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The order completeness of some spaces of vector-valued functions

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Let E be a Banach lattice. Necessary and sufficient conditions are given for the order completeness of the Banach lattices $C(X, E)$ and $L^1(\mu, E)$ in terms of the compactness of the order intervals in E. The results have interpretations in terms of spaces of compact and nuclear operators.

1.

Let E be a Banach lattice. If X is a compact Hausdorff space, then the space $C(X, E)$ of norm-continuous E-valued functions on X is a Banach lattice, where the norm of $f \in C(X, E)$ is given by $\|f\| = \sup\{\|f(t)\| : t \in X\}$, and where $f \geq 0$ means that $f(t) \geq 0$ in E for each $t \in X$. Certain spaces of compact operators are isomorphic to spaces of the type $C(X, E)$. (See [2], [3] and [7] for these results and for the notation used here.) Specifically, the space of compact operators from a space $L^1(\mu)$ into E , ordered in the natural way, is order and norm isomorphic to $C(X, E)$ for some X . Also, the space of compact operators $E \to C(X)$ is order and norm isomorphic to $C(X, E')$. Moreover, any space $C(X, E)$ is order and norm isomorphic to the space of weak*-weak continuous compact operators $C(X)' \to E$ $\left(\text{or } E' \to C(X)\right)$. The space $C(X, E)$ is also isomorphic to $C(X) \hat{\hat{\otimes}} E$.

Similarly, if (Ω, Σ, μ) is any measure space, then the space $L^1(\mu, E)$ of E-valued Bochner integrable functions on Ω is a Banach

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lattice for the usual norm and the natural order. Certain spaces of nuclear operators are order isomorphic to spaces of the type $L^1(\mu, E)$. (See [3].) The space $L^1(\mu, E)$ is isomorphic to $L^1(\mu)\hat{\otimes} E$.

In this paper necessary and sufficient conditions are given for the order completeness of the spaces $C(X, E)$ and $L^1(\mu, E)$, in terms of the compactness of the order intervals in E .

2.

A Banach lattice E has the *countable interpolation property* if, given sequences $\{x_n\}$ and $\{y_n\}$ in E such that $x_n \leq x_{n+1} \leq y_{n+1} \leq y_n$ for each n , there exists an element z in E such that $x_n \leq z \leq y_n$ for each n . Seever [6] showed that if X is compact and Hausdorff, then $C(X)$ has the countable interpolation property if and only if X is an F space (that is, the closures of any two disjoint open F_σ sets in X are disjoint).

LEMMA. *Suppose that E is a Banach lattice and that X is an infinite compact Hausdorff space. Suppose that $C(X, E)$ has the countable interpolation property. Then every order interval in E is compact.*

Proof. Let $x \in E$ be positive, and let $\{x_n\}$ be any sequence in $[0, x]$. There is a sequence $\{O_n\}$ of non-empty disjoint open sets in X . For each n , pick $t_n \in O_n$, and then $\varphi_n \in C(X)$ such that $\varphi_n(t_n) = 1$, and $\varphi_n = 0$ outside O_n , and $0 \leq \varphi_n \leq 1$ on X . Define $f_n, g_n \in C(X, E)$ by

$$f_n(t) = \sum_1^n \varphi_j(t)x_j \ , \quad g_n(t) = x - \sum_1^n \varphi_j(t)\left(x-x_j\right) \ .$$

It is clear that $f_n \leq f_{n+1} \leq g_{n+1} \leq g_n$ for each n , and so there exists $h \in C(X, E)$ such that $f_n \leq h \leq g_n$ for each n . Now $f_n\left(t_n\right) = g_n\left(t_n\right) = x_n$, and so $x_n = h\left(t_n\right) \in h(X)$. Since $h(X)$ is compact, $\{x_n\}$ has a convergent subsequence, and so $[0, x]$ is compact.

COROLLARY (Rudin, see [1]). *If X and Y are infinite compact Hausdorff spaces, then $X \times Y$ is not an F space. In particular, $X \times Y$ is not stonian.*

Proof. This is immediate from the lemma and from Seever's result, since (see [7, p. 357]) $C(X \times Y) \cong C(X, C(Y))$.

PROPOSITION. *Let X be an infinite compact Hausdorff space, and let E ($\neq \{0\}$) be a Banach lattice. Then $C(X, E)$ is order complete (respectively σ-order complete) if and only if X is stonian (respectively σ-stonian) and every order interval in E is compact.*

Proof. If $C(X, E)$ is σ-order complete, then, by the lemma, the order intervals in E are all compact. Conversely, if the order intervals in E are all compact, then Walsh [8] has shown that there is a family $\{e_i : i \in I\}$ of atoms of norm one in E such that each $x \in E$ has a unique unconditionally norm convergent expansion $x = \sum_i \alpha_i e_i$, where $\alpha_i \in \mathbb{R}$ for each i . (An element $x \in E$ is an *atom* if $x \geq 0$ and if $y \in E$ and $0 \leq y \leq x$ imply that $y = \alpha x$ for some $\alpha \in \mathbb{R}$.) It is easy to see from this that $C(X, E)$ is isomorphic to the space of families $\{f_i\}$ of functions f_i in $C(X)$ for which $\sum_i f_i(t)e_i$ converges unconditionally and uniformly with respect to $t \in X$. It is now clear that $C(X, E)$ is [σ-] order complete if and only if $C(X)$ is [σ-] order complete, that is, if and only if X is [σ-] stonian. (See [5].)

The next result deals with the situation dual to that in the above proposition. In the proof, we make use of the fact that every order interval in a Banach lattice E is weakly compact if and only if every majorized increasing sequence in E converges in norm. (See [4].)

PROPOSITION. *Let (Ω, Σ, μ) be a measure space, and let E be a Banach lattice. If every order interval in E is weakly compact, then the same is true in $L^1(\mu, E)$, and so $L^1(\mu, E)$ is order complete. Conversely, if $L^1(\mu, E)$ is σ-order complete and if μ is not purely atomic and if $L^1(\mu) \neq \{0\}$, then the order intervals in E are all weakly compact.*

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Proof. Suppose that $f_n, f : \Omega \to E$ are Bochner integrable and that $0 \le f_1(\omega) \le f_2(\omega) \le \ldots \le f(\omega)$ for each $\omega \in \Omega$. If the order intervals in E are weakly compact, then $g(\omega) = \lim f_n(\omega)$ (norm limit) exists for each $\omega \in \Omega$. Then [2, p. 151], since $\|g(\omega)-f_n(\omega)\| \le \|f(\omega)\|$, the function g is in $L^1(\mu, E)$ and $\|g-f_n\| \to 0$. It follows that the order intervals in $L^1(\mu, E)$ are weakly compact.

Conversely, if $L^1(\mu, E)$ is σ-order complete, and if μ is not purely atomic, then the order intervals in E are weakly compact. To see this, suppose the contrary, and first note that E contains a sublattice order isomorphic to l^∞. (Since E is clearly σ-order complete, we may apply Proposition 2.1 ((k) $\Rightarrow$ (c)) of [4].) Also, because μ is not purely atomic, $L^1(\mu)$ contains a sublattice order isomorphic to $L^1[0, 1]$. It is thus not difficult to see that it is enough to show that $L^1(\mu, E)$ is not σ-order complete, where $E = l^\infty$ and where $(\Omega, \Sigma, \mu) = [0, 1)$ with Lebesgue measure. For each $t \in [0, 1)$, let $t = \sum_1^\infty \varepsilon_i(t)/2^i$ be the binary expansion of t which doesn't end in a string of 1's. Define $f_n : [0, 1) \to l^\infty$ by $f_n(t) = (\varepsilon_1(t), \ldots, \varepsilon_n(t), 0, 0, \ldots)$. Then $\{f_n\}$ is a majorized increasing sequence of simple functions. If $\{f_n\}$ has a supremum in $L^1(\mu, E)$, it would have to be almost everywhere equal to the function $g : t \to (\varepsilon_1(t), \varepsilon_2(t), \ldots)$. But if $t, t' \in [0, 1)$ are distinct, then $\varepsilon_i(t) \ne \varepsilon_i(t')$ for some i, and so $\|g(t)-g(t')\| = 1$. Therefore [2, p. 147], g is not measurable. (This example was suggested to the author by J.J. Uhl.)

REMARKS. The function $g : [0, 1) \to l^\infty$ defined above is not even weakly measurable. To see this, take any free ultrafilter $\underline{\underline{F}}$ on $\mathbb{N}$, and denote by $\varphi(t)$ the limit of the sequence $\{\varepsilon_i(t)\}$ with respect to $\underline{\underline{F}}$. The function φ is not Lebesgue measurable. (See [7, p. 247].)

The preceding proposition and its proof are also valid for $L^p(\mu, E)$ if $1 < p < \infty$.

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