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THE ORDER OF ENTIRE FUNCTIONS WITH RADIALLY DISTRIBUTED ZEROS

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ABSTRACT. It is shown that an entire function with radially distributed zeros has finite order λ if it has finite lower order μ. It is then shown that functions with real negative zeros only are extremal for the problem of maximizing the Nevanlinna characteristic in the class of entire functions satisfying $\lambda - \mu > 1$.

Let λ, μ, ρ be the order, lower order and the exponent of convergence of the zeros of an entire function f. Whittaker [**8**, p. 130] has shown that if μ and ρ are finite, then λ is finite and $\lambda = \max(\mu, \rho)$. The finiteness of μ by itself, however, is not enough to make λ finite. It is a rather interesting fact, that a radial distribution of the zeros of f makes λ finite if μ is finite. We point out that the theorem whose statement constitutes the title of Whittaker's paper [8], is an immediate corollary of earlier and more informative results of Edrei and Fuchs [**2**, p. 298], [**3**, pp. 261, 264].

Using rather difficult estimates of $T(r, f)$, Edrei and Fuchs [**2**, p. 308] have shown that $q \leqslant \mu$ for a *canonical product* f of finite genus q ($\geqslant 1$) having only real negative zeros. Their result implies that $q \leqslant \mu \leqslant \lambda \leqslant q + 1$ for such functions provided that λ is assumed finite. Years later Shea [**6**, p. 204], in studying the Valiron deficiencies of meromorphic functions, obtained as a corollary a bound on λ in terms of μ only, for entire functions f having only real negative zeros and *finite* order λ.

Our first result (Theorem 1 below) generalizes the above results and the proof extends to subharmonic (and δ-subharmonic) functions in space. In addition, our proof may be of interest because of its simplicity.

THEOREM 1. *Let f be an entire function of order λ and lower order μ. Assume that all the zeros of f lie on the radii defined by*

$$re^{i\omega_0}, re^{i\omega_1}, \dots, re^{i\omega_m} \qquad (r > 0, m \geqslant 0),$$

where the ω's are real.

Then λ is finite if and only if μ is finite.

If $m = 0$ and μ is finite then $\lambda \leqslant [\mu] + 1$.

Entire functions whose zeros lie on a ray are believed to be extremal for a large class of problems in Nevanlinna theory. Let f be entire with zeros $\{a_n\}$ and nonintegral order λ, and let F be the canonical product with zeros $\{-|a_n|\}$. If

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$0 < \lambda < 1$, then it is a consequence of Gol'dberg's lemma [**4**, p. 106] that $T(r,f) \leqslant T(r, F)$, but nothing is known if $\lambda > 1$. In this direction the following result may be of interest.

THEOREM 2. *Let f be entire of finite nonintegral order* λ *and lower order* μ. *If* $\lambda - \mu > 1$, *then there exists a sequence* $\{x_n\}$ *increasing to infinity and a positive* γ (< 1) *such that*

$$T(r,f) \leqslant T(r, F), \qquad x_n^{\gamma} \leqslant r \leqslant x_n. \tag{1}$$

Connected to our Theorem 1 is the following unpublished result of I-Lok Chang.

THEOREM A. *Let f be entire* $f(0) = 1$, *and let* $\{a_j\}$ *be the sequence of its zeros. Take* $N(r, 1/f)$ *to be the counting function that appears in Nevanlinna's theory.*

Let $k \geqslant 1$ *be an integer and let*

$$\sum_{j=1}^{\infty} |a_j|^{-k} = +\infty. \tag{2}$$

Consider the point-set

$$\Delta_k = \{z: |\arg z^k| \leqslant \beta < \pi/2\}, \tag{3}$$

and let

$$\sum_{a_j \notin \Delta_k} |a_j|^{-k} < +\infty. \tag{4}$$

Then, Nevanlinna's characteristic $T(r,f)$ *satisfies the relation*

$$T(r,f) \geqslant \tfrac{1}{2} N(r, 1/f) + r^{-k}\Omega(r) \tag{5}$$

with $\Omega(r) \to +\infty$ *as* $r \to +\infty$.

Since $r^{-k}T(r,f)$ always tend to a limit (possibly infinite) when $\sum_{j=1}^{\infty}|a_j|^{-k} < +\infty$, we may 'append' the obvious corollary of Theorem A to obtain

THEOREM B. *If f is an entire function satisfying* (3) *and* (4) *for some positive integer k, and if* $T(r,f)$ *is its Nevanlinna characteristic, then* $\lim_{r\to\infty} r^{-k}T(r,f)$ *exists as a finite or infinite limit.*

The next corollary shows the connection between Chang's result and Theorem 1.

COROLLARY OF CHANG'S THEOREM. *Let f be an entire function having all its zeros on the two rays*

$$r, re^{i\pi m/\alpha} \qquad (r > 0,\ \alpha\ (\geqslant 1),\ m \text{ integers}).$$

If the lower order μ *of f is finite, then its order* λ *is finite and* $\lambda \leqslant [\mu] + 2\alpha$.

PROOF OF COROLLARY. Let f satisfy the conditions of the corollary and suppose first that m is even and has no common factors with α. Let k be the (unique) multiple of α in the set $[\mu] + 1, [\mu] + 2, \ldots, [\mu] + \alpha$. Then all the zeros of f lie in Δ_k and condition (4) of Theorem A is satisfied. By Theorem B the $\lim_{r\to\infty} r^{-k}T(r,f)$ exists. Since $k > \mu$, this limit must be finite. It follows that $\lambda \leqslant k$ and so $\lambda \leqslant k \leqslant [\mu] + \alpha \leqslant [\mu] + 2\alpha$. The case when m is odd may be proved similarly, but k must be taken to be a multiple of 2α. We remark that examples of Edrei and Fuchs [**2**, p. 295] show that the bound $[\mu] + \alpha$, obtained when m is even, is sharp. We also note that, if instead of one ray, we have a finite number of rays of arguments $m_1\pi/\alpha_1$, $m_2\pi/\alpha_2, \ldots, m_s\pi/\alpha_s$, then a function having all its zeros on these rays and having lower order μ will have order λ bounded above by $[\mu] + 2$ (lowest common multiple of $\alpha_1, \alpha_2, \ldots, \alpha_s$). We finally point out that the corollary may be proved directly from Theorem A.

Proof of Theorem 2. Let f be of finite nonintegral order λ, then $N(r) = N(r, 1/f)$ has order λ. Then F has order λ. By Theorem 1, the lower order μ' of F satisfies $\lambda - \mu' \leqslant 1$ and so $\mu < \mu'$. Choose ε (> 0) so that $\mu < \mu' - \varepsilon$ and then choose γ such that $\mu/(\mu' - \varepsilon) < \gamma < 1$. By Whittaker's Lemma [**8**, p. 130] there exists a sequence $\{x_n\}$ increasing to infinity such that

$$T(r, f) \leqslant r^{\mu' - \varepsilon} \qquad (x_n^\gamma \leqslant r \leqslant x_n). \tag{6}$$

Since $T(r, F) \geqslant r^{\mu' - \varepsilon}$ for all large r, (1) follows.

Proof of Theorem 1. Let f be an entire function satisfying the conditions of Theorem 1 and assume that its lower order μ is finite. We first show that the condition of the theorem implies that the zeros of f are located in 'suitable' sectors. This we do by following, step by step, an argument of Edrei, Fuchs and Hellerstein [**1**, p. 149]. Consider the set of arguments ω_j and assume that $\omega_0 = 2\pi$; this is clearly no restriction. Choose k ($0 \leqslant k \leqslant m$), and relabel if necessary, so that $\{2\pi, \omega_1, \ldots, \omega_k\}$ is a maximal linearly independent set. If $k < m$, there exists integers n_{l_j} and σ (> 0) such that

$$\sigma\omega_l = 2\pi n_{l_0} + \sum_{j=1}^{k} n_{l_j}\omega_j \qquad (l = k+1, \ldots, m). \tag{7}$$

Put

$$M_l = \sum_{j=1}^{k} |n_{l_j}|, \qquad M = \sup\{\sigma, M_{k+1}, M_{k+2}, \ldots, M_m\}. \tag{8}$$

By Weyl's equidistribution theorem [7], there exists a sequence $\{\lambda_s\}$ of positive integers satisfying

$$|\lambda_s\omega_j - L_{sj}2\pi| \leqslant \frac{\pi}{(2+\varepsilon)M} \qquad (j = 1, 2, \ldots, k;\ s = 1, \ldots;\ \varepsilon > 0), \tag{9}$$

where the L_{sj} are integers.

Choose s_0 so that $\sigma\lambda_{s_0} > \mu$ and put $q = \sigma\lambda_{s_0}$. We are now ready to show that the zeros of f lie in "suitable" sectors: In (9) take $s = s_0$, multiply by $|n_{l_j}|$ and sum over j from 1 to k. In view of (7) and (8) we get

$$\left|\omega_l - \frac{\Delta_{hl}2\pi}{q}\right| \leqslant \frac{\pi}{(2+\varepsilon)q} \qquad (l = k+1, k+2, \ldots, m), \tag{10}$$

where the Δ's are integers.

By (8) and (9), it is clear that (10) holds also for $l = 1, 2, \ldots, k$, with $\Delta_{hl} = \sigma L_{hl}$. Hence we have

$$\left|\omega_l - \frac{\Delta_{hl}2\pi}{q}\right| \leqslant \pi/(2+\varepsilon)q \qquad (l = 1, 2, \ldots, m;\ q > \mu,\ h = s_0). \tag{11}$$

To continue we write $\log|f(re^{i\theta})| = \Sigma_{m=-\infty}^{\infty} c_m(r)e^{im\theta}$. Then we have [5, p. 379]

$$c_m(r) = -\frac{1}{2m}\left\{ \sum_{r<r_k<R} (r/z_k)^m + \sum_{r_k<r} (\bar{z}_k/r)^m \right\} + (r/R)^m O(T(2R)), \quad (12)$$

where $\{z_k\}$ are the zeros of f and $r_k = |z_k|$.

In (12) we put $m = q$. Since $q > \mu$, the last term in (12) will tend to zero as $R \to \infty$ through a suitable sequence $\{R_n\}$. It follows that $\Sigma_{r<r_k<R} z_k^{-q}$ tends to a limit as R $(= R_n)$ tends to infinity. If we write $z_k = r_k e^{i\theta_k}$, it follows that $\mathrm{Re}\{\Sigma_{r<r_k<R} z_k^{-q}\} = \Sigma_{r<r_k<R} r_k^{-q} \cos(q\theta_k)$ tends to a limit as R $(= R_n)$ tends to infinity. Since the arguments θ_k satisfy (11) we have $\cos(q\pi/(2+\varepsilon)q)\Sigma_{r<r_k<R} r_k^{-q} < \Sigma_{r<r_k<R} r_k^{-q}\cos(q\theta_k)$. It follows that $\Sigma_{r<r_k<R} r_k^{-q}$ is bounded as R $(= R_n)$ tends to infinity, and being an increasing function of R, it will have a limit as $R \to \infty$ unrestricted. Thus Σr_k^{-q} converges and so, the exponent of convergence of the zeros of f is $< q$. By Whittaker's result, $\lambda < \max(\mu, q)$.

When the zeros of f all lie on a ray, we may choose $q = [\mu] + 1$. Using this in (22) we obtain $\rho < [\mu] + 1$ from which follows that $\lambda < [\mu] + 1$. This completes the proof of Theorem 1.

PROOF OF THEOREM B. Let f be an entire function whose zeros satisfy (3) and (4) for some integer s (> 1). If $\liminf_{r\to\infty} r^{-s}T(r, f) = \infty$ then $\lim_{r\to\infty} r^{-s}T(r, f) = \infty$. Suppose then that $\liminf_{r\to\infty} r^{-s}T(r, f) < +\infty$. Then the lower order μ of f is finite and $\mu < s$. In (12), take $m = s$ and let R tend to infinity through a sequence R_n such that $R_n^{-s}T(R_n, f)$ tends to a finite limit. By taking subsequences if necessary and repeating the same arguments after (12), we conclude as before, that $\Sigma|a_j|^{-s} < +\infty$. It follows that f is of finite order $\lambda < s$. Thus we may write $f(z) = e^{Q(z)}P(z)$ where Q is a polynomial of degree $d < s$ and P is a Weierstrass product of genus $s - 1$. Since for such products P, even when not canonical, $T(r, P) = o(r^s)$ as $r \to \infty$, we conclude by the elements of the theory that $\lim_{r\to\infty} r^{-s}T(r, f)$ exists and is $< +\infty$. This completes the proof of Theorem B.

REMARK. The possibility that $T(r, f) < T(r, F)$ for a set that contains arbitrarily large values of r is further supported by the following: Let

$$f(z) = e^{p(z)} \prod_{n=1}^{\infty} e(z/z_n, q) \quad \text{and} \quad F(z) = e^{P(z)} \prod_{n=1}^{\infty} E(z/-|z_n|, q)$$

be two entire functions, with $p(z) = a_0 + a_1 z + \cdots + a_q z^q$, and $P(z) = |a_0| + \cdots + |a_q|z^q$ and $q =$ the greatest integer less than or equal to the order λ of f which we assume *finite*. Then we have [5, p. 380]

$$|c_m(r; f)| < |c_m(r; F)| < 2T(r, F) - N\left(r, \frac{1}{F}\right), \quad \text{for all } m. \quad (13)$$

In the proof of the approximation lemma of Edrei and Fuchs [2, p. 312] we apply inequality (13) in place of their inequality (8.8). The result is that in the error term appearing in their lemma, we may replace $T(r, f)$ by $T(r, F)$.

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