any $c \in N$, let σ be the minimal cone in Σ containing $\overline{c}$, then there exists a unique expression*

$$c = v + \sum_{\rho_i \subset \sigma} m_i b_i,$$

where $m_i \in \mathbb{Z}_{\geq 0}$, and $v \in Box(\sigma)$.

LEMMA 4.6. *Let τ is a cone in the complete simplicial fan Σ and $\{\rho_1, \dots, \rho_s\} \subset link(\tau)$. Suppose $\rho_1, \dots, \rho_s$ are contained in a cone $\sigma \subset \Sigma$. Then $\sigma \cup \tau$ is contained in a cone of Σ.*

Proof. Using the following result, see [F], [O]. Let $\rho_1, \dots, \rho_s$ be rays in the complete simplicial fan Σ. If for any i, j, ρ_i, ρ_j generate a cone, then $\rho_1, \dots, \rho_s$ generate a cone. □

PROPOSITION 4.7. *Let ${}^P\mathcal{X}(\mathbf{\Sigma^e}) \longrightarrow B$ be a toric stack bundle over a smooth variety B whose fibre $\mathcal{X}(\mathbf{\Sigma^e})$ is the toric Deligne–Mumford stack associated to the extended stacky fan $\mathbf{\Sigma^e}$, then we have an isomorphism of $A^*({}^P\mathcal{X}(\mathbf{\Sigma^e}))$-modules:*

$$\bigoplus_{v \in Box(\mathbf{\Sigma^e})} A^*\big({}^P\mathcal{X}\big(\mathbf{\Sigma^e}/\sigma(\overline{v})\big)\big)[deg(y^v)] \cong \frac{A^*(B)[N]^{\mathbf{\Sigma^e}}}{\mathcal{I}({}^P\mathbf{\Sigma^e})}.$$

Proof. From the definition of $A^*(B)[N]^{\mathbf{\Sigma^e}}$ and Lemma 4.5, we see that $A^*(B)[N]^{\mathbf{\Sigma^e}} = \bigoplus_{v \in Box(\mathbf{\Sigma^e})} y^v \cdot S_\Sigma$. Since $\mathcal{I}({}^P\mathbf{\Sigma^e})$ is the ideal in $A^*(B)[N]^{\mathbf{\Sigma^e}}$ defined in (2), then $\bigoplus_{v \in Box(\mathbf{\Sigma^e})} y^v \cdot \mathcal{I}({}^P\mathbf{\Sigma^e})$ is the ideal $\mathcal{I}({}^P\mathbf{\Sigma^e})$ in $\bigoplus_{v \in Box(\mathbf{\Sigma^e})} y^v \cdot S_\Sigma = A^*(B)[N]^{\mathbf{\Sigma^e}}$. So, we obtain the isomorphism of $A^*({}^P\mathcal{X}(\mathbf{\Sigma^e}))$-modules:

$$\frac{A^*(B)[N]^{\mathbf{\Sigma^e}}}{\mathcal{I}({}^P\mathbf{\Sigma^e})} \cong \bigoplus_{v \in Box(\mathbf{\Sigma^e})} \frac{y^v \cdot S_\Sigma}{y^v \cdot \mathcal{I}({}^P\mathbf{\Sigma^e})}. \tag{9}$$

For any $v \in Box(\mathbf{\Sigma^e})$, let $\sigma(\overline{v})$ be the minimal cone in Σ containing $\overline{v}$. Let $\rho_1, \ldots, \rho_l \in link(\sigma(\overline{v}))$, and $\tilde{\rho}_i$ be the image of ρ_i under the natural map $N \longrightarrow N(\sigma(\overline{v})) = N/N_{\sigma(\overline{v})}$. Then $S_{\Sigma/\sigma(\overline{v})} \subset A^*(B)[N(\sigma(\overline{v}))]^{\mathbf{\Sigma^e}/\sigma(\overline{v})}$ is the subring given by: $\tilde{x}_i \longmapsto y^{\tilde{b}_i}$, for $\rho_i \in link(\sigma(\overline{v}))$. Consider the morphism: $i : A^*(B)[\tilde{x}_1, \ldots, \tilde{x}_l] \longrightarrow A^*(B)[x_1, \ldots, x_n]$ given by $\tilde{x}_i \longrightarrow x_i$. From Lemma 4.6, it is easy to check that the ideal $I_{\Sigma/\sigma(\overline{v})}$ goes to the ideal I_Σ, so we have a morphism $S_{\Sigma/\sigma(\overline{v})} \longrightarrow S_\Sigma$. Since S_Σ is a subring of $A^*(B)[N]^{\mathbf{\Sigma^e}}$ given by $x_i \longmapsto y^{b_i}$, we use the notations y^{b_i}. Let $\tilde{\Psi}_v : S_{\Sigma/\sigma(\overline{v})}[deg(y^v)] \longrightarrow y^v \cdot S_\Sigma$ be the morphism given by: $y^{\tilde{b}_i} \longmapsto y^v \cdot y^{b_i}$. If $\sum_{i=1}^{l} \tilde{\theta}(\tilde{b}_i) y^{\tilde{b}_i} + c_1(\xi_{\tilde{\theta}})$ belongs to the ideal $\mathcal{I}({}^P\mathbf{\Sigma^e}/\sigma(\overline{v}))$, then

$$\begin{aligned} \tilde{\Psi}_v\left(\sum_{i=1}^{l} \tilde{\theta}(\tilde{b}_i) y^{\tilde{b}_i} + c_1(\xi_{\tilde{\theta}})\right) &= y^v \cdot \left(\sum_{i=1}^{l} \tilde{\theta}(\tilde{b}_i) y^{\tilde{b}_i} + c_1(\xi_{\tilde{\theta}})\right) \\ &= y^v \cdot \left(\sum_{i=1}^{n} \theta(b_i) y^{b_i} + c_1(\xi_\theta)\right), \end{aligned}$$

where θ is determined by the diagram:

$$\begin{array}{ccc} N & & \\ {\scriptstyle\pi}\downarrow & \searrow^{\theta} & \\ N(\sigma(\overline{v})) & \xrightarrow{\tilde{\theta}} & \mathbb{Z}. \end{array} \tag{10}$$

So, $\theta(b_i) = \tilde{\theta}(\tilde{b}_i)$. From the definition of the line bundle ξ_θ, we have $\xi_\theta \cong \xi_{\tilde{\theta}}$. We obtain that $\tilde{\Psi}_v(\sum_{i=1}^{l} \tilde{\theta}(\tilde{b}_i) y^{\tilde{b}_i} + c_1(\xi_{\tilde{\theta}})) \in y^v \cdot \mathcal{I}({}^P\mathbf{\Sigma^e})$. So, $\tilde{\Psi}_v$ induce a morphism $\Psi_v : \frac{S_{\Sigma/\sigma(\overline{v})}}{\mathcal{I}({}^P\mathbf{\Sigma^e}/\sigma(\overline{v}))}[deg(y^v)] \longrightarrow \frac{y^v \cdot S_\Sigma}{y^v \cdot \mathcal{I}({}^P\mathbf{\Sigma^e})}$, such that $\Psi_v([y^{\tilde{b}_i}]) = [y^v \cdot y^{b_i}]$.

Conversely, for such $v \in Box(\mathbf{\Sigma^e})$ and $\rho_i \subset \sigma(\overline{v})$, choose $\theta_i \in Hom(N, \mathbb{Q})$ such that $\theta_i(b_i) = 1$ and $\theta_i(b_{i'}) = 0$ for $b_{i'} \neq b_i \in \sigma(\overline{v})$. We consider the following morphism $p : A^*(B)[x_1, \ldots, x_n] \longrightarrow A^*(B)[\tilde{x}_1, \ldots, \tilde{x}_l]$, where p is given by:

$$x_i \longmapsto \begin{cases} \tilde{x}_i & \text{if } \rho_i \subseteq link(\sigma(\overline{v})), \\ -\sum_{j=1}^{l} \theta_i(b_j)\tilde{x}_j & \text{if } \rho_i \subseteq \sigma(\overline{v}), \\ 0 & \text{if } \rho_i \nsubseteq \sigma(\overline{v}) \cup link(\sigma(\overline{v})). \end{cases}$$

For any $x_{i_1} \cdots x_{i_s}$ in I_Σ, also from Lemma 4.6, we prove that $p(x_{i_1} \cdots x_{i_s}) \in I_{\Sigma/\sigma(\overline{v})}$. We also use the notations y^{b_i} to replace x_i. The map p induces a surjective map: $S_\Sigma \longrightarrow S_{\Sigma/\sigma(\overline{v})}$ and a surjective map: $\tilde{\Phi}_v : y^v \cdot S_\Sigma \longrightarrow S_{\Sigma/\sigma(\overline{v})}[deg(y^v)]$. Let $y^v \cdot (\sum_{i=1}^{n} \theta(b_i) y^{b_i} + c_1(\xi_\theta))$ belong to the ideal $y^v \cdot \mathcal{I}({}^P\mathbf{\Sigma^e})$. For $\theta \in M$, we have $\theta = \theta_v + \theta'_v$, where $\theta_v \in N(\sigma(\overline{v}))^* = M \cap \sigma(\overline{v})^\perp$ and θ'_v belongs to the orthogonal complement of the subspace $\sigma(\overline{v})^\perp$ in M.

From (10), we have:

$$\begin{aligned}&\tilde{\Phi}_v\left(y^v\cdot\left(\sum_{i=1}^n\theta(b_i)y^{b_i}+c_1(\xi_\theta)\right)\right)\\&\quad=\sum_{i=1}^l\theta_v(\tilde{b}_i)y^{\tilde{b}_i}+c_1(\xi_{\theta_v})+\sum_{\rho_i\subset\sigma(\overline{v})}\theta'_v(b_i)\left(-\sum_{j=1}^l\theta_i(b_j)y^{\tilde{b}_j}\right)\\&\qquad+c_1(\xi_{\theta'_v})+\sum_{i=1}^l\theta'_v(b_i)y^{\tilde{b}_i}.\end{aligned}$$

Note that $(\sum_{i=1}^l\theta_v(\tilde{b}_i)y^{\tilde{b}_i}+c_1(\xi_{\theta_v}))\in\mathcal{I}({}^P\mathbf{\Sigma^e}/\sigma(\overline{v}))$. From the definition of $\xi_{\tilde{\theta}}$ over $\mathcal{X}(\mathbf{\Sigma^e}/\sigma(\overline{v}))$, $\xi_{\theta'_v}=0$. Now, let $\theta'_v=\sum_{\rho_i\subset\sigma(\overline{v})}a_i\theta_i$ where $a_i\in\mathbb{Q}$, then $\sum_{\rho_i\subset\sigma(\overline{v})}\theta'_v(b_i)=\sum_{\rho_i\subset\sigma(\overline{v})}a_i\theta_i(b_i)$. We have: $\sum_{\rho_i\subset\sigma(\overline{v})}a_i\theta_i(b_i)(-\sum_{j=1}^l\theta_i(b_j)y^{\tilde{b}_j})+\sum_{\rho_i\subset\sigma(\overline{v})}\sum_{j=1}^l a_i\theta_i(b_j)y^{\tilde{b}_j}=0$, so we have $\tilde{\Phi}_v(y^v\cdot(\sum_{i=1}^n\theta(b_i)y^{b_i}+c_1(\xi_\theta)))\in\mathcal{I}({}^P\mathbf{\Sigma^e}/\sigma(\overline{v}))$. So, $\tilde{\Phi}_v$ induces a morphism

$$\Phi:\frac{y^v\cdot S_\Sigma}{y^v\cdot\mathcal{I}({}^P\mathbf{\Sigma^e})}\longrightarrow\frac{S_{\Sigma/\sigma(\overline{v})}}{\mathcal{I}({}^P\mathbf{\Sigma^e}/\sigma(\overline{v}))}[deg(y^v)].$$

Note that $\Phi_v\Psi_v=1$ is easy to check. For any $[y^v\cdot y^{b_i}]\in\frac{y^v\cdot S_\Sigma}{y^v\cdot\mathcal{I}({}^P\mathbf{\Sigma^e})}$, since $y^v\cdot(-\sum_{j=1}^l\theta_i(b_j)y^{b_j}+\sum_{j=1}^n\theta_i(b_j)y^{b_j})=y^v\cdot y^{b_i}$, we have $[y^v\cdot(-\sum_{j=1}^l\theta_i(b_j)y^{b_j})]=[y^v\cdot y^{b_i}]$, we check that $\Psi_v\Phi_v=1$. So Φ_v is an isomorphism. From Lemma 4.4, for any $v\in Box(\mathbf{\Sigma^e})$, we have an isomorphism of Chow rings: $\frac{S_{\Sigma/\sigma(\overline{v})}}{\mathcal{I}({}^P\mathbf{\Sigma^e}/\sigma(\overline{v}))}\cong A^*({}^P\mathcal{X}(\mathbf{\Sigma^e}/\sigma(\overline{v})))$. Taking into account all the v in $Box(\mathbf{\Sigma^e})$ and (9) we have the isomorphism: $\bigoplus_{v\in Box(\mathbf{\Sigma^e})}A^*({}^P\mathcal{X}(\mathbf{\Sigma^e}/\sigma(\overline{v})))[deg(y^v)]\cong\frac{A^*(B)[N]^{\mathbf{\Sigma^e}}}{\mathcal{I}({}^P\mathbf{\Sigma^e})}$. Note that both sides of (9) are $S_\Sigma/\mathcal{I}({}^P\mathbf{\Sigma^e})=A^*({}^P\mathcal{X}(\mathbf{\Sigma^e}))$-modules, we complete the proof. □

Remark. In Proposition 5.2 of [BCS], the authors give a proof of Proposition 4.7 for toric Deligne–Mumford stacks. We give a more explicit proof of this isomorphism for the toric stack bundle.

4.2. The orbifold cup product. In this section, we consider the orbifold cup product on $A^*_{orb}({}^P\mathcal{X}(\mathbf{\Sigma^e}))$. First, we determine the 3-twisted sectors of ${}^P\mathcal{X}(\mathbf{\Sigma^e})$ which are the components of the double inertia stack $\mathcal{I}_2({}^P\mathcal{X}(\mathbf{\Sigma^e}))$ of ${}^P\mathcal{X}(\mathbf{\Sigma^e})$; see [CR2]. It follows that all 3-twisted sectors of ${}^P\mathcal{X}(\mathbf{\Sigma^e})$ are:

$$\coprod_{(g_1,g_2,g_3)\in Box(\mathbf{\Sigma^e})^3,g_1g_2g_3=1}{}^P\mathcal{X}(\mathbf{\Sigma^e}/\sigma(\overline{g}_1,\overline{g}_2,\overline{g}_3)),\tag{11}$$

where $\sigma(\overline{g}_1,\overline{g}_2,\overline{g}_3)$ is the minimal cone in Σ containing $\overline{g}_1,\overline{g}_2,\overline{g}_3$. For any 3-twisted sector ${}^P\mathcal{X}(\mathbf{\Sigma^e})_{(g_1,g_2,g_3)} = {}^P\mathcal{X}(\mathbf{\Sigma^e}/\sigma(\overline{g}_1,\overline{g}_2,\overline{g}_3))$, we have an inclusion $e\colon {}^P\mathcal{X}(\mathbf{\Sigma^e}/\sigma(\overline{g}_1,\overline{g}_2,\overline{g}_3)) \longrightarrow {}^P\mathcal{X}(\mathbf{\Sigma^e})$ because ${}^P\mathcal{X}(\mathbf{\Sigma^e}/\sigma(\overline{g}_1,\overline{g}_2,\overline{g}_3))$ is a substack of ${}^P\mathcal{X}(\mathbf{\Sigma^e})$. Let H be the subgroup generated by g_1,g_2,g_3, then the genus zero, degree zero orbifold stable map to ${}^P\mathcal{X}(\mathbf{\Sigma^e})$ determines a Galois covering $\pi : C \longrightarrow \mathbb{P}^1$ branching over three marked points $0,1,\infty$, such that the transformation group of this covering is H. We have the definition below.

Definition 4.8 ([CR1]). The obstruction bundle $O_{(g_1,g_2,g_3)}$ over ${}^P\mathcal{X}(\mathbf{\Sigma^e}/\sigma(\overline{g}_1,\overline{g}_2,\overline{g}_3))$ is defined as the H-invariant bundle:

$$\left(e^*T({}^P\mathcal{X}(\mathbf{\Sigma^e}))\otimes H^1(C,\mathcal{O}_C)\right)^H.$$

Proposition 4.9. *Let ${}^P\mathcal{X}(\mathbf{\Sigma^e})_{(g_1,g_2,g_3)} = {}^P\mathcal{X}(\mathbf{\Sigma^e}/\sigma(\overline{g}_1,\overline{g}_2,\overline{g}_3))$ be a 3-twisted sector of the stack ${}^P\mathcal{X}(\mathbf{\Sigma^e})$. Let $g_1+g_2+g_3 = \sum_{\rho_i\subset\sigma(\overline{g}_1,\overline{g}_2,\overline{g}_3)} a_i b_i$, $a_i = 1,2$, then the Euler class of the obstruction bundle $O_{(g_1,g_2,g_3)}$ over ${}^P\mathcal{X}(\mathbf{\Sigma^e})_{(g_1,g_2,g_3)}$ is:*

$$\prod_{a_i=2} c_1(\mathcal{L}_i)|_{{}^P\mathcal{X}(\mathbf{\Sigma^e}/\sigma(\overline{g}_1,\overline{g}_2,\overline{g}_3))},$$

where $\mathcal{L}_i$ is the line bundle over ${}^P\mathcal{X}(\mathbf{\Sigma^e})$ in Definition 4.3.

Proof. Let $\mathcal{X}(\mathbf{\Sigma^e})$ be the toric Deligne–Mumford stack corresponding to the extended stacky fan $\mathbf{\Sigma^e}$. Let $\sigma(\overline{g}_1,\overline{g}_2,\overline{g}_3)$ be the minimal cone in Σ containing $\overline{g}_1,\overline{g}_2,\overline{g}_3$. From (11), we have the 3-twisted sector $\mathcal{X}(\mathbf{\Sigma^e})_{(g_1,g_2,g_3)} = \mathcal{X}(\mathbf{\Sigma^e}/\sigma(\overline{g}_1,\overline{g}_2,\overline{g}_3))$ and ${}^P\mathcal{X}(\mathbf{\Sigma^e})_{(g_1,g_2,g_3)} = {}^P\mathcal{X}(\mathbf{\Sigma^e}/\sigma(\overline{g}_1,\overline{g}_2,\overline{g}_3))$. Since $e : \mathcal{X}(\mathbf{\Sigma^e})_{(g_1,g_2,g_3)} \longrightarrow \mathcal{X}(\mathbf{\Sigma^e})$ is an inclusion, we have an exact sequence:

$$\begin{aligned} 0 \longrightarrow T\mathcal{X}\big(\mathbf{\Sigma^e}/\sigma(\overline{g}_1,\overline{g}_2,\overline{g}_3)\big) &\longrightarrow e^*T\mathcal{X}(\mathbf{\Sigma^e}) \\ &\longrightarrow N\big(\mathcal{X}\big(\mathbf{\Sigma^e}/\sigma(\overline{g}_1,\overline{g}_2,\overline{g}_3)\big)/\mathcal{X}(\mathbf{\Sigma^e})\big) \longrightarrow 0, \end{aligned}$$

where $N(\mathcal{X}(\mathbf{\Sigma^e}/\sigma(\overline{g}_1,\overline{g}_2,\overline{g}_3))/\mathcal{X}(\mathbf{\Sigma^e}))$ is the normal bundle of $\mathcal{X}(\mathbf{\Sigma^e}/\sigma(\overline{g}_1,\overline{g}_2,\overline{g}_3))$ in $\mathcal{X}(\mathbf{\Sigma^e})$.

Since $\mathcal{X}(\mathbf{\Sigma^e}) = [Z^e/G^e]$, the tangent bundle $T(\mathcal{X}(\mathbf{\Sigma^e})) = [T(Z^e)/T(G^e)]$ is a quotient stack. Z^e is an open subvariety of $\mathbb{A}^n\times(\mathbb{C}^\times)^{m-n}$, so $T(Z^e) = \mathcal{O}^n_{Z^e}$. Now, from the construction of the line bundle L_k over $\mathcal{X}(\mathbf{\Sigma^e})$, we have a canonical map: $\bigoplus_{k=1}^n L_k \longrightarrow T(\mathcal{X}(\mathbf{\Sigma^e}))$. Since, we have a natural map $T(\mathcal{X}(\mathbf{\Sigma^e})) \longrightarrow N(\mathcal{X}(\mathbf{\Sigma^e}/\sigma(\overline{g}_1,\overline{g}_2,\overline{g}_3))/\mathcal{X}(\mathbf{\Sigma^e}))$, we obtain a map of vector bundles over $\mathcal{X}(\mathbf{\Sigma^e}/\sigma(\overline{g}_1,\overline{g}_2,\overline{g}_3))$:

$$\varphi : \bigoplus_{\rho_k\subset\sigma(\overline{g}_1,\overline{g}_2,\overline{g}_3)} L_k \longrightarrow N\big(\mathcal{X}\big(\mathbf{\Sigma^e}/\sigma(\overline{g}_1,\overline{g}_2,\overline{g}_3)\big)/\mathcal{X}(\mathbf{\Sigma^e})\big).$$

Then from the definition of the line bundle $\mathcal{L}_k$ over ${}^P\mathcal{X}(\mathbf{\Sigma^e})$, we have the map:

$$\tilde{\varphi} : \bigoplus_{\rho_k\subset\sigma(\overline{g}_1,\overline{g}_2,\overline{g}_3)} \mathcal{L}_k \longrightarrow N\big({}^P\mathcal{X}\big(\mathbf{\Sigma^e}/\sigma(\overline{g}_1,\overline{g}_2,\overline{g}_3)\big)/{}^P\mathcal{X}(\mathbf{\Sigma^e})\big),$$

where $N({}^P\mathcal{X}(\mathbf{\Sigma^e}/\sigma(\overline{g}_1,\overline{g}_2,\overline{g}_3))/{}^P\mathcal{X}(\mathbf{\Sigma^e}))$ is the normal bundle of ${}^P\mathcal{X}(\mathbf{\Sigma^e}/\sigma(\overline{g}_1,\overline{g}_2,\overline{g}_3))$ in ${}^P\mathcal{X}(\mathbf{\Sigma^e})$. For any point map:

$$x: Spec\mathbb{C} \hookrightarrow \mathcal{X}\big(\mathbf{\Sigma^e}/\sigma(\overline{g}_1,\overline{g}_2,\overline{g}_3)\big) \hookrightarrow {}^P\mathcal{X}\big(\mathbf{\Sigma^e}/\sigma(\overline{g}_1,\overline{g}_2,\overline{g}_3)\big),$$

note that $x^*\tilde{\varphi}$ is an isomorphism, so $\tilde{\varphi}$ is an isomorphism. We have the exact sequence:

$$0 \longrightarrow T\big({}^P\mathcal{X}\big(\mathbf{\Sigma^e}/\sigma(\overline{g}_1,\overline{g}_2,\overline{g}_3)\big)\big) \longrightarrow e^*T({}^P\mathcal{X}(\mathbf{\Sigma^e})) \longrightarrow \bigoplus_{\rho_k\subset\sigma(\overline{g}_1,\overline{g}_2,\overline{g}_3)} \mathcal{L}_k \longrightarrow 0.$$

Now, using the result in the proof of Proposition 6.3 in [BCS], we have:

$$dim_{\mathbb{C}}\big(\mathcal{L}_k\otimes H^1(C,\mathcal{O}_C)\big)^H = \begin{cases} 0 & \text{if } a_k=1,\\ 1 & \text{if } a_k=2.\end{cases}$$

So, from Definition 4.8, we have:

$$e\big(O_{(g_1,g_2,g_3)}\big) \cong \prod_{a_i=2} c_1(\mathcal{L}_i)|_{{}^P\mathcal{X}(\mathbf{\Sigma^e}/\sigma(\overline{g}_1,\overline{g}_2,\overline{g}_3))}. \qquad \square$$

4.3. Proof of Theorem 1.1. From the definition of the orbifold cohomology in [CR1], we have that $A^*_{orb}({}^P\mathcal{X}(\mathbf{\Sigma^e})) = \bigoplus_{g\in Box(\mathbf{\Sigma^e})} A^*({}^P\mathcal{X}(\mathbf{\Sigma^e}/\sigma(\overline{g})))[deg(y^g)]$. From Proposition 4.7, we have an isomorphism between $A^*({}^P\mathcal{X}(\mathbf{\Sigma^e}))$-modules:

$$\bigoplus_{g\in Box(\mathbf{\Sigma^e})} A^*\big({}^P\mathcal{X}\big(\mathbf{\Sigma^e}/\sigma(\overline{g})\big)\big)[deg(y^g)] \cong \frac{A^*(B)[N]^{\mathbf{\Sigma^e}}}{\mathcal{I}({}^P\mathbf{\Sigma^e})}.$$

So, we have an isomorphism of $A^*({}^P\mathcal{X}(\mathbf{\Sigma^e}))$-modules: $A^*_{orb}({}^P\mathcal{X}(\mathbf{\Sigma^e})) \cong \frac{A^*(B)[N]^{\mathbf{\Sigma^e}}}{\mathcal{I}({}^P\mathbf{\Sigma^e})}$. Next, we show that the orbifold cup product defined in [CR1] coincides with the product in ring $A^*(B)[N]^{\mathbf{\Sigma^e}}/\mathcal{I}({}^P\mathbf{\Sigma^e})$. From the above isomorphisms, it suffices to consider the canonical generators y^{b_i}, y^g where $g\in Box(\mathbf{\Sigma^e})$ and $\gamma\in A^*(B)$. Since $b_i\in N$, the twisted sector determined by b_i is the whole toric stack bundle ${}^P\mathcal{X}(\mathbf{\Sigma^e})$, $y^{b_i}\cup_{orb}\gamma$ is the usual product $y^{b_i}\cdot\gamma$ in the deformed ring because y^{b_i} and γ belong to the ordinary Chow ring of ${}^P\mathcal{X}(\mathbf{\Sigma^e})$.

For $y^g\cup_{orb} y^{b_i}$ and $y^g\cup_{orb}\gamma$, where $g\in Box(\mathbf{\Sigma^e})$. g determines a twisted sector ${}^P\mathcal{X}(\mathbf{\Sigma^e}/\sigma(\overline{g}))$. The corresponding twisted sectors to b_i and γ are the whole toric stack bundle ${}^P\mathcal{X}(\mathbf{\Sigma^e})$. It is easy to see that the 3-twisted sector corresponding to (g,b_i) and (g,γ) are ${}^P\mathcal{X}(\mathbf{\Sigma^e})_{(g,1,g^{-1})} \cong {}^P\mathcal{X}(\mathbf{\Sigma^e}/\sigma(\overline{g}))$, where g^{-1} is the inverse of g in the local group. From the dimension formula in [CR1], the obstruction bundle over ${}^P\mathcal{X}(\mathbf{\Sigma^e})_{(g,1,g^{-1})}$ has rank zero. So, from the definition of orbifold cup product in [CR1] it is easy to check that $y^g\cup_{orb} y^{b_i} = y^g\cdot y^{b_i}$, $y^g\cup_{orb}\gamma = y^g\cdot\gamma$.

For the orbifold product $y^{g_1}\cup_{orb} y^{g_2}$, where $g_1,g_2\in Box(\mathbf{\Sigma^e})$. From (11), we see that if there is no cone in Σ containing $\overline{g}_1,\overline{g}_2$, then there is no 3-twisted sector corresponding to the elements g_1, g_2, so the orbifold cup product is zero from the definition. On the other hand, from the definition of the group ring $A^*(B)[N]^{\mathbf{\Sigma^e}}$, $y^{g_1} \cdot y^{g_2} = 0$, so $y^{g_1} \cup_{orb} y^{g_2} = y^{g_1} \cdot y^{g_2}$. If there is a cone $\sigma \in \Sigma$ such that $\overline{g}_1, \overline{g}_2 \in \sigma$, let $g_3 \in Box(\mathbf{\Sigma^e})$ such that $\overline{g}_3 \in \sigma(\overline{g}_1, \overline{g}_2)$ and $g_1 g_2 g_3 = 1$ in the local group. Using the same method in the proof of main theorem in [BCS], we get: $y^{g_1} \cup_{orb} y^{g_2} = y^{g_1} \cdot y^{g_2}$. The theorem is proved.

5. The μ-gerbe

In this section, we study the degenerate case of toric Deligne–Mumford stacks. In this case, N is a finite Abelian group, the simplicial fan Σ is 0. The toric stack bundle is a μ-gerbe $\mathcal{X}$ over B for a finite Abelian group μ.

Let $N = \mathbb{Z}_{p_1^{n_1}} \oplus \cdots \oplus \mathbb{Z}_{p_s^{n_s}}$ be a finite Abelian group, where $p_1, \ldots, p_s$ are prime numbers and $n_1, \ldots, n_s > 1$. Let $\beta^e : \mathbb{Z} \longrightarrow N$ be given by the vector $(1, 1, \ldots, 1)$. $N_{\mathbb{Q}} = 0$ implies that $\Sigma = 0$, then $\mathbf{\Sigma^e} = (N, \Sigma, \beta^e)$ is an extended stacky fan from Section 2. Let $n = lcm(p_1^{n_1}, \ldots, p_s^{n_s})$, then $n = p_{i_1}^{n_{i_1}} \cdots p_{i_t}^{n_{i_t}}$, where $p_{i_1}, \ldots, p_{i_t}$ are the distinct prime numbers which have the highest powers $n_{i_1}, \ldots, n_{i_t}$. Note that the vector $(1, 1, \ldots, 1)$ generates an order n cyclic subgroup of N. We calculate the Gale dual $(\beta^e)^\vee : \mathbb{Z} \longrightarrow \mathbb{Z} \oplus \bigoplus_{i \notin \{i_1, \ldots, i_t\}} \mathbb{Z}_{p_i}^{n_i}$, where $DG(\beta^e) = \mathbb{Z} \oplus \bigoplus_{i \notin \{i_1, \ldots, i_t\}} \mathbb{Z}_{p_i}^{n_i}$. We have the following exact sequence:

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \xrightarrow{\beta^e} N \longrightarrow \bigoplus_{i \notin \{i_1, \ldots, i_t\}} \mathbb{Z}_{p_i}^{n_i} \longrightarrow 0,$$

$$0 \longrightarrow 0 \longrightarrow \mathbb{Z} \xrightarrow{(\beta^e)^\vee} \mathbb{Z} \oplus \bigoplus_{i \notin \{i_1, \ldots, i_t\}} \mathbb{Z}_{p_i}^{n_i} \longrightarrow \mathbb{Z}_n \oplus \bigoplus_{i \notin \{i_1, \ldots, i_t\}} \mathbb{Z}_{p_i}^{n_i} \longrightarrow 0.$$

So, we obtain

$$1 \longrightarrow \mu \longrightarrow \mathbb{C}^\times \times \prod_{i \notin \{i_1, \ldots, i_t\}} \mu_{p_i}^{n_i} \xrightarrow{\alpha^e} \mathbb{C}^\times \longrightarrow 1, \tag{12}$$

where the map α^e in (12) is given by the matrix

$$\begin{bmatrix} n \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

and $\mu = \mu_n \times \prod_{i \notin \{i_1, \ldots, i_t\}} \mu_{p_i}^{n_i} \cong N$. The toric Deligne–Mumford stack is $\mathcal{X}(\mathbf{\Sigma^e}) = [\mathbb{C}^\times / \mathbb{C}^\times \times \prod_{i \notin \{i_1, \ldots, i_t\}} \mu_{p_i}^{n_i}] = \mathcal{B}\mu$, the classifying stack of the group μ. Let L be a line bundle over a smooth variety B and $L^\times$ the principal $\mathbb{C}^\times$-bundle induced from L removing the zero section. From our twist, we have ${}^{L^\times}\mathcal{X}(\mathbf{\Sigma^e}) = L^\times \times_{\mathbb{C}^\times} [\mathbb{C}^\times / \mathbb{C}^\times \times \prod_{i \notin \{i_1, \ldots, i_t\}} \mu_{p_i}^{n_i}] = [L^\times / \mathbb{C}^\times \times \prod_{i \notin \{i_1, \ldots, i_t\}} \mu_{p_i}^{n_i}]$, which is exactly a μ-gerbe $\mathcal{X}$ over B. The structure of this gerbe is a μ_n-gerbe coming from the line bundle L plus a trivial $\prod_{i \notin \{i_1, \ldots, i_t\}} \mu_{p_i}^{n_i}$-gerbe over B.

For this toric stack bundle, $Box(\mathbf{\Sigma^e}) = N$, so we have the following proposition for the inertia stack.

PROPOSITION 5.1. *The inertia stack of this toric stack bundle $\mathcal{X}$ is $p_1^{n_1} \cdots p_s^{n_s}$ copies of the μ-gerbe $\mathcal{X}$.*

From our main theorem, we have the following proposition.

PROPOSITION 5.2. *The orbifold cohomology ring of the toric stack bundle $\mathcal{X}$ is given by:*

$$H^*_{orb}(\mathcal{X}, \mathbb{Q}) \cong H^*(B, \mathbb{Q}) \otimes H^*_{orb}(\mathcal{B}\mu, \mathbb{Q}),$$

where $H^*_{orb}(\mathcal{B}\mu; \mathbb{Q}) = \mathbb{Q}[t_1, \dots, t_s]/(t_1^{p_1^{n_1}} - 1, \dots, t_s^{p_s^{n_s}} - 1)$.

Let $N = \mathbb{Z}_r$, and $\beta : \mathbb{Z} \longrightarrow \mathbb{Z}_r$ be the natural projection. The toric Deligne–Mumford stack $\mathcal{X}(\mathbf{\Sigma^e}) = \mathcal{B}\mu_r$. Let $L \longrightarrow B$ be a line bundle, then the toric stack bundle $\mathcal{X} = B_{(L,r)}$ is the μ_r-gerbe over B determined by the line bundle L. We have:

COROLLARY 5.3. *The orbifold cohomology ring of $B_{(L,r)}$ is isomorphic to $H^*(B)[t]/(t^r - 1)$.*

If the variety B is not a toric variety, then the toric stack bundle over B is not a toric Deligne–Mumford stack. But, suppose B is a smooth toric variety, then a μ-gerbe $\mathcal{X}$ can give a toric Deligne–Mumford stack in the sense of [BCS].

EXAMPLE. Let $B = \mathbb{P}^d$ be the d-dimensional projective space. We give stacky fan $\mathbf{\Sigma} = (N, \Sigma, \beta)$ as follows. Let $N = \mathbb{Z}^d \oplus \mathbb{Z}_r$ and $\beta : \mathbb{Z}^{d+1} \longrightarrow N$ be the map determined by the vectors: $\{(1, 0, \dots, 0, 0), (0, 1, \dots, 0, 0), \dots, (0, 0, \dots, 1, 0), (-1, -1, \dots, -1, 1)\}$. Then $DG(\beta) = \mathbb{Z}$, and the Gale dual $\beta^\vee$ is given by the matrix $[r, r, \dots, r]$. So, we have the following exact sequences:

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}^{d+1} \xrightarrow{\beta} \mathbb{Z}^d \oplus \mathbb{Z}_r \longrightarrow 0 \longrightarrow 0,$$

$$0 \longrightarrow \mathbb{Z}^d \longrightarrow \mathbb{Z}^{d+1} \xrightarrow{\beta^\vee} \mathbb{Z} \longrightarrow \mathbb{Z}_r \longrightarrow 0.$$

Then we obtain the exact sequence:

$$1 \longrightarrow \mu_r \longrightarrow \mathbb{C}^\times \xrightarrow{\alpha} (\mathbb{C}^\times)^{d+1} \longrightarrow (\mathbb{C}^\times)^d \longrightarrow 1.$$

The toric Deligne–Mumford stack $\mathcal{X}(\mathbf{\Sigma}) := [\mathbb{C}^{d+1} - \{0\}/\mathbb{C}^\times]$ is the canonical μ_r-gerbe over the projective space $\mathbb{P}^d$ coming from the canonical line bundle, where the $\mathbb{C}^\times$ action is given by $\lambda \cdot (z_1, \dots, z_{d+1}) = (\lambda^r \cdot z_1, \dots, \lambda^r \cdot z_{d+1})$. Denote this toric Deligne–Mumford stack by $\mathcal{G}_r = \mathbb{P}(r, \dots, r)$. If the homomorphism $\beta : \mathbb{Z}^{d+1} \longrightarrow N$ is determined by the vectors: $\{(1, 0, \dots, 0, 0), (0, 1, \dots, 0, 0), \dots, (0, 0, \dots, 1, 0), (-1, -1, \dots, -1, 0)\}$, then $DG(\beta) = \mathbb{Z} \oplus \mathbb{Z}_r$. Comparing to the former exact sequence, we have the exact sequence:

$$1 \longrightarrow \mu_r \longrightarrow \mathbb{C}^\times \times \mu_r \xrightarrow{\alpha} (\mathbb{C}^\times)^{d+1} \longrightarrow (\mathbb{C}^\times)^d \longrightarrow 1.$$

The corresponding toric Deligne–Mumford stack is the trivial μ_r-gerbe $\mathbb{P}^d \times \mathcal{B}\mu_r$ coming from the trivial line bundle over $\mathbb{P}^d$. The coarse moduli spaces of these two stacks are both projective space $\mathbb{P}^d$. From the theorem of this paper or the main theorem in [BCS], the orbifold cohomology rings of these two stacks are isomorphic, although as stacks they are different.

REMARK. Let H represent the hyperplane class of $\mathbb{P}^d$, then $H^*_{orb}(\mathcal{G}_r, \mathbb{Q}) \cong \mathbb{Q}[H]/(H^{d+1}) \otimes \mathbb{Q}[t]/(t^r - 1)$. We conjecture that the orbifold quantum cohomology ring of $\mathcal{G}_r$ defined in [CR2] is isomorphic to $\mathbb{Q}[H]/(H^{d+1} - f(H,q)) \otimes \mathbb{Q}[t]/(t^r - 1 - g(t,q))$, where f, g are two relations and q is the quantum parameter. The orbifold quantum cohomology of trivial gerbe case has been computed in [AJT], where $f(H,q) = q$ and $g(t,q) = 0$.

REMARK. We conjecture that the small orbifold quantum cohomology ring of the nontrivial μ_r-gerbe and trivial μ_r-gerbe over the projective space $\mathbb{P}^d$ should be different. This means that the orbifold quantum cohomology can classify these two different stacks.

6. Application

In this section, we generalize a result of Borisov, Chen, and Smith [BCS] to the toric stack bundle case.

Let $X(\Sigma)$ be a simplicial toric variety, and let $\mathcal{X}(\mathbf{\Sigma})$ be the associated toric Deligne–Mumford stack, where $\mathbf{\Sigma} = (N, \Sigma, \beta)$ is the stacky fan associated to Σ. Let Σ' be a subdivision of Σ such that $X(\Sigma')$ is a crepant resolution of $X(\Sigma)$. Suppose there are m rays in Σ', let $i : (\mathbb{C}^\times)^n \longrightarrow (\mathbb{C}^\times)^m$ be the inclusion. From the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z}^{n-d} & \longrightarrow & \mathbb{Z}^n & \xrightarrow{\beta} & N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle id} & & \\
0 & \longrightarrow & \mathbb{Z}^{m-d} & \longrightarrow & \mathbb{Z}^m & \xrightarrow{\beta'} & N & \longrightarrow & 0,
\end{array}$$

taking Gale dual we get:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N^* & \longrightarrow & (\mathbb{Z}^m)^* & \xrightarrow{(\beta')^\vee} & DG(\beta') & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle id} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & N^* & \longrightarrow & (\mathbb{Z}^n)^* & \xrightarrow{\beta^\vee} & DG(\beta) & \longrightarrow & 0.
\end{array}$$

So, applying the *Hom* functor, we have the following diagram:

$$\begin{array}{ccc}
(\mathbb{C}^\times)^n & \xrightarrow{i} & (\mathbb{C}^\times)^m \\
\downarrow & & \downarrow \\
T & \xrightarrow{id} & T.
\end{array}$$

Let $P \longrightarrow B$ be a principal $(\mathbb{C}^\times)^n$-bundle, we still use P to represent the principal $(\mathbb{C}^\times)^m$-bundle induced by i, then they induce the same principal T bundle E over B. So, ${}^E X(\Sigma') \longrightarrow {}^E X(\Sigma)$ is a crepant resolution. And, ${}^E X(\Sigma)$ is the coarse moduli space of the toric stack bundle ${}^P\mathcal{X}(\mathbf{\Sigma})$ from Proposition 3.3. We have the following result.

PROPOSITION 6.1. *If the Chow ring of the smooth variety B is a Cohen–Macaulay ring. Then there is a flat family $\mathcal{S} \longrightarrow \mathbb{P}^1$ of schemes, such that $\mathcal{S}_0 \cong Spec(A^*_{orb}({}^P\mathcal{X}(\mathbf{\Sigma})))$ and $\mathcal{S}_\infty \cong Spec(A^*({}^E X(\Sigma')))$.*

Proof. We also construct a family of algebras over $\mathbb{P}^1$, such that the fiber over 0 and ∞ are $A^*_{orb}({}^P\mathcal{X}(\mathbf{\Sigma}))$ and $A^*({}^E X(\Sigma'))$, respectively. $X(\Sigma')$ is a smooth variety, and $\{b_1, \ldots, b_n, b_{n+1}, \ldots b_m\}$ generate the whole lattice N, then $A^*(B)[N]^{\mathbf{\Sigma}}$ is the quotient ring of the ring $S := A^*(B)[y^{b_1}, \ldots, y^{b_m}]$ by the binomial ideal determined by (1). Let I_2 denote this ideal. Let I_1 denote the ideal generated by $c_1(\xi_{\theta_j}) + \sum_{i=1}^m \theta_j(b_i) y^{b_i}$ for $1 \leq j \leq d$, where $\theta_1, \ldots, \theta_d$ is a basis of N^*. Since Σ' is a regular subdivision of Σ, then there is a Σ'-linear support function $h : N \longrightarrow \mathbb{Z}$ such that $h(b_i) = 0$ for $1 \leq i \leq n$, $h(b_i) > 0$ for $n+1 \leq i \leq m$. For any lattice points c_1, c_2 lying in the same cone of Σ, $h(c_1 + c_2) \geq h(c_1) + h(c_2)$, and the inequality is strict unless c_1, c_2 lies in the same cone of Σ'.

We describe the family over $\mathbb{P}^1 - \{\infty\}$. Let $\breve{I}_1$ be the ideal in $S[t_1]$ generated by $c_1(\xi_{\theta_j}) t_1^{h(b_i)} + \sum_{i=1}^m \theta_j(b_i) y^{b_i} t_1^{h(b_i)}$ for $1 \leq j \leq d$. So, the choice of h implies that

$$\frac{S[t_1]}{\breve{I}_1 + I_2 + \langle t_1 \rangle} \cong \frac{S}{\langle c_1(\xi_{\theta_j}) + \sum_{i=1}^n \theta_j(b_i) y^{b_i} : 1 \leq j \leq d \rangle + I_2} \cong A^*_{orb}({}^P\mathcal{X}(\mathbf{\Sigma})).$$

The sequence $c_1(\xi_{\theta_j}) + \sum_{i=1}^n \theta_j(b_i) y^{b_i}$ for $1 \leq j \leq d$ is also a homogeneous system of parameters on S/I_2. The Chow ring $A^*(B)$ is a Cohen–Macaulay ring, so S/I_2 is also Cohen–Macaulay. So, the sequence is a regular sequence. Therefore, the Hilbert function of the family $S[t_1/(\breve{I}_1 + I_2)]$ is constant outside a finite set in $\mathbb{Q}^*$.

On the other hand, for the family over $\mathbb{P}^1 - \{0\}$, let $\breve{I}_2$ be the binomial ideal in $S[t_2]$ given by

$$y^{c_1} \cdot y^{c_2} = \begin{cases} y^{c_1+c_2} t_2^{h(c_1+c_2) - h(c_1) - h(c_2)} & \text{if } \exists \sigma \in \Sigma \text{ such that } \overline{c}_1 \in \sigma, \overline{c}_2 \in \sigma, \\ 0 & \text{otherwise.} \end{cases}$$

From the property of the function h, this product becomes (1) for the fan Σ' over $t_2 = 0$. Hence, $S[t_2]/(I_1 + \breve{I}_2 + \langle t_2 \rangle) \cong A^*({}^E X(\Sigma'))$. The sequence $c_1(\xi_{\theta_j}) + \sum_{i=1}^n \theta_j(b_i) y^{b_i}$ for $1 \leq j \leq d$ is a regular sequence on S/I_2 and $S/I_{\Sigma'}$, and we have the same Hilbert function for $S/(I_1 + I_2)$ and $S/(I_1 + I_{\Sigma'})$.

There exists an automorphism φ between these two families so that we construct such a family over $\mathbb{P}^1$. The rest of the proof is the same as in [BCS]. We omit the details. □

REMARK. Ruan [R] conjectured that the cohomology ring of crepant resolution is isomorphic to the orbifold Chow ring of the orbifold if we add some quantum corrections on the ordinary cohomology ring of the crepant resolution coming from the exceptional divisors. Let $\mathbb{P}(1,1,2)$ be the weighted projective plane with one orbifold point whose local group is $\mathbb{Z}_2$. The Hirzburch surface $\mathbb{F}_2$ is the crepant resolution of $\mathbb{P}(1,1,2)$. We can compute the quantum correction of the cohomology ring of the Hirzburch surface and check Ruan's conjecture. This case has been done recently in [Per].

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REFERENCES

[AGV] D. Abramovich, T. Graber and A. Vistoli, *Algebraic orbifold quantum product*, Orbifolds in mathematics and physics (Madison, WI, 2001), Contemp. Math., vol. 310, Amer. Math. Soc., Providence, RI, 2002, pp. 1–24. MR 1950940

[AJT] E. Andreini, Y. Jiang and H. Tseng, *The Kunneth type formula for orbifold Gromov–Witten invariants*, preprint.

[BCS] L. Borisov, L. Chen and G. Smith, *The orbifold Chow ring of toric Deligne–Mumford stacks*, J. Amer. Math. Soc. **18** (2005), 193–215, math.AG/0309229. MR 2114820

[CR1] W. Chen and Y. Ruan, *A new cohomology theory for orbifolds*, Comm. Math. Phys. **248** (2004), 1–31, math.AG/0004129. MR 2104605

[CR2] W. Chen and Y. Ruan, *Orbifold Gromov–Witten theory*, Orbifolds in mathematics and physics (Madison, WI, 2001), Contemp. Math., vol. 310, Amer. Math. Soc. Providence, RI, 2002, pp. 25–85. MR 1950941

[Cox] D. Cox, *The homogeneous coordinate ring of a toric variety*, J. Algebraic Geom. **4** (1995), 17–50. MR 1299003

[DP] R. Donagi and T. Pantev, *Torus fibrations, gerbes and duality*, math.AG/0306213. MR 2399730

[Ed] D. Edidin, *Notes on the construction of the moduli space of curves*, Recent progress in intersection theory (Bologna, 1997), Birkhauser, Boston, 2000, pp. 85–113. MR 1849292

[F] W. Fulton, *Introduction to toric varieties*, Annals of Mathematics Studies, vol. 131, Princeton University Press, Princeton, NJ, 1993. MR 1234037

[Jiang1] Y. Jiang, *The Chen–Ruan cohomology of weighted projective spaces*, Canad. J. Math. **59** (2007), 981–1007. MR 2354399

[Jiang2] Y. Jiang, *A note on finite Abelian gerbes over toric Deligne–Mumford stacks*, Proc. Amer. Math. Soc. **136** (2008), 4151–4156. MR 2431027

[Lang] S. Lang, *Algebra*, Graduate Texts in Mathematics, vol. 211, Springer-Verlag, New York, 2002. MR 1878556

[KM] S. Keel and S. Mori, *Quotients by groupoids*, Ann. of Math. (2) **145** (1997), 193–213. MR 1432041

[O] T. Oda, *Convex bodies and algebraic geometry*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), vol. 15, Springer-Verlag, Berlin, 1988. MR 0922894

[Per] F. Perroni, *Orbifold cohomology of ADE-singularities*, Internat. J. Math. **18** (2007), 1009–1059.

[Po] M. Poddar, *Orbifold Hodge numbers of Calabi–Yau hypersurfaces*, Pacific J. Math. **208** (2003), 151–167. MR 1979377

[R] Y. Ruan, *Cohomology ring of crepant resolutions of orbifolds*, Gromov–Witten theory of spin curves and orbifolds, Contemp. Math., vol. 403, Amer. Math. Soc., Providence, RI, 2006, pp. 117–126. MR 2234886

[SU] P. Sankaran and V. Uma, *Cohomology of toric bundles*, Comment. Math. Helv. **78** (2003), 540–554.MR 1998393

[V] A. Vistoli. *Intersection theory on algebraic stacks and on their moduli spaces*, Invent. Math. **97** (1989), 613–670. MR 1005008

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