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THE Π_3-THEORY OF THE COMPUTABLY ENUMERABLE TURING DEGREES IS UNDECIDABLE

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ABSTRACT. We show the undecidability of the Π_3-theory of the partial order of computably enumerable Turing degrees.

0. INTRODUCTION

Recursively enumerable (henceforth called *computably enumerable*) sets arise naturally in many areas of mathematics, for instance in the study of elementary theories, as solution sets of polynomials or as the word problems of finitely generated subgroups of finitely presented groups. Putting the computably enumerable sets into context with each other in various ways yields structures whose study has for long been a mainstay of computability theory. If the sets are related in the most elementary way, namely by inclusion, one obtains a distributive lattice $\mathcal{E}$ with very complex algebraic properties. Another way to compare sets is to look at the information content. Turing reducibility is a very general, but the most widely accepted concept of relative computability: a set X of natural numbers is Turing-reducible to Y iff the answer to "$n \in X$?" can be determined by a Turing machine computation which can use answers to oracle questions "$y \in Y$?" during the computation. (For more restricted notions of relative computability one would for instance place a priori bounds on the lengths of computations or would limit the access to the oracle.) The Turing degree of a set, i.e. its equivalence class under the equivalence relation given by this preordering, measures the information content of a set while stripping away the features of the set inessential from a computational point of view (for this general concept of computation).

Here we are concerned with the partial order of Turing degrees of computably enumerable sets. This structure has been closely investigated for over fifty years, starting with Post's seminal paper [Po44] and even before. Results of the 1950's and early 1960's, in particular the ground-breaking Sacks Density Theorem [Sa64], led Shoenfield [Sh65] to conjecture a strong homogeneity property (namely, that any extension of embeddings of finite posets consistent with the theory of upper semilattices is always possible). This conjecture would have implied the decidability of the full first-order theory of the computably enumerable degrees. However,

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Shoenfield's conjecture was refuted by the minimal pair theorem of Lachlan [La66] and Yates [Ya66]. Further work in the 1970's and 1980's revealed more and more of the complexity of the poset of the computably enumerable degrees and led to a proof of the undecidability of its first-order theory by Harrington and Shelah, as announced in [HS82].

Once undecidability of a theory has been established, one reasonable next question is at which exact level of quantifier alternations (in brief: quantifier level), if at any, undecidability first occurs. The coding methods traditionally used for showing undecidability of structures arising from computability theory (or elsewhere) establish undecidability at some quantifier level. However, the more indirect the coding is, the higher this level. Sentences actually proved in mathematical investigations of the structures usually have a low quantifier level: for instance, for the partial ordering of computably enumerable degrees, the statement that there is a minimal pair (Σ_2), there is a nonzero degree bounding no minimal pair (Σ_3) and that meet-reducible degrees exist in any open interval (Π_3). So the meaning of the question above is to determine which fragments of the theory experience has shown to be mathematically relevant are undecidable.

By an early result of Sacks [Sa63], the universal, or Π_1-, fragment of the theory of the poset of the computably enumerable degrees is decidable since any existential statement consistent with the theory of partial orderings holds. The Π_2-theory is conjectured by many to be decidable, but, despite many efforts, this still remains an open problem. Two interesting fragments of the Π_2-theory have been considered. The extension of embeddings (of partial orderings) problem was shown to be decidable by Slaman and Soare [SSta]; the lattice embeddings problem, however, remains open (see Lerman [Le96] and Lempp and Lerman [LLta] for recent updates).

On the undecidability side, by work of Harrington and Slaman, [HS82], the Π_4-theory of the poset of the computably enumerable degrees was known to be undecidable. (A much easier proof by Ambos-Spies and Shore [AS93] gave the undecidability of the Π_5-theory.) The present paper establishes undecidability for the Π_3-theory by a very delicate coding so as to minimize quantifier alternations, using the undecidability of the Σ_2-theory of the class of finite bipartite graphs (in the language of just one binary relation, without equality) and Nies's Transfer Lemma [Ni96].

Our paper is organized as follows: In the next section, we present the statement of our theorem establishing our undecidability result and explain the algebraic part of the proof, i.e., the coding. The later sections give the requirements, the intuition, the full construction, and the verification, respectively, for the computability-theoretic part of our result.

For the computability-theoretic argument, we assume the reader to be familiar with $\mathbf{0}'''$-priority arguments. (Chapter XIV of Soare [So87] provides an introduction.)

Our notation generally follows Soare [So87] with some exceptions: The names for the partial computable functionals used follow the "Chicago convention", i.e., those built by us are denoted by upper-case Greek letters near the beginning of the alphabet, those built by the opponent by letters near the end of the alphabet; the use of each functional is denoted by the corresponding lower-case Greek letter. Partial computable functions are denoted by lower-case Greek letters also. Note that we take the use of a functional to be the largest number actually used in the computation (so that changing X at $\varphi(x)$ will destroy a current computation $\Phi^X(x)$). If the oracle of a functional is given as the join of at least two sets, we let the use be the largest number used in one of the sets of this join (so $\Phi^{X_0 \oplus \cdots \oplus X_n}(x)$ is defined iff $\Phi^{X_0 \restriction (\varphi(x)+1) \oplus \cdots \oplus X_n \restriction (\varphi(x)+1)}$ is).

1. The theorems and the algebraic component of the proof

Our main result is

Theorem 1. *The Π_3-theory of the computably enumerable Turing degrees in the language of partial orderings is undecidable.*

Recall that a set of first-order sentences S is *hereditarily undecidable* if there is no computable set of sentences separating $\overline{S}$ and $S \cap V$, where V is the set of all valid sentences in the language of S. Our proof of Theorem 1 uses the following undecidability result for finite bipartite graphs.

Theorem 2 (see Nies [Ni96])**.** *The Σ_2- (and hence the Π_3-) theory of the finite bipartite graphs with nonempty left and right domains in the language of one binary relation, but without equality, is hereditarily undecidable.*

Theorem 2 will be used to prove Theorem 1 via Nies's Transfer Lemma. We first recall a definition.

Definition 3. Let $\mathcal{L}_\mathcal{C}$ and $\mathcal{L}_\mathcal{D}$ be finite relational languages not necessarily containing equality.

(i) A Σ_k*-scheme* s for $\mathcal{L}_\mathcal{C}$ and $\mathcal{L}_\mathcal{D}$ consists of a Σ_k-formula $\varphi_U(\overline{x};\overline{y})$ (in the language $\mathcal{L}_\mathcal{D}$), and, for each m-ary relation symbol $R \in \mathcal{L}_\mathcal{C}$, two Σ_k-formulas $\varphi_R(\overline{x}_0, \dots, \overline{x}_{m-1}; \overline{y})$ and $\varphi_{\neg R}(\overline{x}_0, \dots, \overline{x}_{m-1}; \overline{y})$ (again in $\mathcal{L}_\mathcal{D}$).

(ii) For a Σ_k-scheme s, we define a *correctness condition* $\alpha(\overline{p})$ for a list of parameters $\overline{p}$ as the conjunction of the following formulas:

(a) (coding the universe) $\{\overline{x} \mid \varphi_U(\overline{x};\overline{p})\} \neq \emptyset$, and
(b) (coding the relations) for each m-ary relation symbol R in the language $\mathcal{L}_\mathcal{C}$, the set $\{(\overline{x}_0, \dots, \overline{x}_{m-1}) \mid \forall i < m(\varphi_U(\overline{x}_i;\overline{p})\}$ is the disjoint union of the two sets $\{(\overline{x}_0, \dots, \overline{x}_{m-1}) \mid \varphi_R(\overline{x}_0, \dots, \overline{x}_{m-1};\overline{p})\}$ and $\{(\overline{x}_0, \dots, \overline{x}_{m-1}) \mid \varphi_{\neg R}(\overline{x}_0, \dots, \overline{x}_{m-1};\overline{p})\}$.

$\alpha(\overline{p})$ is then a Π_{k+1}-formula.

(iii) Define a formula $\varphi_{eq(\mathcal{C})}(x,y)$ as the conjunction of all formulas $\forall \overline{z}(R(x,\overline{z}) \leftrightarrow R(y,\overline{z}))$ where R ranges over all relations $R \in \mathcal{L}_\mathcal{C}$ and over all permutations of the arguments of R. (This formula will just redefine equality if the language contains equality.) For an $\mathcal{L}_\mathcal{C}$-structure C, define the induced quotient structure $C/eq(C)$ in the obvious way. Similarly define a formula $\varphi_{eq(\mathcal{D})}(x,y)$ and a quotient structure $D/eq(D)$, using the relations $R \in \mathcal{L}_\mathcal{D}$.

(iv) A class $\mathcal{C}$ of relational structures in $\mathcal{L}_\mathcal{C}$ is Σ_k*-elementarily definable with parameters* in a class of relational structures $\mathcal{D}$ in $\mathcal{L}_\mathcal{D}$ if there is a Σ_k-scheme s such that for each structure $C \in \mathcal{C}$, there are a structure $D \in \mathcal{D}$ and a finite set of parameters $\overline{p} \in D$ satisfying the following:

(a) (correctness condition) $D \models \alpha(\overline{p})$, and
(b) (coding the structure) $C/eq(C) \cong \tilde{C}/eq(\tilde{C})$ where $\tilde{C}$ is the $\mathcal{L}_\mathcal{C}$-structure defined by $\tilde{C} = \{\overline{x} \mid \varphi_U(\overline{x};\overline{p})\}$, and for each m-ary relation symbol $R \in \mathcal{L}_\mathcal{C}$, the relation $\tilde{R}$ on $\tilde{C}$ is defined by $\tilde{R} = \{(\overline{x}_0, \dots, \overline{x}_{m-1}) \mid \varphi_R(\overline{x}_0, \dots, \overline{x}_{m-1};\overline{p})\}$.

Note here that we treat equality just like any other relation symbol in the language $\mathcal{L}_\mathcal{C}$. The following theorem now lets us transfer (hereditary) undecidability.

Theorem 4 (Nies's Transfer Lemma [Ni96])**.** *Fix $k \geq 1$ and $r \geq 2$. Suppose a class of structures $\mathcal{C}$ is Σ_k-elementarily definable with parameters in a class of structures $\mathcal{D}$ (in finite relational languages $\mathcal{L}_\mathcal{C}$ and $\mathcal{L}_\mathcal{D}$, respectively). Then the hereditary undecidability of the Π_{r+1}-theory of $\mathcal{C}$ implies the hereditary undecidability of the Π_{r+k}-theory of $\mathcal{D}$.*

The heart of our argument is then the following:

Theorem 5. *The class of finite bipartite graphs with nonempty left and right domains in the language without equality is Σ_1-elementarily definable in the partial ordering $\mathcal{R}$ of the computably enumerable Turing degrees (i.e. in the class $\{\mathbf{R}\}$).*

Proof of Theorem 1. Apply Nies's Transfer Lemma (Theorem 4, setting $k = 1$ and $r = 2$) to Theorem 2 in order to obtain the hereditary undecidability of the Π_3-theory of the computably enumerable degrees. ☐

A coding of finite bipartite graphs was first used in [LNi95] to establish the undecidability of the Π_4-theory of the computably enumerable wtt–degrees. Here, we also use an ambiguous representation of vertices (as explained below) to ensure the coding is by Σ_1–formulas.

The remaining sections are devoted to the proof of Theorem 5.

2. The requirements for Theorem 5

In this section, we present the computability-theoretic requirements for our construction and show how their satisfaction implies Theorem 5.

Fix a finite bipartite graph G with left domain $L = \{0, 1, \dots, n\}$, right domain $R = \{\hat{0}, \hat{1}, \dots, \hat{n}\}$, and edge relation $E \subseteq L \times R$.

We begin by coding the left domain, using a Σ_1-formula $\psi(x; a, b, c)$. Each node $i \in L$ is represented ambiguously by each degree in a half-open interval $(\mathbf{0}, \mathbf{a}_i]$ of computably enumerable degrees, where the degrees have the following properties:

(6.1) $\mathbf{a}_i$ and $\mathbf{a}_j$ form a minimal pair for $i \neq j$;

(6.2) there are computably enumerable degrees $\mathbf{b}$ and $\mathbf{c}$ such that

$$\forall \mathbf{x} \leq \mathbf{a}(\mathbf{x} \cup \mathbf{b} \ngeq \mathbf{c} \leftrightarrow \exists i \leq n(\mathbf{x} \leq \mathbf{a}_i)),$$

where $\mathbf{a}$ is the join of $\mathbf{a}_0, \dots, \mathbf{a}_n$. See Figure 1.

A Σ_1-formula $\psi(x; a, b, c)$ to code the left domain can now be chosen as

$$0 < x \leq a \,\&\, x \vee b \ngeq c,$$

i.e., as

$$\exists y(y < x) \,\&\, x \leq a \,\&\, \exists y(y \geq x \,\&\, y \geq b \,\&\, y \ngeq c).$$

The above translates into the following requirements: We build computably enumerable sets $A_0, \dots, A_n$, B, and C, and we set $A = \bigoplus_{i \leq n} A_i$. We ensure (6.2) by requiring, for all partial computable functionals Φ and Ψ, all computably enumerable sets X, and all $i \leq n$:

$$\mathcal{N}_{X,\Phi} : X = \Phi^A \rightarrow C = \Gamma^{X \oplus B} \text{ or } \exists i \leq n \exists \Delta(X = \Delta^{A_i})$$

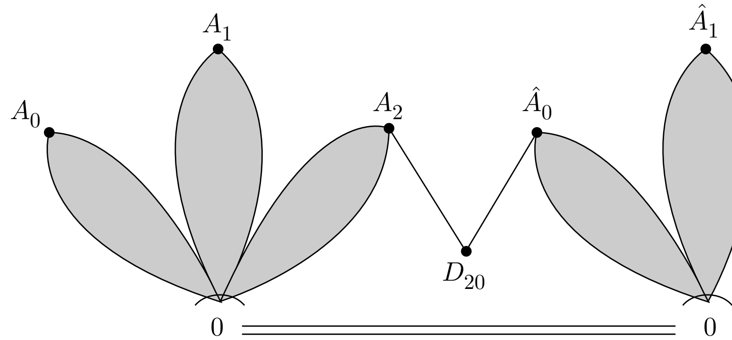

FIGURE 1

(where Γ and Δ are computable functionals built by us and depending on X and Φ), and

$$\mathcal{P}^i_\Psi : C \neq \Psi^{A_i \oplus B}.$$

In order to ensure (6.1), we require furthermore (using "Posner's trick") for all partial computable functionals Ξ, all partial computable functions ν, and all distinct $i, j \leq n$:

$$\mathcal{M}^{i,j}_\Xi : \Xi^{A_i} = \Xi^{A_j} \text{ total } \quad \rightarrow \Xi^{A_i} = \vartheta$$

(where ϑ is a computable function built by us depending on i, j, and Ξ), and

$$\mathcal{Q}^i_\nu : A_i \neq \nu.$$

The right domain is coded similarly using computably enumerable sets $\hat{A}_{\hat{0}}, \dots,$ $\hat{A}_{\hat{n}}, \hat{A}, \hat{B}$, and $\hat{C}$, requirements $\hat{\mathcal{N}}_{\hat{X},\hat{\Phi}}$, $\hat{\mathcal{P}}^{\hat{\imath}}_{\hat{\Psi}}$, etc., and functionals $\hat{\Gamma}$, $\hat{\Delta}$, etc. The Σ_1-formula $\varphi_U(x; a, b, c)$ required by Definition 3 can now be chosen as $\psi(x; a, b, c) \vee \psi(x; \hat{a}, \hat{b}, \hat{c})$.

The point of using an ambiguous representation of the vertices is that the formula φ_U has to be Σ_1. Of course we can define e.g. the set $\{\mathbf{a}_0, \dots, \mathbf{a}_n\}$ as the maximal degrees $\mathbf{x}$ satisfying $\psi(\mathbf{x}; \mathbf{a}, \mathbf{b}, \mathbf{c})$, but this is only a $\Sigma_1\&\Pi_1$-definition. Property (6.1) enables us to recover the vertex from a representing degree.

Now, in defining a copy of the edge relation with parameters, we have to make sure the formula does not depend on the particular representing degrees chosen.

We build two additional computably enumerable degrees $\mathbf{d}$ and $\tilde{\mathbf{d}}$ satisfying for all $i \leq n$ and $\hat{\imath} \leq \hat{n}$:

(6.3) $E(i, \hat{\imath})$ iff not $\mathbf{a}_i \cap \hat{\mathbf{a}}_{\hat{\imath}} \cap \mathbf{d} = \mathbf{0}$ iff $\mathbf{a}_i \cap \hat{\mathbf{a}}_{\hat{\imath}} \cap \tilde{\mathbf{d}} = \mathbf{0}$.

The Σ_1-formula $\varphi_E(x, \hat{x}; d)$ required by Definition 3 can now be chosen as

$$\exists y > 0 \, \exists x_1 \geq x \, \exists \hat{x}_1 \geq \hat{x} \, (\psi(x_1; a, b, c) \,\&\, \psi(\hat{x}_1; \hat{a}, \hat{b}, \hat{c}) \,\&\, (y \leq x_1 \,\&\, y \leq \hat{x}_1 \,\&\, y \leq d)),$$

i.e., as $\exists y \exists z \, (z < y \,\&\, \dots)$.

The Σ_1-formula $\varphi_{\neg E}(x, \hat{x}; \tilde{d})$ is chosen similarly using $\tilde{d}$ in place of d.

In order to ensure the first equivalence of (6.3), we build, for each $(i, \hat{\imath})$, a computably enumerable set $D_{i,\hat{\imath}}$, define $F_{i,\hat{\imath}} = \bigoplus_{(j,\hat{\jmath}) \neq (i,\hat{\imath})} D_{j,\hat{\jmath}}$, and require (again using

Posner's trick) for all partial computable functionals Ω, all partial computable functions χ, and all $(i,\hat{\imath}) \in L \times R$:

$$\mathcal{R}_\Omega^{i,\hat{\imath}} : \Omega^{A_i} = \Omega^{\hat{A}_{\hat{\imath}}} = \Omega^{F_{i,\hat{\imath}}} \text{ total } \to \Omega^{F_{i,\hat{\imath}}} = \lambda$$

(where λ is a computable function built by us depending on $i, \hat{\imath}$, and Ω), as well as

$$\mathcal{S}^{j,\hat{\jmath}} : D_{j,\hat{\jmath}} \leq_T A_j, \hat{A}_{\hat{\jmath}},$$
$$\mathcal{T}_\chi^{j,\hat{\jmath}} : D_{j,\hat{\jmath}} \neq \chi.$$

We then simply let $\mathbf{d}$ and $\tilde{\mathbf{d}}$ be the degrees of the sets $D = \bigoplus_{E(j,\hat{\jmath})} D_{j,\hat{\jmath}}$ and $\tilde{D} = \bigoplus_{\neg E(j,\hat{\jmath})} D_{j,\hat{\jmath}}$, respectively.

To verify, e.g., that the first equivalence in (6.3) is satisfied, note that the direction from left to right holds by the definition of D, and the other direction since, if not $E(i,\hat{\imath})$, then $D \leq_T F_{i,\hat{\imath}}$. Also note that the requirements $\mathcal{T}_\chi^{j,\hat{\jmath}}$ and $\mathcal{S}^{j,\hat{\jmath}}$ together imply that the requirements $\mathcal{Q}_\nu^i$ are satisfied, so we can omit the latter ones from now on.

We now see that the above requirements ensure (6.1)–(6.3), and that the latter establish Theorem 5 via the formulas φ_U, φ_E, and $\varphi_{\neg E}$ given above. So we are reduced to showing the satisfaction of our requirements.

3. The intuition for Theorem 5

The crucial part of the construction, and the part which makes this a $\mathbf{0}'''$-priority argument, is the interplay between the $\mathcal{N}$- and the $\mathcal{P}$-strategies. (This interaction was first considered by Harrington and Slaman [HSta] in their work on the undecidability of the theory of the computably enumerable degrees.) So we will explain this part first and then gradually add the rest of the requirements. Since we will add the (fairly easy) $\mathcal{R}$-, $\mathcal{S}$-, and $\mathcal{T}$-requirements last (which constitute the only connection between the unhatted and the hatted side), we will concentrate on the unhatted side first, the hatted side being entirely analogous.

3.1. $\mathcal{P}$-strategies below one $\mathcal{N}$-strategy. An $\mathcal{N}$-strategy α starts out by building $\Gamma^{X \oplus B} = C$ as long as the length of agreement between X and Φ^A keeps increasing. The basic $\mathcal{P}$-strategy β is merely a Friedberg-Muchnik strategy and is thus only dangerous to the $\mathcal{N}$-strategy α if α is of higher priority than β. In that case, β typically enumerates a witness x into C while restraining $B\restriction(\psi(x)+1)$, where typically $\psi(x)$ was defined much later than $\gamma(x)$, and thus possibly $\psi(x) > \gamma(x)$, whence α cannot correct $\Gamma^{X \oplus B}(x)$ via B. But note that an $X\restriction(\gamma(x)+1)$-change after $\Psi^{A_i \oplus B}(x)$ is defined would allow α to reset $\gamma(x) > \psi(x)$ without changing B, and then β can later enumerate x into C and the new $\gamma(x)$ into B without injuring $\Psi^{A_i \oplus B}(x)$ or making $\Gamma^{X \oplus B}(x)$ incorrect. Alternatively, if $X\restriction(\gamma(x)+1)$ does not change, then β will enumerate $\gamma(x)$ into B (but *not* x into C so as to retain this witness), and so destroy $\Psi^{A_i \oplus B}(x)$ while defining $\Delta^{A_i}\restriction(\gamma(x)+1) = X\restriction(\gamma(x)+1)$. Intuitively, X has two choices: to change and allow $\gamma(x) > \psi(x)$ at some point, or not to change and risk $\Delta^{A_i} = X$ (while $\Gamma^{X \oplus B}(x)$ is undefined since $\gamma(x)$ tends to infinity).

To be more precise, we distinguish three phases in the action of β, depending on which of the three sets $\bigoplus_{j\neq i} A_j$, B, and A_i is unrestrained. The $\mathcal{P}^i_\Psi$-strategy proceeds as follows:

1. Pick a fresh witness x (for $C(x) \neq \Psi^{A_i \oplus B}(x)$).
2. Wait for $\Psi^{A_i \oplus B}(x) \downarrow= 0$.
3. ($\bigoplus_{j\neq i} A_j$-phase) Restrain $(A_i \oplus B)\restriction(\psi(x)+1)$ and request that $\gamma(x)$ be set $> \psi(x)$ if $X\restriction(\gamma(x)+1)$ changes. Wait for this until the next β-stage (which we assume to be α-expansionary).
4. If now $\gamma(x) > \psi(x)$, then enumerate x into C and $\gamma(x)$ into B and stop. Otherwise:
5. (B-phase) Enumerate $\gamma(x)$ into B (so $\gamma(x)$ will increase), define
$$\Delta^{A_i}\restriction(\gamma(x)+1) = X\restriction(\gamma(x)+1),$$
and restrain $A\restriction(\varphi\gamma(x)+1)$ (to prevent $X\restriction(\gamma(x)+1)$ from changing now when this X-change is useless to us). Wait for the next β-stage (which we again assume to be α-expansionary).
6. (A_i-phase) Allow A_i to change and correct Δ^{A_i} via A_i (if an X-change makes this necessary). Go back to Step 2. (The role of the A_i-phase will become clearer later when we add more requirements.)

The possible outcomes and effects of the $\mathcal{P}^i_\Psi$-strategy β are as follows:

A. β eventually waits at Step 2 forever or stops at Step 4: Then $\Psi^{A_i \oplus B}(x) \neq C(x)$, and β's effect is finitary.

B. β goes from Step 6 to Step 2 infinitely often: Then β enumerates an increasing (and thus computable) sequence of values $\gamma(x)$ into B (and thus destroys $\Gamma^{X\oplus B}$); alternately drops the restraint on $\bigoplus_{j\neq i} A_j$, B, and A_i, and achieves $\Delta^{A_i} = X$ since $X\restriction(\gamma(x)+1)$ does not change during $\bigoplus_{j\neq i} A_j$-phases by hypothesis, cannot change during B-phases by the $A\restriction(\varphi\gamma(x)+1)$-restraint; and may change during A_i-phases but then Δ^{A_i} is corrected by A_i-enumeration.

Note that a $\mathcal{P}$-strategy below the finite outcome A of β is in the same situation as β once β stops acting, while a $\mathcal{P}$-strategy below the infinite outcome B of β is completely finitary since it does not have to deal with $\Gamma^{X\oplus B}$-correction.

3.2. $\mathcal{P}$-strategies below several $\mathcal{N}$-strategies. The situation for a $\mathcal{P}$-strategy β below several $\mathcal{N}$-strategies (we will restrict ourselves here to two of them, α_0 and α_1, say) is more complicated since now one of X_0 and X_1 may change while the other does not. We proceed in the usual $\mathbf{0}'''$-priority fashion by "blaming" the lowest-priority $\mathcal{N}$-strategy without X-change.

More formally, the $\mathcal{P}$-strategy β proceeds as follows:

1. Pick a fresh witness x (for $C(x) \neq \Psi^{A_i \oplus B}(x)$).
2. Wait for $\Psi^{A_i \oplus B}(x) \downarrow= 0$.
3. ($\bigoplus_{j\neq i} A_j$-phase) Restrain $(A_i \oplus B)\restriction(\psi(x)+1)$ and request that $\gamma_k(x)$ be set $> \psi(x)$ if $X_k\restriction(\gamma(x)+1)$ changes (for $k \leq 1$). Wait until the next β-stage (which, as usual, we assume to be α_0- and α_1-expansionary).

4. If now $\gamma_1(x) > \psi(x)$, then proceed to Step 6. Otherwise: (B-phase) Enumerate $\gamma_1(x)$ into B (but neither $\gamma_0(x)$ into B nor x into C), define
$$\Delta_1^{A_i}\restriction(\gamma_1(x)+1) = X_1\restriction(\gamma_1(x)+1),$$
and restrain $A\restriction(\varphi\gamma_1(x)+1)$. Wait for the next β-stage.
5. (A_i-phase) Allow A_i to change and correct $\Delta_1^{A_i}$ via A_i if necessary. Go back to Step 2.
6. (still $\bigoplus_{j\neq i} A_j$-phase) If now (also) $\gamma_0(x) > \psi(x)$, then enumerate x into C and both $\gamma_0(x)$ and $\gamma_1(x)$ into B and stop. Otherwise:
7. (B-phase) Enumerate *both* $\gamma_0(x)$ and $\gamma_1(x)$ into B (killing *both* $\Gamma_0^{X_0\oplus B}(x)$ and $\Gamma_1^{A_i\oplus B}(x)$), discard (the current version of) $\Delta_1^{A_i\oplus B}$ (since it existed on the assumption that X_1 never changes during a $\bigoplus_{j\neq i} A_j$-phase), define
$$\Delta_0^{A_i}\restriction(\gamma_0(x)+1) = X_0\restriction(\gamma_0(x)+1),$$
and restrain $A\restriction(\varphi\gamma_0(x)+1)$. Wait for the next β-stage.
8. Allow A_i to change and correct $\Delta_0^{A_i}$ via A_i if necessary. Go back to Step 2.

The possible outcomes and effects of the $\mathcal{P}^i_\Psi$-strategy β are as follows:

A. β eventually waits at Step 2 forever or stops at Step 6: Then $\Psi^{A_i\oplus B}(x) \neq C(x)$, and β's effect is finitary.

B. β goes from Step 5 to Step 2 infinitely often but only finitely often from Step 8 to Step 2: Then β enumerates an increasing sequence of values $\gamma_1(x)$ and a finite sequence of values $\gamma_0(x)$ into B (and thus destroys $\Gamma_1^{X_1\oplus B}(x)$ but not $\Gamma_0^{X_0\oplus B}(x)$); alternately drops the restraints on $\bigoplus_{j\neq i} A_j$, B, and A_i (imposed in Steps 3–4) while those of Step 7 are finite; achieves $\Delta_1^{A_i} = X_1$ since $X_1\restriction(\gamma_1(x)+1)$ does not change after the last time Step 8 is visited; and $\Gamma_0^{X_0\oplus B}$ is not affected.

C. β goes from Step 8 to Step 2 infinitely often: Then β enumerates two increasing sequences of values $\gamma_0(x)$ and $\gamma_1(x)$ (and thus destroys both $\Gamma_0^{X_0\oplus B}(x)$ and $\Gamma_1^{X_1\oplus B}(x)$); alternately drops the restraints on $\bigoplus_{j\neq i} A_j$, B, and A_i; achieves $\Delta_0^{A_i} = X_0$ since $X_0\restriction(\gamma_0(x)+1)$ does not change unless $\Delta_0^{A_i}$-correction via A_i is allowed by Step 8; and both $\Gamma_1^{X_1\oplus B}$ and $\Delta_1^{A_i}$ are destroyed.

$\mathcal{P}$-strategies below the finite outcome A of β are in the same position as β itself once β stops acting. Below the infinite outcomes B and C of β, the situation is more complicated.

Below outcome B, a $\mathcal{P}$-strategy β' is only faced with one Γ, namely $\Gamma_0^{X_0\oplus B}$, which is handled as in Section 3.1. Furthermore, β' also has to deal with the A_i-enumeration of β during the latter's A_i-phase; but β' can then simply restrain $A\restriction(\varphi_1\gamma_1(x')+1)$ to prevent dangerous $X\restriction(\gamma_1(x')+1)$-changes causing $\Delta_1^{A_i}$-correction via A_i. And finally, the different phases of β and β' have to be coordinated by occasional finite delays for either of them so that either of β and β' enumerates into a set Z only when the other is also in a Z-phase. (In the formal construction, we will use *Z-stages* to ensure this.)

On the other hand, below the infinite outcome C of β, we must first introduce a new $\mathcal{N}_1$-strategy $\hat\alpha_1$ (trying to build a functional $\hat\Gamma_1^{X_i\oplus B} = C$) since $\Gamma_1^{X_1\oplus B}$ has been destroyed by β without building a Δ_1 to replace it. A $\mathcal{P}$-strategy β' below $\hat\alpha_1$

now acts as in Section 3.1 with respect to $\hat{\Gamma}_1^{X_1 \oplus B}$ while preventing A_i-injury (for $\Delta_0^{A_i}$-correction) as explained in the previous paragraph.

3.3. Adding the $\mathcal{M}$-requirements. For the $\mathcal{M}$-strategies (trying to make each pair $(\deg A_i, \deg A_j)$ a minimal pair), observe that each strategy discussed so far (or to be discussed later, as will be seen) which enumerates numbers into sets only enumerates numbers (or the uses of such numbers) chosen by itself and thus "large" with respect to higher-priority restraint. (This fails in many other $\mathbf{0}'''$-priority arguments.)

This special feature of our construction implies that the usual minimal pair strategy can be used to ensure the $\mathcal{M}$-requirements: At each $\mathcal{M}$-expansionary stage, numbers may be enumerated into either A_i or A_j but not both, and no "small" numbers may enter either set between $\mathcal{M}$-expansionary stages.

3.4. Adding the $\mathcal{R}$-, $\mathcal{S}$-, and $\mathcal{T}$-requirements. The $\mathcal{T}$-strategies are the only strategies enumerating into the sets $D_{j,\hat{\jmath}}$. The $\mathcal{S}$-requirements (for which there will be no separate strategies) are ensured by direct coding, i.e., any number entering $D_{j,\hat{\jmath}}$ also enters A_j and $\hat{A}_{\hat{\jmath}}$ at the same time, so that the $\mathcal{T}$-strategies may only enumerate into $D_{j,\hat{\jmath}}$ at stages at which enumeration into both A_j and $\hat{A}_{\hat{\jmath}}$ is allowed. The $\mathcal{T}$-strategies pick a fresh witness x, wait for $\chi(x) \downarrow = 0$, and then enumerate x at the next such stage.

The $\mathcal{R}$-requirements are ensured by modified minimal pair strategies. Note that an $\mathcal{R}$-strategy has three possibilities to preserve the common computation, namely, restraining A_i, $\hat{A}_{\hat{\imath}}$, or $F_{i,\hat{\imath}} = \bigoplus_{(i,\hat{\imath}) \neq (j,\hat{\jmath})} D_{j,\hat{\jmath}}$. There is no strategy (it would have to be a $\mathcal{T}$-strategy) wanting to enumerate into all of A_i, $\hat{A}_{\hat{\imath}}$, and $F_{i,\hat{\imath}}$ simultaneously. So we can proceed as for the $\mathcal{M}$-strategies in Section 3.3, allowing at $\mathcal{R}$-expansionary stages enumeration into at most two of A_i, $\hat{A}_{\hat{\imath}}$, and $F_{i,\hat{\imath}}$.

3.5. The $(Z, \hat{Z})$-stage mechanism. There is one detail that we have glossed over so far, namely, the coordination of stages at which numbers may be enumerated into the various sets. We use the feature of $(A_i, A_{\hat{\imath}})$-*stages* (i.e., stages at which numbers may enter only the sets $A_i, A_{\hat{\imath}}$, and $D_{i,\hat{\imath}}$), and $(B, \hat{B})$-stages (i.e., stages at which numbers may enter only the sets B and $\hat{B}$). (These "$(Z, \hat{Z})$-stages", as we will call them, impose *no restrictions on C or $\hat{C}$,* i.e., numbers may enter these sets at any time.) It should be clear that requiring that any stage be a $(Z, \hat{Z})$-stage (for $Z = (B, \hat{B})$ or $(A_i, \hat{A}_{\hat{\imath}})$ for some $(i, \hat{\imath}) \in L \times R$), combined with usual restraint during nonexpansionary stages, ensures the satisfaction of the $\mathcal{M}$- and $\mathcal{R}$-requirements as explained in Sections 3.3 and 3.4.

The only thing we have to ensure for the strategies that enumerate numbers (i.e., the $\mathcal{P}$-, and $\mathcal{T}$-strategies) is that each has infinitely many chances to enumerate a number into whatever set it wants. For this, we require:

(7) Any strategy along the true path can act at infinitely many $(Z, \hat{Z})$-stages for any $(Z, \hat{Z}) = (B, \hat{B})$ or $(A_i, \hat{A}_{\hat{\imath}})$ for any $(i, \hat{\imath}) \in L \times R$,

where the true path (to be defined precisely later) roughly corresponds to the set of strategies whose guess about the outcomes of all higher-priority strategies is correct.

We ensure (7) by induction roughly as follows: Let $T \subseteq \Lambda^{<\omega}$ be the tree of strategies, where Λ is the set of possible outcomes of the strategies. Now the strategy $\emptyset$ (the empty node) can act at any stage, so (7) can easily be ensured for

$\emptyset$ by effectively declaring each stage a $(Z, \hat{Z})$-stage for some $(Z, \hat{Z})$ such that each occurs infinitely often. Given a strategy $\xi\hat{\ }\langle o\rangle \in T$, we distinguish two cases: If outcome o must occur cofinitely often as the true outcome (because o is the only possible or is the finite outcome of ξ) then (7) for ξ clearly implies (7) for $\xi\hat{\ }\langle o\rangle$. On the other hand, if o is an infinite outcome of ξ, then we must slow down the construction by ending the stage for ξ rather than going on to a new substage whenever ξ has outcome o but at a $(Z, \hat{Z})$-stage when it is not "$(Z, \hat{Z})$'s turn for $\xi\hat{\ }\langle o\rangle$" so as to ensure (7).

4. The full construction for Theorem 5

Fix an arbitrary *effective priority ordering* of all requirements of order type ω, and let $\{\mathcal{N}_l\}_{l\in\omega}$ be the *priority ordering of all $\mathcal{N}$-requirements* under this ordering.

4.1. The tree of strategies. Let $\Lambda = \{\infty, f\} \cup \omega$ be the *set of possible outcomes* of strategies. (Intuitively, ∞ and f denote the infinite and finite outcome of a strategy, while the outcome $l \in \omega$ of a $\mathcal{P}$-strategy denotes that an $\mathcal{N}_l$-strategy's functional Γ was destroyed by the $\mathcal{P}$-strategy.)

We now define satisfaction of requirements (along a node) and the tree of strategies $T \subseteq \Lambda^{<\omega}$ in the next two definitions by simultaneous induction.

Definition 8. Fix a node $\xi \in T$.

(i) If $\xi = \emptyset$ then no requirement is active or satisfied along ξ.

Now assume $\xi \neq \emptyset$ and let $\eta = \xi^-$. Assume that satisfaction of requirements along any node $\subset \xi$ and the assignment of any node $\subset \xi$ to a requirement have already been defined.

(ii) Any requirement that is not an $\mathcal{N}$-requirement and that is satisfied along η is also *satisfied along* ξ (via the same strategy).
(iii) If $\eta\hat{\ }\langle\infty\rangle = \xi$ or $\eta\hat{\ }\langle f\rangle = \xi$, then any requirement satisfied or active along η is also *satisfied or active along* ξ (via the same strategy), respectively.
(iv) If $\eta\hat{\ }\langle l\rangle = \xi$ for some $l \in \omega$ and $\mathcal{N}_l$ is active along η via α, then $\mathcal{N}_l$ is *satisfied along ξ via α*, and any $\mathcal{N}$-requirement active or satisfied along α is also *active or satisfied along* ξ (via the same strategy), respectively.
(v) If $\eta\hat{\ }\langle\infty\rangle = \xi$ or $\eta\hat{\ }\langle f\rangle = \xi$, then the requirement assigned to η is *active along ξ via η* (if it is an $\mathcal{N}$-requirement and $\eta\hat{\ }\langle\infty\rangle = \xi$) or *satisfied along ξ via η* (otherwise).
(vi) Any requirement not satisfied or active along ξ by (ii)-(v) is *not satisfied or active along* ξ.

Definition 9. The *tree of strategies* $T \subseteq \Lambda^{<\omega}$ is defined as follows. Fix a node $\xi \in T$ and assume that satisfaction of requirements along any node $\subseteq \xi$ and the assignment of any node $\subset \xi$ to a requirement have already been defined.

(i) The strategy ξ is *assigned to the requirement* of highest priority that is neither active nor satisfied along ξ.
(ii) The set of *possible outcomes of* ξ is $\{\infty, f\}$ if ξ is assigned to an $\mathcal{N}$-, $\mathcal{M}$-, or $\mathcal{R}$-requirement; $\{f\}$ if ξ is assigned to a $\mathcal{T}$-requirement; and $\{l \in \omega \mid \mathcal{N}_l$ is active along $\xi\} \cup \{f\}$ if ξ is assigned to a $\mathcal{P}$-requirement.
(iii) The *immediate successors* of ξ on T are $\xi\hat{\ }\langle o\rangle$, where o ranges over the possible outcomes of ξ.

Informally, we call a node $\xi \in T$ an *$\mathcal{X}$-strategy* if it is assigned to requirement $\mathcal{X}$.

Summarizing the above intuitively, we see the following possibilities: An $\mathcal{N}$-, $\mathcal{M}$-, $\mathcal{R}$-, or $\mathcal{T}$-strategy merely satisfies its own requirement and can have the infinite outcome (in addition to the finite outcome) iff it is an $\mathcal{N}$-strategy or a minimal pair strategy. An $\mathcal{N}$-strategy merely introduces a functional Γ (which may be destroyed later, at which point $\mathcal{N}$ switches from "active" to "satisfied" by a functional Δ). A $\mathcal{P}$-strategy β can either satisfy its own requirement and thus have finite outcome; or destroy the $\mathcal{N}_l$-strategy α_l's functional Γ (where $\mathcal{N}_l$ is active along β via α_l) and satisfy the $\mathcal{N}_l$-requirement by building a functional Δ while injuring all $\mathcal{N}$-strategies and all infinite $\mathcal{P}$-strategies between α_l and β.

We summarize the static properties of the tree of strategies in the following:

Lemma 10 (Assignment Lemma)**.** *Along any path $p \in [T]$, each requirement is assigned at most finitely often and is eventually active, or satisfied, along all sufficiently long nodes $\subset p$ via a fixed strategy.*

Proof. An easy but tedious induction on the priority of requirements. ☐

4.2. $(Z, \hat{Z})$**-stages.** Set $m = (n+1) \cdot (\hat{n}+1) + 1$, and fix an arbitrary *indexing* of $\{(A_i, \hat{A}_{\hat{\imath}}) \mid (i, \hat{\imath}) \in L \times R\} \cup \{(B, \hat{B})\}$, denoted as $\{(Z_j, \hat{Z}_j) \mid j < m\}$.

Definition 11. A stage $s \in \omega$ is a *$(Z_j, \hat{Z}_j)$-stage* if $j < m$ and $s \equiv j \mod m$.

As remarked in Section 3.5, this ensures (7) for the node $\emptyset \in T$; for all other $\xi \in T$, (7) has to be ensured dynamically by the construction using the following:

Definition 12. Fix a nonempty subset $S \subseteq \{(Z_j, \hat{Z}_j) \mid j < m\}$, a strategy $\xi \in T$, a possible outcome o of ξ, and a stage $s \in \omega$. Fix the maximal stage $t < s$ (if it exists) such that t is a $(Z_j, \hat{Z}_j)$-stage for some $(Z_j, \hat{Z}_j) \in S$ and $\xi\hat{\ }\langle o\rangle$ was eligible to act at t. Then the *next type of stage for the quadruple* (S, ξ, o, s) is a $(Z_k, \hat{Z}_k)$-stage, where $k > j$ is minimal such that $(Z_k, \hat{Z}_k) \in S$ (if t and such k exist) or k is minimal such that $(Z_k, \hat{Z}_k) \in S$ (otherwise).

Intuitively, we have the following situation: We need to ensure (7) for a strategy $\xi\hat{\ }\langle o\rangle$ (which can act infinitely often) and for a set of types of stages S (e.g., depending on whether a $\mathcal{P}^i$-strategy is in its B-phase, $\bigoplus_{j \neq i} A_j$-phase, or A_i-phase). In that case, we will delay the next time $\xi\hat{\ }\langle o\rangle$ can act until the current stage s is of the next type of stage for the quadruple (S, ξ, o, s).

4.3. The construction, stage by stage. A strategy is *initialized* by making all its parameters undefined and by making totally undefined all its functions and functionals (i.e., Γ for an $\mathcal{N}$-strategy, various Δ's for a $\mathcal{P}$-strategy, ϑ for an $\mathcal{M}$-strategy, and λ for an $\mathcal{R}$-strategy). A number is *picked big* by choosing it larger than any number mentioned so far in the construction.

The construction now proceeds in stages. All parameters are measured at the current (sub)stage and remain unchanged unless explicitly specified otherwise. At the beginning, all strategies are initialized. A *stage* s consists of *substages* $t \leq s$ (possibly not all $t \leq s$). At each substage t, a strategy $\xi \in T$ of length t is eligible to act and, after completing its action, decides which strategy $\xi\hat{\ }\langle o\rangle$ should be eligible to act at substage $t+1$ or whether to end the stage (because $t = s$ or s is not the next type of stage for $\xi\hat{\ }\langle o\rangle$). So fix a stage s, a substage t, and a strategy $\xi \in T$ eligible to act at substage t of stage s. Distinguishing cases depending on the type of requirement to which ξ is assigned, we describe ξ's action and the choice of the next strategy eligible to act.

Case $\mathcal{N}$: ξ is an $\mathcal{N}_{X,\Phi}$-strategy: Define ξ's *length of agreement* by

$$\ell(\xi) = \max\{x \mid \forall y < x(\Phi^A(y) \downarrow= X(y))\},$$

and say s is ξ-*expansionary* if ξ is eligible to act at s and

$$\forall s' < s(\xi\hat{\ }\langle\infty\rangle \text{ eligible to act at } s' \to \ell_{s'}(\xi) < \ell(\xi)).$$

(Note that this definition is nonstandard, due to the delay feature of Section 3.5.)

If s is not ξ-expansionary then simply let $\xi\hat{\ }\langle f\rangle$ be eligible to act next. Otherwise:

Step 1: For each $y < \ell(\xi)$ for which $\Gamma^{X\oplus B}(y)$ is currently undefined, define $\Gamma^{X\oplus B}(y) = C(y)$ with the previous use (if $\Gamma^{X\oplus B}(y)$ was previously defined and no $\mathcal{P}$-strategy $\supseteq \xi\hat{\ }\langle\infty\rangle$ has requested that $\gamma(y')$ be lifted for some $y' \leq y$ at the previous ξ-expansionary stage) or with big use (otherwise).

Step 2: If s is of the next type of stage for the quadruple $(\{(Z_j, \hat{Z}_j) \mid j < m\}, \xi, \infty, s)$, then let $\xi\hat{\ }\langle\infty\rangle$ be eligible to act at the next substage; otherwise, end the stage.

Case $\hat{\mathcal{N}}$: Analogous to Case $\mathcal{N}$, using hatted parameters.

Case $\mathcal{P}$: ξ is a $\mathcal{P}^i_\Psi$-strategy: Follow the first subcase which applies. (When checking for computations $\Psi^{A_i\oplus B}(x)$, only accept computations that have existed at the previous stage at which α was eligible to act.)

Subcase 1: ξ's witness is currently undefined: Pick a big witness x and let $\xi\hat{\ }\langle f\rangle$ be eligible to act next.

Subcase 2: ξ's witness x is already in C: Let $\xi\hat{\ }\langle f\rangle$ be eligible to act next.

Remaining subcases: Let $\alpha_0 \subset \ldots \subset \alpha_{k-1} \subset \xi$ be all the $\mathcal{N}$-strategies α_l such that their requirement $\mathcal{N}_{i_l}$, say, is active along ξ via α_l, and let $\Gamma_l^{X_l\oplus B}$ be the functional built by each α_l. (This *indexing* of α, Γ, and X saves a bit on notation.) Let s' be the most recent stage (if any) at which $\xi\hat{\ }\langle o\rangle$ was eligible to act for some outcome $o \in \Lambda$, which we also fix.

Subcase 3: $\Psi^{A_i\oplus B}(x) \downarrow= 0$ and for all $l < k$, $\gamma_l(x) > \psi(x)$. Enumerate x into C and $\gamma_l(x)$ into B for all such α_l, initialize all strategies $\geq_L \xi\hat{\ }\langle f\rangle$, and end the stage. (This corresponds to stopping in Step 6 of the module of Section 3.2.)

Subcase 4: $o = f$ and $\Psi^{A_i}(x) \downarrow= 0$: If s is of the next type of stage for the quadruple $(\{A_j, \hat{A}_{\hat{\imath}}) \mid j \neq i \wedge \hat{\imath} \leq \hat{n}\}, \xi, i_{k-1}, s)$, then request that $\gamma_{k-1}(x)$ be lifted and let $\xi\hat{\ }\langle i_{k-1}\rangle$ be eligible to act next; otherwise, end the stage. (This corresponds to Step 3 of the module of Section 3.2 with the delay of Section 3.5.)

Subcase 5: $o = f$ (and so $\Psi^{A_i}(x) \downarrow= 0$ fails): Let $\xi\hat{\ }\langle f\rangle$ be eligible to act next. (This corresponds to Step 2 of the module of Section 3.2.)

Subcase 6: $o = i_l$ (for some $l < k$), s' is an $(A_j, A_{\hat{\imath}})$-stage (for some $j \neq i$ and $\hat{\imath} \leq \hat{n}$), and $\gamma_l(x) > \psi(x)$ (so $l > 0$, else Subcase 3 would have applied): If s is of the next type of stage for the quadruple $(\{A_j, \hat{A}_{\hat{\imath}}) \mid j \neq i \wedge \hat{\imath} \leq \hat{n}\}, \xi, i_{l-1}, s)$, then make $\Delta_l^{A_i}$ totally undefined, request that $\gamma_{l-1}(x)$ be lifted, and let $\xi\hat{\ }\langle i_{l-1}\rangle$ be eligible to act next; otherwise, end the stage. (This corresponds to proceeding to Step 6 in the module of Section 3.2 with the delay of Section 3.5.)

Subcase 7: $o = i_l$ (for some $l < k$), s' is an $(A_j, \hat{A}_{\hat{\imath}})$-stage (for some $j \neq i$ and $\hat{\imath} \leq \hat{n}$), and $\gamma_l(x) \leq \psi(x)$: If s is not a $(B, \hat{B})$-stage, then end the stage; otherwise, enumerate $\gamma_{l'}(x)$ into B (for all $l' \in [l, k)$), define $\Delta_l^{A_i}(y) = X_l(y)$ (for all $y \leq \gamma_l(x)$ for which $\Delta_l^{A_i}(y)$ is currently undefined) with the previous use (if $\Delta_l^{A_i}(y)$ was defined before and $\delta_l(y') \notin A_i$ for all $y' \leq y$) or with big use (otherwise), and let $\xi\,\hat{}\,\langle i_l\rangle$ be eligible to act next. (This corresponds to Step 4 or 7 of the module of Section 3.2 with the usual delay.)

Subcase 8: $o = i_l$ (for some $l < k$) and s' is a $(B, \hat{B})$-stage: If s is of the next type of stage for the quadruple $(\{(A_i, \hat{A}_{\hat{\imath}}) \mid \hat{\imath} \leq n\}, \xi, i_l, s)$, then let $\xi\,\hat{}\,\langle i_l\rangle$ be eligible to act next; otherwise, end the stage. (This corresponds to Step 5 or 8 of the module of Section 3.2 with the usual delay.)

Subcase 9: $o = i_l$ (for some $l < k$), s' is an $(A_i, \hat{A}_{\hat{\imath}})$-stage (for some $\hat{\imath} \leq \hat{n}$), and for some (least) y, $\Delta_l^{A_i}(y) \downarrow\neq X_l(y)$: If s is an $(A_i, \hat{A}_{\hat{\imath}})$-stage for some $\hat{\imath} \leq \hat{n}$, then enumerate $\delta_l(y)$ into A_i. In either case, end the stage.

Subcase 10: $o = i_l$ (for some $l < k$), s' is an $(A_i, \hat{A}_{\hat{\imath}})$-stage (for some $\hat{\imath} \leq \hat{n}$), and for all y, $\Delta_l^{A_i}(y)$ is undefined or equals $X_l(y)$: Let $\xi\,\hat{}\,\langle f\rangle$ be eligible to act next. (This corresponds to going back to Step 2 of the module of Section 3.2.)

Subcase 11: Otherwise: End the stage.

Case $\hat{\mathcal{P}}$: Analogous to Case $\mathcal{P}$, interchanging the roles of unhatted and hatted parameters.

Case $\mathcal{M}$: ξ is an $\mathcal{M}_{\Xi}^{i,j}$-strategy: Define ξ's *length of agreement* by

$$\ell(\xi) = \max\{x \mid \forall y < x(\Xi^{A_i}(y) = \Xi^{A_j}(y))\},$$

and define ξ-expansionary stages as in Case $\mathcal{N}$.

If s is not ξ-expansionary, then simply let $\xi\,\hat{}\,\langle f\rangle$ be eligible to act next.

Otherwise, for each $y < \ell(\xi)$ for which $\vartheta(y)$ is currently undefined, define $\vartheta(y) = \Xi^{A_i}(y)$. If s is of the next type of stage for the quadruple $(\{Z_j, \hat{Z}_j) \mid j < m\}, \xi, \infty, s)$, then let $\xi\,\hat{}\,\langle\infty\rangle$ be eligible to act at the next substage; otherwise, end the stage.

Case $\hat{\mathcal{M}}$: Analogous to Case $\mathcal{M}$, interchanging the roles of unhatted and hatted parameters.

Case $\mathcal{R}$: ξ is an $\mathcal{R}_{\Omega}^{i,\hat{\imath}}$-strategy: Define ξ's *length of agreement* by

$$\ell(\xi) = \max\{x \mid \forall y < x(\Omega^{A_i}(y) = \Omega^{\hat{A}_{\hat{\imath}}}(y) = \Omega^{F_{i,\hat{\imath}}}(y))\}.$$

Now proceed as in Case $\mathcal{M}$, replacing Ξ and ϑ by Ω and λ.

Case $\mathcal{T}$: ξ is a $\mathcal{T}_{\chi}^{j,\hat{\jmath}}$-strategy: Follow the first subcase which applies:

Subcase 1: ξ's witness is currently undefined: Pick a big witness x and let $\alpha\,\hat{}\,\langle f\rangle$ be eligible to act next.

Subcase 2: ξ's witness x is currently not in $D_{j,\hat{\jmath}}$, $\chi(x)\downarrow= 0$, and s is an $(A_j, \hat{A}_{\hat{\jmath}})$-stage: Enumerate x into $D_{j,\hat{\jmath}}$, A_j, and $\hat{A}_{\hat{\jmath}}$, and end the stage.

Subcase 3: Otherwise: Let $\alpha\,\hat{}\,\langle f\rangle$ be eligible to act next.

Case $\hat{\mathcal{T}}$: Analogous to Case $\mathcal{T}$, interchanging the roles of unhatted and hatted parameters.

At the end of the stage, initialize all strategies $>_L$ the strategy eligible to act at the last substage of stage s.

5. The verification of Theorem 5

We first analyze the possible injury in the construction. We begin by defining the computations relevant to a strategy, i.e., the computations that a strategy might want to protect.

Definition 13. For an $\mathcal{N}$-, $\mathcal{P}$-, $\mathcal{M}$-, or $\mathcal{R}$-strategy α and a stage s at which α is eligible to act (at substage $t = |\alpha|$), we call a computation *relevant to α at stage s* if

(i) α is an $\mathcal{N}_{X,\Phi}$-strategy and the computation is of the form $\Phi^A(x)$ for some $x < \ell(\alpha)$; or

(ii) α is a $\mathcal{P}^i_\Psi$-strategy and the computation is of the form $\Psi^{A_i \oplus B}(x)$ (for α's witness x) which also existed at a previous stage at which α was eligible to act; or

(iii) α is an $\mathcal{M}^{i,j}_\Xi$-strategy and the computation is of the form $\Xi^{A_i}(x)$ or $\Xi^{A_j}(x)$ for some $x < \ell(\alpha)$; or

(iv) α is an $\mathcal{R}^{i,\hat{\imath}}_\Omega$-strategy and the computation is of the form $\Omega^{A_i}(x)$, $\Omega^{\hat{A}_{\hat{\imath}}}(x)$, or $\Omega^{F_{i,\hat{\imath}}}(x)$ for some $x < \ell(\alpha)$.

(We analogously define computations relevant to $\hat{\mathcal{N}}$-, $\hat{\mathcal{P}}$- and $\hat{\mathcal{M}}$-strategies.)

Lemma 14 (Non-Injury Lemma)**.** *Suppose α is an $\mathcal{N}$-, $\mathcal{P}$-, $\mathcal{M}$-, or $\mathcal{R}$-strategy eligible to act at stages s_0 and $s_1 > s_0$ but not between these stages. Also assume that α is not initialized between substage $t = |\alpha|$ of stage s_0 and substage t of stage s_1.*

(i) *If α ends the stage at stage s_0, or if $\alpha\hat{}\langle f\rangle$ is eligible to act at stage s_0, then no computation relevant to α at stage s_0 existing at the beginning of substage t of stage s_0 is destroyed between the beginning of substage t of stage s_0 and the beginning of substage t of stage s_1.*

(ii) *If α is an $\mathcal{M}$- or $\mathcal{R}$-strategy and $\alpha\hat{}\langle\infty\rangle$ is eligible to act at stage s_0, then for each $x < \ell(\alpha)$, at least one of the computations at x relevant to α at stage s_0 is not destroyed between the beginning of substage t of stage s_0 and the beginning of substage t of stage s_1.*

(iii) *If α is a $\mathcal{P}^i$-strategy, $\alpha\hat{}\langle i_l\rangle$ is eligible to act for some $l < k_\alpha$, and s_0 is an $(A_j, \hat{A}_{\hat{\imath}})$-stage for some $j \neq i$ and $\hat{\imath} \leq \hat{n}$, then the computation relevant to α at stage s_0 is not destroyed between the beginning of substage t of stage s_0 and the beginning of substage t of stage s_1.*

(An analogous statement applies to strategies for hatted requirements.)

Proof. We distinguish cases for the strategy ξ which might destroy a computation relevant to α during the "critical interval" (i.e., between the beginning of substage t of stage s_0 and the beginning of substage t of stage s_1).

Case 1: $\xi <_L \alpha$: No such ξ can be eligible to act without α being initialized at the same stage.

Case 2: $\xi >_L \alpha$: Every such ξ is initialized at stage s_0 and thus cannot destroy a computation existing at stage s_0.

Case 3: $\xi = \alpha$: Only a $\mathcal{P}$-strategy may destroy a computation relevant to itself, and then only at a $(B, \hat{B})$-stage when $\alpha\hat{}\langle i_l\rangle$ is eligible to act for some $l < k_\alpha$. But, during the critical interval, α is eligible to act only at stage s_0, yielding the desired contradiction to our hypotheses.

Case 4: $\xi \supseteq \alpha\hat{}\langle f\rangle$: During the critical interval, ξ can only be eligible to act at stage s_0, so $\alpha\hat{}\langle f\rangle$ must be eligible to act at stage s_0. We distinguish two subcases.

Subcase 4.1: α is an $\mathcal{N}$-, $\mathcal{M}$-, or $\mathcal{R}$-strategy: Then s_0 is not α-expansionary, so let s be the greatest (α-expansionary) stage at which $\alpha\hat{}\langle\infty\rangle$ was eligible to act (set $s = 0$ if no such stage exists). By induction on s and part (i) of this lemma, $\ell(\alpha)[s] \geq \ell(\alpha)[s_0]$, and for each $x < \ell(\alpha)[s_0] = \ell(\alpha)$, at least one relevant computation for x cannot be destroyed between the beginning of substage t of stage s and the beginning of substage t of stage s_0. But, by initialization at stage s, all of ξ's parameters were chosen after stage s and thus cannot destroy this computation.

Subcase 4.2: α is a $\mathcal{P}$-strategy: Suppose that $\Psi^{A_i \oplus B}(x) = 0$ at the beginning of substage t of stage s_0 and that this computation also existed at a previous stage at which α was eligible to act. Then, at stage s_0, Subcase 2, 3, or 4 of Case $\mathcal{P}$ must apply to α. If Subcase 4 applies then s_0 is an $(A_j, \hat{A}_{\hat{\imath}})$-stage for some $j \neq i$ and $\hat{\imath} \leq \hat{n}$, so ξ cannot destroy $\Psi^{A_i \oplus B}(x)$ at stage s_0. Otherwise, let $s \leq s_0$ be the stage at which x is enumerated into C. Then ξ is initialized at stage s, and so, when it is eligible to act at stage s_0, cannot destroy $\Psi^{A_i \oplus B}(x)$ which already existed at stage s (by induction on s and part (i) of this lemma).

Case 5: $\xi \supseteq \alpha\,\hat{}\,\langle\infty\rangle$: Then α is an $\mathcal{M}^{i,j}$- or $\mathcal{R}^{i,\hat{\imath}}$-strategy (since the lemma was not claimed for an $\mathcal{N}$-strategy in this case), and s_0 is α-expansionary. But numbers cannot enter A_i and A_j at the same stage for $i \neq j$; and numbers cannot enter all of A_i, $\hat{A}_{\hat{\imath}}$, and $\Omega^F_{i,\hat{\imath}}$ at the same stage. So (ii) has been established in this case.

Case 6: $\xi \supseteq \alpha\,\hat{}\,\langle l\rangle$ for some $l \in \omega$: Then α is a $\mathcal{P}^i_\Psi$-strategy, and, by the hypothesis of (iii), s_0 must be an $(A_j, \hat{A}_{\hat{\imath}})$-stage for some $j \neq i$ and $\hat{\imath} \leq \hat{n}$, so ξ cannot destroy $\Psi^{A_i \oplus B}(x)$ at stage s_0.

Case 7: $\xi \subset \alpha$: Then ξ is eligible to act at stage s_0 only before substage t, i.e., before the critical interval. If a $\mathcal{T}$-strategy $\xi \subset \alpha$ destroys a computation (relevant to α at s_0) during the critical interval, then α is initialized during the critical interval, contrary to hypothesis. $\mathcal{N}$-, $\mathcal{M}$-, and $\mathcal{R}$-strategies do not enumerate any numbers. So assume that ξ is a $\mathcal{P}^j_\Psi$-strategy. Then ξ may enumerate into B (in Subcase 3 or 7 of Case $\mathcal{P}$) or into A_j (in Subcase 9 of Case $\mathcal{P}$), as well as into C (which is irrelevant here).

So, assume, for the sake of a contradiction, that ξ enumerates into B or A_j at a stage $s \in (s_0, s_1]$, destroying a computation relevant to α at stage s_0. We distinguish two subcases.

Subcase 7.1: ξ enumerates into B at stage s: Then ξ destroys $\Gamma^{X \oplus B}(x)$ for some $\mathcal{N}$-strategy $\subset \xi$ (where x is ξ's current witness). Since we assume that the computation relevant to α at stage s_0 has existed at a previous stage at which α was eligible to act, the use of this computation must be less than $\gamma(x)$ (which was lifted since then), a contradiction.

Subcase 7.2: ξ enumerates into A_j at stage s_0: Then, at stage s, $\xi_0 = \xi$ corrects its computation $\Delta^{A_j}(y_0)$ for some y_0 and ends the stage. This happens because y_0 has entered the corresponding set X_0, made possible by a change in $A\restriction(\varphi(y_0)+1)$. If this A-change is due to enumeration by some $\mathcal{P}$-strategy $\xi_1 \subset \alpha$ (possibly but not necessarily $\xi_1 = \xi_0$), then we trace this back to a previous A-change, etc. Eventually, we find some strategy ξ_p (which is either not a $\mathcal{P}$-strategy or $\not\subset \alpha$) which triggers these Δ-corrections by an A-change at z, say, at a stage s'.

We now distinguish subcases by the relative location of ξ_p:

Subcase 7.2.1: $\xi_p < \alpha$: Then α is initialized when ξ_p enumerates z into A. (For $\xi_p \subset \alpha$, we use the fact that ξ_p must be a $\mathcal{T}$-strategy by our hypotheses.) But then α cannot be eligible to act between stages s' and s, since each ξ_q ends the stage for $q < p$. (For $\xi_p \subset \alpha$, this also holds for $q = p$.) Thus α must be initialized during the critical interval, a contradiction.

Subcase 7.2.2: $\xi_p \supseteq \alpha\,\hat{}\,\langle\infty\rangle$ or $\supseteq \alpha\,\hat{}\,\langle l\rangle$ for some $l \in \omega$: By the same argument as in the previous subcase, we must then have $s' = s_0$. Since all ξ_q (for $q \leq p$)

are comparable and assume an infinitary outcome of each other, they must all enumerate into the same set A_k (for some fixed $k \leq n$); otherwise, some ξ_q (for $q < p$) would use the X_q-change to lift a Γ-use instead of enumerating into A. If α is a $\mathcal{P}$-strategy, then by the same argument $k = j$, so s_0 is an $(A_j, \hat{A}_i)$-stage, contrary to the hypothesis of part (iii). Otherwise, α must be an $\mathcal{M}$- or $\mathcal{R}$-strategy; but then enumeration into one fixed set A_k cannot destroy all computations relevant to α at s_0, establishing part (ii) in this case.

Subcase 7.2.3: $\xi_p \geq \alpha\hat{\ }\langle f\rangle$: By an argument as in Subcase 7.2.1, we have $s_0 < s'$. Let $\eta \subseteq \xi_p$ be the least strategy which is not $\subseteq \alpha$. Let s^* be the least stage $\leq s'$ at which η is eligible to act and such that η is not initialized between stages s^* and s_0. Then clearly $s_0 < s^*$. But then, by initialization, no strategy can enumerate, during the interval $[s^*, s]$, any number into any set that was picked before stage s^*, so the above sequence of enumerations by the ξ_q cannot lead to the destruction of a computation relevant to α at stage s_0. □

We now define the true path of the construction.

Definition 15. (i) The *true path* f of the construction is the leftmost path through T such that every node along it is eligible to act infinitely often.

(ii) For any requirement $\mathcal{X}$, we say $\mathcal{X}$ is *active*, or *satisfied, along* f via a strategy $\xi \subset f$ if $\mathcal{X}$ is active, or satisfied, via ξ along all sufficiently long $\eta \subset f$.

The following lemma shows that the true path is well-defined (and in particular infinite) and that the $(Z_j, \hat{Z}_j)$-stages mechanism works correctly.

Lemma 16 (True Path Lemma). *Let $\xi \subset f$. Then:*

(i) *ξ is initialized at most finitely often;*
(ii) *ξ is eligible to act at infinitely many $(Z_j, \hat{Z}_j)$-stages for all $j < m$; and*
(iii) *ξ declares one of its successors to be eligible to act next (instead of ending the stage) infinitely often.*

Proof. We proceed by induction on the length of ξ.

(i) is clear, since only strategies to the right of strategies currently eligible to act are initialized.

(ii) is clear if $\xi = \emptyset$ (since ξ is eligible to act at any stage) or if $\xi = \eta\hat{\ }\langle f\rangle$ for some η (since then at all but finitely many stages, ξ is eligible to act iff η is). If $\xi = \eta\hat{\ }\langle\infty\rangle$ or $\eta\hat{\ }\langle i\rangle$ for some η and $i \in \omega$, then (ii) follows by (i) for $\xi^- = \xi\lceil|\xi|$ and by the way the next type of stage is determined in the construction.

(iii) is clear if ξ is a $\mathcal{T}$- or $\hat{\mathcal{T}}$-strategy; if ξ an $\mathcal{N}$-, $\mathcal{M}$-, or $\mathcal{R}$-strategy (or its hatted equivalent) and there are only finitely many ξ-expansionary stages; and if ξ is a $\mathcal{P}$ or $\hat{\mathcal{P}}$-strategy and eventually only Subcases 2 or 5 are invoked (since in all these cases ξ ends the stage at most finitely often and $\xi\hat{\ }\langle f\rangle \subset f$).

Now suppose ξ is an $\mathcal{N}$-, $\mathcal{M}$-, or $\mathcal{R}$-strategy (or its hatted equivalent) and there are infinitely many ξ-expansionary stages. We want to show $\xi\hat{\ }\langle\infty\rangle \subset f$. Assume, for the sake of a contradiction, that at any stage $\geq$ some fixed ξ-expansionary stage s_0, ξ is not initialized and does not declare $\xi\hat{\ }\langle\infty\rangle$ eligible to act. By Lemma 14, no computation for any $y < \ell(\xi)$ involved in the definition of $\ell(\xi)$ can be injured at any stage $\geq s_0$; so, by our nonstandard definition of ξ-expansionary stages, all stages $\geq s_0$ at which ξ is eligible to act are also ξ-expansionary, so that $\xi\hat{\ }\langle\infty\rangle$ must eventually be eligible to act by (ii) for ξ.

Finally, assume that ξ is a $\mathcal{P}$- or $\hat{\mathcal{P}}$-strategy which is not eventually stuck at Step 2 or 5. Then, again by Lemma 14, ξ's computation $\Psi^{A_i}(x) = 0$ cannot be destroyed by any $\eta \neq \xi$; so for some least $l < k$, ξ invokes Subcases 7, 8, and 10 (in this order) infinitely often, slowed down by the $(Z_j, \hat{Z}_j)$-stages mechanism. $\square$

The satisfaction of each requirement is now established fairly easily.

Lemma 17 (Noncomputability Lemma)**.** *The $\mathcal{T}$-requirements are satisfied.*

Proof. By Lemma 10, we may fix a strategy $\xi \subset f$ such that some fixed such requirement is satisfied along f via ξ. The rest is now routine, using Lemma 16 (i) and (ii). $\square$

Lemma 18 (Minimal Pair/Triple Lemma)**.** *The $\mathcal{M}$-, $\hat{\mathcal{M}}$- and $\mathcal{R}$-requirements are satisfied.*

Proof. By Lemma 10, we may fix a strategy $\xi \subset f$ such that some fixed such requirement is satisfied along f via ξ. (By symmetry, assume this requirement is unhatted.) For an $\mathcal{M}^{i,j}$-requirement, we now observe that numbers cannot enter both A_i and A_j at the same stage. For an $\mathcal{R}^{i,\hat{\imath}}$-requirement, we observe that numbers cannot enter all of $A_i, \hat{A}_{\hat{\imath}}$, and $F_{i,\hat{\imath}}$ at the same stage. The rest is now a routine minimal pair argument, using Lemmas 14 and 16. $\square$

Lemma 19 ($\mathcal{P}/\hat{\mathcal{P}}$ Lemma)**.** *The $\mathcal{P}$- and $\hat{\mathcal{P}}$-requirements are satisfied.*

Proof. By Lemma 10, we may fix a strategy $\xi \subset f$ such that some fixed such requirement is satisfied along f via ξ. By Definition 8(v), we have $\xi\hat{\ }\langle f\rangle \subset f$. Thus, by Lemma 16 (i) and (ii), ξ is eventually stuck waiting at Step 2 or 5 (and, at Step 2, cannot be injured, by Lemma 14), ensuring the $\mathcal{P}$- or $\hat{\mathcal{P}}$-requirements. $\square$

Lemma 20 ($\mathcal{N}/\hat{\mathcal{N}}$ Lemma)**.** *The $\mathcal{N}$- and $\hat{\mathcal{N}}$-requirements are satisfied.*

Proof. By Lemma 10, we may fix a strategy $\xi \subset f$ such that some fixed such requirement (by symmetry, say, an $\mathcal{N}$-requirement) is active or satisfied along f via ξ. We distinguish three cases.

Case 1: The $\mathcal{N}$-requirement is active along f via ξ: Then ξ builds a functional $\Gamma^{X\oplus B}$. For fixed y, $\gamma(y)$ is lifted by $\mathcal{P}$-strategies $\eta \supset \xi$ at most finitely often; so $\Gamma^{X\oplus B}$ is total. And by the way the $\mathcal{P}$-strategies enumerate numbers into C, $\Gamma^{X\oplus B}$ must also correctly compute C.

Case 2: The $\mathcal{N}$-requirement is satisfied along f via an $\mathcal{N}$-strategy ξ. Then, by Definition 8(v), $\xi\hat{\ }\langle f\rangle \subset f$; so, by Lemma 14 (i), $\Phi^A \neq X$.

Case 3: The $\mathcal{N}$-requirement is satisfied along f via a $\mathcal{P}^i$-strategy ξ: Then, by Definition 8(iv), $\xi\hat{\ }\langle l\rangle \subset f$, and ξ kills a functional $\Gamma^{X\oplus B}$ built by an $\mathcal{N}$- (i.e., an $\mathcal{N}_l$-) strategy $\eta \subset \xi$. In $\Gamma^{X\oplus B}$'s stead, ξ builds a functional Δ^{A_i}. Then Δ^{A_i} is total, and by the way the $\mathcal{P}^i$-strategy ξ corrects Δ^{A_i}, this functional correctly computes X. $\square$

Lemmas 17–20 now establish the satisfaction of all requirements and, by Section 2, also Theorem 5. $\square$

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