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The p-optimal martingale measure and its asymptotic relation with the minimal-entropy martingale measure

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We prove convergence of the p-optimal martingale measures to the minimal-entropy martingale measure for $p \to 1$. This is done for bounded stochastic processes in a discrete-time setting with a finite horizon. We also investigate in detail an example of an unbounded process, where we do not find this convergence.

Keywords: entropy; martingale measures

1. Introduction

In recent years the problem of finding martingale measures for a stochastic process has found applications in the field of mathematical finance, e.g. the famous Black–Scholes formula for evaluating a European call option can be seen as the expectational value of a random variable with respect to the (in this case unique) martingale measure for the discounted stock price process. In general there is *no unique* martingale measure for a stochastic process. So one is confronted with the problem of choosing a proper martingale measure. Very popular possibilities are the so-called minimal martingale measure, which has been introduced by Föllmer and Schweizer (1991), or the variance-optimal measure (Schweizer 1995; Delbaen and Schachermayer 1996; Delbaen *et al*. 1997). The latter is characterized by minimizing the L^2 norm of the Radon–Nikodym derivative of the new measure with respect to the original measure among all signed martingale measures for the process. The former exhibits this feature locally (for a more exact description see Föllmer and Schweizer (1991)). Another possibility is the minimal-entropy martingale measure. It has been shown by Frittelli (1996) that for a bounded process a unique martingale measure, which minimizes relative entropy between the original measure and the martingale measure, always exists. In addition, if the relative entropy is finite, the two measures are equivalent. For an economic interpretation of the variance-optimal and minimal-entropy measures see Delbaen *et al*. (1997) and see Frittelli (1996) and Platen and Rebolledo (1995) respectively.

The aim of this paper is to find a connection between these two concepts in discrete time with a finite horizon. It turns out that the missing link is given by martingale measures, which we call p optimal and which are characterized by minimizing the L^p norm instead of the L^2 norm. For the role of the p-optimal measures in connection with the closedness of the space of stochastic integrals see Grandits and Krawczyk (1996). We prove that for bounded processes the p-optimal measures converge to the minimal-entropy measure in $L^1(P)$, if p tends to 1. As the minimal-entropy measure is always positive, the p-optimal measures (for $p-1$ small enough) do not share the drawback of the variance-optimal measure, namely that the price of a positive contingent claim is sometimes negative, if determined in the variance-optimal framework.

For unbounded processes, Frittelli and Lakner (1996) have already given an example, where the minimal-entropy measure does not exist. We give an example of an unbounded one-step process, where the minimal-entropy martingale measure does exist but, depending on the expectational value of the process, the above convergence appears or not.

2. Preliminary results

We consider in this paper a stochastic process $(S_k)_{k=0}^T$ on the stochastic basis $(\Omega, \mathscr{F}, (\mathscr{F}_k)_{k=0}^T, P)$ in discrete time, which is adapted, $\mathbb{R}^d$ valued and bounded. Using the notation

$$\mathscr{M}^e(S) \equiv \{Q | Q \text{ is a probability measure, } Q \sim P \text{ and } S \text{ is a } Q \text{ martingale}\}$$

for all equivalent martingale measures for S, we assume that $\mathscr{M}^e(S) \neq \emptyset$, and that $\mathscr{F}_0$ is trivial. Whenever we use the process S, these assumptions above should hold, unless something else is indicated.

By the famous Dalang–Morton–Willinger theorem (Theorem (2.6) of Dalang *et al.* (1990)) the existence of an equivalent martingale measure for S is equivalent to the *no-arbitrage condition* (NA) (the measure may be chosen s.t. the density is uniformly bounded (Schachermayer 1992, Theorem 1.1)). (NA) can be formulated in the following way: for $k = 1, \ldots, T$ and each $\mathscr{F}_{k-1}$-measurable bounded $\mathbb{R}^d$-valued function h s.t.

$$(h(\omega), S_k(\omega) - S_{k-1}(\omega)) \geq 0, \quad P \text{ a.s.}$$

we have

$$(h(\omega), S_k(\omega) - S_{k-1}(\omega)) = 0, \quad P \text{ a.s.},$$

where (., .) denotes the inner product in $\mathbb{R}^d$. One can replace $(h, \Delta S_k)$ by $(H \cdot S)_T$, where H is a predictable process, and $\cdot$ denotes d-dimensional (discrete) stochastic integration.

The notion of entropy and minimal-entropy martingale measure are introduced in the following definitions.

Definition 2.1. *Let Q and P be two probability measures on $(\Omega, \mathscr{F})$; then the relative entropy $I(Q, P)$ of Q with respect to P is given by*

$$I(Q, P) = \begin{cases} \int \frac{\mathrm{d}Q}{\mathrm{d}P} \ln\left(\frac{\mathrm{d}Q}{\mathrm{d}P}\right) \mathrm{d}P & \text{if } Q \ll P, \\ +\infty & \text{otherwise.} \end{cases}$$

Definition 2.2. *Q^E is the solution of the following minimum problem for Q:*

$$\mathrm{E}_Q[S_k|\mathscr{F}_{k-1}] = S_{k-1}, \quad k = 1, 2, \ldots, T,$$

$$\mathrm{E}_P\left[\frac{\mathrm{d}Q}{\mathrm{d}P}\right] = 1,$$

$$I(Q, P) \to \textit{minimum}.$$

By Theorem 7 and 11 of Frittelli (1996) and our assumptions on S this problem has a unique solution of the form

$$Z_T^E = \frac{\mathrm{d}Q^E}{\mathrm{d}P} = c\,\mathrm{e}^f,$$

where $f = (H \cdot S)_T$. H denotes a predictable process, and c is a normalizing constant.

Remark 2.1. If $Z_T = c\,\mathrm{e}^f$, where $f = (H \cdot S)_T$, is the density of a martingale measure for S, then c and f are uniquely determined by Proposition 9 of Frittelli (1996). This holds in contrast with H_k, which are *not* unique on sets, where the support of the on $\mathscr{F}_{k-1}$ conditioned law of ΔS_k is not $\mathbb{R}^d$.

Finally we give the concept of p-optimal martingale measures. In order to do this, we need the following definition:

$\mathscr{M}^s(S) \equiv$

$$\left\{Q|Q \text{ is a signed measure, } Q \ll P,\ S \text{ is a } Q \text{ martingale, } \frac{\mathrm{d}Q}{\mathrm{d}P} \in L^1(P) \text{ and } \mathrm{E}\left(\frac{\mathrm{d}Q}{\mathrm{d}P}\right) = 1\right\},$$

and we call it the set of *signed* martingale measures for S. Since S is bounded, $\mathscr{M}^s(S)$ is closed with respect to $\|\cdot\|_{L^1(P)}$ (we identify measures with densities here).

Definition 2.3. *For $1 < p < \infty$, $Q^p \in \mathscr{M}^s(S)$ is the solution of the minimum problem (for $p = 2$ compare Delbaen and Schachermayer (1996) or Schweizer (1995))*

$$\mathrm{E}_Q[S_k|\mathscr{F}_{k-1}] = S_{k-1}, \qquad k = 1, 2, \ldots, T,$$

$$\mathrm{E}\left[\frac{\mathrm{d}Q}{\mathrm{d}P}\right] = 1,$$

$$\mathrm{E}\left[\left|\frac{\mathrm{d}Q}{\mathrm{d}P}\right|^p\right] \to \textit{minimum}.$$

Note that all expectations, where we do not indicate the measure, are taken with respect to the original measure P.

There exists a unique solution of the minimum problem because firstly, by the Dalang–Morton–Willinger theorem, we always have even a positive martingale measure with bounded density for S and therefore one in L^p, secondly $\mathscr{M}^s(S)$ is closed with respect to

$\|\cdot\|_{L^1(P)}$ and finally the spaces L^p are uniformly convex. However, note that in general the p-optimal measures are only signed measures.

Remark 2.2. In the sequel we use the function $n(p) \equiv 1/(p-1)$ in order to avoid too complicated notation, and we even drop the argument of the function n, if the meaning is clear.

Before we give an explicit formula for the density of the p-optimal measures, we need the concept of alignment (Luenberger 1969).

Definition 2.4. *Let F be a Banach space. Then two vectors $x \in F$, $x^* \in F^*$ are aligned, if $\langle x, x^* \rangle = \|x\|\|x^*\|$ holds. For L^p spaces this means equality in the Hölder inequality.*

Lemma 2.1. *The density of the p-optimal martingale measure $Z_T(p)$ $(1 < p < \infty)$ for S is aligned to $(1 + f_p)$, i.e.*

$$Z_T(p) = C_p \operatorname{sgn}(1 + f_p)|1 + f_p|^{n(p)},$$

where $f_p \in \overline{\mathcal{G}^q_T}$, $\mathcal{G}^q_T = \{(H \cdot S)_T \cap L^q(P),\ H$ predictable$\}$, the closure is understood in the sense of L^q (q conjugate to p) and C_p is a normalizing constant.

The proof is standard in the theory of minimum norm problems (Luenberger 1969, Theorem 5.8.1).

Remark 2.3. Note that, if Z_T is given by the formula above and if it is a martingale measure for S, then it is the p-optimal measure by Corollary 5.8.1 of Luenberger (1969).

As a corollary we give another form of the density, which is more convenient in the limiting process $p \to 1$.

Corollary 2.1. *The density of the p-optimal martingale measure $Z_T(p)$ $(1 < p < \infty)$ for S can also be written as*

$$Z_T(p) = C_p \operatorname{sgn}\left(1 + \frac{f_p}{n(p)}\right)\left|1 + \frac{f_p}{n(p)}\right|^{n(p)},$$

where $f_p \in \mathcal{G}^q_T$.

Proof. The additional n is of course a matter of taste, which makes life easier, when going with p to 1. As the space of stochastic integrals is already closed with respect to the topology of convergence in measure (a proof of this result is given in Appendix 1), we can skip the symbol for closing the space of stochastic integrals. □

In the proof of our main result we need a further formula for $Z_T(p)$, which we give in the next lemma.

Lemma 2.2. *There exists a predictable $\mathbb{R}^d$-valued process β_p, s.t. the density of the p-optimal martingale measure for S is given by*

$$Z_T(p) = C_p \prod_{k=1}^{T} \left| 1 + \frac{(\beta_{p,k}, \Delta S_k)}{n(p)} \right|^{n(p)} \operatorname{sgn}\left(1 + \frac{(\beta_{p,k}, \Delta S_k)}{n(p)} \right).$$

We also have P a.s.

$$\begin{aligned} 0 < \mathrm{E}&\left[\prod_{r=k}^{T} \left| 1 + \frac{(\beta_{p,r}, \Delta S_r)}{n(p)} \right|^{n(p)} \operatorname{sgn}\left(1 + \frac{(\beta_{p,r}, \Delta S_r)}{n(p)} \right) | \mathscr{F}_{k-1} \right] \\ &= \mathrm{E}\left[\prod_{r=k}^{T} \left| 1 + \frac{(\beta_{p,r}, \Delta S_r)}{n(p)} \right|^{n(p)+1} | \mathscr{F}_{k-1} \right] \\ &\leqslant 1 \end{aligned} \tag{1}$$

for $k = 1, \ldots, T$.

Proof. To simplify the notation we shall fix p in the proof of this lemma and skip therefore the index p, i.e. we shall write C for C_p, β_r for $\beta_{p,r}$ and n for $n(p)$.

The proof is by induction, and we start with the construction of the process β. Setting $-|1 + x/n|^{n+1}$ for $U(x)$ in the proof of Theorem 1 of Rogers (1994), we infer that there exists an $\mathscr{F}_{T-1}$-*measurable* function β_T, which minimizes

$$\mathrm{E}\left[\left| 1 + \frac{(\beta_T, \Delta S_T)}{n} \right|^{n+1} | \mathscr{F}_{T-1} \right].$$

Note that $-|1 + x/n|^{n+1}$ is not strictly increasing as demanded by Rogers (1994), but the proof there works for our U as well. The equation

$$\mathrm{E}\left[\left| 1 + \frac{(\beta_T, \Delta S_T)}{n} \right|^{n} \operatorname{sgn}\left(1 + \frac{(\beta_T, \Delta S_T)}{n} \right) \Delta S_T | \mathscr{F}_{T-1} \right] = 0$$

holds by (2.11) of Rogers (1994). Note that β_T is not uniquely determined, but $(\beta_T, \Delta S_T)$ is (see also Lemma 3.1 below).

We show now (1) for $k = T$:

$$\begin{aligned} \mathrm{E}&\left[\left| 1 + \frac{(\beta_T, \Delta S_T)}{n} \right|^{n+1} | \mathscr{F}_{T-1} \right] \\ &= \mathrm{E}\left[\left| 1 + \frac{(\beta_T, \Delta S_T)}{n} \right|^{n} \operatorname{sgn}\left(1 + \frac{(\beta_T, \Delta S_T)}{n} \right) \left(1 + \frac{(\beta_T, \Delta S_T)}{n} \right) | \mathscr{F}_{T-1} \right] \\ &= \mathrm{E}\left[\left| 1 + \frac{(\beta_T, \Delta S_T)}{n} \right|^{n} \operatorname{sgn}\left(1 + \frac{(\beta_T, \Delta S_T)}{n} \right) | \mathscr{F}_{T-1} \right], \end{aligned}$$

where we have used the predictability of β. As the middle terms in (1), evaluated for $\beta_T \equiv 0$, are equal to 1, and β_T is the minimizer, the right-hand side of (1) is clear. Assuming the contrary for the left-hand side, namely the existence of a set $A \in \mathscr{F}_{T-1}$ with $P(A) > 0$, s.t. $\mathrm{E}[|1 + (\beta_T, \Delta S_T)/n|^{n+1} | \mathscr{F}_{T-1}] = 0$ holds on A, yields $(\beta_T, \Delta S_T) = -n$ on A. This is impossible, since $E_Q[\Delta S_T | \mathscr{F}_{T-1}] = 0$ should hold for some equivalent martingale measure Q by our (NA) assumption, concluding our proof for $k = T$. We proceed with the induction and start again to construct β_k. β_k is defined as the solution of the extremal problem

$$\min_{\beta_k} E\left[\left|1 + \frac{(\beta_k, \Delta S_k)}{n}\right|^{n+1} \prod_{r=k+1}^{T} \left|1 + \frac{(\beta_r, \Delta S_r)}{n}\right|^{n+1} | \mathscr{F}_{k-1}\right].$$

Because $\prod_{r=k+1}^{T} |1 + (\beta_r, \Delta S_r)/n|^{n+1}$ is never identically equal to zero on $\mathscr{F}_k$-measurable sets with positive measure by the induction assumption, the existence of an $\mathscr{F}_{k-1}$-*measurable* solution β_k can be verified as in the case $k = T$. We also get the validity of

$$\mathrm{E}\left[\left|1 + \frac{(\beta_k, \Delta S_k)}{n}\right|^{n} \operatorname{sgn}\left(1 + \frac{(\beta_k, \Delta S_k)}{n}\right) \Delta S_k \prod_{r=k+1}^{T} \left|1 + \frac{(\beta_r, \Delta S_r)}{n}\right|^{n+1} | \mathscr{F}_{k-1}\right] = 0,$$

which can be written as

$$\mathrm{E}\left[\left|1 + \frac{(\beta_k, \Delta S_k)}{n}\right|^{n} \operatorname{sgn}\left(1 + \frac{(\beta_k, \Delta S_k)}{n}\right) \Delta S_k \,\mathrm{E}\left[\prod_{r=k+1}^{T} \left|1 + \frac{(\beta_r, \Delta S_r)}{n}\right|^{n+1} | \mathscr{F}_k\right] | \mathscr{F}_{k-1}\right] = 0 \tag{2}$$

or

$$\mathrm{E}\left[\Delta S_k \prod_{r=k}^{T} \left|1 + \frac{(\beta_r, \Delta S_r)}{n}\right|^{n} \operatorname{sgn}\left(1 + \frac{(\beta_r, \Delta S_r)}{n}\right) | \mathscr{F}_{k-1}\right] = 0.$$

It remains to prove (1). Using (2) and the induction assumption we get

$$\begin{aligned}
&\mathrm{E}\left[\prod_{r=k}^{T} \left|1 + \frac{(\beta_r, \Delta S_r)}{n}\right|^{n+1} | \mathscr{F}_{k-1}\right] \\
&\quad = \mathrm{E}\left[\left|1 + \frac{(\beta_k, \Delta S_k)}{n}\right|^{n} \operatorname{sgn}\left(1 + \frac{(\beta_k, \Delta S_k)}{n}\right)\left(1 + \frac{(\beta_k, \Delta S_k)}{n}\right)\right. \\
&\qquad \left.\times \mathrm{E}\left[\prod_{r=k+1}^{T} \left|1 + \frac{(\beta_r, \Delta S_r)}{n}\right|^{n+1} | \mathscr{F}_k\right] | \mathscr{F}_{k-1}\right] \\
&\quad = \mathrm{E}\left[\left|1 + \frac{(\beta_k, \Delta S_k)}{n}\right|^{n} \operatorname{sgn}\left(1 + \frac{(\beta_k, \Delta S_k)}{n}\right) \mathrm{E}\left[\prod_{r=k+1}^{T} \left|1 + \frac{(\beta_r, \Delta S_r)}{n}\right|^{n+1} | \mathscr{F}_k\right] | \mathscr{F}_{k-1}\right] \\
&\quad = \mathrm{E}\left[\left|1 + \frac{(\beta_k, \Delta S_k)}{n}\right|^{n} \operatorname{sgn}\left(1 + \frac{(\beta_k, \Delta S_k)}{n}\right)\right.
\end{aligned}$$

$$\times \mathrm{E}\left[\prod_{r=k+1}^{T}\left|1+\frac{(\beta_r, \Delta S_r)}{n}\right|^n \operatorname{sgn}\left(1+\frac{(\beta_r, \Delta S_r)}{n}\right)|\mathscr{F}_k\right]|\mathscr{F}_{k-1}\Bigg]$$

$$= \mathrm{E}\left[\prod_{r=k}^{T}\left|1+\frac{(\beta_r, \Delta S_r)}{n}\right|^n \operatorname{sgn}\left(1+\frac{(\beta_r, \Delta S_r)}{n}\right)|\mathscr{F}_{k-1}\right].$$

The inequalities in (1) are proven in completely the same way as for $k = T$.

C is a normalizing constant and, as $Z_T(p)$ is the density of a martingale measure for S by construction, and because we may write it as $C|1+f|^n \operatorname{sgn}(1+f)$ with $f \in \mathscr{G}^q_T$, it is the p-optimal martingale measure by Remark 2.3. □

Remark 2.4. A similar result has been given by Schweizer (1995) for $p = 2$. In this case, one can give even explicit formulae for $\beta_{2,k}$.

For our next preparatory result we need some further notation (Bennet and Sharpley 1988).

Definition 2.5. *$L \log L$ consists of all P-measurable real functions f for which*

$$\int |f| \ln^+ |f| \, \mathrm{d}P < \infty$$

(here $\ln^+ x = \max(\ln x, 0)$).

Proposition 2.1. *For the densities $Z_T(p)$ of the p-optimal martingale measures for S we have*

$$(Z_T(p))_{1<p<\infty} \text{ is bounded in } L \log L.$$

Proof. We assume that P is *not* a martingale measure for S. This assumption is justified because, if P is a martingale measure for S, we have $Z_T(p) = 1$ for all p and the assertion is trivial.

In view of the form of $Z_T(p)$ in Corollary 2.1 we define the sets

$$A_p = \{f_p \leqslant -n\},$$

$$B_p = \{-n < f_p \leqslant 0\},$$

$$D_p = \{f_p > 0\}.$$

First of all we need an estimate for $|\int_{B_p}(1 + f_p/n)^n f_p \, \mathrm{d}P|$. Noting that the function $(1 - x/n)^n x$ in the interval $[0, n]$ can be estimated above by $1/e$, gives

$$\left|\int_{B_p}\left(1+\frac{f_p}{n}\right)^n f_p \, \mathrm{d}P\right| \leqslant \frac{1}{e}.$$

Because of the relations

$$\operatorname{sgn}\left\{\int_{A_p}\left(1+\frac{f_p}{n}\right)f_p\,\mathrm{d}P\right\} = \operatorname{sgn}\left\{\int_{D_p}\left(1+\frac{f_p}{n}\right)f_p\,\mathrm{d}P\right\} = -\operatorname{sgn}\left\{\int_{B_p}\left(1+\frac{f_p}{n}\right)f_p\,\mathrm{d}P\right\},$$

$$\int_\Omega \operatorname{sgn}\left(1+\frac{f_p}{n}\right)\left|1+\frac{f_p}{n}\right|^n f_p\,\mathrm{d}P = 0,$$

we conclude that

$$\int_\Omega \left|1+\frac{f_p}{n}\right|^n |f_p|\,\mathrm{d}P \leqslant \frac{2}{e} \tag{3}$$

holds. In the following estimates we denote by M positive constants, which do not depend on p, but which are not necessarily identical:

$$\int_{D_p}\left(1+\frac{f_p}{n}\right)^n \ln^+\left(1+\frac{f_p}{n}\right)^n \mathrm{d}P \leqslant \int_{D_p}\left(1+\frac{f_p}{n}\right)^n f_p\,\mathrm{d}P \leqslant M,$$

where we have used $\alpha - n\ln(1+\alpha/n) > 0$ for $\alpha > 0$ and (3). Using, instead of this relation, $\alpha - n\ln(\alpha/n - 1) > 0$ for $\alpha \geqslant 2n$, we get

$$\int_{A_p}\left|1+\frac{f_p}{n}\right|^n \ln^+\left|1+\frac{f_p}{n}\right|^n \mathrm{d}P \leqslant M$$

in the same way, and we end up with

$$\int_\Omega\left|1+\frac{f_p}{n}\right|^n \ln^+\left|1+\frac{f_p}{n}\right|^n \mathrm{d}P \leqslant M. \tag{4}$$

It remains to show that $|C_p| \leqslant M$ holds.

We claim that

$$\exists\delta > 0 \ s.t. \int_{D_p} f_p\,\mathrm{d}P \geqslant \delta \qquad \forall p. \tag{5}$$

Assuming the contrary, we get a sequence $\{p_k\}_{k=1}^\infty$ s.t. $\int_{D_k} f_k\,\mathrm{d}P \to 0$ for $k\to\infty$ with the obvious meaning of D_k and f_k. We claim that this in turn implies that

$$f_k^- \to 0 \qquad \text{in probability for } k\to\infty. \tag{6}$$

If (6) were false, we could extract a subsequence, which we denote again by f_k, s.t. $P[f_k^- > \alpha] > \alpha > 0$ holds for all k and some α. By Lemma A.1.1 of Delbaen and Schachermayer (1994) we can now find that $g_k \in \operatorname{conv}(f_k^-, f_{k+1}^-, \ldots)$ s.t. $g_k \to g$ in probability with $P[g > 0] > 0$. Applying the same convex combinations to f_k^+, we get a sequence $h_k \to 0$ in the norm of L^1 and therefore in probability. Since $g_k - h_k$ are elements of the space of stochastic integrals $\mathscr{G}_T \equiv \{(H\cdot S)_T | H \text{ predictable}\}$, and $\mathscr{G}_T$ is closed with respect to the topology induced by convergence in probability (see Proposition A1.1 in Appendix 1), we get a contradiction to our (NA) assumption. Therefore our claim (6) is true.

After a further extraction of a subsequence we get $f_k \to 0$ P a.s. and therefore

$$\operatorname{sgn}\left(1+\frac{f_k}{n(p_k)}\right)\left|1+\frac{f_k}{n(p_k)}\right|^{n(p_k)} \to 1 \quad P\ a.s.$$

As we have already shown the boundedness of $|1 + f_k/n(p_k)|^{n(p_k)}$ in $L \log L$,

$$\operatorname{sgn}\left(1+\frac{f_k}{n(p_k)}\right)\left|1+\frac{f_k}{n(p_k)}\right|^{n(p_k)} \to 1 \qquad \text{for } k \to \infty$$

holds with respect to the norm of L^1. We finally get

$$\|Z_T(p_k) - 1\|_{L^1} \to 0 \qquad \text{for } k \to \infty,$$

which is a contradiction, because the space of martingale measures for our *bounded* S is closed with respect to the L^1 norm, and the constant function 1 is not a density of a martingale measure for S under our assumptions. We conclude that (5) is valid.

Now on the one hand we have

$$\int\left|1+\frac{f_p}{n}\right|^n \mathrm{d}P \geqslant \int_{D_p}\left(1+\frac{f_p}{n}\right)^n \mathrm{d}P \geqslant \int_{D_p}(1+f_p) \geqslant \delta,$$

but on the other hand

$$C_p\int\left|1+\frac{f_p}{n}\right|^n \mathrm{d}P = \int |Z_T(p)|\,\mathrm{d}P \leqslant M$$

holds true. The last inequality follows from

$$\|Z_T(p)\|_{L^1} \leqslant \|Z_T(p)\|_{L^p} \leqslant M,$$

where the first inequality is trivial, and the second follows from the fact that we have always a martingale measure for S with bounded density. So we end up with $C_p \leqslant M$ for all p and our proof is complete. □

3. Main results

The aim of this section is to prove that the p-optimal measures converge to the minimal-entropy martingale measure with respect to the norm of $L^1(P)$ under our assumptions for the process S. In order to do this, we need some concepts, which have been developed by Schachermayer (1992). These concepts are necessary to prove our results for the $\mathbb{R}^d$-valued case, which is slightly more technical then the real-valued case.

Definition 3.1. *Let $\mathscr{G} \subset \mathscr{F}$ be two σ-algebras on the probability space (Ω, P). Let Y be an $\mathbb{R}^d$-valued bounded $\mathscr{F}$-measurable random variable. Then we define the following subspaces of $L^0(\Omega, \mathscr{G}, P; \mathbb{R}^d)$:*

$$N(Y) = \{k \in L^0(\Omega, \mathscr{G}, P; \mathbb{R}^d)\colon (k(\omega), Y(\omega)) = 0 \quad P \text{ a.s.}\},$$

$$N^{\perp}(Y) = \{h \in L^0(\Omega, \mathscr{G}, P; \mathbb{R}^d)\colon (k(\omega), h(\omega)) = 0 \quad P \text{ a.s. for each } k \in N(Y)\}.$$

For an interpretation of this definition and for a proof of the following lemma we refer to Lemma 2.4 of Schachermayer (1992).

Lemma 3.1. *There is a continuous surjective projection*

$$\pi\colon L^0(\Omega, \mathscr{G}, P; \mathbb{R}^d) \to N(Y)^\perp$$

with $\ker(\pi) = N(Y)$. *In other words,*

$$L^0(\Omega, \mathscr{G}, P; \mathbb{R}^d) = N(Y) \oplus N(Y)^\perp.$$

We then have, for each $h \in L^0(\Omega, \mathscr{G}, P; \mathbb{R}^d)$,

$$(h(\omega), Y(\omega)) = (\pi(h)(\omega), Y(\omega)) \quad P \ a.s.$$

The first step in the proof of our main theorem is to find arbitrary large sets, on which the p-optimal measures are bounded. We need the following lemmata.

Lemma 3.2. *Let* $\mathscr{G} \subset \mathscr{F}$ *be two* σ-*algebras on the probability space* (Ω, P). *Let Y be an* $\mathbb{R}^d$-*valued bounded* $\mathscr{F}$-*measurable random variable which (seen as an one-step process with the filtration* $\mathscr{F}_0 \equiv \mathscr{G}$ *and* $\mathscr{F}_1 \equiv \mathscr{F}$*) satisfies* (NA). *Then*

$$g 1_A > 0 \quad P \ a.s.$$

holds, where g is defined by

$$g(\omega) = \inf_{h \in N^\perp, \|h(\omega)\| \equiv 1} \mathrm{E}[(h, Y)^+ | \mathscr{G}]$$

and A by

$$A \equiv \{\omega | \mathrm{E}[\|Y\| | \mathscr{G}] > 0\}.$$

$\|\cdot\|$ *denotes the maximum norm in* $\mathbb{R}^d$.

Proof. Assuming the contrary, namely the existence of a set $B \in \mathscr{G}$ with $B \subset A$ and $P(B) > 0$, s.t. $g 1_B = 0$ holds, yields the existence of a sequence $\{h_k\}_{k=1}^\infty \in N^\perp$ with $\|h_k(\omega)\| \equiv 1$, s.t.

$$g_k 1_B \leqslant \frac{1}{k} 1_B \quad P \text{ a.s.},$$

where g_k denotes $\mathrm{E}[(h_k, Y)^+ | \mathscr{G}]$. Taking the expectational value of this inequality gives

$$\mathrm{E}[(h_k, Y)^+ 1_B] \leqslant \frac{1}{k} P[B] \quad \forall k$$

or

$$\mathrm{E}[(\tilde{h}_k, Y)^+] \leqslant \frac{1}{k} P[B] \quad \forall k,$$

where we have defined $\tilde{h}_k = h_k 1_B$. This is a contradiction to Lemma 2.5. of Schachermayer (1992). □

Lemma 3.3. *Let* $\{p_i\}_{i=1}^{i=\infty}$ *be a sequence with* $\lim_{i\to\infty} p_i = 1$. *Then for the densities of the* p_i-*optimal martingale measures* $Z_T(p_i)$

$$|Z_T(p_i)| \leqslant M(\omega) \qquad P\ a.s.$$

holds $\forall i \in \mathbb{N}$, *and for some positive* $\mathscr{F}_T$-*measurable function* M.

Proof. By Lemma 2.2, $Z_T(p_i)$ has the form

$$Z_T(p_i) = C_i \prod_{k=1}^{T} \left| 1 + \frac{(\beta_{i,k},\, \Delta S_k)}{n(p_i)} \right|^{n(p_i)} \operatorname{sgn}\left(1 + \frac{(\beta_{i,k},\, \Delta S_k)}{n(p_i)} \right).$$

C_i are normalizing constants, which are bounded as has been shown in the second part of the proof of Proposition 2.1. Note that we write $\beta_{i,k}$ for $\beta_{p_i,k}$ and C_i for C_{p_i}. The proof is by induction.

(a) We claim the existence of an $\mathscr{F}_{T-1}$-measurable positive function m_T s.t.

$$\|\pi(\beta_{i,T})\| \leqslant m_T(\omega) \quad P\ a.s. \quad \forall i \in \mathbb{N},$$

holds, where π is the projection introduced in Lemma 3.1 for the random variable ΔS_T and the σ-algebras $\mathscr{F}_{T-1}$, $\mathscr{F}_T$.

In the sequel we denote $\pi(\beta_{i,k})$ by $\pi_{i,k}$. Using Lemma 2.2, the definitions $\Omega^+_{i,k} \equiv \{\omega | (\pi_{i,k},\, \Delta S_k) \geqslant 0\}$, $\hat{\pi}_{i,k} \equiv \|\pi_{i,k}\|$ and $e_{i,k} \equiv (\pi_{i,k}/\hat{\pi}_{i,k}) 1_{\{\hat{\pi}_{i,k} > 0\}}$, we get

$$\begin{aligned}
1 &\geqslant \mathrm{E}\left[\left| 1 + \frac{(\pi_{i,T},\, \Delta S_T)}{n(p_i)} \right|^{n(p_i)+1} | \mathscr{F}_{T-1} \right] \\
&\geqslant \mathrm{E}\left[\left| 1 + \frac{(\pi_{i,T},\, \Delta S_T)^+}{n(p_i)} \right|^{n(p_i)+1} 1_{\Omega^+_{i,T}} | \mathscr{F}_{T-1} \right] \\
&\geqslant \mathrm{E}\left[\left(1 + \frac{(\pi_{i,T},\, \Delta S_T)^+}{n(p_i)} \right)^{n(p_i)+1} | \mathscr{F}_{T-1} \right] - 1 \\
&\geqslant \mathrm{E}[(\pi_{i,T},\, \Delta S_T)^+ | \mathscr{F}_{T-1}] \\
&= \hat{\pi}_{i,T} \mathrm{E}[(e_{i,T},\, \Delta S_T)^+ | \mathscr{F}_{T-1}].
\end{aligned}$$

Now the last term is equal to $\hat{\pi}_{i,T} g_T(\omega)$, where g_T is strictly positive on $A_{i,T} \equiv \{\omega | \mathrm{E}[\|\Delta S_T\| | \mathscr{F}_{T-1}] > 0\} \cap \{\omega | \hat{\pi}_{i,T} > 0\}$ by Lemma 3.2, and we end up with

$$\hat{\pi}_{i,T} \leqslant m_T \equiv g_T^{-1} \quad P \text{ a.s. on } A_{i,T} \qquad \forall i \in \mathbb{N}.$$

On $(A_{i,T})^c$ we define $\pi_{i,T} \equiv 0$, finishing the case $k = T$.

(b) We claim the existence of an $\mathscr{F}_{k-1}$-measurable positive function m_k s.t.

$$\|\pi_{i,k}\| \leqslant m_k(\omega) \quad P \text{ a.s.} \quad \forall i \in \mathbb{N}.$$

Let $l_{i,k}$ be defined by

$$l_{i,k} \equiv \mathrm{E}\left[\prod_{r=k+1}^{T}\left|1+\frac{(\pi_{i,r},\,\Delta S_r)}{n(p_i)}\right|^{n(p_i)+1}|\mathscr{F}_k\right].$$

By Lemma 2.2, $l_{i,k} > 0$ holds P a.s. for all $i \in \mathbb{N}$ and $k = 0, \ldots, T-1$. In addition we have

$$\begin{aligned}
\liminf_i l_{i,k} &= \liminf_i \mathrm{E}\left[\prod_{r=k+1}^{T}\left|1+\frac{(\pi_{i,r},\,\Delta S_r)}{n(p_i)}\right|^{n(p_i)+1}|\mathscr{F}_k\right] \\
&\geqslant \mathrm{E}\left[\liminf_i \prod_{r=k+1}^{T}\left|1+\frac{(\pi_{i,r},\,\Delta S_r)}{n(p_i)}\right|^{n(p_i)+1}|\mathscr{F}_k\right] \\
&\geqslant \mathrm{E}\left[\liminf_i \left|1-\frac{\sum_{r=k+1}^{T} dm_r\|\Delta S_r\|_{L^\infty}}{n(p_i)}\right|^{n(p_i)+1}|\mathscr{F}_k\right] \\
&\geqslant \mathrm{E}[\mathrm{e}^{-2(\sum_{r=k+1}^{T} dm_r\|\Delta S_r\|_{L^\infty})}|\mathscr{F}_k] \\
&> 0 \quad P \text{ a.s.},
\end{aligned}$$

where we have used the induction assumption in the second inequality. Hence $l_k \equiv \inf_i l_{i,k} > 0$ holds P a.s. Using this and Lemma 2.2, we conclude that, similarly as in (a),

$$\begin{aligned}
1 &\geqslant \mathrm{E}\left[\left|1+\frac{(\pi_{i,k},\,\Delta S_k)}{n(p_i)}\right|^{n(p_i)+1} \mathrm{E}\left[\prod_{r=k+1}^{T}\left|1+\frac{(\pi_{i,r},\,\Delta S_r)}{n(p_i)}\right|^{n(p_i)+1}|\mathscr{F}_k\right]|\mathscr{F}_{k-1}\right] \\
&\geqslant \mathrm{E}\left[\left|1+\frac{(\pi_{i,k},\,\Delta S_k)}{n(p_i)}\right|^{n(p_i)+1} l_k|\mathscr{F}_{k-1}\right] \\
&\geqslant \mathrm{E}\left[\left(1+\frac{(\pi_{i,k},\,\Delta S_k)^+}{n(p_i)}\right)^{n(p_i)+1} 1_{\Omega^+_{i,k}} l_k|\mathscr{F}_{k-1}\right] \\
&\geqslant \mathrm{E}\left[\left(1+\frac{(\pi_{i,k},\,\Delta S_k)^+}{n(p_i)}\right)^{n(p_i)+1} l_k|\mathscr{F}_{k-1}\right] - \mathrm{E}[l_k|\mathscr{F}_{k-1}] \\
&\geqslant \mathrm{E}[(\pi_{i,k},\,\Delta S_k)^+ l_k|\mathscr{F}_{k-1}] \\
&= \hat{\pi}_{i,k}\mathrm{E}[(e_{i,k},\,\Delta S_k)^+ l_k|\mathscr{F}_{k-1}]
\end{aligned}$$

holds. Now denote the last term by $\hat{\pi}_{i,k}\overline{g}_k(\omega)$, where $\overline{g}_k$ is strictly positive on $A_{i,k} \equiv \{\omega | E[\|\Delta S_k\||\mathscr{F}_{k-1}] > 0\} \cap \{\omega | \hat{\pi}_{i,k} > 0\}$, and we end up with

$$\hat{\pi}_{i,k} \leqslant m_k \equiv \overline{g}_k^{-1} \quad P \text{ a.s. on } A_{i,k} \qquad \forall i \in \mathbb{N}.$$

On $(A_{i,k})^c$ we define $\pi_{i,k} \equiv 0$, finishing the proof. $\square$

An immediate consequence of the proof is the following.

Corollary 3.1. *Let $\{p_i\}_{i=1}^{i=\infty}$ be a sequence with $\lim_{i\to\infty} p_i = 1$. Then the densities of the p_i-optimal martingale measures $Z_T(p_i)$ can be written as*

$$Z_T(p_i) = C_i \left|1 + \frac{(H_{p_i} \cdot S)_T}{n(p_i)}\right|^{n(p_i)} \operatorname{sgn}\left(1 + \frac{(H_{p_i} \cdot S)_T}{n(p_i)}\right)$$

and

$$|H_{p_i,k}| \leqslant L(\omega) \qquad \text{for } k = 1, \ldots, T,\ i \in \mathbb{N}$$

holds for some positive $\mathscr{F}_{T-1}$-measurable function L.

The next lemma is crucial for the limiting process $p \to 1$.

Lemma 3.4. *Let $(\Omega, \mathscr{F}, P)$ be a probability space. Let $\{r_n\}_{n=1}^{\infty}$ be measurable functions, which are uniformly bounded and let $l \in L^\infty(P)$. Further $r_n \to 0$ in the weak* topology, but* not *in the topology of convergence in probability. Then the following holds true:*

$$\exists \delta > 0 \ s.t. \int e^{l+r_n} r_n \, dP > \delta$$

for a subsequence n_k, which we have again denoted by n, and for $n \geqslant N(\delta)$.

Proof. Defining $dQ/dP = e^l/E[e^l]$ yields that r_n does not converge to 0 in probability with respect to the measure Q. Therefore and because of the *weak** convergence we can find an $\eta > 0$ and an $\epsilon \ll \eta$ ($\epsilon = \eta/100$ will do), s.t.

$$\int |r_n| \, dQ \in [\eta, \eta + \epsilon],$$

$$\left| \int r_n \, dQ \right| < \epsilon$$

hold, after extracting a subsequence and for $n \geqslant N(\epsilon(\eta))$. Hence

$$\int r_n^+ \, dQ \in \left[\frac{\eta - \epsilon}{2}, \frac{\eta}{2} + \epsilon\right],$$

$$\int r_n^- \, dQ \in \left[\frac{\eta - \epsilon}{2}, \frac{\eta}{2} + \epsilon\right],$$

and using the Jensen inequality yields

$$\begin{aligned}\int e^{r_n}\, r_n \,\mathrm{d}Q &= \int e^{r_n^+}\, r_n^+ \,\mathrm{d}Q - \int e^{-r_n^-}\, r_n^- \,\mathrm{d}Q \\ &\geqslant \exp\left(\int r_n^+ \,\mathrm{d}Q\right)\left(\int r_n^+ \,\mathrm{d}Q\right) - \int e^{-r_n^-}\, r_n^- \,\mathrm{d}Q \\ &\geqslant e^{(\eta-\epsilon)/2}\left(\frac{\eta-\epsilon}{2}\right) - \left(\frac{\eta}{2}+\epsilon\right) \\ &\geqslant \frac{\eta^2}{8}.\end{aligned}$$

We finally find that

$$\int e^{l+r_n}\, r_n \,\mathrm{d}P \geqslant \frac{\eta^2}{8}\,\mathrm{E}[e^l] \equiv \delta$$

for $n \geqslant N(\epsilon(\eta))$. □

Our next lemma is of purely technical nature.

Lemma 3.5. *Let* $(\Omega, \mathscr{F}, P)$ *be a probability space and* $\{g_n\}_{n=1}^{\infty}$, $\{h_n\}_{n=1}^{\infty}$ *be real measurable functions with absolute values uniformly bounded by M. Then* $\forall \epsilon > 0\ \exists \overline{N}(\epsilon)$, *s.t.*

$$\left|\int\left(1+\frac{g_n}{n}\right)^n h_n \,\mathrm{d}P - \int e^{g_n}\, h_n \,\mathrm{d}P\right| < \epsilon$$

holds for all $n \geqslant N \equiv \max(\overline{N}, M)$.

Proof. As the $\{g_n\}$ are uniformly bounded, we have

$$\left(1+\frac{g_m}{n}\right)^n \to e^{g_m}$$

for $n \to \infty$ uniformly in m with respect to the norm of L^{∞}. Since the $\{h_n\}$ are uniformly bounded, we conclude that

$$\lim_{n\to\infty}\int\left(1+\frac{g_m}{n}\right)^n h_m \,\mathrm{d}P = \int e^{g_m}\, h_m \,\mathrm{d}P$$

holds uniformly in m. Hence

$$\left|\int\left(1+\frac{g_n}{n}\right)^n h_n \,\mathrm{d}P - \int e^{g_n}\, h_n \,\mathrm{d}P\right| < \epsilon$$

for n large enough, and the proof is complete. □

Returning to our original filtered probability space we can now formulate the following.

Proposition 3.1. *Let* $\{p_i\}_{i=1}^{i=\infty}$ *be a sequence with* $\lim_{i\to\infty} p_i = 1$. *Then there exists a* *subsequence, again denoted by* p_i*, s.t.*

$$\lim_{i\to\infty} Z_T(p_i) \to C\,\mathrm{e}^{(H\cdot S)_T}$$

holds in the norm of $L^1(P)$ *for some predictable process* H.

Proof. Remember that $Z_T(p_i)$ has the form

$$Z_T(p_i) = C_{p_i}\,\mathrm{sgn}\left(1+\frac{f_{p_i}}{n(p_i)}\right)\left|1+\frac{f_{p_i}}{n(p_i)}\right|^{n(p_i)}$$

with $f_{p_i} = (H_{p_i}\cdot S)_T$. Owing to Corollary (3.1) we may write $\Omega = \bigcup_{l=1}^{\infty} K_l$, where equality means that the symmetric difference is a P null set, s.t. $\|H_{p_i}\|_{L^\infty} < l$ holds on K_l for $i\in\mathbb{N}$. Of course the $C_{p_i}\in\mathbb{R}$ are bounded.

We prove that the asserted convergence holds P a.s. on K, where K denotes K_l for some fixed l. After an extraction of a subsequence, which we denote again by p_i, we simultaneously have $C_{p_i}\equiv C_i \to C$ and $1_K f_{p_i} \equiv 1_K f_i \to 1_K f$ in the *weak** topology for a constant C and a function f bounded on K. This is possible, since $L^1(P)$ is weakly compactly generated, and therefore the closed unit ball in $L^\infty(P)$ is *weak** sequentially compact (Diestel 1975, Section 5.2, Corollary 3; 1975, p. 143). Defining $f_i \equiv r_i + f$ yields $1_K r_i \to 0$ *weak**. Our claim is now

$$1_K r_i \to 0 \quad \text{in probability.} \tag{7}$$

Because $Z_T(p_i)$ is the density of a martingale measure for S, f_i is a stochastic integral with respect to S and K is $\mathscr{F}_{T-1}$ measurable, we have for m and i large enough

$$\int_K \left(1+\frac{f_i}{n(p_i)}\right)^{n(p_i)} f_m\,\mathrm{d}P = 0$$

or

$$\int_K \left(1+\frac{f_i}{n(p_i)}\right)^{n(p_i)} (r_i - r_m)\,\mathrm{d}P = 0$$

and, after $m\to\infty$, we end up with

$$\int_K \left(1+\frac{f_i}{n(p_i)}\right)^{n(p_i)} r_i\,\mathrm{d}P = 0. \tag{8}$$

Using now Lemma 3.5 with $g_i \equiv f_i$, $h_i \equiv r_i$ and the measure P restricted to the set K, we get $\forall\epsilon>0\ \exists N_1(\epsilon)$ s.t.

$$\left|\int_K \left(1+\frac{f_i}{n(p_i)}\right)^{n(p_i)} r_i\,\mathrm{d}P - \int_K \mathrm{e}^{f_i}\, r_i\,\mathrm{d}P\right| < \epsilon$$

$\forall i \geqslant N_1(\epsilon)$. If (7) were false, we could find by Lemma 3.4 ($l\equiv f$) a $\delta>0$, s.t., after extracting a subsequence,

$$\int_K e^{f_i}\, r_i \, dP > \delta$$

holds for $i \geqslant N_2(\delta)$. Choosing now ϵ small enough and $i \geqslant \max(N_1, N_2)$, we arrive, by combining the last two inequalities, at

$$\int_K \left(1 + \frac{f_i}{n(p_i)}\right)^{n(p_i)} r_i \, dP > \delta - \epsilon > 0,$$

which is a contradiction to (8). Therefore (7) is true, and we find after a further extraction of a subsequence

$$\lim_{i\to\infty} 1_K f_i = 1_K f \quad P \text{ a.s.}$$

Since we have this for each $K = K_l$, diagonalization yields a subsequence with $f_i \to f$ P a.s., where $f = (H \cdot S)_T$ holds for some predictable H, because of the closedness of the space of stochastic integrals in $L^0(P)$ with respect to convergence in probability (see Proposition A1.1 in Appendix 1). This gives

$$\lim_{i\to\infty} C_i \,\mathrm{sgn}\left(1 + \frac{f_i}{n(p_i)}\right)\left|1 + \frac{f_i}{n(p_i)}\right|^{n(p_i)} = C\, e^{(H\cdot S)_T} \quad P \text{ a.s.}$$

and, because of boundedness of the $Z_T(p)$ in $L \log L$ (Proposition 2.1), we get the existence of a sequence $\{p_i\}_{i=1}^{\infty}$ with $p_i \to 1$, s.t.

$$\lim_{p_i\to 1} Z_T(p_i) = C\, e^{(H\cdot S)_T}$$

with respect to the norm of $L^1(P)$, and our proof is finished. □

Remark 3.1. $C\, e^{(H\cdot S)_T}$ is the density of a martingale measure for S, because the space $\mathscr{M}^s(S)$ is closed in L^1, since S is assumed to be bounded.

The final theorem says that we need not extract subsequences in the previous proposition.

Theorem 3.1.

$$\lim_{p\to 1} Z_T(p) = Z_T^E \equiv C\, e^{(H\cdot S)_T},$$

where Z_T^E is the minimal-entropy martingale measure, and the convergence holds with respect to the norm of $L^1(P)$.

Proof. Assuming the contrary, we find a $\delta > 0$, s.t. $\|Z_T(p_k) - Z_T^E\|_{L^1} > \delta$ for all k and a sequence $\{p_k\}$, tending to 1. By Proposition 3.1 these $Z_T(p_k)$ have a convergent subsequence with limit $\tilde{Z}_T^E = \tilde{C}\, e^{\tilde{f}} \neq Z_T^E, \tilde{f} = (\tilde{H} \cdot S)_T$ for some predictable $\tilde{H}$ and $\tilde{Z}_T^E \in \mathscr{M}^s(S)$, but there is only one martingale measure for S of this form (see Remark 2.1), and we have a contradiction. □

4. A counterexample

It has been shown by Frittelli and Lakner (1996) that for an unbounded process S, which has an equivalent martingale measure, the minimal-entropy martingale measure need not exist. We now give an example, where this measure exists, but the convergence of Theorem 3.1 does not hold.

Example 4.1. Let $(\Omega, (\mathscr{F}_0, \mathscr{F}_1), P, (S_0, S_1))$ be a one-step process with $\Omega = (0, 1]$, $\mathscr{F}_0 = \{\emptyset, \Omega\}$, $\mathscr{F}_1 = \sigma(S_1)$, P the Lebesgue measure, $S_0 = 0$, $S_1(\omega) = \ln\omega + \kappa$, where $\omega \in \Omega$ and $\kappa \in \mathbb{R}$. This is easily seen to be an exponentially distributed random variable plus a constant. We confine ourselves here to $0 < \kappa < 1$, since this turns out to be the most interesting case. For brevity we write S for S_1.

It is simple to calculate $\sigma \equiv \sup\{\rho | \int e^{\rho|S|} \, dP < \infty\} = 1 > 0$, which is equivalent to say $S \in L_{\exp}$ (Bennet and Sharpley 1988). Solution of the equation

$$\int e^{(\ln\omega+\kappa)\rho^E} (\ln\omega + \kappa) \, dP = 0$$

yields $\rho^E = 1/\kappa - 1$ which, together with the normalizing constant $C^E = e^{\kappa-1}/\kappa$, determines the density of the minimal-entropy measure $Z^E = C^E e^{\rho^E S}$ (see Remark 2.1).

The p-optimal martingale measure for S exists for all $p > 1$ by the same reasons as in the case of bounded S (see the arguments after Definition 2.3), except that we use now the closedness of $\mathscr{M}^s(S)$ with respect to $\|\cdot\|_{L^p(P)}$ instead of $\|\cdot\|_{L^1(P)}$. $Z(p)$ is given by

$$Z(p) = C_p \left| 1 + \frac{\rho_p S}{n(p)} \right|^{n(p)} \operatorname{sgn}\left(1 + \frac{\rho_p S}{n(p)}\right),$$

where C_p is a normalizing constant, and ρ_p is determined by

$$\int \left| 1 + \frac{\rho_p S}{n(p)} \right|^{n(p)} \operatorname{sgn}\left(1 + \frac{\rho_p S}{n(p)}\right) S \, dP = 0.$$

Note that $\rho_p > 0$ by our assumption $E[S] < 0$.

For later use we prove the following easy lemmata.

Lemma 4.1. *Let S be as in Example* 4.1*, and*

$$Z(p) = C_p \left| 1 + \frac{\rho_p S}{n(p)} \right|^{n(p)} \operatorname{sgn}\left(1 + \frac{\rho_p S}{n(p)}\right)$$

the p-optimal martingale measures. Then the normalizing constants C_p of $Z(p)$ fulfil

$$C_p \leqslant \mu \qquad \forall p > 1$$

and some constant $\mu > 0$.

Proof. We have

$$\gamma \geqslant \int |Z(p)|\,\mathrm{d}P = C_p \int \left|1+\frac{\rho_p S}{n}\right|^n \mathrm{d}P \geqslant C_p \int_{\{S>0\}} \left|1+\frac{\rho_p S}{n}\right|^n \mathrm{d}P \geqslant C_p \int_{\{S>0\}} 1\,\mathrm{d}P = C_p\alpha,$$

where the first inequality holds by the arguments used for the last inequality in the proof of Proposition 2.1, $\alpha \equiv P[S>0]>0$ holds, and γ is some positive constant. Defining $\mu \equiv \gamma/\alpha$ concludes the proof. □

Lemma 4.2. *Let S be as in Example 4.1, and*

$$Z(p) = C_p \left|1+\frac{\rho_p S}{n(p)}\right|^{n(p)} \operatorname{sgn}\left(1+\frac{\rho_p S}{n(p)}\right)$$

the p-optimal martingale measures. Then we have

$$\int \left|\left|1+\frac{\rho_p S}{n(p)}\right|^{n(p)} \operatorname{sgn}\left(1+\frac{\rho_p S}{n(p)}\right) S\right| \mathrm{d}P \leqslant 2\|S\|_{L^1(P)} \qquad \forall p>1.$$

Proof. Similarly as in Proposition 2.1 we define

$$A_p \equiv \left\{\omega \middle| 1+\frac{\rho_p S}{n(p)} \leqslant 0\right\},$$

$$B_p \equiv \left\{\omega \middle| 1+\frac{\rho_p S}{n(p)} > 0\right\} \cap \{\omega | S \leqslant 0\},$$

$$C_p \equiv \{\omega | S>0\}.$$

We have

$$\begin{aligned}
&\operatorname{sgn}\left(\int_{A_p} \left|1+\frac{\rho_p S}{n(p)}\right|^{n(p)} \operatorname{sgn}\left(1+\frac{\rho_p S}{n(p)}\right) S\,\mathrm{d}P\right) \\
&\qquad = \operatorname{sgn}\left(\int_{C_p} \left|1+\frac{\rho_p S}{n(p)}\right|^{n(p)} \operatorname{sgn}\left(1+\frac{\rho_p S}{n(p)}\right) S\,\mathrm{d}P\right) \\
&\qquad = -\operatorname{sgn}\left(\int_{B_p} \left|1+\frac{\rho_p S}{n(p)}\right|^{n(p)} \operatorname{sgn}\left(1+\frac{\rho_p S}{n(p)}\right) S\,\mathrm{d}P\right)
\end{aligned}$$

and therefore, using $\int |1+\rho_p S/n(p)|^{n(p)} \operatorname{sgn}(1+\rho_p S/n(p))\, S\,\mathrm{d}P = 0$, we conclude that

$$\begin{aligned}
\int \left|\left|1+\frac{\rho_p S}{n(p)}\right|^{n(p)} \operatorname{sgn}\left(1+\frac{\rho_p S}{n(p)}\right) S\right| \mathrm{d}P &\leqslant 2\left|\int_{B_p} \left|1+\frac{\rho_p S}{n(p)}\right|^{n(p)} \operatorname{sgn}\left(1+\frac{\rho_p S}{n(p)}\right) S\,\mathrm{d}P\right| \\
&\leqslant 2\int |S|\,\mathrm{d}P
\end{aligned}$$

holds. □

In the sequel we write $a_n \sim b_n$, if $\gamma_1 a_n \leqslant b_n \leqslant \gamma_2 a_n$ holds for all n and some $\gamma_1, \gamma_2 > 0$. Our next lemma is of purely technical nature.

Lemma 4.3. *Let S be as in Example 4.1. Assume that $\lim_{k\to\infty} \sigma_k = \sigma$, $\lim_{k\to\infty} n_k = \infty$ and $\sigma_k > 0$ hold. Then we have*

$$\lim_{k\to\infty} \int_{A_k} \left|1 + \frac{\sigma_k S}{n_k}\right|^{n_k} \operatorname{sgn}\left(1 + \frac{\sigma_k S}{n_k}\right) S \, \mathrm{d}P = 0 \qquad \text{for } \sigma < \sigma_0,$$

$$\lim_{k\to\infty} \int_{A_k} \left|1 + \frac{\sigma_k S}{n_k}\right|^{n_k} \operatorname{sgn}\left(1 + \frac{\sigma_k S}{n_k}\right) S \, \mathrm{d}P = \infty \qquad \text{for } \sigma > \sigma_0,$$

where $A_k \equiv \{\omega | 1 + \sigma_k S/n_k \leqslant 0\}$, and σ_0 is the solution of the equation $\mathrm{e}^{-1/\sigma_0 - 1}\sigma_0 = 1$, which gives $\sigma_0 \approx 3.6$.

Proof. A lengthy but straightforward computation yields, if we denote n_k by n for the moment

$$\begin{aligned}\int_{A_k} \left|1 + \frac{\sigma_k S}{n}\right|^{n} \operatorname{sgn}\left(1 + \frac{\sigma_k S}{n}\right) S \, \mathrm{d}P &= \mathrm{e}^{-\kappa}\, \mathrm{e}^{-n/\sigma_k}\, \frac{n!}{(n/\sigma_k)^{n-1}} \left(\frac{n+1}{n/\sigma_k} + 1\right) \\ &\sim \mathrm{e}^{-\kappa}\, \mathrm{e}^{-n/\sigma_k}\, \frac{n!}{(n/\sigma_k)^{n-1}} \\ &\sim \mathrm{e}^{-\kappa}\, \mathrm{e}^{-n/\sigma_k}\, \frac{(n/e)^n n^{1/2}}{(n/\sigma_k)^{n-1}} \\ &\sim (\mathrm{e}^{-1/\sigma_k - 1}\, \sigma_k)^n\, \mathrm{e}^{-\kappa}\, n^{1/2} \frac{n}{\sigma_k},\end{aligned}$$

which gives the desired result. □

Using Lemma 4.2 and Lemma 4.3, we can now prove the following.

Lemma 4.4. *Let S be as in Example 4.1, and*

$$Z(p) = C_p \left|1 + \frac{\rho_p S}{n(p)}\right|^{n(p)} \operatorname{sgn}\left(1 + \frac{\rho_p S}{n(p)}\right)$$

the p-optimal martingale measure. Then

$$\limsup_{k\to\infty} \rho_{p_k} \leqslant \sigma_0$$

holds for all sequences $\{p_k\}_{k=1}^{k=\infty}$ s.t. $\lim_{k\to\infty} p_k = 1$. ($\sigma_0$ is defined in Lemma 4.3.)

Proof. Assuming the contrary, we can find a sequence $\{p_k\}_{k=1}^{k=\infty}$ s.t. $\lim_{k\to\infty} p_k = 1$, $\lim_{k\to\infty} \rho_{p_k} = \hat{\sigma} > \sigma_0$ and $\hat{\sigma} \in \overline{\mathbb{R}} \equiv \mathbb{R} \cup \infty$ holds. By Lemma 4.3 this implies that

$$\lim_{k\to\infty}\int_{A_{p_k}}\left|1+\frac{\rho_{p_k}S}{n(p_k)}\right|^{n(p_k)}\operatorname{sgn}\left(1+\frac{\rho_{p_k}S}{n(p_k)}\right)S\,\mathrm{d}P=\infty,$$

which is a contradiction to Lemma 4.2. □

Finally we have the following theorem.

Theorem 4.1. *In Example* 4.1 *we find that*

$$\lim_{p\to 1} Z(p)=C_0\,\mathrm{e}^{\sigma_0 S}\qquad \textit{for } 0<\kappa<\kappa_0=\frac{1}{\sigma_0+1}\qquad (\textit{i.e. } \rho^E>\sigma_0)$$

$$\lim_{p\to 1} Z(p)=Z^E\qquad \textit{for } \kappa_0\leqslant\kappa<1\qquad (\textit{i.e. } \rho^E\leqslant\sigma_0),$$

where the convergence holds in $\|\cdot\|_{L^1(P)}$, *and* σ_0 *is the solution of* $\mathrm{e}^{-1/\sigma_0-1}\,\sigma_0=1$. *Note that* $C_0\,\mathrm{e}^{\sigma_0 S}$ *is not a martingale measure for* S.

Proof. In Proposition 2.1 we have proved boundedness of the p-optimal martingale measures in $L\log L$ for bounded processes S, but the boundedness of S is used in the proof only for the estimates of the normalizing constants C_p. Therefore combining Proposition 2.1 and Lemma 4.1 we get boundedness in $L\log L$ of the $Z(p)$ for S in Example 4.1. To prove the claimed convergence in $L^1(P)$, we therefore have to show only P a.s. convergence, or the convergence of ρ_p and C_p respectively as $p\to 1$.

Once we have shown convergence of ρ_p, convergence of $Z(p)/C_p$ in $L^1(P)$ follows and, by the normalization condition $\int Z(p)\,\mathrm{d}P=1$, convergence of C_p follows. Summarizing, we have to show that $\lim_{p\to 1}\rho_p$ is either σ_0 or ρ^E. We distinguish between two cases.

Case 1: $0<\kappa<\kappa_0\equiv 1/(\sigma_0+1)$ *(i.e.* $\rho^E>\sigma_0$*).* Our claim is $\lim_{p\to 1}\rho_p=\sigma_0$. By Lemma 4.4 it suffices to show $\liminf_{k\to\infty}\rho_{p_k}=\sigma_0$ for all sequences $\{p_k\}_{k=1}^{k=\infty}$ tending to 1. Assuming the contrary, namely the existence of a sequence $\{p_k\}_{k=1}^{k=\infty}$ s.t. $\lim_{k\to\infty}p_k=1$ and $\lim_{k\to\infty}\rho_{p_k}=\hat\rho<\sigma_0$ hold, yields by dominated convergence

$$\lim_{k\to\infty}\int_{(A_{p_k})^c}\left(1+\frac{\rho_{p_k}S}{n(p_k)}\right)^{n(p_k)}S\,\mathrm{d}P=\int\mathrm{e}^{\hat\rho S}S\,\mathrm{d}P<0,$$

where we have used the notation A_{p_k} from Lemma 4.2. Note that the last inequality holds, because the function $f(\rho)=\int\mathrm{e}^{\rho S}S\,\mathrm{d}P$ is strictly increasing, and $f(\rho^E)=0$.

On the other hand

$$\lim_{k\to\infty}\int_{A_{p_k}}\left|1+\frac{\rho_{p_k}S}{n(p_k)}\right|^{n(p_k)}\operatorname{sgn}\left(1+\frac{\rho_{p_k}S}{n(p_k)}\right)S\,\mathrm{d}P=0$$

holds by Lemma 4.3, which gives a contradiction, since this implies that

$$\int\left|1+\frac{\rho_{p_k}S}{n(p_k)}\right|^{n(p_k)}\operatorname{sgn}\left(1+\frac{\rho_{p_k}S}{n(p_k)}\right)S\,\mathrm{d}P<0$$

for k large enough.

Case 2: $\kappa_0 \leqslant \kappa < 1$ *(i.e.* $\rho^E \leqslant \sigma_0$*).* Our claim now is $\lim_{p\to 1} \rho_p = \rho^E$. Assuming the contrary, namely first the existence of a sequence $\{p_k\}_{k=1}^{k=\infty}$ s.t. $\lim_{k\to\infty} p_k = 1$ and $\lim_{k\to\infty} \rho_{p_k} = \hat{\rho} < \rho^E$ hold, yields in completely the same way as in case 1 a contradiction. Finally assuming the existence of a sequence $\{p_k\}_{k=1}^{k=\infty}$ tending to 1 with $\lim_{k\to\infty} \rho_{p_k} = \hat{\rho}$, s.t. $\rho^E < \hat{\rho} \leqslant \sigma_0$ holds (the upper bound for $\hat{\rho}$ comes from Lemma 4.4), gives

$$\lim_{k\to\infty} \int_{(A_{p_k})^c} \left(1 + \frac{\rho_{p_k} S}{n(p_k)}\right)^{n(p_k)} S \, \mathrm{d}P = \int \mathrm{e}^{\hat{\rho} S} S \, \mathrm{d}P > 0,$$

but

$$\int_{A_{p_k}} \left|1 + \frac{\rho_{p_k} S}{n(p_k)}\right|^{n(p_k)} \operatorname{sgn}\left(1 + \frac{\rho_{p_k} S}{n(p_k)}\right) S \, \mathrm{d}P \geqslant 0 \qquad \forall k$$

yields again a contradiction as in case 1. □

Remark 4.1. Noting that in case 1

$$\lim_{k\to\infty} \int_{A^c_{p_k}} \left(1 + \frac{\rho_{p_k} S}{n(p_k)}\right)^{n(p_k)} S \, \mathrm{d}P = \int \mathrm{e}^{\sigma_0 S} S \, \mathrm{d}P = -\alpha < 0$$

holds, we infer that

$$\int_{A_{p_k}} Z(p_k) S \, \mathrm{d}P \geqslant \beta > 0$$

is valid for k large enough. Since $S \in L_{\exp}$, which can be identified with the Banach space dual of $L \log L$ (Bennet and Sharpley 1988, Theorem 6.5), we get $\lim_{k\to\infty} \|Z(p_k)^-\|_{L \log L} \neq 0$.

Appendix 1

Proposition A1.1. *Let S be a stochastic process on the stochastic basis* $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t=0}^T, P)$ *in discrete time, which is adapted and* $\mathbb{R}^d$ *valued. Suppose that S does not admit arbitrage or, equivalently, that there is an equivalent martingale measure for S.*

Then the space of stochastic integrals $\mathscr{G}_T \equiv \{(H \cdot S)_T | H \text{ predictable}\}$ *is closed in* $L^0(P)$ *with respect to convergence in measure and therefore* $\mathscr{G}_T^p$ *is norm closed in* $L^p(P)$ *for each* $1 \leqslant p < \infty$.

Proof. The latter assertion is an immediate consequence of the former assertion and the observation that $\mathscr{G}_T^p = \mathscr{G}_T \cap L^p(P)$.

For the proof of the closedness of $\mathscr{G}_T$ we shall prove a lemma, which might have some independent interest. Admitting the subsequent lemma for the moment, we can finish the proof as follows. Let $(H^n)_{n=1}^{\infty}$ be a sequence of predictable processes such that the sequence $(f_n)_{n=1}^{\infty} = ((H^n \cdot S)_T)_{n=1}^{\infty}$ converges in measure to some f_0. We have to show that there is a predictable integrand H^0 such that $f_0 = (H^0 \cdot S)_T$.

From the subsequent lemma we infer that, for each $1 \leqslant t \leqslant T$, the sequence $((H^n \cdot S)_t)_{n=1}^{\infty} - ((H^n \cdot S)_{t-1})_{n=1}^{\infty}$ converges in measure to some $f_{0,t}$; hence we may apply Stricker (1990, Proposition 2) to find, for $1 \leqslant t \leqslant T$, $\mathscr{F}_{t-1}$-measurable functions H_t^0 such that

$$(H_t^0, S_t - S_{t-1}) = f_{0,t},$$

which just means that the predictable integrand $(H_t^0)_{t=1}^T$ does the job. □

Lemma A1.1. *Under the assumptions of Proposition* A.1.1 *let* $(H^n)_{n=1}^{\infty}$ *be a sequence of predictable processes such that*

$$f_n = (H^n \cdot S)_T$$

converges in measure; then

$$g_n = (H^n \cdot S)_t$$

converges in measure for all $0 \leqslant t \leqslant T$.

Proof. It suffices to show the assertion for $t = T - 1$. Suppose that the lemma were false; then we could find a sequence $(f_n)_{n=1}^{\infty} = ((H^n \cdot S)_T)_{n=1}^{\infty}$ as above tending to zero in measure, while $(g_n)_{n=1}^{\infty} = ((H^n \cdot S)_{T-1})_{n=1}^{\infty}$ does not so.

Hence, by passing to a subsequence and changing signs, if necessary, we may find an $\alpha > 0$ such that

$$P\{g_n \leqslant -\alpha\} > \alpha \qquad \text{for } n \in \mathbb{N}.$$

Consider the predictable integrands

$$A_t^n(\omega) = H_t^n 1_T(t) 1_{\{g_n \leqslant -\alpha\}}(\omega)$$

and let a_n denote the random variables

$$a_n = (A^n \cdot S)_T \wedge 1 = ((f_n - g_n) 1_{\{g_n \leqslant -\alpha\}}) \wedge 1.$$

Note that each $a_n \in \mathscr{G} - L_+^0(\Omega, \mathscr{F}_T, P)$, where $\mathscr{G}$ is defined by $\mathscr{G} = \{(h, \Delta S_T) | h \in L^0(\Omega, \mathscr{F}_{T-1}, P, \mathbb{R}^{\mathrm{d}})\}$, and that the negative parts $((a_n)^-)_{n=1}^{\infty}$ tend to zero in measure or, by passing to a subsequence, even P a.s. On the other hand, for each $\epsilon > 0$,

$$\lim_{n \to \infty} P\{a_n \geqslant \alpha - \epsilon\} \geqslant \alpha.$$

By lemma A1.1 of Delbaen and Schachermayer (1994) we infer that there is a sequence of convex combinations of $(a_n^+)_{n=1}^{\infty}$, denoted by $(c_n)_{n=1}^{\infty}$, converging almost surely to some $c\colon \Omega \to [0, \infty)$, for which we have $\mathrm{E}[c] > 0$. Applying the *same* convex combinations to $(a_n^-)_{n=1}^{\infty}$, denoting the result by $(d_n)_{n=1}^{\infty}$, (d_n) tend to zero P a.s. Summing up, we get a $c\colon \Omega \to [0, \infty)$, for which we have $\mathrm{E}[c] > 0$ and which is in $\mathscr{G} - L_+^0(\Omega, \mathscr{F}_T, P)$ by

Schachermayer (1992, Lemma 2.1). This is clearly a contradiction to the (NA) assumption. □

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