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THE ORDER OF APPEARANCE OF THE PRODUCT OF FIVE CONSECUTIVE LUCAS NUMBERS

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ABSTRACT. Let F_n be the nth Fibonacci number and let L_n be the nth Lucas number. The order of appearance $z(n)$ of a natural number n is defined as the smallest natural number k such that n divides F_k. For instance, $z(F_n) = n = z(L_n)/2$ for all $n > 2$. In this paper, among other things, we prove that

$$z(L_nL_{n+1}L_{n+2}L_{n+3}L_{n+4}) = \frac{n(n+1)(n+2)(n+3)(n+4)}{12}$$

for all positive integers $n \equiv 0, 8 \pmod{12}$.

1. Introduction

Let $(F_n)_n$ be the Fibonacci sequence given by $F_{n+2} = F_{n+1} + F_n$, for $n \geq 0$, where $F_0 = 0$ and $F_1 = 1$. Let $(L_n)_n$ be the Lucas sequence which follows the same recursive pattern as the Fibonacci numbers, but with initial values $L_0 = 2$ and $L_1 = 1$. These numbers are well-known for possessing amazing properties (for example consult [4]). The period $k(m)$ of the Fibonacci sequence modulo a positive integer m is the smallest positive integer n such that

$$F_n \equiv 0 \pmod{m} \quad \text{and} \quad F_{n+1} \equiv 1 \pmod{m}.$$

The study of the divisibility properties of Fibonacci numbers has always been a popular area of research. Let n be a positive integer, the *order* (*or rank*) *of appearance* of n in the Fibonacci sequence, denoted by $z(n)$, is defined as the smallest positive integer k, such that $n \mid F_k$ (some authors also call it *order of apparition*, or *Fibonacci entry point*). There are several results about $z(n)$ in the literature. For example, $z(m) \leq 2m$, for all $m \geq 1$ (see [15] and [10] for improvements) and in the case of a prime number p, one has the better upper bound $z(p) \leq p + 1$, which is a consequence of the known congruence

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$F_{p-(\frac{p}{5})} \equiv 0 \pmod p$ for $p \neq 2, 5$, where $\left(\frac{a}{q}\right)$ denotes the Legendre symbol of a with respect to a prime $q > 2$. We will use Pochhammer polynomial $n^{(k)} = n(n+1)(n+2)\dots(n+k-1)$ for the simplification of notation in the following text.

In recent papers, the first author [5]–[9] found explicit formulas for the order of appearance of integers related to Fibonacci and Lucas numbers, such as $C_m \pm 1, C_nC_{n+1}C_{n+2}C_{n+3}$ and C_n^k, where C_n represents F_n or L_n.

In this paper, we continue this program by studying the order of appearance of the product of five consecutive Lucas numbers. Our main result is the following.

Theorem 1.1. *Let n be any nonnegative integer. Then*

$$z(L_nL_{n+1}L_{n+2}L_{n+3}L_{n+4}) = \begin{cases} n^{(5)}, & n \equiv 1 \pmod 6; \\ \frac{1}{2}n^{(5)}, & n \equiv 2, 10, 14, 18, 22, 30, 34 \pmod{36}; \\ \frac{1}{3}n^{(5)}, & n \equiv 3, 5 \pmod 6; \\ \frac{1}{4}n^{(5)}, & n \equiv 4 \pmod{12}; \\ \frac{1}{6}n^{(5)}, & n \equiv 6, 26 \pmod{36}; \\ \frac{1}{12}n^{(5)}, & n \equiv 0, 8 \pmod{12}. \end{cases} \tag{1.1}$$

Remark 1. The completeness of cases in Theorem 1.1 follows from the fact that the first case and the third case together include all positive odd integers n and the other cases include all nonnegative even integers n.

2. Auxiliary results

Before proceeding further, we recall some facts on Fibonacci numbers for the convenience of the reader.

The p-adic valuation (or order) of r, $\nu_p(r)$, is the exponent of the highest power of a prime p which divides r. The p-adic order of the Fibonacci and Lucas numbers was completely characterized, see [14], [16] and [12]. For instance, from the main results of L e n g y e l [12], we extract the following result.

Lemma 2.1. *Let p be any prime. For $n \geq 1$, we have*

$$\nu_2(F_n) = \begin{cases} 0 & \text{if } n \equiv 1, 2 \pmod 3; \\ 1 & \text{if } n \equiv 3 \pmod 6; \\ 3 & \text{if } n \equiv 6 \pmod{12}; \\ \nu_2(n) + 2 & \text{if } n \equiv 0 \pmod{12}, \end{cases}$$

$$\nu_5(F_n) = \nu_5(n),$$

and for any prime $p \neq 5$ and $p > 2$

$$\nu_p(F_n) = \begin{cases} \nu_p(n) + \nu_p\big(F_{z(p)}\big) & \text{if } n \equiv 0 \pmod{z(p)}; \\ 0 & \text{if } n \not\equiv 0 \pmod{z(p)}. \end{cases}$$

Lemma 2.2. *Let p be any prime, let $k(p)$ be the period modulo p of the Fibonacci sequence. For $n \geq 1$, we have*

$$\nu_2(L_n) = \begin{cases} 0, & n \equiv 1, 2 \pmod 3; \\ 2, & n \equiv 3 \pmod 6; \\ 1, & n \equiv 0 \pmod 6, \end{cases}$$

and for any prime $p > 2$,

$$\nu_p(L_n) = \begin{cases} \nu_p(n) + \nu_p\big(F_{z(p)}\big), & k(p) \neq 4z(p) \text{ and } n \equiv \frac{z(p)}{2} \pmod{z(p)}; \\ 0, & \text{otherwise.} \end{cases}$$

Remark 2. Since $k(5) = 20$ and $z(5) = 5$, we have $k(5) = 4z(5)$ and so the previous lemma yields $\nu_5(L_n) = 0$. In fact, the same happens for the primes

$$13, 17, 37, 53, 61, 73, 89, 97, 109, 113, 137, 149, 157, 173, 193, 197, \ldots$$

which is the OEIS sequence A053028. We point out an interesting result of Lagarias [11] concerning the density of this set of primes.

Lemma 2.3 (Cf. Lemma 2.1 [9])**.** *We have*

(a) $F_n \mid F_m$ *if and only if* $n \mid m$.
(b) $L_n \mid F_m$ *if and only if* $n \mid m$ *and* m/n *is even.*
(c) $L_n \mid L_m$ *if and only if* $n \mid m$ *and* m/n *is odd.*
(d) $F_{2n} = F_n L_n$.
(e) $\gcd(L_n, L_{n+1}) = \gcd(L_n, L_{n+2}) = 1$.

Lemma 2.4 (Cf. Lemma 2.2 of [9])**.** *We have*

(a) *If* $F_n \mid m$, *then* $n \mid z(m)$.
(b) *If* $L_n \mid m$, *then* $2n \mid z(m)$.
(c) *If* $n \mid F_m$, *then* $z(n) \mid m$.

Lemma 2.5. *Let k, n, m be any positive integers. We have*

(a) *If* $n \equiv 0, 3 \pmod 6$, *then* $2F_n \mid F_{2n}$.
If $n \equiv 2 \pmod 4$, *then* $3F_n \mid F_{2n}$.
(b) *If* $n \equiv 1, 2 \pmod 3$, *then* $2F_n \mid F_{3n}$.
(c) *If* $n \equiv 6 \pmod{12}$, *then* $6F_n \mid F_{2n}$.
(d) *If* $n \equiv 2 \pmod 4$, *then* $6F_n \mid F_{6n}$.

(e) *If* $m \nmid F_{kn}$, *then* $m \nmid F_n$.

(f) *If* $n \equiv 2, 10, 14, 18, 22, 30 \pmod{36}$, *then* $L_n L_{n+4} \nmid F_{\frac{1}{3}n^{(5)}}$.

(g) *If* $n \equiv 2 \pmod 4$, *then* $L_{n+2} \nmid F_{\frac{1}{4}n^{(5)}}$.

(h) *If* $1 \le k \le 5$, *then* $\nu_2\big(\prod_{i=0}^{k} L_{n+i}\big) \le 3$.

Proof.

(a) Using the identity $F_{2n} = F_n L_n$ and Lemma 2.2, we clearly obtain the assertion.

(b) Using the identity $F_{3n} = F_n\big(L_{2n} + (-1)^n\big)$, see [4, p. 92], the fact that L_{2n} is odd for $n \equiv 1, 2 \pmod 3$ by Lemma 2.2 we clearly obtain the assertion.

(c) Using the identity $F_{2n} = F_n L_n$ and Lemma 2.2 we obtain the assertion.

(d) Using the identity $F_{6n} = F_{3n} L_{3n} = L_{3n} F_n\big(L_{2n} + (-1)^n\big)$, see [4, p. 92], and the fact that $6 \mid L_{3n}$ for $n \equiv 2 \pmod 4$, with respect to Lemma 2.2, we have the assertion.

(e) Let us consider that $m \mid F_n$. Using the well-known property $F_n \mid F_{kn}$ we obtain $m \mid F_{kn}$.

(f) To prove the assertion it is suffice to show that $\nu_3(L_n L_{n+4}) > \nu_3(F_{\frac{1}{3}n^{(5)}})$ for $n \equiv 2, 10, 14, 18, 22, 30 \pmod{36}$ (thus $n \equiv 2 \pmod 4$ and $n \not\equiv 6, 26, 34$ (mod 36)). Using Lemmas 2.1, 2.2 and the clear fact that $4 \mid \frac{1}{3}n^{(5)}$ for any nonnegative integer n we obtain

$$\nu_3(L_n) = \begin{cases} \nu_3(n) + 1, & n \equiv 2 \pmod 4; \\ 0, & n \not\equiv 2 \pmod 4, \end{cases} \tag{2.1}$$

$$\nu_3(F_n) = \begin{cases} \nu_3(n) + 1, & n \equiv 0 \pmod 4; \\ 0, & n \not\equiv 0 \pmod 4, \end{cases}$$

hence

$$\begin{aligned} \nu_3(L_n L_{n+4}) &= \nu_3(L_n) + \nu_3(L_{n+4}) \\ &= \big(\nu_3(n) + 1\big) + \big(\nu_3(n+4) + 1\big) \\ &= \nu_3(n) + \nu_3(n+4) + 2 \end{aligned}$$

and

$$\begin{aligned} \nu_3(F_{\frac{1}{3}n^{(5)}}) &= \nu_3\left(\frac{1}{3}n^{(5)}\right) + 1 = \nu_3\big(n^{(5)}\big) \\ &= \nu_3(n) + \nu_3(n+4) + 1, \end{aligned}$$

as clearly $\nu_3\big((n+1)(n+2)(n+3)\big) = 1$ holds for $n \not\equiv 6, 7, 8 \pmod 9$ and all cases from the assertion are in this form.

(g) For $n \equiv 2 \pmod 4$ we have $\frac{1}{4}n^{(5)}/(n+2) \equiv 1 \pmod 2$, hence the assertion follows from Lemma 2.3 (b).

(h) Since there are unique ϵ and δ belonging to $\{0,\dots,5\}$ such that

$$n+\epsilon \equiv 3 \pmod 6 \quad \text{and} \quad n+\delta \equiv 0 \pmod 6,$$

we have

$$\nu_2\left(\prod_{i=0}^{k} L_{n+i}\right) \le \sum_{i=0}^{5} \nu_2(L_{n+i}) = \nu_2(L_{n+\epsilon}) + \nu_2(L_{n+\delta}) = 2+1 = 3.$$

Thus the lemma follows. □

Remark 3. The reader may be wondering why this paper deals with Lucas numbers, but it does not study the Fibonacci case. The reason is exactly that the previous item (h) does not hold for Fibonacci numbers. Actually, $\nu_2\big(\prod_{i=0}^{k} F_{n+i}\big)$ can be sufficiently large which causes the substantial increasing in the number of cases to be studied.

3. The proof of Theorem 1.1

Since there are at least two even numbers among $n, n+1, n+2, n+3, n+4$, we conclude (using Lemma 2.3 (b)) that

$$L_{n+i} \mid F_{n^{(5)}} \quad \text{for } i = 0,1,2,3,4. \tag{3.1}$$

We will consider these cases:

- Let $n \equiv 1 \pmod 6$. Then $\gcd(L_n, L_{n+3}) = \gcd(L_n, L_{n+4}) = 1$. This together with Lemma 2.3 (e) implies that the numbers $L_n, L_{n+1}, L_{n+2}, L_{n+3}$ and L_{n+4} are pairwise coprime. Thus (3.1), together with Lemma 2.4 (c), leads to
$$z(L_nL_{n+1}L_{n+2}L_{n+3}L_{n+4}) \mid n^{(5)}. \tag{3.2}$$
On the other hand, for $i=0,1,2,3,4$ clearly $L_{n+i} \mid L_nL_{n+1}L_{n+2}L_{n+3}L_{n+4}$, hence $2(n+i) \mid z(L_nL_{n+1}L_{n+2}L_{n+3}L_{n+4})$ with respect to Lemma 2.4 (b). Since $n, \frac{n+1}{2}, 2(n+2), \frac{n+3}{2}, n+4$ are pairwise coprime, then
$$2n\frac{n+1}{2}2(n+2)\frac{n+3}{2}(n+4) \mid z(L_nL_{n+1}L_{n+2}L_{n+3}L_{n+4}). \tag{3.3}$$
Combining (3.2) and (3.3)
$$z(L_nL_{n+1}L_{n+2}L_{n+3}L_{n+4}) = n^{(5)}.$$

- Let $n \equiv 4 \pmod{12}$. Using Lemma 2.3 (b) we clearly have that
$$L_{n+i} \mid F_{\frac{1}{4}n^{(5)}} \tag{3.4}$$
for $i = 0, 1, 2, 3, 4$. Further $\gcd(L_n, L_{n+3}) = \gcd(L_n, L_{n+4}) = 1$. This together with Lemma 2.3 (e) yields that the numbers $L_n, L_{n+1}, L_{n+2}, L_{n+3}$, and L_{n+4} are pairwise coprime. Thus Lemma 2.4 (c) implies that
$$z(L_nL_{n+1}L_{n+2}L_{n+3}L_{n+4}) \mid \frac{1}{4}n^{(5)}. \tag{3.5}$$
On the other hand, for $i=0, 1, 2, 3, 4$ clearly $L_{n+i} \mid L_nL_{n+1}L_{n+2}L_{n+3}L_{n+4}$, hence $2(n+i) \mid z(L_nL_{n+1}L_{n+2}L_{n+3}L_{n+4})$. Since $n/2, n+1, (n+2)/2, n+3, (n+4)/2$ are pairwise coprime, then
$$2\frac{n}{4}(n+1)\frac{n+2}{2}(n+3)\frac{n+4}{2} \mid z(L_nL_{n+1}L_{n+2}L_{n+3}L_{n+4}). \tag{3.6}$$
Thus, combining (3.5) and (3.6) we have
$$z(L_nL_{n+1}L_{n+2}L_{n+3}L_{n+4}) \in \left\{\frac{1}{8}n^{(5)}, \frac{1}{4}n^{(5)}\right\}.$$
Now, we show that
$$L_nL_{n+1}L_{n+2}L_{n+3}L_{n+4} \nmid F_{\frac{1}{8}n^{(5)}}.$$
In fact, by using Lemma 2.3 (b) we have
$$\begin{aligned} L_n &\nmid F_{\frac{1}{8}n^{(5)}} \quad \text{for } n \equiv 16 \pmod{24}, \\ L_{n+4} &\nmid F_{\frac{1}{8}n^{(5)}} \quad \text{for } n \equiv 4 \pmod{24}. \end{aligned}$$

- Let $n \equiv 8 \pmod{12}$. Using Lemma 2.3 (b) we clearly have that
$$L_{n+i} \mid F_{\frac{1}{12}n^{(5)}} \tag{3.7}$$
for $i = 0, 1, 2, 3, 4$. Further $\gcd(L_n, L_{n+3}) = \gcd(L_n, L_{n+4}) = 1$ and together with Lemma2.3 (e), we observe that the numbers L_n, L_{n+1}, L_{n+2}, L_{n+3} and L_{n+4} are pairwise coprime. Thus Lemma 2.4 (c) implies that
$$z(L_nL_{n+1}L_{n+2}L_{n+3}L_{n+4}) \mid \frac{1}{12}n^{(5)}. \tag{3.8}$$
On the other hand, for $i=0, 1, 2, 3, 4$ clearly $L_{n+i} \mid L_nL_{n+1}L_{n+2}L_{n+3}L_{n+4}$, hence $2(n+i) \mid z(L_nL_{n+1}L_{n+2}L_{n+3}L_{n+4})$. Observe that there exist $a, b, c, d \in \{0, 1\}$, with $a+b = c+d = 1$, such that $n/4^a$, $(n+1)/3^c$, $(n+2)/2$, $n+3, (n+4)/(4^b \cdot 3^d)$ are pairwise coprime, then
$$\frac{1}{24}n^{(5)} \mid z(L_nL_{n+1}L_{n+2}L_{n+3}L_{n+4})/2. \tag{3.9}$$
Thus, using (3.8) and (3.9) we have
$$z(L_nL_{n+1}L_{n+2}L_{n+3}L_{n+4}) = \frac{1}{12}n^{(5)}.$$

- Let $n \equiv 0 \pmod{12}$. Using Lemma 2.3 (b) we clearly have that

$$L_{n+i} \mid F_{\frac{1}{12}n^{(5)}} \tag{3.10}$$

for $i = 0, 1, 2, 3, 4$. Further $\gcd\left(\frac{L_n}{2}, L_{n+3}\right) = 1$ and $\gcd(L_n, L_{n+4}) = 1$. Hence using Lemma 2.5 (a) we obtain

$$L_n L_{n+1} L_{n+2} L_{n+3} L_{n+4} \mid 2F_{\frac{1}{12}n^{(5)}} \mid F_{\frac{1}{6}n^{(5)}}$$

and

$$z(L_n L_{n+1} L_{n+2} L_{n+3} L_{n+4}) \mid \frac{1}{6}n^{(5)}. \tag{3.11}$$

On the other hand, for $i=0, 1, 2, 3, 4$ clearly $L_{n+i} \mid L_n L_{n+1} L_{n+2} L_{n+3} L_{n+4}$, hence $2(n+i) \mid z(L_n L_{n+1} L_{n+2} L_{n+3} L_{n+4})$. Observe that there exist $a, b, c, d \in \{0, 1\}$, with $a+b = c+d = 1$, such that $n/(4^a \cdot 3^c)$, $n+1$, $(n+2)/2$, $(n+3)/3^d$, $(n+4)/4^b$ are pairwise coprime, then

$$\frac{1}{24}n^{(5)} \mid z(L_n L_{n+1} L_{n+2} L_{n+3} L_{n+4})/2 \tag{3.12}$$

and therefore

$$z(L_n L_{n+1} L_{n+2} L_{n+3} L_{n+4}) \in \left\{\frac{1}{12}n^{(5)}, \frac{1}{6}n^{(5)}\right\}.$$

We show that

$$L_n L_{n+1} L_{n+2} L_{n+3} L_{n+4} \mid F_{\frac{1}{12}n^{(5)}}.$$

The proof will be based on comparing p-adic orders of

$$L_n L_{n+1} L_{n+2} L_{n+3} L_{n+4} \quad \text{and} \quad F_{\frac{1}{12}n^{(5)}} \quad \text{for all primes } p.$$

Thus we shall prove that

$$\nu_p(F_{\frac{1}{12}n^{(5)}}) \geq \nu_p(L_n L_{n+1} L_{n+2} L_{n+3} L_{n+4}) \tag{3.13}$$

holds for all primes p.

Using Lemma 2.5 (h), Lemma 2.1 and the clear fact that

$$\frac{1}{12}n^{(5)} \equiv 0 \pmod{48},$$

we have

$$\begin{aligned} \nu_2(F_{\frac{1}{12}n^{(5)}}) &= \nu_2\left(\frac{1}{12}n^{(5)}\right) + 2 \\ &= \nu_2(n^{(5)}) - 2 + 2 \geq 6 \\ &> \nu_2(L_n L_{n+1} L_{n+2} L_{n+3} L_{n+4}). \end{aligned}$$

Now, we will consider $p \neq 2$. Suppose that $\nu_p(L_n L_{n+1} L_{n+2} L_{n+3} L_{n+4}) \neq 0$ (otherwise the desired inequality is directly proved). Since $\frac{L_n}{2}$, L_{n+1}, L_{n+2}, L_{n+3} and L_{n+4} are pairwise coprime, p divides only one of $L_n, L_{n+1}, L_{n+2}, L_{n+3}, L_{n+4}$, say $p \mid L_{n+\delta}$ for some $\delta \in \{0, 1, 2, 3, 4\}$. Thus $p \mid L_{n+\delta} \mid F_{\frac{1}{12}n^{(5)}}$

implying $z(p) \mid \frac{1}{12}n^{(5)}$, by Lemma 2.4 (c). Therefore using Lemma 2.1 and Lemma 2.2 we obtain

$$\begin{aligned}\nu_p(L_nL_{n+1}L_{n+2}L_{n+3}L_{n+4}) &= \nu_p(L_{n+\delta})\\ &\leq \nu_p(n+\delta)+\nu_p\big(F_{z(p)}\big)\\ &\leq \nu_p\left(\frac{1}{12}n^{(5)}\right)+\nu_p\big(F_{z(p)}\big)\\ &= \nu_p\left(F_{\frac{1}{12}n^{(5)}}\right).\end{aligned}$$

- Let $n \equiv 3 \pmod 6$. Using Lemma 2.3 (b) we clearly have that

$$L_{n+i} \mid F_{\frac{1}{3}n^{(5)}} \quad \text{for } i = 0,1,2,3,4. \tag{3.14}$$

Further $\gcd(\frac{L_n}{2}, L_{n+3}) = \gcd(L_n, L_{n+4}) = \gcd(L_{n+1}, L_{n+4}) = 1$ and together with Lemma 2.3 (e) the numbers $\frac{L_n}{2}, L_{n+1}, L_{n+2}, L_{n+3}$ and L_{n+4} are pairwise coprime. Hence using Lemma 2.5 (a) we obtain

$$L_nL_{n+1}L_{n+2}L_{n+3}L_{n+4} \mid 2F_{\frac{1}{3}n^{(5)}} \mid F_{2\frac{1}{3}n^{(5)}}.$$

In particular,

$$z(L_nL_{n+1}L_{n+2}L_{n+3}L_{n+4}) \mid \frac{2}{3}n^{(5)}. \tag{3.15}$$

On the other hand, for $i=0,1,2,3,4$ clearly $L_{n+i} \mid L_nL_{n+1}L_{n+2}L_{n+3}L_{n+4}$, hence $2(n+i) \mid z(L_nL_{n+1}L_{n+2}L_{n+3}L_{n+4})$. Observe that there exist $a,b,c,d \in \{0,1\}$, with $a+b=c+d=1$, such that $n/3^a, (n+1)/2^c, n+2$, $(n+3)/(3^b \cdot 2^d), n+4$ are pairwise coprime, then

$$\frac{1}{6}n^{(5)} \mid z(L_nL_{n+1}L_{n+2}L_{n+3}L_{n+4})/2. \tag{3.16}$$

Thus, using (3.15) and (3.16) we have

$$z(L_nL_{n+1}L_{n+2}L_{n+3}L_{n+4}) \in \left\{\frac{1}{3}n^{(5)}, \frac{2}{3}n^{(5)}\right\}.$$

We show that

$$L_nL_{n+1}L_{n+2}L_{n+3}L_{n+4} \mid F_{\frac{1}{3}n^{(5)}}.$$

The proof will be again based on comparing p-adic orders of

$$L_nL_{n+1}L_{n+2}L_{n+3}L_{n+4} \quad \text{and} \quad F_{\frac{1}{3}n^{(5)}}$$

for all primes p. Thus we prove that

$$\nu_p\left(F_{\frac{1}{3}n^{(5)}}\right) \geq \nu_p(L_nL_{n+1}L_{n+2}L_{n+3}L_{n+4}) \tag{3.17}$$

holds for all primes p. This relation clearly holds for $p=5$ with respect to Lemma 2.1 and Lemma 2.2. Using Lemma 2.5 (h), Lemma 2.1 and the clear fact that $\frac{1}{3}n^{(5)} \equiv 0 \pmod{24}$ we obtain

$$\nu_2(F_{\frac{1}{3}n^{(5)}}) = \nu_2\left(\frac{1}{3}n^{(5)}\right) + 2 = \nu_2\big(n^{(5)}\big) + 2$$
$$\geq 5 > \nu_2(L_nL_{n+1}L_{n+2}L_{n+3}L_{n+4}).$$

When $p \neq 2$ and $p \neq 5$, the proof of (3.17) can be done by the same way as in the case $n \equiv 0 \pmod{12}$.

- Let $n \equiv 5 \pmod 6$. Using Lemma 2.3 (b) we clearly have that

$$L_{n+i} \mid F_{\frac{1}{3}n^{(5)}} \quad \text{for } i = 0, 1, 2, 3, 4.$$

Further $\gcd(L_n, L_{n+3}) = \gcd(L_n, L_{n+4}) = \gcd(\frac{L_{n+1}}{2}, L_{n+4}) = 1$, and together with Lemma 2.3 (e) the numbers $L_n, \frac{L_{n+1}}{2}, L_{n+2}, L_{n+3}$ and L_{n+4} are pairwise coprime. Hence using Lemma 2.5 (b) we obtain

$$L_nL_{n+1}L_{n+2}L_{n+3}L_{n+4} \mid 2F_{\frac{1}{3}n^{(5)}} \mid F_{3\frac{1}{3}n^{(5)}}$$

and then

$$z(L_nL_{n+1}L_{n+2}L_{n+3}L_{n+4}) \mid n^{(5)}. \tag{3.18}$$

On the other hand, for $i = 0, 1, 2, 3, 4$ clearly $L_{n+i} \mid L_nL_{n+1}L_{n+2}L_{n+3}L_{n+4}$, hence $2(n+i) \mid z(L_nL_{n+1}L_{n+2}L_{n+3}L_{n+4})$. Observe that there exist $a, b, c, d \in \{0, 1\}$ with $a+b = c+d = 1$, such that $n, (n+1)/(2^a \cdot 3^b), n+2, (n+3)/2^b, (n+4)/3^b$ are pairwise coprime, then

$$\frac{1}{6}n^{(5)} \mid z(L_nL_{n+1}L_{n+2}L_{n+3}L_{n+4})/2. \tag{3.19}$$

Thus, using (3.18) and (3.19) we have

$$z(L_nL_{n+1}L_{n+2}L_{n+3}L_{n+4}) \in \left\{\frac{1}{3}n^{(5)}, \frac{2}{3}n^{(5)}, n^{(5)}\right\}.$$

The fact that $L_nL_{n+1}L_{n+2}L_{n+3}L_{n+4} \mid F_{\frac{1}{3}n^{(5)}}$ holds can be proved in the same way as in the case $n \equiv 3 \pmod 6$.

- Let $n \equiv 6 \pmod{36}$. Using Lemma 2.3 (b) we clearly have that

$$L_{n+i} \mid F_{\frac{1}{6}n^{(5)}} \quad \text{for } i = 0, 1, 2, 3, 4.$$

Further $\gcd\big(\frac{L_n}{2}, L_{n+3}\big) = \gcd\big(\frac{L_n}{2}, \frac{L_{n+4}}{3}\big) = \gcd\big(L_{n+1}, \frac{L_{n+4}}{3}\big) = 1$ and together with Lemma 2.3 (e) the numbers $\frac{L_n}{2}, L_{n+1}, L_{n+2}, L_{n+3}$ and $\frac{L_{n+4}}{3}$ are pairwise coprime. Hence using Lemma 2.5 (c) we obtain

$$L_nL_{n+1}L_{n+2}L_{n+3}L_{n+4} \mid 6F_{\frac{1}{6}n^{(5)}} \mid F_{\frac{1}{3}n^{(5)}}$$

and

$$z(L_nL_{n+1}L_{n+2}L_{n+3}L_{n+4}) \mid \frac{1}{3}n^{(5)}. \tag{3.20}$$

On the other hand, for $i=0,1,2,3,4$ clearly $L_{n+i} \mid L_nL_{n+1}L_{n+2}L_{n+3}L_{n+4}$, hence $2(n+i) \mid z(L_nL_{n+1}L_{n+2}L_{n+3}L_{n+4})$. Since $n/6, n+1, n+2, n+3$, $(n+4)/2$ are pairwise coprime, we have that

$$\frac{1}{12}n^{(5)} \mid z(L_nL_{n+1}L_{n+2}L_{n+3}L_{n+4})/2. \tag{3.21}$$

Thus, using (3.20) and (3.21) yields

$$z(L_nL_{n+1}L_{n+2}L_{n+3}L_{n+4}) \in \left\{\frac{1}{6}n^{(5)}, \frac{1}{3}n^{(5)}\right\}.$$

So, it remains to prove that

$$L_nL_{n+1}L_{n+2}L_{n+3}L_{n+4} \mid F_{\frac{1}{6}n^{(5)}}.$$

Using Lemma 2.5 (h) and the clear fact $\frac{1}{6}n^{(5)} \equiv 0 \pmod{12}$ by Lemma 2.1, we get

$$\begin{aligned}\nu_2\left(F_{\frac{1}{6}n^{(5)}}\right) = \nu_2\left(\frac{1}{6}n^{(5)}\right) + 2 &= \nu_2(n^{(5)}) + 1\\ &\geq 5 > \nu_2(L_nL_{n+1}L_{n+2}L_{n+3}L_{n+4}).\end{aligned}$$

For $p > 2$ we can prove that $\nu_p(L_nL_{n+1}L_{n+2}L_{n+3}L_{n+4}) \leq \nu_p\big(F_{\frac{1}{6}n^{(5)}}\big)$ in the same way as in the case $n \equiv 0 \pmod{12}$.

- Let $n \equiv 26 \pmod{36}$. Using Lemma 2.3 (b) we clearly have that

$$L_{n+i} \mid F_{\frac{1}{6}n^{(5)}} \quad \text{for } i = 0,1,2,3,4.$$

Further $\gcd(\frac{L_n}{3}, L_{n+3}) = \gcd(\frac{L_n}{3}, \frac{L_{n+4}}{2}) = \gcd(L_{n+1}, \frac{L_{n+4}}{2}) = 1$ and this together with Lemma 2.3 (e) implies that the numbers $\frac{L_n}{3}, L_{n+1}, L_{n+2}$, L_{n+3} and $\frac{L_{n+4}}{2}$ are pairwise coprime. Hence using Lemma 2.5 (d) we obtain

$$L_nL_{n+1}L_{n+2}L_{n+3}L_{n+4} \mid 6F_{\frac{1}{6}n^{(5)}} \mid F_{n^{(5)}}$$

and

$$z(L_nL_{n+1}L_{n+2}L_{n+3}L_{n+4}) \mid n^{(5)}. \tag{3.22}$$

On the other hand, for $i=0,1,2,3,4$ clearly $L_{n+i} \mid L_nL_{n+1}L_{n+2}L_{n+3}L_{n+4}$, hence $2(n+i) \mid z(L_nL_{n+1}L_{n+2}L_{n+3}L_{n+4})$. Since $n/2, n+1, n+2, n+3$, $(n+4)/6$ are pairwise coprime, we have that

$$\frac{1}{12}n^{(5)} \mid z(L_nL_{n+1}L_{n+2}L_{n+3}L_{n+4})/2. \tag{3.23}$$

Thus, (3.22) and (3.23) yield

$$z(L_nL_{n+1}L_{n+2}L_{n+3}L_{n+4}) \in \left\{\frac{1}{6}n^{(5)}, \frac{1}{3}n^{(5)}, \frac{1}{2}n^{(5)}, \frac{2}{3}n^{(5)}, \frac{5}{6}n^{(5)}, n^{(5)}\right\}.$$

So, it remains to prove that $L_nL_{n+1}L_{n+2}L_{n+3}L_{n+4} \mid F_{\frac{1}{6}n^{(5)}}$, but the proof is the same as in the previous case.

- $n \equiv 2, 10, 14, 18, 22, 30, 34 \pmod{36}$. Using Lemma 2.3 (b) we clearly have for $i = 0, 1, 2, 3, 4$:
  - ∗ If $n \equiv 2, 14, 18, 30 \pmod{36}$, then $L_{n+i} \mid F_{\frac{1}{6}n^{(5)}}$.
  - ∗ If $n \equiv 10, 22, 34 \pmod{36}$, then $L_{n+i} \mid F_{\frac{1}{2}n^{(5)}}$.

  It can be seen that:
  - ∗ If $n \equiv 2, 14 \pmod{36}$, then $\gcd\big(\frac{L_n}{3}, L_{n+3}\big) = \gcd\big(\frac{L_n}{3}, \frac{L_{n+4}}{2}\big) = \gcd\big(L_{n+1}, \frac{L_{n+4}}{2}\big) = 1$.
  - ∗ If $n \equiv 10 \pmod{36}$, then $\gcd\big(\frac{L_n}{3}, L_{n+3}\big) = \gcd\big(\frac{L_n}{3}, L_{n+4}\big) = \gcd\big(L_{n+1}, L_{n+4}\big) = 1$.
  - ∗ If $n \equiv 18, 30 \pmod{36}$, then $\gcd\big(\frac{L_n}{2}, L_{n+3}\big) = \gcd\big(\frac{L_n}{2}, \frac{L_{n+4}}{3}\big) = \gcd\big(L_{n+1}, \frac{L_{n+4}}{3}\big) = 1$.
  - ∗ If $n \equiv 22, 34 \pmod{36}$, then $\gcd(L_n, L_{n+3}) = \gcd\big(L_n, \frac{L_{n+4}}{3}\big) = \gcd\big(L_{n+1}, \frac{L_{n+4}}{3}\big) = 1$.

  By Lemma 2.5 (a), (d) we obtain:
  - ∗ If $n \equiv 2, 14, 18, 30 \pmod{36}$, then $L_nL_{n+1}L_{n+2}L_{n+3}L_{n+4} \mid 6F_{\frac{1}{6}n^{(5)}} \mid F_{6\frac{1}{6}n^{(5)}}$.
  - ∗ If $n \equiv 10, 22, 34 \pmod{36}$, then $L_nL_{n+1}L_{n+2}L_{n+3}L_{n+4} \mid 3F_{\frac{1}{2}n^{(5)}} \mid F_{2\frac{1}{2}n^{(5)}}$.

  Thus in all cases we have
  $$z(L_nL_{n+1}L_{n+2}L_{n+3}L_{n+4}) \mid n^{(5)}. \tag{3.24}$$
  On the other hand, for $i = 0, 1, 2, 3, 4$ clearly $L_{n+i} \mid L_nL_{n+1}L_{n+2}L_{n+3}L_{n+4}$, hence $2(n+i) \mid z(L_nL_{n+1}L_{n+2}L_{n+3}L_{n+4})$. Since:
  - ∗ $n \equiv 2, 14 \pmod{36}$, then $n/2, (n+1)/3, n+2, n+3, (n+4)/2$ are pairwise coprime.
  - ∗ $n \equiv 10, 22, 34 \pmod{36}$, then $n/2, n+1, n+2, n+3, (n+4)/2$ are pairwise coprime.
  - ∗ $n \equiv 18, 30 \pmod{36}$, then $n/2, n+1, n+2, (n+3)/3, (n+4)/2$ are pairwise coprime.

  Thus:
  - ∗ $n \equiv 2, 14, 18, 30 \pmod{36}$ implies $\frac{1}{6}n^{(5)} \mid z(L_nL_{n+1}L_{n+2}L_{n+3}L_{n+4})$.
  - ∗ $n \equiv 10, 22, 34 \pmod{36}$ implies $\frac{1}{2}n^{(5)} \mid z(L_nL_{n+1}L_{n+2}L_{n+3}L_{n+4})$.

  Summarizing, we have:
  - ∗ $n \equiv 2, 14, 18, 30 \pmod{36}$ implies
  $$z(L_nL_{n+1}L_{n+2}L_{n+3}L_{n+4}) \in \left\{\frac{k}{6}n^{(5)} : k \in \{0, \ldots, 5\}\right\},$$
  - ∗ $n \equiv 10, 22, 34 \pmod{36}$ implies
  $$z(L_nL_{n+1}L_{n+2}L_{n+3}L_{n+4}) \in \left\{\frac{1}{2}n^{(5)}, n^{(5)}\right\}.$$

To prove the assertion for this case we must prove that:

* $L_nL_{n+1}L_{n+2}L_{n+3}L_{n+4} \mid F_{\frac{1}{2}n^{(5)}}$ if $n \equiv 10, 22, 34 \pmod{36}$, and
* $L_nL_{n+1}L_{n+2}L_{n+3}L_{n+4} \nmid F_{\frac{1}{3}n^{(5)}}$ and $L_nL_{n+1}L_{n+2}L_{n+3}L_{n+4} \mid F_{\frac{1}{2}n^{(5)}}$ if $n \equiv 2, 14, 18, 30 \pmod{36}$.

First, we shall prove that

$$L_nL_{n+1}L_{n+2}L_{n+3}L_{n+4} \nmid F_{\frac{1}{3}n^{(5)}} \quad \text{for } n \equiv 2, 14, 18, 30 \pmod{36}. \tag{3.25}$$

In fact, if $n \equiv 2, 30 \pmod{36}$, then

$$\nu_3(L_nL_{n+4}) = 3 > 2 = \nu_3\big(F_{\frac{1}{3}n^{(5)}}\big).$$

If $n \equiv 14 \pmod{36}$, then

$$\begin{aligned}\nu_3(L_nL_{n+4}) &= \nu_3(n+4) + 2 > \nu_3(n+4) + 1\\ &= \nu_3(n) + \nu_3(n+4) = \nu_3\big(F_{\frac{1}{3}n^{(5)}}\big).\end{aligned}$$

If $n \equiv 18 \pmod{36}$, then

$$\begin{aligned}\nu_3(L_nL_{n+4}) &= \nu_3(n) + 2 > \nu_3(n) + 1\\ &= \nu_3(n) + \nu_3(n+4) = \nu_3\big(F_{\frac{1}{3}n^{(5)}}\big).\end{aligned}$$

In summary, $L_nL_{n+4} \nmid F_{\frac{1}{3}n^{(5)}}$ for $n \equiv 2, 14, 18, 30 \pmod{36}$.

Now, we shall prove that

$$L_nL_{n+1}L_{n+2}L_{n+3}L_{n+4} \mid F_{\frac{1}{2}n^{(5)}}$$

for all $n \equiv 2, 10, 14, 18, 22, 30, 34 \pmod{36}$. For that, we use the same p-adic valuation argument as before. For $p \neq 2$ and $p \neq 5$, we proceed exactly as in the case $n \equiv 0 \pmod{12}$. For the case $p = 2$, we have

$$\begin{aligned}\nu_2(F_{\frac{1}{2}n^{(5)}}) &= \nu_2(n) + \nu_2(n+2) + \nu_2(n+4) + 1\\ &\geq 5 > 3 \geq \nu_2(L_nL_{n+1}L_{n+2}L_{n+3}L_{n+4}).\end{aligned}$$

Therefore, the proof is complete. □

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