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THE p-TH MOMENT EXPONENTIAL STABILITY OF NEUTRAL STOCHASTIC FUNCTIONAL DIFFERENTIAL EQUATIONS

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ABSTRACT. Since neutral stochastic functional differential equations are essentially complex, it is not easy to study stability problems of their solutions by applying usual procedures based on Lyapunov functionals. In this paper we present criteria on the bases of which it is relatively easy to verify the p-th moment exponential stability.

1. Introduction and preliminary results

It is well-known that stochastic modelling including Gaussian white noise perturbations has played an important role in many areas of science and engineering for a long time. Having in mind that the Gaussian white noise is an abstraction, not a real phenomenon, which is mathematically described as a formal derivative of a Brownian motion process, such stochastic modelling is based on various stochastic differential equations of the Ito type. Some of the most frequent and most important stochastic models are described by very complex neutral stochastic functional differential equations. These equations were introduced by Kolmanovskii and Nosov [3, 4] to study the behavior of chemical engineering systems in which the physical and chemical processes were distinguished by their complexity, and to explore the theory of aeroelasticity in which aeroelastic efforts present an interaction between

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aerodynamic, elastic and inertial forces. Bearing in mind the complexity of these equations, the main interest in the field has often been directed to the existence, uniqueness and stability of their solutions. We refer the reader to some of the papers and books by X. Mao [5, 7, 8, 9, 10], as well as [2, 4, 6, 12] among others.

It is also well known that one of the classical and powerful techniques applied in the study of stability problems is a stochastic version of the Lyapunov direct method. However, this method is not very convenient in applications due to the difficulty it causes on effectively finding Lyapunov functionals. In this paper we present some direct and more applicable criteria to study the p-th moment exponential stability, $p \geq 2$, for a very general neutral stochastic functional differential equation. In fact, we generalize the results from paper [8] referring only to the exponential stability in mean square. It should be pointed out that we shall use an elementary inequality, basically different than the one from paper [8] and from other ones treating a similar subject. This will enable us to study the p-th moment exponential stability by applying the technique from [8].

The paper is organized as follows: In the continuation of this section we introduce some basic notions and notations, analogous to the ones from [8]. Then we present the neutral stochastic functional differential equation which will be considered in the sequel. In Section 2 we give our main results, sufficient conditions under which the trivial solution is the p-th moment exponentially stable. We also give an example to illustrate the presented theory.

Our initial assumption is that all random variables and processes are defined on a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ with a natural filtration $\{\mathcal{F}_t\}_{t\geq 0}$ generated by a standard m-dimensional Brownian motion $w = \{w(t), t \geq 0\}$, $w(t) = (w_1(t), w_2(t), \dots, w_m(t))$, i.e. $\mathcal{F}_t = \sigma\{w(s), 0 \leq s \leq t\}$. Let the Euclidean norm be denoted by $|\cdot|$. For a matrix A, let $||A||$ be the operator norm of A, where $||A|| = \sup\{|Ax| : |x| = 1, x \in R^n\}$, and A^T the transpose of a vector or matrix A.

For a given $\tau > 0$, let $L^p([-\tau, 0]; R^n)$ be the family of Borel measurable R^n-valued functions $\varphi(s), -\tau \leq s \leq 0$, with the norm

$$||\varphi||_{L^p} = \left(\int_{-\tau}^{0} |\varphi(s)|^p \, ds \right)^{1/p} < \infty.$$

Let also $\mathcal{W}([-\tau, 0]; R_+)$ be the family of Borel measurable bounded non-negative functions $\eta(s), -\tau \leq s \leq 0$, such that $\int_{-\tau}^{0} \eta(s)\, ds = 1$ (the weighting functions), and $\mathcal{GB}_{\mathcal{F}_0}([-\tau, 0]; R^n)$ be the family of continuous bounded R^n-valued stochastic processes $\xi = \{\xi(s), -\tau \leq s \leq 0\}$, such that $\xi(s)$ is $\mathcal{F}_0$-measurable for every s (we require that $\mathcal{F}_s = \mathcal{F}_0$ for $-\tau \leq s \leq 0$).

The topic of our analysis is the following n-dimensional neutral stochastic functional differential equation

$$(1) \qquad d[x(t) - G(x_t)] = [f(t, x(t)) + g(t, x_t)]\, dt + \sigma(t, x_t)\, dw(t), \quad t \geq 0$$

with an initial data $x_0 = \xi = \{\xi(s), -\tau \le s \le 0\} \in \mathcal{GB}_{\mathcal{F}_0}([-\tau, 0]; R^n)$. The coefficients of this equation are

$$G : L^p([-\tau, 0]; R^n) \to R^n, \quad f : R_+ \times R^n \to R^n,$$
$$g : R_+ \times L^p([-\tau, 0]; R^n) \to R^n, \quad \sigma : R_+ \times L^p([-\tau, 0]; R^n) \to R^n \times R^m,$$

and $x_t = \{x(t + s), -\tau \le s \le 0\}$ is an $L^p([-\tau, 0]; R^n)$-valued stochastic process.

An $\mathcal{F}_t$-adapted process $x = \{x(t), -\tau \le t \le \infty\}$ is said to be the solution of Eq. (1) if it satisfies the initial condition and the corresponding integral equation holds a.s., i.e. for every $t \ge 0$,

$$x(t) - G(x_t) = \xi(0) - G(x_0) + \int_0^t [f(s, x(s)) + g(s, x_s)]\, ds + \int_0^t \sigma(s, x_s)\, dw(s) \quad \text{a.s.} \tag{2}$$

Remember that in [4] Kolmanovskii and Nosov proved the basic existence-and-uniqueness theorem under the following conditions: $|G(\varphi) - G(\phi)|^2 \le k \int_{-\tau}^0 \eta(s)|\varphi(s) - \phi(s)|^2 ds$ for all $\varphi, \phi \in L^2([-\tau, 0]; R^n)$ and for a constant $k \in (0, 1)$, $\eta(\cdot) \in \mathcal{W}([-\tau, 0; R_+)$; the linear growth condition and the usual global condition, or in a weakened version local Lipschitz condition in the second argument hold for f, g and σ. Moreover, if there exists the p-th moment for ξ, then $\sup_{-\tau \le t < \infty} E|x(t; \xi)|^p < \infty$ (for more details see [4, 9, 12]). Since our investigation is devoted to stability problems, we always assume, with no special emphasis on the conditions, that Eq. (1) has a unique solution $x(t; \xi)$ for arbitrary initial data $\xi \in \mathcal{GB}_{\mathcal{F}_0}([-\tau, 0]; R^n)$ satisfying $\sup_{-\tau \le t < \infty} E|x(t; \xi)|^p < \infty$, and that all Lebesgue's and Ito's integrals employed further are well defined.

We also need the following inequality in our investigation: For $p \ge 1$, $x, y \in R^n$ and $\theta \in (0, 1)$,

$$|x + y|^p \le \frac{|x|^p}{(1 - \theta)^{p-1}} + \frac{|y|^p}{\theta^{p-1}}. \tag{3}$$

The proof immediately holds by putting $\alpha = \frac{1-\theta}{\theta}$ in the inequality (X. Mao [11], Lemma 4.1): If $p \ge 1$, $x, y \in R^n$ and $\alpha > 0$, then $|x + y|^p \le (1 + \alpha)^{p-1}(|x|^p + \alpha^{-(p-1)}|y|^p)$.

2. Main results

As it was mentioned in Section 1, we will study the p-th moment exponential stability of the solutions of Eq. (1), by mainly applying the procedures from paper [8]. As usual, we assume that $G(0) = 0$, $f(t, 0) \equiv 0$, $g(t, 0) \equiv 0$ and $\sigma(t, 0) \equiv 0$, so that Eq. (1) admits a trivial solution $x(t; 0) \equiv 0$.

We first present an auxiliary result which is basically used in the proofs of the forthcoming assertions. Note again that we suppose $p \ge 2$.

Lemma 1. *Let there exist a constant* $k\in(0,1)$ *and a function* $\eta\in\mathcal{W}([-\tau,0];R_+)$ *such that the functional* $G: L^p([-\tau,0];R^n)\to R^n$ *satisfies the condition*

$$|G(\varphi)|^p \le k\int_{-\tau}^0 \eta(s)\,|\varphi(s)|^p\,ds \tag{4}$$

for all $\varphi\in L^p([-\tau,0];R^n)$*. Let also* $\{x(t),-\tau\le t<\infty\}$ *be an* n*-dimensional stochastic process satisfying* $\sup_{-\tau\le t<\infty} E|x(t)|^p<\infty$*, and for some constants* $\alpha>0$*,* $\delta\in[0,1)$ *and* $c>0$

$$E|x(t)-G(x_t)|^p \le c\,e^{-\alpha t}+\delta\sup_{t-\tau\le s\le t}E|x(s)|^p \tag{5}$$

for all $t\ge 0$*. If* $\delta^{\frac1p}+k^{\frac1p}<1$*, then*

$$\limsup_{t\to\infty}\frac{\ln E|x(t)|^p}{t}\le -(\alpha\wedge\beta), \tag{6}$$

where $\beta=-\frac{p}{\tau}\ln(\delta^{\frac1p}+k^{\frac1p})>0$*.*

Proof. By applying the inequality (3) for an arbitrary $\theta\in(0,1)$, then (4) and (5), we obtain

$$\begin{aligned} E|x(t)|^p &\le \frac{1}{(1-\theta)^{p-1}}E|x(t)-G(x_t)|^p+\frac{1}{\theta^{p-1}}E|G(x_t)|^p \\ &\le \frac{1}{(1-\theta)^{p-1}}\Big[c\,e^{-\alpha t}+\delta\sup_{t-\tau\le s\le t}E|x(s)|^p\Big] \\ &\quad+\frac{k}{\theta^{p-1}}E\int_{-\tau}^0\eta(s)|x(s+t)|^p ds \\ &\le \frac{c}{(1-\theta)^{p-1}}\,e^{-\alpha t}+\Big[\frac{\delta}{(1-\theta)^{p-1}}+\frac{k}{\theta^{p-1}}\Big]\sup_{t-\tau\le s\le t}E|x(s)|^p. \end{aligned} \tag{7}$$

Let $\delta>0$ and $\theta=\dfrac{k^{\frac1p}}{\delta^{\frac1p}+k^{\frac1p}}$. Then

$$\begin{aligned} E|x(t)|^p &\le \frac{c\,(\delta^{\frac1p}+k^{\frac1p})^{p-1}}{\delta^{\frac{p-1}{p}}}\,e^{-\alpha t}+(\delta^{\frac1p}+k^{\frac1p})^p\sup_{t-\tau\le s\le t}E|x(s)|^p \\ &= c_1\,e^{-\alpha t}+(\delta^{\frac1p}+k^{\frac1p})^p\sup_{t-\tau\le s\le t}E|x(s)|^p, \end{aligned}$$

where c_1 is a generic constant.

Let us denote that

$$\psi_k=\sup_{(k-1)\tau\le t\le k\tau}E|x(t)|^p$$

for all $k=0,1,2,\dots$ Then, for $k\ge 1$

$$\psi_k\le c_1e^{-\alpha(k-1)\tau}+(\delta^{\frac1p}+k^{\frac1p})^p\,(\psi_{k-1}\vee\psi_k).$$

Let $0 < \varepsilon < \alpha \wedge \beta$ be arbitrary. Then,

$$e^{\varepsilon k\tau}\psi_k < c_1 e^{\alpha\tau-(\alpha-\varepsilon)k\tau} + (\delta^{\frac{1}{p}} + k^{\frac{1}{p}})^p e^{\varepsilon\tau}(e^{\varepsilon(k-1)\tau}\psi_{k-1} \vee e^{\varepsilon k\tau}\psi_k).$$

Since

$$\max_{1\le i\le k}(e^{\varepsilon i\tau}\psi_i) \le c_1 e^{\alpha\tau} + (\delta^{\frac{1}{p}} + k^{\frac{1}{p}})^p e^{\varepsilon\tau}[\psi_0 + \max_{1\le i\le k}(e^{\varepsilon i\tau}\psi_i)],$$

and since $(\delta^{\frac{1}{p}} + k^{\frac{1}{p}})^p e^{\varepsilon\tau} < (\delta^{\frac{1}{p}} + k^{\frac{1}{p}})^p e^{\beta\tau} = 1$, then

$$\max_{1\le i\le k}(e^{\varepsilon i\tau}\psi_i) \le C,$$

where $C = \left[c_1 e^{\alpha\tau} + (\delta^{\frac{1}{p}} + k^{\frac{1}{p}})^p e^{\varepsilon\tau}\psi_0\right]/\left[1 - (\delta^{\frac{1}{p}} + k^{\frac{1}{p}})^p e^{\varepsilon\tau}\right] > 0$. Therefore,

$$\psi_k \le C e^{-\varepsilon k\tau}$$

for all $k \ge 1$, which implies that

$$\limsup_{k\to\infty}\frac{\ln\psi_k}{k\tau} \le -\varepsilon.$$

By the definition of ψ_k it follows that

$$\limsup_{t\to\infty}\frac{\ln E|x(t)|^p}{t} \le -\varepsilon.$$

Now the proof of this lemma, i.e., the relation (6), immediately holds letting $\varepsilon \to \alpha \wedge \beta$.

If $\delta = 0$, we choose $k < \theta < 1$, start form (7) and repeat the previous procedure. Thus, the proof becomes complete. □

We can now give the conditions under which the trivial solution of Eq. (1) is the p-th moment exponentially stable (for simplicity, we denote that $trace\,[B^T B] = |B|^2$ for any matrix B).

Theorem 1. *Let there exist a function* $\eta_1(\cdot) \in \mathcal{W}([-\tau,0];R_+)$ *and a pair of constants* $0 \le \lambda_2 < \lambda_1$ *such that*

$$\begin{aligned}&\frac{p}{2}|\varphi(0)-G(\varphi)|^{p-4}\Big\{|\varphi(0)-G(\varphi)|^2\Big[2[\varphi(0)-G(\varphi)]^T[f(t,\varphi(0))+g(t,\varphi)]\\ &\qquad +|\sigma(t,\varphi)|^2\Big] + (p-2)|[\varphi(0)-G(\varphi)]^T\sigma(t,\varphi)|^2\Big\}\\ &\qquad \le -\lambda_1|\varphi(0)|^p + \lambda_2\int_{-\tau}^0 \eta_1(s)\,|\varphi(s)|^p\,ds\end{aligned} \tag{8}$$

for all $t \ge 0$ *and* $\varphi \in L^p([-\tau,0];R^n)$. *Let also the condition (4) hold with a constant* $k \in (0,1)$ *and a function* $\eta \in \mathcal{W}([-\tau,0];R_+)$. *Then the p-th moment Lyapunov exponent of the solution of Eq. (1) is not greater than* $-(\alpha\wedge\beta)$, *that is,*

$$\limsup_{t\to\infty}\frac{\ln E|x(t;\xi)|^p}{t} \le -(\alpha\wedge\beta), \tag{9}$$

where $\alpha \in (0, \lambda_1 - \lambda_2)$ *is the unique root of the equation*

$$\lambda_1 - 2^{p-1}\alpha - (2^{p-1}\alpha k + \lambda_2)\, e^{\alpha\tau} = 0, \tag{10}$$

and $\beta = -\frac{1}{\tau}\ln k > 0$.

Proof. For reasons of notational simplicity, we use $x(t)$ instead of $x(t,\xi)$ to denote the solution of Eq. (1) for given initial data $\xi \in \mathcal{GB}_{\mathcal{F}_0}([-\tau,0];R^n)$. By applying the Ito formula [1, 9] we have

$$\begin{aligned}
&E(e^{\alpha t}|x(t) - G(x_t)|^p) = E|\xi(0) - G(\xi)|^p \\
&\quad + E\int_0^t \alpha e^{\alpha s}|x(s) - G(x_s)|^p\, ds \\
&\quad + p\, E\int_0^t e^{\alpha s}|x(s) - G(x_s)|^{p-2}[x(s) - G(x_s)]^T[f(s,x(s)) + g(s,x_s)]\, ds \\
&\quad + p\, E\int_0^t e^{\alpha s}|x(s) - G(x_s)|^{p-2}[x(s) - G(x_s)]^T\sigma(s,x_s)\, dw_s \\
&\quad + \frac{p}{2}\, E\int_0^t e^{\alpha s}|x(s) - G(x_s)|^{p-2}|\sigma(s,x_s)|^2\, ds \\
&\quad + \frac{p(p-2)}{2}\, E\int_0^t e^{\alpha s}|x(s) - g(x_s)|^{p-4}|[x(s) - G(x_s)]^T\sigma(s,x_s)|^2\, ds.
\end{aligned}$$

Then (8) and (4) imply that

$$\begin{aligned}
E(e^{\alpha t}|x(t) - G(x_t)|^p) &\le E|\xi(0) - G(\xi)|^p \\
&\quad + 2^{p-1}\alpha\, E\int_0^t e^{\alpha s}[|x(s)|^p + |G(x_s)|^p]\, ds \\
&\quad + E\int_0^t e^{\alpha s}\left(-\lambda_1|x(s)|^p + \lambda_2\int_{-\tau}^0 \eta_1(v)|x(s+v)|^p dv\right) ds \\
&\le E|\xi(0) - G(\xi)|^p + [-(\lambda_1 - 2^{p-1}\alpha)]\, E\int_0^t e^{\alpha s}|x(s)|^p ds \\
&\quad + 2^{p-1}\alpha k\, E\int_0^t e^{\alpha s}\int_{-\tau}^0 \eta(v)|x(s+v)|^p\, dv\, ds \\
&\quad + \lambda_2\, E\int_0^t e^{\alpha s}\int_{-\tau}^0 \eta_1(v)|x(s+v)|^p\, dv\, ds.
\end{aligned} \tag{11}$$

By replacing the order of integration we have

$$
\begin{aligned}
&\int_0^t e^{\alpha s} \int_{-\tau}^0 \eta_1(v)|x(s+v)|^p dv\, ds \\
&\quad = \int_0^t e^{\alpha s} \int_{s-\tau}^{s} \eta_1(v-s)|x(v)|^p dv\, ds \\
&\quad = \int_{-\tau}^t \left(\int_{v\vee 0}^{(v+\tau)\wedge t} e^{\alpha s}\eta_1(v-s)\, ds \right) |x(v)|^p dv \qquad (12)\\
&\quad \le \int_{-\tau}^t e^{\alpha(v+\tau)}|x(v)|^p dv \int_{-\tau}^0 \eta_1(s)\, ds \\
&\quad = \int_{-\tau}^t e^{\alpha(v+\tau)}|x(v)|^p dv,
\end{aligned}
$$

and similarly,

$$
\int_0^t e^{\alpha s} \int_{-\tau}^0 \eta(v)|x(s+v)|^p dv\, ds \le \int_{-\tau}^t e^{\alpha(v+\tau)}|x(v)|^p dv. \tag{13}
$$

By taking (12) and (13) into (11) we deduce that

$$
E(e^{\alpha t}|x(t)-G(x_t)|^p) \le C_1-[\lambda_1-2^{p-1}\alpha-(2^{p-1}\alpha k+\lambda_2)\, e^{\alpha\tau}] \int_0^t e^{\alpha s} E|x(s)|^p ds,
$$

where $C_1 = E|\xi(0) - G(\xi)|^p + (2^{p-1}\alpha k + \lambda_2)\, e^{\alpha\tau} \int_{-\tau}^0 E|\xi(s)|^p ds$. Now, (10) implies that

$$
E|x(t) - G(x_t)|^p \le C_1 e^{-\alpha t},
$$

so that the condition (5) is satisfied for $\delta = 0$. Therefore, (6) follows immediately by applying Lemma 1, which completes the proof. □

The fact that there exists a unique root $\alpha \in (0, \lambda_1 - \lambda_2)$ of the equation (10) follows from the properties of the function $h(\alpha) = 2^{p-1}\alpha + (2^{p-1}\alpha k + \lambda_2)e^{\alpha\tau} - \lambda_1$: $h(0) = \lambda_2 - \lambda_1 < 0$, $h(\lambda_1 - \lambda_2) > 0$ and $h'(\alpha) > 0$.

Theorem 2. *Let there exist a function $\eta_1(\cdot) \in \mathcal{W}([-\tau, 0]; R_+)$ and a pair of constants $0 \le \lambda_1 < \lambda_2$ such that*

$$
\begin{aligned}
&\frac{p}{2}|\varphi(0) - G(\varphi)|^{p-4}\Big\{|\varphi(0) - G(\varphi)|^2\Big[2[\varphi(0) - G(\varphi)]^T[f(t, \varphi(0)) + g(t, \varphi)] \\
&\qquad +|\sigma(t, \varphi)|^2\Big] + (p-2)|[\varphi(0) - G(\varphi)]^T\sigma(t, \varphi)|^2\Big\} \qquad (14)\\
&\quad \le \lambda_1|\varphi(0)|^p - \lambda_2 \int_{-\tau}^0 \eta_1(s)\, |\varphi(s)|^p\, ds
\end{aligned}
$$

for all $t \ge 0$ and $\varphi \in L^p([-\tau, 0]; R^n)$. Let also the condition (4) hold with a constant $k \in (0, 1)$ and a function $\eta \in \mathcal{W}([-\tau, 0]; R_+)$. If

$$
\lambda_2 < \big(1 - k^{\frac{1}{p}}\big)^p/\tau, \tag{15}
$$

then the p-th moment Lyapunov exponent of the solution of Eq. (1) is not greater than $-(\alpha \wedge \beta)$*, that is,*

$$\limsup_{t\to\infty} \frac{\ln E|x(t;\xi)|^p}{t} \le -(\alpha \wedge \beta), \tag{16}$$

where $\alpha \in (0, \lambda_2 - \lambda_1)$ *is the unique root of the equation*

$$\lambda_1 + 2^{p-1}\alpha(1 + ke^{\alpha\tau}) - \lambda_2 = 0, \tag{17}$$

and $\beta = -\frac{p}{\tau}\ln[(\lambda_2\tau)^{\frac{1}{p}} + k^{\frac{1}{p}}] > 0$.

Proof. Again, by applying the Ito formula, then (4) and (14), we find

$$\begin{aligned} (18)\qquad E(e^{\alpha t}|x(t) - G(x_t)|^p) \le\; & E|\xi(0) - G(\xi)|^p \\ & + (\lambda_1 + 2^{p-1}\alpha)\, E\int_0^t e^{\alpha s}|x(s)|^p ds \\ (19)\qquad & + 2^{p-1}\alpha k\, E\int_0^t e^{\alpha s}\int_{-\tau}^0 \eta(v)|x(s+v)|^p dv\, ds \\ & - \lambda_2\, E\int_0^t e^{\alpha s}\int_{-\tau}^0 \eta_1(v)|x(s+v)|^p dv\, ds. \end{aligned}$$

Then,

$$\begin{aligned} &\int_0^t e^{\alpha s}\int_{-\tau}^0 \eta_1(v)|x(s+v)|^p dv\, ds \\ &\quad = \int_{-\tau}^t \left(\int_{v\vee 0}^{(v+\tau)\wedge t} e^{\alpha s}\eta_1(v-s)\, ds\right)|x(v)|^p dv \\ (20)\qquad &\quad \ge \int_0^{(t-\tau)\vee 0}\left(\int_v^{v+\tau} e^{\alpha s}\eta_1(v-s)\, ds\right)|x(v)|^p dv \\ &\quad \ge \int_0^{(t-\tau)\vee 0} e^{\alpha v}|x(v)|^p dv. \end{aligned}$$

By taking (13) and (20) into (19) we finally obtain that

$$\begin{aligned} E(e^{\alpha t}|x(t) - G(x_t)|^p) &\le C_2 + [\lambda_1 + 2^{p-1}\alpha(1 + ke^{\alpha\tau})]\int_0^t e^{\alpha s}E|x(s)|^p ds \\ &\quad - \lambda_2\, E\int_0^{(t-\tau)\vee 0} e^{\alpha s}|x(s)|^p ds \\ &\le C_2 + [\lambda_1 + 2^{p-1}\alpha(1+ke^{\alpha\tau}) - \lambda_2]\int_0^t e^{\alpha s}E|x(s)|^p ds \\ &\quad + \lambda_2\tau e^{\alpha t}\sup_{t-\tau\le s\le t} E|x(s)|^p, \end{aligned}$$

where $C_2 = E|\xi(0) - G(\xi)|^p + 2^{p-1}\alpha k e^{\alpha\tau}\int_{-\tau}^0 E|\xi(s)|^p ds$. But (17) implies that

$$E|x(t) - G(x_t)|^p \le C_2 e^{-\alpha t} + \lambda_2\tau \sup_{t-\tau\le s\le t} E|x(s)|^p,$$

and since (5) holds, the proof of this theorem immediately follows by virtue of Lemma 1. □

To conclude that α is a unique root of the equation (17) over $(0, \lambda_2 - \lambda_1)$, we will proceed as above: $h(\alpha) = \lambda_2 - \lambda_1 - 2^{p-1}\alpha(1 + ke^{\alpha\tau})$ and $h(0) = \lambda_2 - \lambda_1 > 0$, $h(\lambda_2 - \lambda_1) < 0$ and $h'(\alpha) < 0$.

We now use Theorem 1 to estabilish an applicable corollary.

Corollary 1. *Let (4) hold with $k \in (0,1)$ and $\eta(\cdot) \in \mathcal{W}([-\tau,0];R_+)$. Let there also exist positive constants l_1, l_2, l_3, l_4 and functions $\eta_1(\cdot), \eta_2(\cdot) \in \mathcal{W}([-\tau,0];R_+)$ such that*

$$x^T f(t,x) \le -l_1|x|^2, \quad |f(t,x)|^2 \le l_2|x|^2, \tag{21}$$

$$|g(t,\varphi)|^p \le l_3 \int_{-\tau}^{0} \eta_1(s)|\varphi(s)|^p ds, \quad |\sigma(t,\varphi)|^p \le l_4 \int_{-\tau}^{0} \eta_2(s)|\varphi(s)|^p ds \tag{22}$$

for all $t \ge 0$, $x \in R^n$ and $\varphi \in L^p([-\tau,0];R^n)$. Let also the following condition be valid:

$$\begin{aligned} 0 &\le (p+2)\big(\sqrt{kl_2} + \sqrt{kl_3}\big) + 4\big[\sqrt{kl_3} + \sqrt{l_3} + (p-1)l_4\big] \\ &\quad + k(p-2)\left(-2l_1 + \sqrt{kl_2} + \sqrt{l_3} + \frac{1+\sqrt{k}}{\sqrt{l_3}} + p - 1 \right) \\ &< (p+2)\big(2l_1 - \sqrt{kl_2} - \sqrt{l_3}\big) - (p-2)\left(\frac{\sqrt{l_2}+\sqrt{l_3}}{\sqrt{k}} + \frac{1+\sqrt{k}}{\sqrt{l_3}} + p - 1 \right) \end{aligned} \tag{23}$$

Then the trivial solution of Eq. (1) is the p-th moment exponentially stable.

Proof. If we put

$$\begin{aligned} I_1(t) &\equiv [\varphi(0) - G(\varphi)]^T [f(t,\varphi(0)) + g(t,\varphi)] \\ &\le \varphi^T(0) \cdot f(t,\varphi(0)) + |G(\varphi)| \cdot |f(t,\varphi(0))| + |\varphi(0)| \cdot |g(t,\varphi)| \\ &\quad + |G(\varphi)| \cdot |g(t,\varphi)|, \\ I_2(t) &\equiv |\sigma(t,\varphi)|^2, \\ I_3(t) &\equiv |[\varphi(0) - G(\varphi)]^T \sigma(t,\varphi)|^2 \\ &\le |\varphi(0) - G(\varphi)|^2 \cdot |\sigma(t,\varphi)|^2, \end{aligned}$$

the left hand side in (8) becomes

$$\begin{aligned} I(t) &\equiv \frac{p}{2} |\varphi(0) - G(\varphi)|^{p-4} \big[|\varphi(0) - G(\varphi)|^2 \big(2I_1(t) + I_2(t)\big) + (p-2)I_3(t) \big] \\ &\le p\, |\varphi(0) - G(\varphi)|^{p-2} \big[\varphi^T(0) \cdot f(t,\varphi(0)) + |G(\varphi)| \cdot |f(t,\varphi(0))| \\ &\quad + |\varphi(0)| \cdot |g(t,\varphi)| + |G(\varphi)| \cdot |g(t,\varphi)| + 2(p-1)|\sigma(t,\varphi)|^2 \big]. \end{aligned}$$

By using the inequalities $(|a|+|b|)^{p-2} \le 2^{p-3}(|a|^{p-2}+|b|^{p-2})$ and $|a|\cdot|b| \le (|a|^2+|b|^2)/2$, we deduce that

$$I(t) \le p\,2^{p-4}\big[|\varphi(0)|^{p-2}+|G(\varphi)|^{p-2}\big]\Bigg[-2l_1|\varphi(0)|^2+\sqrt{\frac{l_2}{k}}|G(\varphi)|^2 +\sqrt{\frac{k}{l_2}}|f(t,\varphi(0))|^2+\sqrt{l_3}|\varphi(0)|^2+\frac{1}{\sqrt{l_3}}|g(t,\varphi)|^2+\sqrt{\frac{l_3}{k}}|G(\varphi)|^2 +\sqrt{\frac{k}{l_3}}|g(t,\varphi)|^2+(p-1)|\sigma(t,\varphi)|^2\Bigg].$$

Now, we can apply Young inequality $(|a|^p)^{\frac{2}{p}}(|b|^p)^{\frac{p-2}{p}} \le \frac{2}{p}|a|^p+\frac{p-2}{p}|b|^p$ to estimate the terms of the form $|a|^2|b|^{p-2}$, then use (21) and (22) and finally obtain

$$\begin{aligned}I(t) \le 2^{p-4}\Bigg\{&\Bigg[(p+2)\big(-2l_1+(\sqrt{kl_2}+\sqrt{l_3}\big)\\&+(p-2)\Bigg(\frac{\sqrt{l_2}+\sqrt{l_3}}{\sqrt{k}}+\frac{1+\sqrt{k}}{\sqrt{l_3}}+p-1\Bigg)\Bigg]|\varphi(0)|^p\\&+\Bigg[k(p-2)\Bigg(-2l_1+\sqrt{kl_2}+\sqrt{l_3}+\frac{1+\sqrt{k}}{\sqrt{l_3}}+p-1\Bigg)\\&+(p+2)\big(\sqrt{kl_2}+\sqrt{kl_3}\big)\Bigg]\int_{-\tau}^0\eta(s)|\varphi(s)|^p ds\\&+4\big(\sqrt{kl_3}+\sqrt{l_3}\big)\int_{-\tau}^0\eta_1(s)|\varphi(s)|^p ds\\&+4(p-1)l_4\int_{-\tau}^0\eta_2(s)|\varphi(s)|^p ds\Bigg\}\\ \equiv d_1|\varphi(0)|^p+&\int_{-\tau}^0\big[d_2\eta(s)+d_3\eta_1(s)+d_4\eta_2(s)\big]\,|\varphi(s)|^p ds,\end{aligned}$$

where d_1, d_2, d_3, d_4 are some generic constants. Therefore,

$$I(t) \le d_1|\varphi(0)|^p+(d_2+d_3+d_4)\int_{-\tau}^0\eta_3(s)|\varphi(s)|^p ds,$$

where $\eta_3(s)=\frac{d_2\eta(s)+d_3\eta_1(s)+d_4\eta_2(s)}{d_2+d_3+d_4}$ and $\eta_3(\cdot)\in\mathcal{W}([-\tau,0];R_+)$.

On the basis of (23) we deduce that $d_1 \le 0$ and $d_2+d_3+d_4>0$. If we put $\lambda_1=-d_1$, $\lambda_2=d_2+d_3+d_4$, we also find that $0\le\lambda_2<\lambda_1$. Thus, all the conditions of Theorem 1 are valid and, therefore, the trivial solution of Eq. (1) is the p-th moment exponentially stable. This completes the proof. □

Note that for $p = 2$ the condition (23) is reduced to $2l_1 > 2\sqrt{kl_2} + 2\sqrt{kl_3} + 2\sqrt{l_3} + l_4$, i.e. to the one from paper [8].

Note also that a corollary based on Theorem 2 cannot be analogously formulated because it is not possible to obtain $d_1 + d_2 + d_3 < 0$ by using the previous procedure.

We conclude the paper by an example to illustrate the previous theoretical considerations.

Example. Let us determine sufficient conditions under which the trivial solution of Eq. (1) is the p-th moment exponentially stable. We suppose that $w(t)$ is a one-dimensional Brownian motion and

$$\begin{aligned}
f(t, x(t)) &= -\begin{pmatrix} \left[a + \frac{1}{1+|x^2(t)|}\right] x^1(t) \\ a\, x^2(t) \end{pmatrix}, \\
G(x_t) &= \frac{k^{\frac{1}{p}}}{\tau} \int_{-\tau}^{0} \begin{pmatrix} \sin(x_t^1(s)) \\ -\sin(x_t^2(s)) \end{pmatrix} ds, \\
g(t, x_t) &= \theta \int_{-\tau}^{0} \begin{pmatrix} \ln[1 + x_t^1(s)] \\ -x_t^2(s) \end{pmatrix} ds, \\
\sigma(t, x_t) &= \frac{\gamma 3^{\frac{1}{p}}}{\tau^{\frac{2}{p}}} \int_{-\tau}^{0} s^{\frac{2}{p}} \begin{pmatrix} x_t^1(s) \\ x_t^2(s) \end{pmatrix} ds,
\end{aligned}$$

where $a > 0$, $0 < k < 1$, θ, γ are constants and $x(t) = (x^1(t), x^2(t))^T$, $x_t(s) = (x_t^1(s), x_t^2(s))^T$. It is easy to check that

$$\begin{aligned}
(x(t))^T f(t, x(t)) &\le -a\, |x(t)|^2, \\
|f(t, x(t))|^2 &\le (a+1)^2 |x(t)|^2,
\end{aligned}$$

so that $l_1 = a, l_2 = (a+1)^2$. Likewise, by applying Hölder inequality we have

$$\begin{aligned}
|G(x_t)|^p &\le \frac{k}{\tau^p}\, \tau^{p-1} \int_{-\tau}^{0} |x_t(s)|^p\, ds = k \int_{-\tau}^{0} \frac{1}{\tau}\, |x_t(s)|^p\, ds, \\
|g(t, x_t)|^p &\le (\theta\tau)^p \int_{-\tau}^{0} \frac{1}{\tau}\, |x_t(s)|^p\, ds, \\
|\sigma(t, x_t))|^p &\le (\gamma\tau)^p \int_{-\tau}^{0} \frac{3}{\tau^3} s^2 |x_t(s)|^p\, ds,
\end{aligned}$$

so that $l_3 = (\theta\tau)^p$, $l_4 = (\gamma\tau)^p$, $\eta(s) = \eta_1(s) \equiv 1/\tau$, $\eta_3(s) = 3s^2/\tau^3$. By using the condition (23) from Corollary 1, we conclude that the trivial solution will be the p-the moment exponentially stable if

$$\begin{aligned}0 &\le (p+2)\sqrt{k}\left[a+1+(\theta\tau)^{\frac{p}{2}}\right]+4\left[\sqrt{k}(\theta\tau)^{\frac{p}{2}}+(\theta\tau)^{\frac{p}{2}}+(p-1)(\gamma\tau)^{p}\right]\\ &\quad +(p-2)\sqrt{k}\left[-2a+\sqrt{k}(a+1)+(\theta\tau)^{\frac{p}{2}}+\frac{1+\sqrt{k}}{(\theta\tau)^{\frac{p}{2}}}+p-1\right]\\ &< (p+2)\left[2a-\sqrt{k}(a+1)-(\theta\tau)^{\frac{p}{2}}\right]\\ &\quad -(p-2)\left[\frac{a+1+(\theta\tau)^{\frac{p}{2}}}{\sqrt{k}}+\frac{1+\sqrt{k}}{(\theta\tau)^{\frac{p}{2}}}+p-1\right].\end{aligned}$$

Let us specify $p=4$, $\theta=\gamma=1/4$, for example. Then, from the previous relation we find that

$$\begin{aligned}a\equiv h(k,\tau) > \frac{1}{2(1-\sqrt{k})(5\sqrt{k}+k\sqrt{k}-k-1)}\cdot\Bigg[\frac{32}{\tau^2}(k^2+k+k\sqrt{k}+\sqrt{k})\\ +\frac{\tau^2}{8}\Big(5k+k\sqrt{k}+\frac{43}{8}\sqrt{k}+1\Big)+2k^2+12k+6k\sqrt{k}+6\sqrt{k}+2\Bigg],\end{aligned}$$

where $5\sqrt{k}+k\sqrt{k}-k-1>0$, i.e. $k\in(0.04271,1)$. In Figure 1 the plot

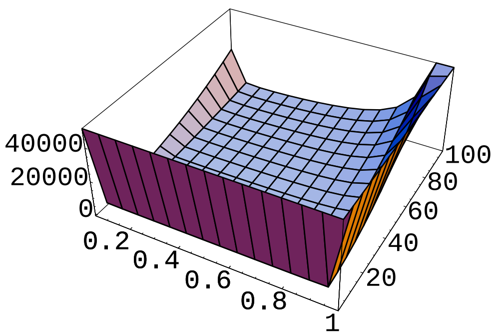

FIGURE 1. Graph of the function $a=h(k,\tau)$, for $k\in(0.0427,1)$ and $\tau\in(0,100)$

$a=h(k,\tau)$ is given for $k\in(0.0427,1)$, $\tau\in(0,100)$. Clearly, all points (k,τ,a) over this plot represent the area of the fourth moment exponential stability of the considered equation. Let us analyze this area.

For a fixed $k\in(0.04271,1)$, if τ goes to zero, then a increases very quickly. In Figure 2 a dependence between τ and a is represented for $k=0.5$. So, for $\tau=0.5$, for example, we find $a>16571.2$. It can be shown that for every $k\in(0.04271,1)$ the least a is obtained for $\tau\in(2,4)$. Thus, for $k=0.5$, $\tau=2.4$, for example, we find $a>21.2617$. If k increases, then a increases too, which can be seen in Figure 1 and Figure 3.

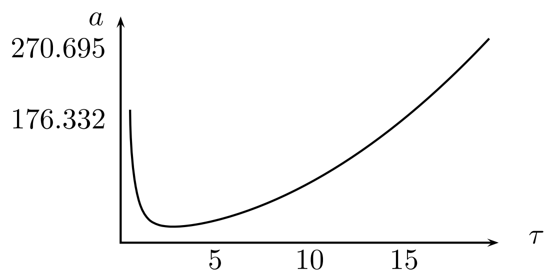

FIGURE 2. Graph of the function $a = h(0.5, \tau)$

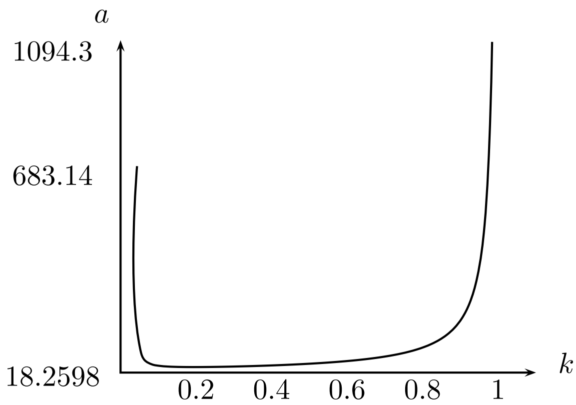

FIGURE 3. Graph of the function $a = h(k, 5)$

For a fixed τ, $\tau = 5$, for example, and for k close to 0.04271, parameter a is very big, but it decreases very quickly if k increases up to the minimum point $k = 0.192764$, $a = 18.2598$ (see Figure 3). After that, if $k > 0.192764$ parameter a increases and, clearly, the stability area is $a > h(k, 5)$.

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