THE PAINLEVÉ APPROACH TO NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS

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La propriété de Painlevé, un siècle après
The Painlevé property, one century later

Abstract. The "Painlevé analysis" is quite often perceived as a collection of tricks reserved to experts. The aim of this course is to demonstrate the contrary and to unveil the simplicity and the beauty of a subject which is in fact *the* theory of the (explicit) integration of nonlinear differential equations.

To achieve our goal, we will *not* start the exposition with a more or less precise "Painlevé test". On the contrary, we will finish with it, after a gradual introduction to the rich world of singularities of nonlinear differential equations, so as to remove any cooking recipe.

The emphasis is put on embedding each method of the test into the well known theorem of perturbations of Poincaré. A summary can be found at the beginning of each chapter.

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Chapter 1

Introduction

This course is a major revision of a previous one delivered in Chamonix [9]. Let us start with a few realistic applications.

1.1 A few elementary examples

1.1.1 Linearization of the Riccati equation

The Riccati equation

$$u' = a_2(x)u^2 + a_1(x)u + a_0(x),\ a_2 \neq 0, \quad \text{with } ' = \frac{\mathrm{d}}{\mathrm{d}x} \tag{1.1}$$

is known to be linearizable, but how to retrieve the explicit formula which maps it onto a linear equation? One just looks for the *first* coefficient u_0 of an expansion for u which describes the dependence on the integration constant (a simple pole), i. e. a Laurent series in some expansion function $\varphi(x)$: $u = \varphi^{-1}(u_0 + u_1\varphi + \ldots)$. It is given by balancing the two lowest degree terms $-u_0\varphi'\varphi^{-2} = a_2(u_0\varphi^{-1})^2$, and the linearizing transformation is then simply the change of function $u \to \varphi$ defined by the *singular part transformation* $u = u_0\varphi^{-1}$

$$u = -\frac{\varphi'}{a_2\varphi},\ \varphi'' - \left[\frac{a_2'}{a_2} + a_1\right]\varphi' + a_0 a_2 \varphi = 0. \tag{1.2}$$

This will be justified at the end of this course, in section 7.3.

1.1.2 A first integral of the Lorenz model

The Lorenz model of atmospheric circulation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \sigma(y - x),\ \frac{\mathrm{d}y}{\mathrm{d}t} = rx - y - xz,\ \frac{\mathrm{d}z}{\mathrm{d}t} = xy - bz(x - y) \tag{1.3}$$

admits for $(b, \sigma, r) = (0, 1/3, \text{arbitrary})$ the first integral

$$\left[-\frac{3}{4}x^4 + \frac{4}{3}x(y-x) + (z-r+1)x^2\right] e^{4t/3} = K, \tag{1.4}$$

but can one go further, i. e. can one obtain more first integrals or even explicitly integrate? The answer is yes [106]. By elimination of (y, z), one first builds the second order equation for $x(t)$

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = \frac{1}{x}\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 - \frac{x^3}{4} - \frac{K}{3x}e^{-4t/3}. \tag{1.5}$$

For $K = 0$ this equation admits the first integral

$$\frac{1}{x^2}\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \frac{x^2}{4} = A^2, \tag{1.6}$$

and the general solution $x = (1/(2A))\cosh(t - t_0)$. For $K \neq 0$ this equation for $x(t)$ is equivalent, as shown below, to the following equation for $X(T)$

$$X'' = \frac{X'^2}{X} - \frac{X'}{T} + \frac{\alpha X^2 + \beta}{T} + \gamma X^3 + \frac{\delta}{X},\ (\alpha, \beta, \gamma, \delta) \text{ constant}, \tag{1.7}$$

which is the third of six irreducible equations discovered between 1900 and 1906 by Paul Prudent Painlevé and his student Bertrand Gambier and, according to the theory of Painlevé developed in these lectures, the integration is then achieved ("parfaite", says Painlevé). Two words may not be familiar to the reader : "equivalent" and "irreducible". "Equivalent" means that the transformation law from the physical variables (x, t) to the variables (X, T) which satisfy (P3) should not result from a good guess, but should be looked for within a precise set of transformations (mathematically the homographic transformations (3.5) defined section 3.3) designed so as not to alter the structure of singularities (poles, branch points, ...). In this case, one finds that the transformation

$$x = a(t)X,\ T = \tau(t), \quad \text{with } a = \frac{2ic}{3}e^{-t/3},\ \tau = ce^{-t/3},\ c^4 = \frac{27}{4}K, \tag{1.8}$$

maps the equation for $x(t)$ to the equation (P3) for $X(T)$ with the parameter values for (P3) $\alpha = \beta = 0,\ \gamma = \delta = 1$.

As to "irreducible", it means that there exist no transformation, again within a precise class (Drach, Umemura, see section 3.5), reducing any of the six (Pn) equations either to a linear equation or to a first order equation. Consequently, the general solution of (Pn) has no "explicit expression", it is just *defined by the equation itself.* There is absolutely no difference between defining the "exponential function" from the general solution of $u' = u$ and defining the "P3 function" from the general solution of the equation (P3).

1.1.3 A reduction of the Boussinesq equation

The Boussinesq equation of fluid mechanics

$$u_{tt} + \left[u^2 + \frac{1}{3}u_{xx}\right]_{xx} = 0, \tag{1.9}$$

in the stationary case where u does not depend on time, reduces to an ordinary differential equation (EDO) which admits two first integrals

$$u'' + 3u^2 + K_1 x + K_2 = 0 \tag{1.10}$$

and, depending on K_1, this ODE is equivalent either to the (P1) equation

$$u'' = 6u^2 + x \tag{1.11}$$

or, after one more integration, to an equation introduced by Weierstrass, the *elliptic equation*

$$u'^2 = 4u^3 - g_2 u - g_3,\ (g_2, g_3) \text{ complex constants.} \tag{1.12}$$

Both equations have a general solution single valued in the whole complex plane.

1.2 "Solvable" models, "integrable" equations and so on

Two main fields contributed to the recent interest for the Painlevé theory. The first one is statistical physics. When Ising solved his one-dimensional model and found the partition function $F = -(1/\beta)\,\mathrm{Log}(2\cosh(\beta J)), \beta = k_B T$, the result was *a posteriori* not surprising. But, when Barouch, McCoy and Wu [67] expressed the correlation function of the two-dimensional Ising model with a (P3) function, this strongly contributed to revive the interest for these six functions, which now appear in any "solvable model" of statistical physics (see Di Francesco, this volume). Retrospectively, the cosh function of Ising is a quite elementary output of the Painlevé theory.

The second field is that of partial differential equations (PDE), as shown by the above Boussinesq example. After the extension of the Fourier transform to nonlinear PDEs [57], called *inverse spectral transform* (IST), Ablowitz and Segur [2] noticed a link between those "IST–integrable PDEs" and the theory of Painlevé, link expressed by Ablowitz, Ramani and Segur [4] as the conjecture : "Every ODE obtained by an exact reduction of a nonlinear PDE solvable by the IST method has the Painlevé property". For more details, see the book by Ablowitz and Clarkson [1] and [84].

1.3 Insufficiency of quadratures, the need for a theory

Let us return to our main subject, the explicit, analytic integration of ODEs. An exceptionally clear introduction *ad usum Delphini* is the *Leçon d'ouverture* (Œuvres vol. I p. 199) given by Painlevé in 1895 before starting the *Leçons de Stockholm*. For centuries, the question of integration has been formulated as : find enough first integrals in order to reduce the problem to a sequence of *quadratures*. But even the simple example of the pendulum shows the insufficiency of this point of view. Its motion is reducible to a quadrature defined by the integral

$$t - t_0 = \int_{u_0}^{u} \frac{\mathrm{d}u}{\sqrt{(1-u^2)(1-k^2u^2)}},\ k \text{ constant}, \tag{1.13}$$

giving the time t as a "function" of the position u. However, this *elliptic integral* does not provide the desired result, i. e. the position as a function of time, and, worse, nothing ensures the existence of such an expression. This classical problem (the inversion of the elliptic integral) could be solved by Abel and Jacobi only by going to the complex domain, leading to a unique value $u(t,t_0,u_0) = u_0 + \mathrm{sn}(t-t_0,k)$. The symbol sn does deserve the name of *function* (this is one of the twelve Jacobi elliptic functions, equivalent to the unique Weierstrass function) because, for any complex k, the application $t \to \mathrm{sn}(t,k)$ is single valued.

Following an idea of Briot and Bouquet, this led Painlevé to remark (*Leçon* no. 1) : "Mais l'importance de cette classe d'équations [à solution générale uniforme] apparaît mieux encore si on observe que la plupart des transcendantes auxiliaires, dont le rôle est si considérable (fonctions exponentielle, elliptiques, fuchsiennes, etc), intègrent des équations différentielles algébriques très simples. Les équations différentielles apparaissent donc comme la source des transcendantes uniformes les plus remarquables, susceptibles notamment de contribuer à l'intégration d'autres équations différentielles dont l'intégrale n'est plus uniforme."

This is the famous "double interest" of differential equations : one may consider them either as the source for defining new functions, or as a class of equations to be integrated with the existing functions available.

1.4 What can "to integrate" mean? The Painlevé property

Any converging Taylor series defined on some part of the real line, representing for instance a solution of an ODE on some interval $-R < x < R$, defines in fact an analytic function inside the disk $|x| < R$. Therefore, even when their variables are real, differential equations and their solutions are naturally defined in the complex plane.

To integrate an ODE is to acquire a global knowledge of its general solution, not only the local knowledge ensured by the existence theorem of Cauchy. So, the most demanding possible definition for the "integrability" of an ODE is the single valuedness of its general solution, so as to adapt this solution to any kind of initial conditions. Since even linear equations may fail to have this property, e.g. $2u' + xu = 0,\ u = cx^{-1/2}$, a more reasonable definition is the following one.

Definition. The *Painlevé property* (PP) of an ODE is the uniformizability of its general solution.

Following Bureau [13], we will call *stable* an equation with the PP. In the above example, uniformization is achieved for instance by removing from the complex plane any line joining the two branch points 0 and ∞.

1.5 Singularities of ordinary differential equations

"Les fonctions, comme les êtres vivants, sont caractérisées par leurs singularités" (Paul Montel). Singularities are responsible for the limitation of the domain of validity of Taylor or Laurent expansions, so their study is mandatory.

There exits a deep difference between the singularities of solutions of differential equations according as whether these equations are linear or nonlinear. In the linear case, the general solution (GS) has no other singularities than those of the coefficients of the equation once solved for the highest derivative. These singularities have a location independent of the arbitrary coefficients of integration and they are called *fixed*.

On the contrary, solutions of nonlinear ODEs may have other singularities, then called *movable*, at locations depending on the arbitrary coefficients. Thus, the equations [89]

$$\frac{\mathrm{d}u}{\mathrm{d}x} + \frac{u}{x^2} = 0,\ u = ce^{1/x}, \tag{1.14}$$

$$\frac{\mathrm{d}u}{\mathrm{d}x} + \frac{u^2}{x} = 0,\ u = \frac{1}{c + \mathrm{Log}\, x}, \tag{1.15}$$

$$\frac{\mathrm{d}u}{\mathrm{d}x} - \frac{\sqrt{1-u^2}}{x} = 0,\ u = \sin(c + \mathrm{Log}\, x), \tag{1.16}$$

where c is the arbitrary constant of integration, all have a fixed singularity in their general solution at $x = 0$ (isolated essential singular point for the first one, logarithmic branch point for the two others). In addition, among the last two ones, which are nonlinear, the second one has movable simple poles, and the third one has no movable singularity. All three have the PP.

The possible singularities of differential equations have been classified by Mittag-Leffler : in addition to the familiar ones (poles, branch points, essential singular points), there can exist essential singular lines, analytic or not, or perfect sets of singular points, as illustrated by the Fuchsian and Kleinean functions of Poincaré. One example is Chazy's equation of class III

$$u''' - 2uu'' + 3u'^2 = 0, \tag{1.17}$$

whose general solution is only defined inside or outside a circle characterized by the three initial conditions (two for the center, one for the radius); this solution is holomorphic in its domain of definition and cannot be analytically continued beyond it. This equation therefore has the PP, and the only singularity is a movable analytic essential singular line which is a natural boundary.

1.6 Outline and basic references

For the outline, we refer the reader to the detailed table of contents; the choice made is to develop the construction of necessary conditions, at the expense of the explicit integration methods, only briefly introduced in chapter 7.

The basic texts are due to Painlevé and his students and we abbreviate their references as *Leçons* ([89] *Leçons de Stockholm*, 1895, a high level course on nonlinear ODEs), BSMF ([90] first memoir, 1900, on first and second order ODEs), Acta ([92] second memoir, 1902, on second and higher order ODEs), CRAS ([94], 1906, an addendum after Gambier discovered the functions (P4), (P5), (P6)), Gambier ([56] thèse, 1909, on second order ODEs), Chazy ([22] thèse, 1910, on third and higher order ODEs), Garnier ([58] thèse, 1911, on higher order ODEs). Most Painlevé works are reprinted in Œuvres ([95] three volumes 1973, 74, 76, again available from CNRS-Éditions, e-mail editions@edition.cnrs.fr). For a global overview of these results, see the book of Hille [63]. preferably to the one of Ince [66]. For a detailed exposition (indeed, in the classical period, it was kind of fashionable to avoid details) and additional results, see the three memoirs of Bureau M. I [14], M. II [15], M. III [16]. Peter Clarkson maintains an extensive bibliography [26] covering both the classical and the recent period, reproduced in [1].

Chapter 2

The meromorphy assumption

2.1 Specificity of the elliptic function

A very deep result of L. Fuchs, Poincaré ([98], cf. Œuvres de Painlevé III p. 189) and Painlevé (*Leçon* no. 7 p. 107) is that the class of first order ODEs $F(u', u, x) = 0$, with F polynomial in u' and u, analytic in x, defines one and only one function, from the general solution of (1.12). This function is not historically new since this is precisely the *elliptic function* $\wp$ introduced earlier by Weierstrass, i. e. the particular solution of

$$\wp'^2 = 4\wp^3 - g_2\wp - g_3 = 4(\wp - e_1)(\wp - e_2)(\wp - e_3),\ (g_2, g_3, e_\alpha) \in \mathcal{C}, \tag{2.1}$$

which admits a pole at the origin

$$\wp(x, g_2, g_3) = x^{-2} + \frac{g_2}{20}x^2 + \frac{g_3}{28}x^4 + O(x^6). \tag{2.2}$$

The novelty of $\wp$ is elsewhere : this is the transcendental dependence of the general solution $\wp(x - x_0, g_2, g_3)$ on the arbitrary constant x_0, which makes impossible the reducibility of the elliptic equation to a linear equation. Among the many nice properties of elliptic functions (see e.g. [5]), the most interesting to us is their structure of singularities. These are doubly periodic meromorphic functions (which is their usual definition), and there exists an *entire* function σ, i. e. without any singularity at a finite distance, whose $-\wp$ is the second logarithmic derivative

$$\wp = -\frac{\mathrm{d}}{\mathrm{d}x}\zeta,\ \zeta = \frac{\mathrm{d}}{\mathrm{d}x}\operatorname{Log}\sigma,\ \zeta''^2 + 4\zeta'^3 - g_2\zeta' + g_3 = 0. \tag{2.3}$$

Therefore the only singularities of the general solution $\wp(x-x_0,g_2,g_3)$ of (1.12) come from the zeroes of σ and are a lattice of movable double poles located at $x_0+2m\omega+2n\omega'$, with m and n integers, ω,ω' the two half-periods.

2.2 The meromorphy assumption

The Laurent expansion (2.2) certainly motivated two students of Weierstrass, Paul Hoyer and Sophie Kowalevski, to investigate further the possibility for the GS of an ODE to be represented by a Laurent series with a finite *principal part*, so as to exclude essential singularities. This meromorphy assumption, briefly said, consists in checking the existence of the Laurent series and its ability to represent the GS, i. e. to contain enough arbitrary parameters. But, since a Laurent series is only defined inside its annulus of convergence, this study is only local and it cannot dispense of a further study in order to explicitly integrate, using completely different means.

The first attempt is due to Hoyer in 1879 [64] with the system

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}a_1&a_2&a_3\\b_1&b_2&b_3\\c_1&c_2&c_3\end{pmatrix}\begin{pmatrix}x_2x_3\\x_3x_1\\x_1x_2\end{pmatrix}, \tag{2.4}$$

under the restriction that neither the determinant nor any of its first or second order diagonal minors vanishes; he even generalized the assumption to the Puiseux series

$$x_i=\sum_{j=0}^{+\infty}A_{ij}\{(t-t_0)^{\frac{1}{r}}\}^{-n+j},\ i=1,2,3,$$

with n and r positive integers and $(A_{10},A_{20},A_{30})\neq(0,0,0)$, but in fact the numerous cases of integrability by elliptic functions which he discovered were found by a direct Ansatz, and not as necessary conditions for the Laurent series to exist. Continued by Kowalevski with a quite similar system except that it is six-dimensional, which will be seen section 2.5, the method will only get its final shape with Gambier in 1910 (see pages 9 and 49 of his thesis).

2.3 A flavor of the meromorphy test

We must warn the reader that this section is *not* the algorithm to apply, but just a flavor of it; the final algorithm will only be given section 6.6.

Let us start with a single equation; the case of a system is not different, apart from technical complications.

Consider the equation

$$E(x,u)\equiv-\frac{\mathrm{d}^2u}{\mathrm{d}x^2}+6u^2+g(x)=0, \tag{2.5}$$

with g analytic.

Assume that u has a polar behavior at some location x_0 distinct from any of the possible singularities of the coefficients of the equation, here $g(x)$; such a pole is therefore movable. One has to check the existence of *all* possible Laurent series with a finite principal part

$$u = \sum_{j=0}^{+\infty} u_j \chi^{j+p},\ \chi = x - x_0,\ u_0 \neq 0, \tag{2.6}$$

in which $-p$ is the order of the pole, which must be an integer, and the coefficients u_j are independent of x.

After insertion of this series in (2.5), which is polynomial in u and its derivatives, and replacement of $g(x)$ by its Taylor series in the neighborhood of x_0, the left-hand side, as a sum of Laurent series, is itself a Laurent series with a finite principal part

$$\begin{aligned} E &= [p(p-1)u_0\chi^{p-2} + (p+1)pu_1\chi^{p-1} + \ldots] + 6[u_0^2\chi^{2p} + 2u_0u_1\chi^{2p+1} + \ldots] \\ &\quad +[g(x_0) + g'(x_0)\chi + \ldots], \end{aligned} \tag{2.7}$$

which we denote more generally

$$E = \sum_{j=0}^{+\infty} E_j \chi^{j+q}, \tag{2.8}$$

q being the smallest integer of the list $(p-2, 2p, 0)$. The method consists of expressing the conditions for this series to identically vanish : $\forall j \in \mathcal{N}\ :\ E_j = 0$.

First step. Determine all possible *families of movable singularities* (u_0, p). This is expressed with three conditions :

- 1) (condition $u_0 \neq 0$) equality of at least two elements of the list $(p-2, 2p, 0)$ (q denotes their common value), the involved terms of E being called *dominant* and denoted $\hat{E}$,
- 2) (dominance condition) inferiority of q to the other elements of the list,
- 3) (vanishing Laurent series condition) vanishing of the coefficient E_0 of the lowest power χ^q, which involves only the dominant terms

$$E_0 \equiv \lim_{\chi \to 0} \chi^{-q} \hat{E}(x, u_0\chi^p) = 0,\ u_0 \neq 0, \tag{2.9}$$

i. e. respectively : one linear equation for p by pair of terms considered, several linear inequations for p, one algebraic equation for (u_0, p).

A necessary condition to prevent multivaluedness is then

- **C0**. All possible values for p are integer.

If there exists no family which is truly singular (p negative), the method stops without concluding.

Here, the various possibilities for these linear equations and inequations are

$$q = p - 2 = 2p \quad \text{and } q \le 0, \tag{2.10}$$
$$q = 2p = 0 \quad \text{and } q \le p - 2, \tag{2.11}$$
$$q = p - 2 = 0 \quad \text{and } q \le 2p. \tag{2.12}$$

Their geometric representation is known as the *Puiseux diagram* or *Newton's polygon* (ref. [63] sec. 3.3, [66] sec. 12.61). The two solutions $(p, q) = (-2, -4)$, $(2, 0)$ satisfy the condition **C0** and the second one must be rejected, as being nonsingular. So, the dominant part is here $\hat{E} = -u'' + 6u^2$. The algebraic equation $E_0 = 0$

$$E_0 \equiv -6u_0 + 6u_0^2 = 0, \ u_0 \ne 0, \tag{2.13}$$

has only one root $u_0 = 1$.

For $j = 1, 2, \dots$, each successive equation $E_j = 0$ has then the form

$$\forall j \ge 1 : \ E_j \equiv P(u_0, j) u_j + Q_j(\{u_l \mid l < j\}) = 0, \tag{2.14}$$

here

$$P(u_0, j) = -(j-2)(j-3) + 12u_0 = -(j+1)(j-6), \tag{2.15}$$
$$Q_1 = 0, \ Q_2 = 6u_1^2, \ Q_3 = 12u_1u_2, \tag{2.16}$$
$$\forall j \ge 4 : \qquad Q_j = \frac{g^{(j-4)}(x_0)}{(j-4)!} + 6\sum_{k=1}^{j-1} u_k u_{j-k}. \tag{2.17}$$

So the sequence $E_j = 0, j \ge 1$, is just *one* linear equation with different right-hand sides, and it can be solved recursively for u_j. Whenever the positive integer j is a zero of P, two subcases occur : either Q_j does not vanish and the Laurent series does not exist, or Q_j vanishes and u_j is arbitrary. Since x_0 is already arbitrary, in order to represent the GS, one wants $N - 1$ additional arbitraries to enter the expansion, where N is the order of the ODE. Let us admit for a moment that the value $j = -1$ is always a zero of P, a result whose general proof (given section 5.5) needs prerequisite notions of perturbation theory; this value $j = -1$ will be seen to represent the arbitrary location of x_0. Hence the following steps.

Second step. For each family, determine the polynomial P (do not compute Q_j yet) and require the necessary conditions :

- **C1**. The polynomial P has degree N.
- **C2**. $N - 1$ zeroes of P are positive integers.

- **C3**. The N zeroes of P are simple (i. e. of multiplicity one).

If either **C1**, **C2** or **C3** is violated, the method stops and one concludes to a failure because the general solution cannot be meromorphic.

The zeroes of P are called *indices* and $P = 0$ itself is the *indicial equation*. Indeed, anticipating on the exposition of the general theory sections 5.6 and 5.7, they are the Fuchs indices i near $\chi = 0$ of a linear equation introduced by Darboux [45] under the name "équation auxiliaire", so the indicial equation is computed as follows [52]. Take the derivative of $\hat{E}(x, u)$ with respect to u

$$\forall v : \ \hat{E}'(x,u)v \equiv \lim_{\lambda \to 0} \frac{\hat{E}(x, u + \lambda v) - \hat{E}(x,u)}{\lambda}, \tag{2.18}$$

here

$$\hat{E}'(x,u) \equiv -\partial_x^2 + 12u; \tag{2.19}$$

evaluate this linear operator at point $u = u_0 \chi^p$, which defines the "auxiliary equation" (i. e. the linearized equation at the leading term)

$$\forall v \ : \hat{E}'(x, u_0\chi^p)v = 0, \tag{2.20}$$

here

$$\forall v \ : \hat{E}'(x, u_0\chi^{-2})v \equiv (-\partial_x^2 + 12u_0\chi^{-2})v = 0; \tag{2.21}$$

establish the indicial equation of this linear ODE near its Fuchsian singularity $\chi = 0$

$$P(i) = \lim_{\chi \to 0} \chi^{-i-q} \hat{E}'(x, u_0\chi^p)\chi^{i+p} = 0, \tag{2.22}$$

here

$$\begin{aligned} P(i) &= \lim_{\chi \to 0} \chi^{-i+4}(-\partial_x^2 + 12u_0\chi^{-2})\chi^{i-2} \\ &= -(i-2)(i-3) + 12u_0 \\ &= -(i+1)(i-6). \end{aligned} \tag{2.23}$$

The shift $i \to i + p$ in the above equation is just a convention aimed at not producing an unfortunate difference between the Fuchs index i and the index j of the recursion relation $E_j = 0$.

Now, one just has to check the existence of the Laurent series.

Third step. For every positive integer zero i of P (a Fuchs index), require the condition

- **C4**.

$$\forall i \in \mathcal{N}, \ P(i) = 0 \ : Q_i = 0. \tag{2.24}$$

This is done by successively solving the recursion relation up to the greatest positive integer Fuchs index. As soon as a **C4** condition is violated, one stops and concludes to a failure : the ODE has not the PP. After the greatest positive integer Fuchs index has been checked, the method is finished.

Here, one finds

$$u_0 = 1,\ u_1 = u_2 = u_3 = 0,\ u_4 = -\frac{g_0}{10},\ u_5 = -\frac{g_0'}{6}, \tag{2.25}$$

and the condition **C4** at index $i = 6$ is

$$Q_6 \equiv g_0''/2 = 0. \tag{2.26}$$

i.e., since x_0 is arbitrary, $g'' = 0$. The ODE (2.5) is restricted to be (1.10) which has been seen to have a meromorphic GS, so in this case the generated necessary conditions are sufficient.

Remarks.

1. We have retained the classical vocabulary ("famille" is used by Gambier, p. 38 of his thesis [56], "indices" is used by Gambier, Chazy [22] and Bureau [13]), rather than the one more recently introduced [3, 4] ("branch", "resonances"). Indeed, "branch" has another meaning in classical analysis, where it denotes a determination of a multivalued application, which may create some confusion. As to "resonance", its identification with a basic notion of a linear theory, the Fuchs indices, makes useless the introduction of such a term.

2. Conditions $Q_i = 0$ at Fuchs indices i are often referred to as "no-logarithm conditions" because, if some of them are not satisfied, there exists a generalization of the Laurent series, called $\psi-$series ([63] chap. 7), which is a double expansion in χ and $\text{Log}\,\chi$. This series contains no logarithms (i. e. reduces to the Laurent series) iff all Q_i vanish.

3. One must prove that the radius R of the punctured disk $|x - x_0| < R$ in which the series converges is nonzero.

4. As indicated by Gambier [56] p. 50, there is no need to expand the coefficients $g(x)$ of the equation around x_0. This is achieved [30] by taking for the expansion variable not $x - x_0$, but a mute variable χ with the only property $\chi_x = 1$. Coefficients u_j in eq. (2.25) become dependent on x instead of x_0 :

$$u_0 = 1,\ u_1 = u_2 = u_3 = 0,\ u_4 = -\frac{g(x)}{10},\ u_5 = -\frac{g'(x)}{15},$$
$$Q_6 \equiv \frac{g''(x)}{2} = 0. \tag{2.27}$$

Let us insist again on the danger of using the present test as it is. An example of Chazy makes evident the necessity for a more reliable test : the equation with a single valued general solution ([22] p. 360)

$$(u''' - 2u'u'')^2 + 4u''^2(u'' - u'^2 - 1) = 0,\ u = e^{c_1 x + c_2}/c_1 + \frac{c_1^2 - 4}{4c_1}x + c_3.$$

possesses a logarithmic family $u \sim -\operatorname{Log}(x - x_0)$.

Exercise 2.1 *Handle the equation*

$$2uu'' - 3u'^2 = 0,\ u = c_1/(x - c_2)^2. \tag{2.28}$$

Solution.

$$2p - 2 = 2p - 2,\ E_0 \equiv 2u_0^2 p(p-1) - 3u_0^2 p^2 = 0. \text{ Hence } p = -2,\ u_0 \text{ arbitrary.}\square \tag{2.29}$$

2.4 Extension to a system

If the differential equation is defined by a system

$$\mathbf{E}(x, \mathbf{u}) = 0, \tag{2.30}$$

(boldface characters represent multicomponent quantities), the scalar equations of section 2.3 become systems : a linear system for the components of $\mathbf{p}$, an algebraic system for the components of $\mathbf{u}_0$, a linear system with a rhs for $\mathbf{u}_j$, a determinant for the indicial equation, etc. Take the example of the Euler system (diagonal Hoyer system)

$$E_1 \equiv \frac{\mathrm{d}x_1}{\mathrm{d}t} - \alpha x_2 x_3 = 0,\ E_2 \equiv \frac{\mathrm{d}x_2}{\mathrm{d}t} - \beta x_3 x_1 = 0,\ E_3 \equiv \frac{\mathrm{d}x_3}{\mathrm{d}t} - \gamma x_1 x_2 = 0. \tag{2.31}$$

First step. The necessary condition on $(\mathbf{p}, \mathbf{u}_0)$ is

- **C0**. All components of $\mathbf{p}$ are integer, all components of $\mathbf{u}_0$ are nonzero.

Of course, if a component of $\mathbf{u}_0$ is zero, one must increase by one the associated component of $\mathbf{p}$ until the new component of $\mathbf{u}_0$ becomes nonzero. It there exists no truly singular family (at least one component of $\mathbf{p}$ negative), the method stops without concluding.

Here, the unique solution $\mathbf{p}$ of the linear system $q_1 = p_1 - 1 = p_2 + p_3$ and cyclically is thus $p_1 = p_2 = p_3 = -1, q_1 = q_2 = q_3 = -2$. The algebraic system for $\mathbf{u}_0$

$$\mathbf{E}_0 \equiv \lim_{\chi \to 0} \chi^{-\mathbf{q}} \hat{\mathbf{E}}(x, \mathbf{u}_0 \chi^{\mathbf{p}}) = 0,\ \mathbf{u}_0 \neq \mathbf{0}, \tag{2.32}$$

is written

$$E_{1,0} \equiv -x_{1,0} - \alpha x_{2,0} x_{3,0} = 0 \text{ and cyclically,} \tag{2.33}$$

and it defines four families

$$x_{1,0}^2 = \frac{1}{\beta\gamma},\ x_{2,0}^2 = \frac{1}{\gamma\alpha},\ x_{3,0}^2 = \frac{1}{\alpha\beta},\ x_{1,0}x_{2,0}x_{3,0} = -\frac{1}{\alpha\beta\gamma}, \tag{2.34}$$

which we gather under the unique algebraic writing $x_{1,0} = a, x_{2,0} = b, x_{3,0} = c$.

Second step. The linear system

$$\forall j \geq 1 :\ \mathbf{E}_j \equiv \mathbf{P}(j)\mathbf{u}_j + \mathbf{Q}_j(\{\mathbf{u}_l \mid l < j\}) = 0 \tag{2.35}$$

generates the indicial equation

$$\det \mathbf{P}(i) = 0,\ \mathbf{P}(i) = \lim_{\chi \to 0} \chi^{-i-\mathbf{q}} \hat{\mathbf{E}}'(x, \mathbf{u}_0 \chi^{\mathbf{p}}) \chi^{i+\mathbf{p}}, \tag{2.36}$$

here

$$\mathbf{P}(i) = \begin{pmatrix} i-1 & -\alpha c & -\alpha b \\ -\beta c & i-1 & -\beta a \\ -\gamma b & -\gamma a & i-1 \end{pmatrix},\ \det \mathbf{P}(i) = (i+1)(i-2)^2 = 0. \tag{2.37}$$

Classical results from linear algebra on the resolution of the matrix equation $AX = B$ give the necessary conditions

- **C1**. The polynomial $\det \mathbf{P}$ has degree N.
- **C2**. $N - 1$ zeroes of $\det \mathbf{P}$ are positive integers.
- **C3**. Every positive zero i of $\det \mathbf{P}$ has a multiplicity equal to the dimension of the kernel of $\det \mathbf{P}(i)$.

Here, each of the four families has the same indices $(-1, 2, 2)$ and, for the double index $i = 2$, the three rows of matrix $\mathbf{P}(2)$ are proportional, so its kernel has dimension two.

Third step. For every positive integer zero i of $\mathbf{P}$ (a Fuchs index), require the condition

- **C4**.

$$\forall i \in \mathcal{N},\ \det \mathbf{P}(i) = 0\ :\quad \text{the vector } \mathbf{Q}_i \text{ is orthogonal to the kernel of the adjoint of operator } \mathbf{P}(i). \tag{2.38}$$

Here, the condition **C4** is satisfied at index two, and the Laurent series are finally

$$x_1 = a\chi^{-1} + a_2\chi + O(\chi^2),\ \chi = t - t_0, \tag{2.39}$$

$$x_2 = b\chi^{-1} + b_2\chi + O(\chi^2), \tag{2.40}$$

$$x_3 = c\chi^{-1} + c_2\chi + O(\chi^2),\ a_2 + b_2 + c_2 = 0, \tag{2.41}$$

with (t_0, b_2, c_2) arbitrary.

2.5 Motion of a rigid body around a fixed point

It is ruled by the system

$$\begin{aligned}
&A\frac{\mathrm{d}\omega_1}{\mathrm{d}t} + (C-B)\omega_2\omega_3 + (x_3k_2 - x_2k_3) = 0, \ \frac{\mathrm{d}k_1}{\mathrm{d}t} - \omega_3k_2 + \omega_2k_3 = 0,\\
&B\frac{\mathrm{d}\omega_2}{\mathrm{d}t} + (A-C)\omega_3\omega_1 + (x_1k_3 - x_3k_1) = 0, \ \frac{\mathrm{d}k_2}{\mathrm{d}t} - \omega_1k_3 + \omega_3k_1 = 0,\\
&C\frac{\mathrm{d}\omega_3}{\mathrm{d}t} + (B-A)\omega_1\omega_2 + (x_2k_1 - x_1k_2) = 0, \ \frac{\mathrm{d}k_3}{\mathrm{d}t} - \omega_2k_1 + \omega_1k_2 = 0,
\end{aligned} \tag{2.42}$$

depending on six parameters : the components (A, B, C), positive, of the diagonal inertia momentum I and the components (x_1, x_2, x_3), real, of the vector $\overrightarrow{OG}$ linking the fixed point O to the center of mass G. Because it admits the three first integrals

$$\begin{aligned}
&K_1 = (I\overrightarrow{\Omega}).\overrightarrow{\Omega} - 2\overrightarrow{OG}.\overrightarrow{k} = A\omega_1^2 + B\omega_2^2 + C\omega_3^2 - 2(x_1k_1 + x_2k_2 + x_3k_3),\\
&K_2 = (I\overrightarrow{\Omega}).\overrightarrow{k} = A\omega_1k_1 + B\omega_2k_2 + C\omega_3k_3,\\
&K_3 = \overrightarrow{k}.\overrightarrow{k} = k_1^2 + k_2^2 + k_3^2,
\end{aligned}$$

and a last Jacobi multiplier equal to 1,

$$\sum_{j=1}^{3} \partial_{\omega_j}\left(\frac{\mathrm{d}\omega_j}{\mathrm{d}t}\right) + \sum_{j=1}^{3} \partial_{k_j}\left(\frac{\mathrm{d}k_j}{\mathrm{d}t}\right) = 0, \tag{2.43}$$

a sufficient condition of reducibility to quadratures (i. e. to separation of variables, which implies neither meromorphy nor single valuedness) is the existence of a single additional first integral independent of time.

Before Kowalevski, the only such known cases were

- the isotropy case, with

$$A = B = C \ : \ K_4 = \overrightarrow{OG}.\overrightarrow{\Omega} = x_1\omega_1 + x_2\omega_2 + x_3\omega_3, \tag{2.44}$$

- the case of Euler (1750) and Poinsot(1851) G at the fixed point O ($x_1 = x_2 = x_3 = 0$) with

$$G = O \ : \ K_4 = |I\overrightarrow{\Omega}|^2 = A^2\omega_1^2 + B^2\omega_2^2 + C^2\omega_3^2, \tag{2.45}$$

- the case of Lagrange (1788) and Poisson (1813)

$$A = B, x_1 = x_2 = 0 \ : \ K_4 = \omega_3, \tag{2.46}$$

and for these three cases the general solution is elliptic, hence meromorphic [59].

Let us denote a family as ($t - t_0$ is abbreviated as t)

$$\omega_l = \sum_{j=0}^{+\infty} \omega_{l,j} t^{n_l + j},\ k_l = \sum_{j=0}^{+\infty} k_{l,j} t^{m_l + j},\ \ \omega_{1,0}\omega_{2,0}\omega_{3,0}k_{1,0}k_{2,0}k_{3,0} \neq 0,$$

with $l = 1, 2, 3$ and $(\omega_{l,j}, k_{l,j})$ complex. There exist numerous families, some of them with $n_l - n_k, m_l - m_k$ not integer. To shorten, let us restrict to the case where all the differences $n_l - n_k, m_l - m_k$ are integer, and redefine a family as

$$\begin{aligned} \omega_l &= t^n \sum_{j=0}^{+\infty} \omega_{l,j} t^j,\ (\omega_{1,0}, \omega_{2,0}, \omega_{3,0}) \neq (0, 0, 0), \\ k_l &= t^m \sum_{j=0}^{+\infty} k_{l,j} t^j,\ (k_{1,0}, k_{2,0}, k_{3,0}) \neq (0, 0, 0). \end{aligned} \tag{2.47}$$

One such family is defined [72] by the exponents $n_l = -1, m_l = -2$, and the sextuplets $(\omega_{l,0}, k_{l,0})$ solutions of the algebraic system

$$\begin{aligned} &A\omega_{1,0} + (B - C)\omega_{2,0}\omega_{3,0} + x_2 k_{3,0} - x_3 k_{2,0} = 0,\ 2k_{1,0} + \omega_{3,0}k_{2,0} - \omega_{2,0}k_{3,0} = 0, \\ &B\omega_{2,0} + (C - A)\omega_{3,0}\omega_{1,0} + x_3 k_{1,0} - x_1 k_{3,0} = 0,\ 2k_{2,0} + \omega_{1,0}k_{3,0} - \omega_{3,0}k_{1,0} = 0, \\ &C\omega_{3,0} + (A - B)\omega_{1,0}\omega_{2,0} + x_1 k_{2,0} - x_2 k_{1,0} = 0,\ 2k_{3,0} + \omega_{2,0}k_{1,0} - \omega_{1,0}k_{2,0} = 0, \end{aligned}$$

and the linear system for $j \geq 1$ is

$$\begin{pmatrix} (j-1)A & (C-B)\omega_{3,0} & (C-B)\omega_{2,0} & 0 & x_3 & -x_2 \\ (A-C)\omega_{3,0} & (j-1)B & (A-C)\omega_{1,0} & -x_3 & 0 & x_1 \\ (B-A)\omega_{2,0} & (B-A)\omega_{1,0} & (j-1)C & x_2 & -x_1 & 0 \\ 0 & k_{3,0} & -k_{2,0} & j-2 & -\omega_{3,0} & \omega_{2,0} \\ -k_{3,0} & 0 & k_{1,0} & \omega_{3,0} & j-2 & -\omega_{1,0} \\ k_{2,0} & -k_{1,0} & 0 & -\omega_{2,0} & \omega_{1,0} & j-2 \end{pmatrix} \begin{pmatrix} \omega_{1,j} \\ \omega_{2,j} \\ \omega_{3,j} \\ k_{1,j} \\ k_{2,j} \\ k_{3,j} \end{pmatrix} + \mathbf{Q}_j = 0. \tag{2.48}$$

The determinant $\det \mathbf{P}$ must have five positive zeroes.

In the generic case (A, B, C) all different and $G \neq O$, there exists a unique solution to the algebraic system, depending on one arbitrary parameter and the root of an eighth degree equation [73], but the determinant

$$\det \mathbf{P} = ABC(j+1)j(j-2)(j-4)(j^2 - j - \mu), \tag{2.49}$$

where μ is an algebraic expression of (A, B, C, x_1, x_2, x_3), has five positive integer zeroes iff $\mu = 0$, which corresponds to inadmissible values for the six parameters (A, B, C must be real positive, x_1, x_2, x_3 real).

A thorough discussion of the nongeneric cases of this family $n_l = -1, m_l = -2$ led Kowalevski to retrieve the three known cases, as expected, and finally to find the subcase

$$A = B,\ (x_1, x_2) \neq (0, 0),\ \omega_{1,0}^2 + \omega_{2,0}^2 = 0, \tag{2.50}$$

for which the unique solution is

$$\begin{aligned}
&\omega_{1,0} = -\frac{iC}{2(x_1+ix_2)\lambda},\ \omega_{2,0} = i\omega_{1,0},\ \omega_{3,0} = 2i,\ i^2=-1,\\
&k_{1,0} = -\frac{2C}{x_1+ix_2},\ k_{2,0} = ik_{1,0},\ k_{3,0}=0, \qquad (2.51)\\
&\det \mathbf{P} = ABC(j+1)(j-2)(j-3)(j-4)(j+1-2C/A)(j-2+2C/A),
\end{aligned}$$

in which λ is defined by the relation

$$2C - A - 4\lambda x_3 = 0,\ \lambda \neq 0. \qquad (2.52)$$

There exist five positive integer indices iff $A = 2C, x_3 = 0$, and the first integral

$$A = B = 2C,\ x_3 = 0\ :\ K_4 = |C(\omega_1 + i\omega_2)^2 + (x_1+ix_2)(k_1+ik_2)|^2 \qquad (2.53)$$

terminates the proof of reducibility to quadratures. Sophie Kowalevski then managed to explicitly integrate with hyperelliptic integrals and to prove the meromorphy of the general solution, a feat which won her an instantaneous fame.

Remark. Neither Hoyer nor Kowalevski enforced conditions **C3** and **C4**. This was done for the first time by Appelrot [6], who found another family $n_l = -1, m_l = 0$ with the indices $(-1,-1,0,1,2,2)$ and, despite the absence of five positive integer indices, computed the next terms and found at the double index 2 the no-log condition

$$x_1\sqrt{A(C-B)} + x_2\sqrt{B(A-C)} + x_3\sqrt{C(B-A)} = 0, \qquad (2.54)$$

whose real and imaginary parts yield, with the convention $A > B > C$,

$$x_2 = 0,\ x_1\sqrt{A(B-C)} + x_3\sqrt{C(A-B)} = 0. \qquad (2.55)$$

Nekrasov [86] and Lyapunov then proved the multivaluedness of this case by exhibiting yet another family with complex exponents.

2.6 Insufficiency of the meromorphy

Here is the opinion of Painlevé ([92] pp. 10, 83, Œuvres III pp. 196, 269) : "M^e Kowalevski se propose de trouver tous les cas où le mouvement du solide est défini par des *fonctions méromorphes de t qui possèdent effectivement des pôles.* Son procédé laisse échapper les cas où ces fonctions seraient uniformes sans avoir de pôles, soit qu'elles fussent *holomorphes*, soit que toutes leurs singularités fussent transcendantes.

De plus, après avoir formé les conditions pour qu'il existe des pôles mobiles, M^e Kowalevski remarque que ces conditions entraînent l'intégrabilité des équations du mouvement, ce qui lui permet de mener la question jusqu'au bout. Mais cette remarque laisse échapper un cas où il existe des pôles et qui n'est pas un cas d'intégration. Toutefois les géomètres Russes ont montré, par la suite, que, dans ce cas, les équations du mouvement n'ont pas leur intégrale uniforme.

Les résultats de M^e^ Kowalevski subsistent donc en fait. Mais, si intéressante que soit la voie suivie par M^e^ Kowalevski, il était désirable de reprendre la question d'une façon plus rationnelle. C'est ce que permettent les procédés que j'ai employés pour les équations du second ordre : ils fournissent de la manière la plus naturelle et la plus simple les conditions nécessaires pour que *ce mouvement soit représenté par des fonctions uniformes de t*, sans qu'il soit besoin de faire aucune hypothèse sur ces fonctions. Les conditions auxquelles on parvient ainsi ne diffèrent pas d'ailleurs de celles de M^e^ Kowalevski. Pour ce problème particulier, on n'arrive donc pas à des cas nouveaux."

Painlevé thus insists that poles are not privileged : they are just one kind of singularity among many possible others.

"Une discussion qui écarterait d'avance certaines singularités comme invraisemblables serait *inexistante*." (Painlevé, [92] p. 6, Œuvres III p. 192).

"In the statement of the problem, *poles are not mentioned*; if in the final result the particular integrals prove to be meromorphic, it is a *result* of the research. Likewise, no mention is made of one or another type of critical or singular point." (Bureau [19] p. 105).

Thus, definitely, the meromorphy assumption has to be waived as a *global* property, although it may be, and indeed is, quite useful at the *local* level. The only relevant property based on singularities is the Painlevé property as defined in section 1.4, and the goal is to build a rigorous theory without any *a priori* on the movable singularities. Despite the pessimistic opinion of Picard who thought the task impossible, Painlevé built that theory by a clever application of the theorem of perturbations of Poincaré and Lyapunov.

2.7 A few examples to be settled

The theory to come and the resulting "Painlevé test" should be able to handle the following differential systems for which the meromorphy test of section 2.3 is inconclusive or even erroneous. We give the location where the solution can be found.

1. ("Soit qu'elles fussent *holomorphes*") Extend the test to handle the equation $2uu'' - u'^2 = 0$, with general solution $u = (c_1 x + c_2)^2$. Solution section 5.4.

2. ("Soit que toutes leurs singularités fussent transcendantes"). Extend the test to handle the equations $uu'' - u'^2 = 0$ and $2u^2u'u''' - 3u^2u''^2 + u'^4 = 0$, whose general solution is, respectively, $u = e^{c_1 x + c_2}$ and $u = c_1 e^{1/(c_2 x + c_3)}$.

One may notice that the second equation is the Schwarzian derivative of $\operatorname{Log} u$.

3. The "uncoupled" system with a meromorphic general solution

$$\frac{\mathrm{d}u}{\mathrm{d}x} + u^2 = 0,\ \frac{\mathrm{d}v}{\mathrm{d}x} + v^2 = 0. \tag{2.56}$$

admits two families (modulo the exchange of u and v)

$$(F1)\ :\quad u \sim \chi^{-1},\ v \sim \chi^{-1},\ \text{indices } (-1,-1) \tag{2.57}$$
$$(F2)\ :\quad u \sim \chi^{-1},\ v \sim v_0\chi^0,\ v_0 \text{ arbitrary, indices } (-1,0), \tag{2.58}$$

of which the first one fails the condition **C2**. Solution section 5.4.

4. ("Un cas où il existe des pôles et qui n'est pas un cas d'intégration"). The Bianchi IX cosmological model

$$(\operatorname{Log} A)'' = A^2 - (B-C)^2 \text{ and cyclically, } {}' = \mathrm{d}/\mathrm{d}\tau, \tag{2.59}$$

admits for $B = C$ a particular four-parameter meromorphic solution [108]

$$A = \frac{k_1}{\sinh k_1(\tau-\tau_1)},\ B = C = \frac{k_2^2 \sinh k_1(\tau-\tau_1)}{k_1 \sinh^2 k_2(\tau-\tau_2)}. \tag{2.60}$$

Prove the absence of the PP by studying the family $\mathbf{p} = (0,-2,-2)$ which has only four Fuchs indices $(-1,0,1,2)$. Solution section 5.8.3.

5. The Bianchi IX model (2.59) admits a family $\mathbf{p} = (-1,-1,-1)$ with the indices $(-1,-1,-1,2,2,2)$. Prove the absence of the PP by studying this family. Solution section 5.7.6.

6. The Chazy's equation of class III (1.17) admits a Laurent series which terminates $u = u_0/(x-x_0)^2 - 6/(x-x_0)$, with (x_0,u_0) arbitrary. From the study of this family, decide about the meromorphy of the general solution. Solution section 5.8.1.

7. In a problem in geometry of surfaces, Darboux [44] encountered the system

$$\mathrm{d}x_1/\mathrm{d}t = x_2x_3 - x_1(x_2+x_3) \text{ and cyclically,} \tag{2.61}$$

explicitly excluded by Hoyer, cf. section 2.2, and found the two-parameter meromorphic solution

$$x_1 = c/(t-t_0)^2 + 1/(t-t_0), x_2 = x_3 = 1/(t-t_0),\ (t_0,c) \text{ arbitrary.} \tag{2.62}$$

For $c = 0$, the Fuchs indices are $(-1,-1,-1)$. Extend the test to build no-log conditions at this triple -1 index. Solution section 5.7.

8. The equations

$$-2uu'' + 3u'^2 + d_3 u^3 = 0,\ d_3 \neq 0, \tag{2.63}$$

$$u''' + uu'' - 2u'^2 = 0, \tag{2.64}$$

$$u'''' + 2uu'' - 3u'^2 = 0, \tag{2.65}$$

have no dominant behaviour. Prove the absence of the PP for each of them [24]. Solution section 5.9.

Chapter 3

The true problems

In this chapter, we state the true problems and manage logically to the only correct definition for the Painlevé property (PP) : "absence of movable critical points in the general solution", equivalent to that already given in section 1.4. This includes

– the two classifications of singularities of differential equations (fixed or movable, critical or noncritical),

– the two differences between linear and nonlinear (movable singularity, singular solution),

– the statement of the ambitious program proposed by Painlevé, a first, quick look at the method of resolution (the "double method" and the "double interest") and the results for first order (equation of Riccati, function of Weierstrass) and second order (classification of Gambier, the six Painlevé functions).

All the ODEs considered are defined on $\mathcal{C}$ or on the Riemann sphere (i.e. the complex plane compactified by addition of the unique point at infinity).

Firstly, a more precise definition of the term "to integrate" is required.

Definition. *To integrate* an ODE, in the "modern sense" advocated by Painlevé, is to find for the general solution a finite expression, possibly multivalued, in a finite number of functions, valid in the whole domain of definition.

The important terms in this definition are "finite" and "function".

Example 1 (nonintegrated ODE) : the Taylor series $u = u_0 \sum_{j=0}^{+\infty} [-(x - x_0)u_0]^j$ for the Cauchy solution does represent the general solution of the ODE $u' + u^2 = 0$ but this representation is local and the integration cannot be considered as achieved until one has : found the radius of convergence, performed the summation, analytically continued the sum everywhere this is possible, and identified the analytic continuation with the meromorphic function $(x - x_1)^{-1}$.

Example 2 (integrated ODE) : the ODE $2uu' - 1 = 0$ has for general solution $u = (x - x_0)^{1/2}$, a multivalued finite expression built from the "multivalued function" (see below) $z \to z^{1/2}$.

Representations by an integral, a series or an infinite product are acceptable iff they amount to a *global*, as opposed to *local*, knowledge of the solution.

A prerequisite to the integration in the sense of the above definition is therefore to extend the set of available functions, to serve as a réservoir from which to build finite expressions. At this stage, one must go back to the term "function".

Definition [12]. A *function* is an application of a set of objects into a set of images which applies a given object onto one *and only one* image.

In other words, a function is characterized by its single valuedness, and terms such as "multivalued function" should be carefully avoided. In our context, a function is a single valued application of the Riemann sphere onto itself.

Definition (Painlevé [90] p. 206). A *critical point* of an application of the Riemann sphere onto itself is any singular point, isolated or not, around which at least two determinations are permuted. Common synonyms are : for critical point, branch point, point of ramification; for determination, branch. Such a point is an obstacle for an application to be a function.

Examples : the applications $x \to \sqrt{x-a}$ and $x \to \mathrm{Log}(x-a)$ both have exactly two critical points, a and ∞. Around each of them are permuted respectively two determinations and a countable infinity of determinations.

Remark. An *essential singular point* is not necessarily a critical point, since essential singularities, isolated or not, can be critical or not. Examples of critical essential singularities are : $x = 0$ for $\tan(\mathrm{Log}\, x)$ (nonisolated) or $\sin(C + \mathrm{Log}\, x)$ (isolated and transcendental, *Leçons de Stockholm* pp. 5–6, [92], [66] §14.1 p. 317). Examples of noncritical essential singularities are : $x = \infty$ for e^x or equivalently $x = x_0$ for $e^{1/(x-x_0)}$ (isolated), $x = \infty$ for $\tan x$ (nonisolated). Although, according to a classical theorem of Picard, an analytic function can take any value but at most two (∞ and another one) in the neighborhood of an isolated essential singularity, a noncritical essential singularity is *not* an obstacle to single valuedness.

3.1 First classification of singularities, uniformization

Definition. The *first classification of singularities* is the distinction critical or noncritical between singular points of applications. Note that it does not involve differential equations.

Consider a multivalued application of the Riemann sphere onto itself. There exist two classical methods, called *uniformizations*, to define from it a single valued application, i.e. a function.

The first one is to restrict the object space by subtracting some lines, called *cuts*, so as to forbid local turns around critical points; for the above two examples, one removes any line joining the two points a and ∞.

The second method is to extend the object space to a *Riemann surface*, made of several copies, called sheets, of the Riemann sphere, cut and pasted. A point of the image space may then have several antecedents on the Riemann surface defining the object space. Example : two sheets for $x \to \sqrt{x}$, a countable infinity of sheets for $x \to \operatorname{Log} x$.

As a consequence, to fill the réservoir of functions, one accepts all uniformizable applications, at the price of either restricting the object space by cuts, or defining a Riemann surface in the object space.

Theorem. The general solution of a linear ODE is uniformizable.

Proof. Let

$$E \equiv \sum_{k=0}^{N} a_k(x) \frac{\mathrm{d}^{(k)} u}{\mathrm{d}x^k} = 0,\ a_N(x) = 1 \tag{3.1}$$

be such an N^{th} order ODE. Its general solution

$$u = \sum_{j=1}^{N} c_j u_j,\ c_j \text{ arbitrary constant,} \tag{3.2}$$

has for only singularities those of the N independent particular solutions u_j, a subset of the singularities of the coefficients a_k ([66] §15.1). One knows where to make cuts or to paste the sheets of a Riemann surface in order to uniformize the general solution. QED.

This has important consequences. Firstly, any linear ODE defines a function (Airy, Bessel, Gauss, Legendre, Whittaker, ...). Secondly, in the needed réservoir of functions, one can put all the solutions of all the linear ODEs. Thirdly, a nonlinear ODE is considered as integrated if it is linearizable (of course *via* a finite linearizing expression). Fourthly, in order to extend the list of known functions by means of ODEs, it is necessary to consider nonlinear ODEs.

Hence the problem stated by L. Fuchs and Poincaré.

Problem. Define new functions by means of ODEs, necessarily nonlinear.

3.2 Second classification of singularities, different kinds of solutions

There exist two features of nonlinear ODEs without counterpart in the linear case, they concern the location of singularities of solutions and the possible existence of solutions additional to the general solution.

The singularities of the solutions of nonlinear ODEs may be located at *a priori* unknown locations, which depend on the constants of integration.

Definition (already given in the chapter Introduction). A singular point of a solution of an ODE is called *movable* (resp. *fixed*) if its location in the complex plane depends (resp. does not depend) on the integration constants.

The point at ∞ is to be considered as fixed. A linear ODE has no movable singularities, the zeroes of its general solution depend on the integration constants and are sometimes called for this reason movable zeroes.

Definition. The *second classification of singularities* is the distinction movable or fixed between singularities of solutions of ODEs.

Among the four structures (critical or noncritical) and (fixed or movable) of singularities of solutions of ODEs, only one is an obstacle for this solution to be uniformizable and hence to define a function. This is the presence of singularities at the same time critical and movable. Indeed, in such a case, one knows neither where to make cuts nor where to paste the Riemann sheets, and uniformization is impossible.

But let us come to the second distinction between linear and nonlinear ODEs : contrary to linear ODEs, nonlinear ODEs may have several kinds of solutions.

Definition. The *general solution* (GS) of an ODE of order N is the set of all solutions mentioned in the existence theorem of Cauchy (section 5.3), i.e. determined by the initial value. It depends on N arbitrary independent constants.

Definition. A *particular solution* is any solution obtained from the general solution by giving values to the arbitrary constants. A synonym in English is special solution.

Definition. A *singular solution* is any solution which is not particular. Linear ODEs have no singular solution.

Example. The Clairaut type equation $2u'^2 - xu' + u = 0$ has the general solution $cx - 2c^2$, a particular solution $x - 2$, and the singular solution $x^2/8$.

A singular solution can only exist when the ODE

$$E(u^{(N)}, u^{(N-1)}, \dots, u', u, x) = 0, \tag{3.3}$$

considered as an equation for the highest derivative $u^{(N)}$, possesses at least two determinations (branches), whose coincidence may define a singular solution. This is a generalization of the notion of envelope of a one-parameter family of curves. A practical criterium to detect the singular solutions will be given in section 5.1.

Painlevé stated the following programme ([90] p. 201, Œuvres vol. III p. 123; [92] p. 2, Œuvres vol. III p. 188) :
"Déterminer toutes les équations différentielles algébriques du premier ordre, puis du second ordre, puis du troisième ordre, etc., dont l'intégrale générale est uniforme".

One notices that singular solutions are excluded from this statement. Indeed, they present no interest at all for the theory of integration, for, according to the above theorem, they satisfy an ODE of a strictly lower order than the ODE under consideration and have therefore been encountered at a lower order in the systematic programme stated by Painlevé.

This problem (single valuedness of the general solution) splits into two successive problems whose methods of solution are completely different : absence of movable critical points, then absence of fixed critical points. Hence the final statement.

Problem. Determine all the algebraic differential equations of first order, then second order, then third order, etc., whose general solution has no movable critical points.

This class of equations is often denoted "with fixed critical points". Let us prove that it coincides with the definition of the PP given section 1.4. Out of the four configurations of singularities (critical or noncritical) and (fixed or movable), only the configuration (critical and movable) prevents uniformizability : one does not know where to put the cut since the point is movable.

We have now reached the usual definition, equivalent to the one of section 1.4.

Definition. One calls *Painlevé property* of an ODE the absence of movable critical singularities in its general solution.

3.3 Groups of invariance of the PP

In the fulfillment of the programme of Painlevé, it is sufficient to take one representative equation by class of equivalence of the PP. There exist two relations of equivalence for the PP, defined in sections 3.3.1 and 3.3.2. Other relations of equivalence are defined in section 3.3.3, but they violate the PP.

3.3.1 The homographic group

Theorem. The only bijections (one to one mappings) of the Riemann sphere are the homographic transformations

$$z \to \frac{\alpha z + \beta}{\gamma z + \delta},\ \alpha\delta - \beta\gamma \neq 0,\ (\alpha, \beta, \gamma, \delta) \text{ arbitrary complex constants.} \qquad (3.4)$$

Proof. See any textbook. These transformations define a six-parameter group $\mathcal{H}$ called Möbius group, also denoted $\mathrm{PSL}(2, \mathcal{C})$. This group plays a fundamental rôle in the present theory. Given two triplets of points, there exists a unique homographic transformation applying one triplet onto the other one.

Theorem. The PP of an ODE $E(u, x) = 0$ is invariant under an arbitrary homographic transformation of the dependent variable u and an arbitrary holomorphic change of the independent variable x

$$(u, x) \to (U, X) : u = \frac{\alpha(x)U + \beta(x)}{\gamma(x)U + \delta(x)}, X = \xi(x), \qquad \alpha\delta - \beta\gamma \neq 0, \qquad (3.5)$$

where $\alpha, \beta, \gamma, \delta, \xi$ denote arbitrary analytic (synonym : holomorphic) functions.

Proof. Let x_0 be a regular point of $(\alpha, \beta, \gamma, \delta, \xi)$, and $X_0 = \xi(x_0)$ its transform. In some neighborhood of x_0, the transformation between u and U is close to a homographic transformation with constant coefficients and, according to the previous theorem, the first classification (critical, noncritical) is invariant : if x_0 is critical (resp. noncritical) for u, then X_0 is critical (resp. noncritical) for U, and *vice versa*. Since $(\alpha, \beta, \gamma, \delta, \xi)$ do not depend on x_0, the second classification (fixed or movable) is also invariant. Thus the PP, which only depends on these two classifications, is invariant. QED.

An element of this *homographic group* (3.5) will be denoted $T(\alpha, \beta, \gamma, \delta; \xi)$ or simply $T(\alpha, \beta; \xi)$ in the case $(\gamma = 0, \delta = 1)$. The representative equation is chosen so as to "simplify" some expression, e.g. a three-pole rational fraction the poles of which can be set at predefined locations like $(\infty, 0, 1)$.

Exercise 3.1 *Choose a representative for the Riccati equation (1.1) in its class of equivalence under the homographic group.*

Solution. Two coefficients can be made numeric and the equation reduced to

$$\mathrm{d}U/\mathrm{d}X + U^2 + S(X)/2 = 0, \tag{3.6}$$

under the linear transformation $T(\alpha, \beta; \xi)$

$$u = \alpha U + \beta,\ X = x,\ \alpha = -1/a_2,\ \beta = -(a_2' + a_1 a_2)/(2a_2^2).\square \tag{3.7}$$

This canonical form, in which S is called the *Schwarzian*, will be encountered again in section 6.3.

3.3.2 The birational group

The PP is also invariant under a larger group [89, 56], namely the group of birational transformations, in short the *birational group*,

$$(u, x) \to (U, X) :\ u = r(x, U, \mathrm{d}U/\mathrm{d}X, \dots, \mathrm{d}^{N-1}U/\mathrm{d}X^{N-1}) = 0,\ x = \Xi(X), \tag{3.8}$$
$$(U, X) \to (u, x) :\ U = R(X, u, \mathrm{d}u/\mathrm{d}x, \dots, \mathrm{d}^{N-1}u/\mathrm{d}x^{N-1}) = 0,\ X = \xi(x),$$

(N order of the equation, r and R rational in U, u and their derivatives, analytic in x, X).

For instance, given the ODE $u'' - 2u^3 = 0$ and the new dependent variable $U = u' + u^2$, the algebraic elimination of (u', u'') among these two equations and the derivative of the second one yields the inverse transformation $u = U'/(2U)$, which, once inserted in the direct transformation, yields the transformed equation $UU'' - U'^2/2 - 2U^3 = 0$.

3.3.3 Groups of point transformations (Cartan equivalence classes)

The definition of *to integrate* as given page 25 allows transformations outside the above two groups, which therefore may alter the PP. For instance, the unstable ODE $2uu' - 1 = 0$ is made stable by the change $u^2 \to U$. One such group of point transformations, studied by Roger Liouville [82], Tresse [109] and Cartan [21], is defined as (it includes hodograph transformations)

$$(u,x) \leftrightarrow (U,X) \ : \ u = f(X,U),\ x = g(X,U),\ U = F(x,u),\ X = G(x,u), \tag{3.9}$$

and the variables u and x are two equivalent geometrical coordinates. This geometric approach, in which provides a complementary insight to that of Painlevé, which forbids to exchange the dependent and the independent variables, see section 1.3.

The subgroup of fiber-preserving transformations

$$(u,x) \leftrightarrow (U,X) \ : \ u = f(X,U),\ x = g(X),\ U = F(x,u),\ X = G(x), \tag{3.10}$$

whose equivalence classes are called *Cartan equivalence classes*, has been extensively studied by Kamran *et al.*, see e.g. [65].

3.4 The double interest of differential equations

Let us return to the above problem. At each differential order of the programme, the results are twofold (this is the "double interest" of differential equations) :

1. some *new functions* (defined from the general solution of a stable ODE which is not reducible to a lower order nor to a linear equation),
2. an exhaustive list (i. e. a *classification*) of stable ODEs, which includes the ones defining new functions.

Of course, each equation is characterized by one representative in its equivalence class. Thus, as seen in the introduction, the ODE for $x(t)$ in the case $b = 0, \sigma = 1/3$ of the Lorenz model is not distinct, under the homographic group, from the (P3) equation in the case $\alpha = \beta = 0,\ \gamma = \delta = 1$.

For instance, the result for order one and degree one (the *degree* of an algebraic ODE is the polynomial degree in the highest derivative) is : no new function, one and only one stable equation which is the Riccati equation (1.1).

3.5 The question of irreducibility

The classical definition of irreducibility as given by the "groupe de rationalité" of Jules Drach (Drach, in Œuvres vol. III p. 14, [90] p. 246, [100]) had some weaknesses pointed out by Roger (not Joseph) Liouville in a passionating discussion with Painlevé in the *Comptes Rendus* (see Œuvres vol. III). This is only after further mathematical developments, namely the differential Galois theory, that a precise definition of irreducibility could be given by Umemura [110], see Okamoto, this volume. This definition shares many features with the algorithm of Risch and Norman in computer algebra (which decides if the primitive of a class of expressions, e.g. rational fractions, is inside or outside the class).

3.6 The double method of Painlevé

To solve his problem, Painlevé split it into two parts (this is the "double méthode" [92] p. 11) :

- [1] (a local study) construction of *necessary* conditions for stability,
- [2] (a global study) proof of their *sufficiency*, either by expressing the general solution as a finite expression of a finite number of elementary functions (solutions of linear equations, ...), or by proving the irreducibility of the general solution and its freedom from movable critical points.

The methods pertaining to each part are different. If some necessary condition is violated in the first part, one stops and proceeds to the next equations. If one has exhausted the construction of necessary conditions, or if one believes so (indeed, this process, although probably finite, is sometimes not bounded), one turns to the explicit proof of sufficiency, i.e. practically one tries to integrate (no irreducible equation has been discovered since 1906).

For a good presentation of ideas, see the book of E. Hille [63].

Remark. The reason why movable essential singularities create difficulties lies in the inexistence of methods to express conditions that they be noncritical (*Leçons* pp. 519 sq.).

3.7 The physicist's point of view

The physicist is not interested in establishing a classification nor in finding new functions. Usually, some differential system, whether ordinary or partial, is imposed by physics, and the problem is to "integrate" it in some loose sense. By the way, this loose objective is certainly the main responsible for the numerous, of course divergent, interpretations of "integrable" to be found in the physicists' world.

The best applicability of the present theory arises when one knows nothing or very little on the possible analytical results : first integral, conservation laws of PDEs, particular solutions, ... Then, the first part of the double method of section 3.6 happens to be a precious *integrability detector*. We have already seen a rough version of it : the meromorphy test of section 2.3, the final version of which will be the Painlevé test section 6.6.

The loose objective of the physicist implies performing the test to its end, even if at some point it fails and should be stopped. One thus gathers a lot of information in the form of necessary conditions for a piece of local single valuedness to exist. This *partial integrability detector* can be called the "partial Painlevé test" and will be examplified in section 6.7.

Examining each condition separately, i. e. independently of the others, or simultaneously, one *may* then find pieces of global information like a first integral or a particular closed form solution.

Chapter 4

The classical results (L. Fuchs, Poincaré, Painlevé)

The problem of determining all stable equations has been completely studied for several classes of ODEs, while others are still unfinished. We review here the main results achieved to date.

4.1 ODEs of order one

The completely studied class is (L. Fuchs, Poincaré, Painlevé)

$$E \equiv P(u', u, x) = 0,\ P \text{ polynomial in } (u', u),\ \text{analytic in } x. \tag{4.1}$$

When the degree is one, i.e. for the class $u' = R(u, x)$ with R rational in u and analytic in x, one finds one and only one stable equation, the Riccati equation (1.1). Since it is linearizable, this case defines no new function.

When the degree is greater than one, one finds one and only one new function, the elliptic function $\wp$ of Weierstrass, defined from eq. (1.12). The stable equations are : all the ODEs whose general solution has an algebraic dependence on the arbitrary constant, plus five binomial equations with constant coefficients (i. e. $u'^n = P_m(u),\ (m, n) \in \mathcal{N},\ P_m$ polynomial of degree m). Historically found by Briot and Bouquet, these binomial equations have the following solution (see e.g. [83] Table 1 p. 73)

$$u'^n = (u-a)^{n+1}(u-b)^{n-1},\ n \ge 2,\ \frac{u-b}{u-a} = \left[\frac{b-a}{n}(x-x_0)\right]^n \tag{4.2}$$

$$
\begin{aligned}
u'^2 &= (u-a)^2(u-b)(u-c),\ \frac{1}{u-a} = A\cosh\left[B(x-x_0)\right] + C && (4.3)\\
u'^2 &= (u-a)(u-b)(u-c)(u-d),\ \frac{1}{u-a} = A\wp(x-x_0,g_2,g_3) + B && (4.4)\\
u'^3 &= \left[(u-a)(u-b)(u-c)\right]^2,\ \frac{1}{u-a} = A\wp'(x-x_0,0,g_3) + B && (4.5)\\
u'^4 &= (u-a)^3(u-b)^3(u-c)^2,\ \frac{1}{u-c} - \frac{1}{a-c} = A\wp^2(x-x_0,g_2,0) && (4.6)\\
u'^6 &= (u-a)^5(u-b)^4(u-c)^3,\ \frac{1}{u-a} = A\wp^3(x-x_0,0,g_3) + B, && (4.7)
\end{aligned}
$$

in which a, b, c, d are complex constants and (A, B, C, g_2, g_3) algebraic expressions of (a, b, c, d).

Remarks.

1. Equation (4.3) is a degeneracy of (4.4). The T transformation $u \to a_i + u^{-1}$ (a_i zero of P_{2n}) generates eleven other equations with $m < 2n$, among them the Weierstrass equation (1.12).

2. If the Weierstrass equation had not been known, it would have been discovered at this order one of the systematic process of Painlevé.

4.2 ODEs of order two, degree one

The study of the class

$$u'' = R(u', u, x),\ R \text{ rational in } u',\ \text{algebraic in } u,\ \text{analytic in } x \qquad (4.8)$$

was started by Painlevé [90, 92, 94] and finished by his student Gambier [56].

This class provides six new functions, the functions of Painlevé, defined by the ODEs

$$
\begin{aligned}
\text{(P1) } u'' &= 6u^2 + x\\
\text{(P2) } u'' &= 2u^3 + xu + \alpha\\
\text{(P3) } u'' &= \frac{u'^2}{u} - \frac{u'}{x} + \frac{\alpha u^2 + \beta}{x} + \gamma u^3 + \frac{\delta}{u},\\
\text{(P4) } u'' &= \frac{u'^2}{2u} + \frac{3}{2}u^3 + 4xu^2 + 2(x^2-\alpha)u + \frac{\beta}{u}\\
\text{(P5) } u'' &= \left[\frac{1}{2u} + \frac{1}{u-1}\right]u'^2 - \frac{u'}{x} + \frac{(u-1)^2}{x^2}\left[\alpha u + \frac{\beta}{u}\right] + \gamma\frac{u}{x} + \delta\frac{u(u+1)}{u-1},\\
\text{(P6) } u'' &= \frac{1}{2}\left[\frac{1}{u} + \frac{1}{u-1} + \frac{1}{u-x}\right]u'^2 - \left[\frac{1}{x} + \frac{1}{x-1} + \frac{1}{u-x}\right]u'\\
&\quad + \frac{u(u-1)(u-x)}{x^2(x-1)^2}\left[\alpha + \beta\frac{x}{u^2} + \gamma\frac{x-1}{(u-1)^2} + \delta\frac{x(x-1)}{(u-x)^2}\right],
\end{aligned}
$$

depending on respectively 0,1,2,2,3,4 complex parameters, since the homographic group allows to restrict to $\gamma(\gamma-1)=0$, $\delta(\delta-1)=0$ for (P3), and to $\delta(\delta-1)=0$ for (P5).

The stable equations (4.8) define 53 equivalence classes under the homographic group, including of course the six above ones. They split into 50 with R rational in u and 3 with R algebraic in u, and their list can be found in : the original articles of Gambier [54, 55], [94], Gambier Thèse 1910, Ince 1926 [66] (caution : the numbers 5, 6, 48, 49, 50 of Gambier are changed to 6, 5, 49, 50, 48 in Ince), Murphy 1960 [83], Davis 1961 [46], Bureau M. I 1964, Cosgrove 1993 [38]. In fact, the historical list of Gambier mixes two notions on purpose, namely the irreducibility and the homographic group, which makes this number 53 rather arbitrary; for instance, the classes numbered $1-4, 7-9$ by Gambier have been united by Garnier [58] into the single class

$$u''=\delta(2u^3+xu)+\gamma(6u^2+x)+\beta u+\alpha, \tag{4.9}$$

a stable equation admitting a second order Lax pair.

The 50 stable equations (4.8) with R rational in u define 24 equivalence classes [56] under the birational group and less than 24 Cartan equivalence classes [65] under the group of point transformations (3.10).

This extremely important result (the discovery of six new functions, nowadays frequently encountered in physics, and the exhaustive list of $50+3$ equations) deserves several comments depending on the field of interest.

- (Practical usage). Given an algebraic second order, first degree ODE, either it is transformable by a T transformation (3.5) into one of the $50+3$ equations or not. If it is not, it has movable critical points. If it is, it is *explicitly integrated* and, by looking in the table of Gambier, its general solution is a known finite single valued expression made of the following functions : solutions of linear equations of order at most four, Weierstrass, (P1) to (P6).

- (Movable singularities). The only movable (and of course noncritical) singularities of these ODEs are : poles for $50+2$ of them, in addition a nonisolated essential singularity for 1 of them. This result of Painlevé and Gambier (poles are the only movable singularities of stable second order, first degree equations rational in (u',u) and analytic in x) is often believed to be more general, leading to wrong definitions for the PP; it is no more true for third order, degree one, or even second order, degree higher than one.

- (Fixed critical singularities). (P1), (P2), (P4) have none, (P3) and (P5) have two transcendental critical points $(\infty,0)$, both removable by the uniformizing transformation $x\to e^x$. (P6) has three transcendental critical points $(\infty,0,1)$.

- (Dependence on the arbitrary constants). What characterizes the 6 Painlevé equations among the 24 is the transcendental (i.e. not algebraic) dependence of their general solution on both constants of integration. The $24-6$ equations whose general solution does not involve (P1)–(P6) have either an algebraic dependence on both constants or a semi-transcendental dependence (algebraic for one, transcendental for the other one).

- (Confluence). By a confluence process (Painlevé [94], Gambier Thèse, see Mahoux, this volume), (P6) generates the five others and (P1) generates the Weierstrass equation, so up to now algebraic equations have only defined one master function.

- (Monodromy). (P6) was found independently by R. Fuchs [53] and Schlesinger [105] in the twenty-first problem of Riemann. Given the second order linear ODE $y''(t)+a_1(t,x,u)y'(t)+a_2(t,x,u)y=0$ with four Fuchsian singular points $t=(\infty,0,1,x)$ (see definitions section 5.2.1) and an apparently singular point u, the necessary and sufficient condition for the group of monodromy (see Mahoux, this volume, for definitions) to be independent of x is that $u(x)$ satisfies equation (P6).

4.3 ODEs of higher order or degree

Painlevé's opinion was that no new function should be expected at third order and that one should go to fourth order. In fact, despite huge efforts, no new function has yet been found.

- ODEs of order two, degree higher then one.

  Only some subclasses have been studied, and their classification is nearly finished. See Chazy (Thèse), Bureau (M. III), Cosgrove (1993 [43, 40, 41]).

  Those of degree two have the necessary form

$$\begin{aligned}&\left[u''+E_0u'^2+E_1u'+E_2\right]^2\\&=F_0u'^4+F_1u'^3+F_2u'^2+F_3u'+F_4,\end{aligned}\qquad(4.10)$$

  with (E_k,F_k) rational in u and analytic in x. Its binomial subset ($E_0=E_1=E_2=0$) is classified in Ref. [43].

  The binomial subset $(u'')^n=F(u',u,x)$ of equations of degree $n\geq 3$ is classified in Ref. [40].

- ODEs of order three, degree one.

  The classification is nearly finished. See Garnier (Thèse), Chazy (Thèse), Bureau (M. II, [20]), Cosgrove [41].

- ODEs of order four, degree one.
  The classification is just started. See Chazy (Thèse), Bureau (M. II).

For an account of similar work on PDEs, see Ref. [84].

Chapter 5

Construction of necessary conditions. The theory

The reader only interested in *using* the Painlevé test *may* skip this chapter, whose relevant parts will anyway be referred to in next chapters. By so doing, however, his/her confidence in the Painlevé test will falter at the first encounter of one of the innumerable so-called exceptions, counterexamples, and so on, which are published every year.

This chapter describes all the methods to build necessary conditions (NC) for the absence of movable critical points in the general solution. Most methods are analytic, and we unify their presentation by describing each of them as a perturbation in a small complex parameter ε, to which can then be applied the theorem of perturbations of Poincaré, itself a generalization of the existence theorem of Cauchy. One of them is arithmetic and leads to diophantine conditions on the Fuchs indices of a linear differential equation.

What we try to emphasize is the quite small amount of nonlinear features in these methods. Indeed, most of the information is obtained by well known theories concerning linear equations, whether differential or algebraic.

The detection of singular solutions is first explained in section 5.1. The linear ODEs are then reviewed from the point of view of interest to us. Then we state the fundamental theorems at the origin of all methods. For comparison purposes, two equations are defined which will be later processed by all methods. Finally, we describe each method and apply it to the two examples.

Unless otherwise stated, the class of DEs considered is made of DEs (2.30) polynomial in $\mathbf{u}$ and its derivatives, analytic in x, with $(\mathbf{E}, \mathbf{u})$ multidimensional.

5.1 Removal of singular solutions

Since the PP excludes the consideration of singular solutions, one must discard them as early as possible.

Let us give a practical criterium to detect singular solutions.

Theorem. A necessary condition for a solution of an ODE to be singular is the existence of a common *finite* root $u^{(N)}$ to $E = 0$ and its partial derivative with respect to $u^{(N)}$. If $E(u, x) = 0$ depends polynomially on the two highest derivatives $u^{(N)}, u^{(N-1)}$, after factorization of this polynomial existence condition in $u^{(N-1)}$ (called discriminant), it is necessary that the vanishing factor has an odd multiplicity.

Proof. See e.g. Chazy (Thèse). The condition is not sufficient, and details and examples can be found in [104] chap. 10.

Hence the method : compute the discriminant, factorize it, discard the even factors, test each odd factor to check if it defines a solution to the equation.

Chazy (Thèse p. 358) was the first to notice the absence of correlation between the structure of singularities of the GS and of the SS. Here are such examples.

Single valued GS, SS with a movable critical point (Chazy, Thèse p. 360)

$$\begin{aligned}
&(u''' - 2u'u'')^2 + 4u''^2(u'' - u'^2 - 1) = 0,\\
&\text{discriminant} = -16u''^2(u'' - u'^2 - 1),\\
&\text{GS} : u = e^{c_1 x + c_2}/c_1 + \frac{c_1^2 - 4}{4c_1}x + c_3,\ \text{SS} : u = C_2 - \text{Log}\cos(x - C_1)
\end{aligned} \tag{5.1}$$

GS with a movable critical point, single valued SS (Valiron tome II §148)

$$\begin{aligned}
&27uu'^3 - 12xu' + 8u = 0,\\
&\text{discriminant} = -12^3 \times 27^2 u^2 (27u^3 - 4x^3),\\
&\text{GS} : u^3 = c(x - c)^2,\ \text{SS} : u^3 = (4/27)x^3.
\end{aligned} \tag{5.2}$$

Single valued GS and single valued SS (Painlevé BSMF p. 239)

$$\begin{aligned}
&u''^2 + 4u'^3 + 2(xu' - u) = 0,\\
&\text{discriminant} = -8(2u'^3 + xu' - u),\\
&\text{GS} : u = (1/2)v'^2 - 2v^3 - xv,\ v'' = 6v^2 + x,\ \text{SS} : u = Cx + 2C^3.
\end{aligned} \tag{5.3}$$

5.2 Linear equations near a singularity

Our only interest here is to decide about the local single valuedness near a singularity $x = x_0$, put for convenience at the origin by a homographic transformation ($x \to x - x_0$ or $x \to 1/x$ according as x_0 is at a finite distance or not).

These results are detailed in the course of Reignier, this volume.

Consider the most general linear system, put in a form solved for all first order derivatives (the canonical form of Cauchy)

$$x\frac{\mathrm{d}\mathbf{U}}{\mathrm{d}x} = \mathbf{A}\mathbf{U}, \tag{5.4}$$

with $\mathbf{U}$ a column vector of N components and $\mathbf{A}$ a square matrix rational in x. This can be the representation of the general scalar ODE (3.1), with $U_k = x^k u^{(k)}, k = 0, \dots, N-1$ and $b_k = x^{N-k} a_k$, e. g. for $N = 3$

$$x\frac{\mathrm{d}}{\mathrm{d}x}\begin{pmatrix} U_0 \\ U_1 \\ U_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ -b_0 & -b_1 & 2-b_2 \end{pmatrix}\begin{pmatrix} U_0 \\ U_1 \\ U_2 \end{pmatrix}. \tag{5.5}$$

Definition. The point $x = 0$ is called *Fuchsian* iff all solutions of (5.4) have a polynomial growth near it. It is called *nonFuchsian* if at least one solution has a nonpolynomial growth.

Example. For the ODE $u' + ax^n u = 0,\ n$ integer, a nonzero, the Fuchsian case is $n \geq -1$ and the nonFuchsian one is $n \leq -2$. The solution $u(n)$ for $n = -2, -1, 0$ is $u(-2) = e^{a/x}, u(-1) = x^{-a}, u(0) = e^{-ax}$ and its singularity at $x = 0$ is respectively : an isolated noncritical essential point, a critical point or pole or zero (depending on a), a regular point.

Remarks.

1. This definition is the one of modern authors [7]. It involves a property of the solutions, not of the coefficients of the equation. *Fuchsian* denotes at the same time a case with the solutions $u = (x, x^2)$ (classically called *regular point*) and a case with $u = (x^{-1}, x^2)$ (classically called *singular regular point*). The motivation for such a definition is the difficulty to recognize it on the matricial notation. While in the scalar case (3.1) the canonical form defined by setting $a_{N-1} = 0$ provides an easy criterium to decide about the nature of the singularity, in the matricial case the example

$$\mathbf{A}/x = \begin{pmatrix} n/x & 0 \\ x^{-n} & 0 \end{pmatrix},\ \mathbf{U}_1 = \begin{pmatrix} x^n \\ x \end{pmatrix},\ \mathbf{U}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{5.6}$$

   shows the difficulty to do so.

2. We avoid the usual synonyms *regular singularity* and *irregular singularity* for their built-in conflict.

Definition. Given a point x_0, a *fundamental set of solutions* of a linear ODE of order N is any set of N linearly independent solutions defined in a neighborhood of x_0.

5.2.1 Linear equations near a Fuchsian singularity

Definition. Given a Fuchsian point $x = 0$, the eigenvalues i of $\mathbf{A}(0)$ are called *Fuchs indices.* The *indicial equation* is the characteristic equation of the linear operator $\mathbf{A}(0)$

$$\lim_{x \to 0} \det\ (\mathbf{A}(x) - i) = 0. \tag{5.7}$$

Near a Fuchsian point $x = 0$, there exist a fundamental set of solutions

$$x^{\lambda_i} \sum_{j=0}^{m_i} \varphi_{ij}(x)(\mathrm{Log}\, x)^j,\ \ i = 1, N \tag{5.8}$$

in which the λ_i's are complex numbers (the Fuchs indices), m_i positive integers (their multiplicity), φ_{ij} converging Laurent series of x with finite principal parts.

Series (5.8) are the simplest examples of $\psi-$series.

The necessary and sufficient condition of local single valuedness of the general solution of the linear equation is : λ_i all integer, no Log terms.

In the scalar case (3.1), the indicial equation is

$$\begin{aligned} 0 &= \lim_{x \to 0} x^{N-i} E(x, x^i) = \lim_{x \to 0} \det\ (\mathbf{A}(x) - i) \qquad (5.9)\\ &= b_0(0) + b_1(0)i + b_2(0)i(i-1) + \ldots + b_N(0)i(i-1)\ldots(i-N+1). \end{aligned}$$

Theorem. Given a Fuchsian point of the scalar ODE (3.1), necessary and sufficient conditions for the general solution to be locally single valued near it are

- ∘ the N indices are distinct integers,
- ∘ $N(N-1)/2$ conditions for the absence of logarithms are satisfied.

Proof. See Hille book [63] or any other textbook.

In the matricial case (5.4), these conditions are replaced by

- ∘ the N indices are integers,
- ∘ the multiplicity of each index i is equal to the dimension of the kernel of $\mathbf{A}(0) - i$,
- ∘ all conditions for the absence of logarithms are satisfied.

The search for the no-log conditions can be achieved in one loop, by requiring the existence of a Laurent series extending from the lowest Fuchs index i_1 to $+\infty$

$$x^{i_1} \sum_{j=0}^{+\infty} u_j x^j \tag{5.10}$$

and containing N arbitrary independent coefficients; this is a finite process, which terminates when $j + i_1$ reaches the highest Fuchs index. Consider for instance the third order ODE admitting the three solutions

$$u_1 = x^{-2},\ u_2 = x^{-3} + ax^{-2} \operatorname{Log} x,\ u_3 = x^{-4} + bx^{-3} \operatorname{Log} x + \frac{ab}{2} x^{-2} (\operatorname{Log} x)^2,$$

namely

$$\begin{vmatrix} u & u_1 & u_2 & u_3 \\ u' & u_1' & u_2' & u_3' \\ u'' & u_1'' & u_2'' & u_3'' \\ u''' & u_1''' & u_2''' & u_3''' \end{vmatrix} = 0.$$

Its three Fuchs indices $-4, -3, -2$ are simple, it is sufficient that j runs from 0 to 2 with $i_1 = -4$, the condition $b = 0$ is found at $j = 1$ and the condition $a = 0$ at $j = 2$.

In some of the next sections, we will encounter inhomogeneous ODEs in which the rhs is itself a Laurent series with a finite principal part, so we will have in addition to express the single valuedness of a particular solution as well. This can be incorporated in the single loop described above, provided it starts from the smallest of the two values i_1 and the singularity order of the particular solution, imposed by the rhs.

5.2.2 Linear equations near a nonFuchsian singularity

Near a nonFuchsian singular point $x = 0$, there exist N linearly independent solutions

$$e^{Q_i(1/z_i)} x^{s_i} \sum_{j=0}^{m_i} \varphi_{ij}(z_i) (\operatorname{Log} x)^j,\ z_i = x^{1/q_i},\ i = 1, N \qquad (5.11)$$

in which the q_i's are positive integers, Q_i polynomials, s_i complex numbers called *Thomé indices*, φ_{ij} *formal* Laurent series with a finite principal part. The question of local single valuedness of the general solution cannot be settled so easily, because formal series are generically divergent.

5.3 The two fundamental theorems

Theorem I (Cauchy, Picard). Consider an ODE of order N, of degree one in the highest derivative, defined in the canonical form

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}x} = \mathbf{K}[x, \mathbf{u}],\ x \in \mathcal{C},\ \mathbf{u} \in \mathcal{C}^N. \qquad (5.12)$$

Let $(x_0, \mathbf{u}_0)$ be a point in $\mathcal{C} \times \mathcal{C}^N$ and D be a domain containing $(x_0, \mathbf{u}_0)$. If $\mathbf{K}$ is holomorphic in D,

- there *exists* a solution $\mathbf{u}$ satisfying the initial condition $\mathbf{u}(x_0) = \mathbf{u}_0$,
- it is *unique*,
- it is *holomorphic* in a domain containing $(x_0, \mathbf{u}_0)$.

Proof. See any textbook. For delicate points on this classical theorem, see *Leçons* p. 394. The contribution of Picard is to have moved the holomorphy property from the hypothesis to the conclusion.

There exists an important complement to the theorem of Cauchy, due to Poincaré : the Cauchy solution is also holomorphic in the Cauchy data.

Remark. More practically, the canonical form can also be defined as

$$\frac{\mathrm{d}^N u}{\mathrm{d}x^N} = \mathbf{K}[x, u, u', \dots, u^{(N-1)}]. \tag{5.13}$$

The theorem says nothing whenever the holomorphy of $\mathbf{K}$ is violated, as in the following two cases.

Case 1. $\mathrm{d}u/\mathrm{d}x = u/(u-1)$, at $u_0 = 1$, a point of meromorphy for $\mathbf{K}$. The only way to possibly remove this singularity without altering the structure of singularities is to perform a T transformation (3.5). The homography T : $1/(u-1) = U$ yields a new $\mathbf{K}$, defined by $\mathrm{d}U/\mathrm{d}x = -U^2 - U^3$, which is indeed holomorphic in $\mathcal{C} \times \mathcal{C}$, now making the theorem applicable. In order to shorten the exposition, this step of an homographic transformation will be omitted in the whole chapters 5 and 6, and only reminded for the synthesis of all methods into the Painlevé test section 6.6.

Case 2. $\mathrm{d}u/\mathrm{d}x = \sqrt{4(u-e_1)(u-e_2)(u-e_3)}$, at $u_0 = e_j, j = 1, 2, 3$, critical points for $\mathbf{K}$.

Example. $\mathrm{d}u/\mathrm{d}x + u^2 = 0$, with the datum $u = u_0$ at $x = x_0$. The Cauchy solution is represented by the (infinite) Taylor series $u = u_0 \sum_{j=0}^{+\infty} [-(x-x_0)u_0]^j$, a geometric series whose sum depends on one, not two, arbitrary constants, the arbitrary location $x_1 = x_0 - u_0^{-1}$ of the movable simple pole; it only exists locally, inside the disk of convergence centered at x_0 with radius $|u_0|^{-1}$. This sum is also represented by the Laurent series $(x - x_1)^{-1}$. One notices the enormous advantage of the Laurent series : it reduces to one term, and it has a much larger domain of definition (the whole complex plane but one point).

Lemma (Poincaré, *Mécanique céleste* [99]). Consider an ODE of order N, of degree one in the highest derivative, depending on a small complex parameter ε, defined in the canonical form

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}x} = \mathbf{K}[x, \mathbf{u}, \varepsilon],\ x \in \mathcal{C},\ \mathbf{u} \in \mathcal{C}^N,\ \varepsilon \in \mathcal{C}. \tag{5.14}$$

Let $(x_0, \mathbf{u}_0, 0)$ be a point in $\mathcal{C} \times \mathcal{C}^N \times \mathcal{C}$ and D be a domain containing $(x_0, \mathbf{u}_0, 0)$. If $\mathbf{K}$ is holomorphic in D, the Cauchy solution exists, is unique and holomorphic in a domain containing $(x_0, \mathbf{u}_0, 0)$.

Proof. See any textbook. Note that $\mathbf{K}$ may be independent of ε.

Definition. Given $\mathbf{x}$, the application $\mathbf{u} \to \mathbf{E}(\mathbf{x}, \mathbf{u})$ and some point $\mathbf{u}^{(0)}$, one calls *differential* of $\mathbf{E}$ at point $\mathbf{u}^{(0)}$ the linear application, denoted $\mathbf{E}'(\mathbf{x}, \mathbf{u}^{(0)})$, defined by

$$\forall \mathbf{v} : \ \mathbf{E}'(\mathbf{x}, \mathbf{u}^{(0)})\mathbf{v} = \lim_{\lambda \to 0} \frac{\mathbf{E}(\mathbf{x}, \mathbf{u}^{(0)} + \lambda \mathbf{v}) - \mathbf{E}(\mathbf{x}, \mathbf{u}^{(0)})}{\lambda}. \tag{5.15}$$

This notion is known under various names : Gâteaux derivative, linearized application, tangent map, Jacobian matrix, and sometimes Fréchet derivative.

Definition. Given a DE $\mathbf{E}(\mathbf{x}, \mathbf{u}) = 0$ and a point $\mathbf{u}_0$, the linear DE

$$\mathbf{E}'(\mathbf{x}, \mathbf{u}^{(0)})\mathbf{v} = 0 \tag{5.16}$$

in the unknown $\mathbf{v}$ is called the *linearized equation* in the neighborhood of $\mathbf{u}^{(0)}$ associated to the equation $\mathbf{E}(\mathbf{x}, \mathbf{u}) = 0$.

This is precisely the *équation auxiliaire* (2.20) of Darboux. The auxiliary equation of a linear equation is the linear equation itself.

Let us define the formal Taylor expansions

$$\mathbf{u} = \sum_{n=0}^{+\infty} \varepsilon^n \mathbf{u}^{(n)}, \ \mathbf{K} = \sum_{n=0}^{+\infty} \varepsilon^n \mathbf{K}^{(n)}. \tag{5.17}$$

The single equation (5.14) is equivalent to the infinite sequence

$$n = 0 : \ \frac{\mathrm{d}\mathbf{u}^{(0)}}{\mathrm{d}x} = \mathbf{K}^{(0)} = \mathbf{K}[x, \mathbf{u}^{(0)}, 0] \tag{5.18}$$

$$n \ge 1 : \ \frac{\mathrm{d}\mathbf{u}^{(n)}}{\mathrm{d}x} = \mathbf{K}^{(n)} = \mathbf{K}'[x, \mathbf{u}^{(0)}, 0]\mathbf{u}^{(n)} + \mathbf{R}^{(n)}(x, \mathbf{u}^{(0)}, \dots, \mathbf{u}^{(n-1)}) \tag{5.19}$$

At order zero, the equation is nonlinear.

At order one, the equation, in the particular important case when $\mathbf{K}$ is independent of ε, is the linearized equation (without rhs, $\mathbf{R}^{(1)} = 0$) canonically associated to the nonlinear equation.

At higher orders, this is the same linearized equation with different rhs $\mathbf{R}^{(n)}$ arising from the previously computed terms, and only a particular solution is needed to integrate.

Theorem II (Poincaré 1890, Painlevé BSMF 1900 p. 208, Bureau 1939, M. I). Take the assumptions of previous lemma. If the general solution of (5.14) is single valued in D except maybe at $\varepsilon = 0$, then

- $\varepsilon = 0$ is no exception, i.e. the general solution is also single valued there,
- every $\mathbf{u}^{(n)}$ is single valued.

Proof. See BSMF p. 208. The main difficulty is to prove the convergence of the series. This theorem remains valid if one replaces "single valued" (Painlevé version) by "periodic" (Poincaré version) or "free from movable critical points" (Bureau version).

Remarks.

- This feature (one nonlinear equation (5.18), one linear equation (5.19) with different rhs) is a direct consequence of perturbation theory, it is common to all methods aimed at building necessary stability conditions. The equations may be differential like (5.18)–(5.19), or simply algebraic. Moreover, all the methods which we are about to describe (except the one of Painlevé) will reduce the differential problems to algebraic problems keeping the same feature, and the overall difficulty will be to solve *one* nonlinear algebraic equation, then *one* linear algebraic equation with a countable number (practically, a finite number) of rhs.

- The two theorems and the lemma express a local property, not a global one, therefore they cannot serve to prove integrability as defined page 25. Conversely, they can be used to disprove the PP. In the same spirit, it is generally useless to try and sum the Taylor or Laurent series which will be defined. Indeed, these series only serve as generators of necessary stability conditions. The proof of sufficiency is achieved by completely different methods.

- The two theorems only apply to ODEs written in the canonical form of Cauchy.

As a summary, the equations successively involved are

- the original DE $\mathbf{E}(x, \mathbf{u}) = 0$ also called unperturbed DE because ε will be introduced into the equation from the outside,

- the perturbed DE $\mathbf{E}(X, \mathbf{U}, \varepsilon)$, obtained from the preceding one by some transformation $(x, \mathbf{u}, \mathbf{E}) \to (X, \mathbf{U}, \mathbf{E}, \varepsilon)$ called perturbation,

- a canonical form (it is not unique) $d\mathbf{U}/\mathrm{d}X = \mathbf{K}(X, \mathbf{U}, \varepsilon)$ of the perturbed equation, also called abusively perturbed equation,

- the infinite sequence (5.18)–(5.19) of equations independent of ε.

The methods described in next sections for establishing necessary stability conditions consist of building one or two perturbed equations from the original unperturbed equation, then of applying the theorem II at a point x_0 which is *movable*. This movable point can be either regular (method of Painlevé) or singular noncritical (all the others), which will require its previous transformation to a regular point (by a transformation close to $u \to u^{-1}$) for theorem II to apply. One is thus led to the equations (5.19), i.e. to *one linear* DE with a sequence of rhs. In order to avoid movable critical points in the original equation, one requires single valuedness in a neighborhood of x_0 for : the general solution of the linear homogeneous equation, a particular solution of each of the successive linear inhomogeneous equations.

One must therefore express that a very special class of linear inhomogeneous DEs has a general solution single valued in a neighborhood of x_0. Their lhs (homogeneous part) is the linearized equation (équation auxiliaire) of a nonlinear equation which has already passed the requirement $n = 0$ of theorem II. The rhs (inhomogeneous part) of equation n depends rationally on $\{\mathbf{u}^{(k)}, k = 0, \dots, n-1\}$ and their derivatives, all single valued near x_0 since the necessary conditions have been fulfilled until $n - 1$.

For the coefficients of the homogeneous linear DE, the point x_0 will appear to be either a point of holomorphy (method of Painlevé) or a singular noncritical point (the other methods). In the latter case, both situations (Fuchsian, nonFuchsian, see sections 5.2.1 and 5.2.2) will occur.

5.3.1 Two examples : complete (P1), Chazy's class III

Example 1 ("complete (P1)") (BSMF p. 224, [66] §14.311 p. 329, [14] p. 267, [75]).

$$E \equiv -\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + c\frac{\mathrm{d}u}{\mathrm{d}x} + eu^2 + fu + g = 0, \tag{5.20}$$

with (c, e, f, g) analytic in x, and e nonzero. This equation arises in the systematic study of class (4.8) and has led to the discovery of (P1).

Under a transformation $T(\alpha, \beta; \xi)$ (3.5), equation (5.20) is form-invariant ([14] p. 267)

$$-\frac{\mathrm{d}^2 U}{\mathrm{d}X^2} + \left[c - \frac{\xi''}{\xi'} - 2\frac{\alpha'}{\alpha}\right]\xi'^{-1}\frac{\mathrm{d}U}{\mathrm{d}X} + e\frac{\alpha}{\xi'^2}U^2 + \left[f + c\frac{\alpha'}{\alpha} + 2e\beta - \frac{\alpha''}{\alpha}\right]\frac{U}{\xi'^2}$$
$$+ \left[g + f\beta + c\beta' + e\beta^2 - \beta''\right]\alpha^{-1}\xi'^{-2} = 0,\ \alpha\xi' \ne 0. \tag{5.21}$$

This allows to assign simple predefined values to as many coefficients as gauges in T, i.e. three. For any value of (c, e, f, g) it is possible to choose for the coefficients of $\mathrm{d}U/\mathrm{d}X, U^2, U$ the values $0, 6, 0$, and this requires solving two quadratures and one linear algebraic equation for $(\alpha, \beta; \xi)$

$$(\mathrm{Log}\,\alpha)' = \frac{2}{5}\left[c - \frac{e'}{2e}\right] \tag{5.22}$$

$$\xi'^2 = \frac{e\alpha}{6},\ \beta = \frac{1}{2e}\left[f + c(\mathrm{Log}\,\alpha)' + (\mathrm{Log}\,\alpha)'' + (\mathrm{Log}\,\alpha)'^2\right]. \tag{5.23}$$

Consequently, in what follows, one always assumes $c = 0, e = 6, f = 0$ in (5.20).

Example 2 (Chazy complete equation of class III [22]).

$$-u_{xxx} + \frac{a}{2}(2uu_{xx} - 3u_x^2) + a_1 u_{xx} + c_1 uu_x + c_0 u_x + d_3 u^3 + d_2 u^2 + d_1 u + d_0 = 0, \tag{5.24}$$

where (a, a_i, c_i, d_i) are analytic in x, and a nonzero. This one has led Chazy to the discovery of his equation (1.17).

Under a transformation $T(\alpha, \beta; \xi)$, equation (5.24) is form-invariant. For any value of (a, a_i, c_i, d_i) it is possible to choose the values $2, 0, 0$ for the coefficients of UU_{XX}, U_{XX}, U^2, and this requires solving the coupled ODEs for $(\alpha, \beta; \xi)$ [notation $\Lambda = \alpha'/\alpha$]

$$2\Lambda' - \Lambda^2 + 2a^{-3}[(a^2 c_1 + 18ad_3)\Lambda + a^2 d_2 + 9a' d_3 - 3a^2 a_1 d_3] = 0 \tag{5.25}$$

$$\xi' = \frac{a\alpha}{2},\ \beta = a^{-2}(6a\Lambda + 3a' - aa_1), \tag{5.26}$$

i.e. one Riccati equation followed by two quadratures. Consequently, in what follows, one always assumes $a = 2, a_1 = 0, d_2 = 0$ in (5.24).

None of these two examples has singular solutions.

Exercise 5.1 *Show the impossibility to cancel d_3 in (5.24) by choosing α, β, ξ.*

5.4 The method of pole-like expansions

This is a reliable version of the meromorphy test given in sections 2.3 and 2.4.

Consider a movable singular point x_0 of either the general solution or a particular solution. Since $\mathbf{u}(x_0)$ is not finite, the theorems of section 5.3 cannot be applied. It is nevertheless immediate to check that the perturbation

$$x = x_0 + \varepsilon X,\ u = (\varepsilon X)^p \sum_{n=0}^{+\infty} (\varepsilon X)^n u^{(n)}(x),\ E = (\varepsilon X)^q \sum_{n=0}^{+\infty} (\varepsilon X)^n E^{(n)}(x), \tag{5.27}$$

in which the key point is the dependence of $u^{(n)}$ on x, not X, generates equations $E^{(n)} = 0$ which only differ from the algebraic equations $E_j = 0$ defined by (2.8) by the replacement of x_0 by x. The identification is even complete if χ is defined by $\chi_x = 1$ instead of $\chi = x - x_0$, see Remark page 16.

Fortunately, the method we are about to describe has been made by Bureau (1939) [13] an application of theorems I and II, as will be seen in section 5.6. This *method of pole-like expansions* is the most widely used in Painlevé analysis. Initiated by Paul Hoyer [64] and Sophie Kowalevski [72, 73], it has been formalised by Gambier [56], revived by Ablowitz *et al.* [4] who applied it to wide classes of physical equations, extended to partial differential equations (PDEs) by Weiss *et al.* [111] (WTC), with technical simplifications by Kruskal [68] and Conte [28, 29]. Painlevé himself never used “le procédé connu de Madame

Kowalevski ... dont le caractère nécessaire n'était pas établi" (Acta pp. 10, 83, Œuvres III pp. 196, 269), see section 2.6.

We now rephrase the steps and generated conditions of sections 2.3 and 2.4 so as to adapt them to the new objective : the PP. The expansion is denoted

$$\mathbf{u} = \sum_{j=0}^{+\infty} \mathbf{u}_j \chi^{j+\mathbf{p}},\ \mathbf{u}_0 \neq \mathbf{0},\ \chi' = 1. \tag{5.28}$$

First step. Determine all possible families $(\mathbf{p}, \mathbf{u}_0)$. The necessary condition on $(\mathbf{p}, \mathbf{u}_0)$ is

- **C0**. For each family not describing a singular solution, all components of $\mathbf{p}$ are integer.

It there exists no truly singular family (at least one component of $\mathbf{p}$ negative), the method stops without concluding.

Remarks.

- Some components of $\mathbf{u}_0$ can be zero, or even some components of $\mathbf{u}$.
- To avoid missing some family, one should firstly put the ODE under a canonical form of Cauchy (5.12) or (5.13), so as to enumerate all the points $\mathbf{u}$ which make inapplicable the existence theorem of Cauchy, secondly for each such point build a transformed ODE under an homography making the point regular for the Cauchy theorem, thirdly determine families of the transformed ODE as in section 2.3.
- The derivative of order k of χ^p does not behave like χ^{p-k} if p is positive and $p-k$ negative.

Second step. For each family, compute the indicial polynomial $\det \mathbf{P}$. None of the conditions **C1**, **C2**, **C3** of section 2.4 is required for the existence of the Laurent series since we also accept particular solutions and only exclude singular solutions.

Third step. Unchanged as compared to section 2.4. The condition **C4** is unchanged.

The resulting expansion (5.28) thus contains as many arbitrary coefficients $\mathbf{u}_i$ as the sum of the multiplicities of the distinct positive indices, in addition to the arbitrary location of x_0, associated to the index -1.

For indices which are not positive integers, the method says nothing, not even that they should be integers, and in such a case the expansion (5.28) only represents part of the general solution, without indication about some possible multivaluedness in the missing part.

Remarks.

- The semi-infinite Laurent expansion (5.28) for u about the singular point x_0 is equivalent to an expansion for u^{-1} about a regular point, expansion however different from the Taylor one. This is used in Bureau's method section 5.6.

- We prefer the terms "pole-like singularity" to "pole singularity", for the actual singularity of the *general* solution may not be a pole, as shown by the example of Chazy's equation (1.17), for which it is a movable noncritical essential singularity.

- Index -1 also corresponds to an arbitrary coefficient but, since the general solution cannot depend on more than N such arbitrary coefficients, some renormalisation occurs. In the example $\mathrm{d}u/\mathrm{d}x + u^2 = 0$ already considered in section 3.1, the Cauchy solution near the regular point x_0 can be reexpanded

$$u = \sum_{j=-\infty}^{0} (-u_0)^j (x - x_0)^{j-1} \tag{5.29}$$

[if the example were not so simple, this would be a doubly infinite Laurent series] so as to exhibit an arbitrary coefficient at index $j = i = -1$, the only index of this too simple ODE. Note the "pole-like" singularity x_0, which is in fact an apparent, inessential singularity, in this case a regular point!

5.4.1 The two examples

Example 1. "Complete (P1)" eq. (5.20). Already handled in section 2.3.

Example 2. Chazy's equation (1.17).

First step. The dominant terms are among $-u''', 2uu'' - 3u'^2, d_3 u^3$, hence two possible families

$$\begin{array}{lll} (p,q) = (-1,-4) & u_0 = -6 & \hat{E} \equiv u''' - 2uu'' + 3u'^2 \\ (p,q) = (-2,-6) & d_3 u_0^3 = 0 & \hat{E} \equiv -2uu'' + 3u'^2 + d_3 u^3. \end{array}$$

The second family only exists if $d_3 = 0$. See section 5.9 for a direct proof that $d_3 = 0$ is a necessary stability condition.

Second step. The indicial polynomial of the first family is

$$\begin{aligned} & \chi^{-(i-4)}[-\partial_x^3 + 2u_0\chi^{-1}\partial_x^2 - 6(u_0\chi^{-1})_x\partial_x + 2(u_0\chi^{-1})_{xx}]\chi^{i-1} \\ = & \ -(i-1)(i-2)(i-3) - 12(i-1)(i-2) - 36(i-1) + 2(-1)(-2) \\ = & \ -(i+3)(i+2)(i+1), \end{aligned} \tag{5.30}$$

and the indices are $-3, -2, -1$; the algorithm stops here, due to the absence of positive integer indices.

Provided $d_3 = 0$, the second family has the indices $-1, 0$.

Third step (only for the second family provided $d_3 = 0$). At the index 0, the condition **C4** $Q_0 = 0$ is satisfied and the algorithm stops.

Exercise 5.2 *Find the families and indices of the following equations.*

$$u'' - 2 = 0, \;\; u = (x-a)(x-b) \tag{5.31}$$

$$uu'' - 2u'^2 = 0, \;\; u = a(x-x_0)^{-1} \tag{5.32}$$

$$u'' + 3uu' + u^3 = 0, \;\; u = \frac{1}{x-a} + \frac{1}{x-b}. \tag{5.33}$$

5.4.2 Nongeneric essential-like expansions

Just like (5.28), the expansion

$$\mathbf{u} = \sum_{-j=0}^{\infty} \mathbf{u}_j \chi^{j+\mathbf{p}}, \;\; \mathbf{u}_0 \neq \mathbf{0}, \tag{5.34}$$

valid outside a disk centered at x_0 i.e. in a neighborhood of the point ∞, is locally single valued. From this downward Laurent series, one could conceive a "method of essential-like expansions" quite similar to the method of pole-like expansions, in order to generate necessary stability conditions, this time from the negative integer indices only.

However, for most equations, this method is not applicable. For instance, with the example $-u''' + 2uu'' - 3u'^2 + d_3 u^3 = 0$ (a subset of (1.17)), none of the two expansions (5.34) with $p = -1$ or $p = -2$ exists, unless $d_3 = 0$, which is *not* a reason to conclude that d_3 must vanish.

It only applies to the very restricted class of equations with constant coefficients invariant under a scaling law $(x, \mathbf{u}) \to (kx, k^{\mathbf{p}}\mathbf{u})$, having at least one pole-like family with a negative integer index other than -1. Even then, its failure to detect the movable logarithm in numerous equations which have one makes it of very little use. Such equations are (5.128), (5.144) or the equation $u''' - 7uu'' + 11u'^2 = 0$ whose single family $p = -1, u_0 = -2$ has only negative integer indices $(-6, -1, -1)$.

5.5 The $\alpha-$method of Painlevé

Consider an ordinary differential equation (2.30), a regular point x_0 (i.e. a point of holomorphy of the function $\mathbf{K}$ when (2.30) is written in the canonical form (5.12)), define a small nonzero complex parameter (which Painlevé denoted α) and the perturbation

$$\alpha \neq 0: \;\; x = x_0 + \alpha X, \;\; \mathbf{u} = \alpha^{\mathbf{p}} \sum_{n=0}^{+\infty} \alpha^n \mathbf{u}^{(n)} : \;\; \mathbf{E} = \alpha^{\mathbf{q}} \sum_{n=0}^{+\infty} \alpha^n \mathbf{E}^{(n)} = 0, \tag{5.35}$$

where $\mathbf{p}$ is a sequence of constant integers to be chosen optimally (see example below), $\mathbf{q}$ another sequence of constant integers determined by $\mathbf{p}$, then apply theorem II to the equation for $\mathbf{u}(X, \alpha)$.

At perturbation order zero :

- all the explicit dependence of coefficients on X is removed, i.e. all coefficients of the equation are constant,
- for a suitable choice of $\mathbf{p}$, there only survive a few terms.
- the equation is invariant under the scaling transformation $(X, \mathbf{u}^{(0)}, \mathbf{E}^{(0)}) \to (kX, k^{\mathbf{p}}\mathbf{u}^{(0)}, k^{\mathbf{q}}\mathbf{E}^{(0)})$ (physicists call such an equation scaled, or weighted).

Definition. The *simplified equation* (équation simplifiée) associated to a given perturbation (5.35) is the equation of order zero $\mathbf{E}^{(0)}(x_0, \mathbf{u}^{(0)}) = 0$ in the unknown $\mathbf{u}^{(0)}(X)$.

The simplified equation admits the one-parameter solutions $\mathbf{u}_0^{(0)}(X - X_0)^{\mathbf{p}}$ where $\mathbf{u}_0^{(0)}$ are constants. Its above properties usually make it easy to study.

Definition. The *complete equation* (équation complète), as opposed to the simplified equation, is the equation itself (2.30).

The value $\alpha = 0$ is forbidden in (5.35), but theorem II takes care of that. The constants $\mathbf{p}$ and $\mathbf{q}$ must be integers, chosen so as to satisfy the holomorphy assumption in the small complex parameter of the above lemma. Moreover, since a linear ODE has no movable singularity and since all successive equations $\mathbf{E}^{(n)} = 0, n \geq 1$, are linear, the only way to have movable singularities, in order to test their singlevaluedness, is to select simplified equations which are truly nonlinear.

The successive steps of the $\alpha-$method and the generated necessary conditions for stability are (BSMF p. 209 §7 and footnote 1)

First step. Find all sequences $\mathbf{p}$ of integers satisfying the holomorphy assumptions of Theorem II for the perturbation (5.35). Retain only those defining a truly nonlinear simplified equation. For each sequence $\mathbf{p}$ perform the next steps.

Remark. If the ODE (2.30) has degree one and order N, the holomorphy assumptions of Theorem II require that the highest derivative contributes to the simplified equation.

Second step. Find the general solution $\mathbf{u}^{(0)}$ of the simplified equation.

- **C0**. Require $\mathbf{u}^{(0)}$ to be free from movable critical points.

The general solution $\mathbf{v}$ of the auxiliary equation (5.16) of the simplified equation is then (BSMF p. 209 footnote 1)

$$\mathbf{v} = \sum_{k=1}^{N} d_k \frac{\partial \mathbf{u}^{(0)}}{\partial_{c_k}},\ d_k \text{ arbitrary constants,} \tag{5.36}$$

and, since $\mathbf{u}^{(0)}$ has no movable critical points, $\mathbf{v}$ has no movable critical points either (theorem *Leçons* p. 445).

Third step. For each $n \geq 1$, define $\mathbf{u}^{(n)}$ as a particular solution of equation $\mathbf{E}^{(n)} = 0$ (linear with a rhs), by the classical method of the variation of constants.

- **C1**. Require each $\mathbf{u}^{(n)}$ to be free from movable critical points.

These steps amount to require stability for all the sequence of perturbed equations, exactly as formulated in Theorem II.

Remarks.

- In the second step one can take for $\mathbf{u}^{(0)}$ either the general solution or a particular one, but not a singular solution. The drawback of a particular solution will be a lesser number of generated necessary stability conditions. This may be useful when the quadratures of third step are difficult with the general solution and easy with a particular solution.

- At order $n = 1$ equation $\mathbf{E}^{(1)} = 0$ may contain a rhs, making it different from the auxiliary equation.

- If in the second step only a particular solution has been chosen, it is better that $\mathbf{u}^{(1)}$ be taken as the general solution of equation $\mathbf{E}^{(1)} = 0$.

Many people have intuitively used the $\alpha-$method, let us give two recent examples.

Example 1. In the Lorenz model (1.3), the simultaneous change of variables $(x, y, z) \to (\xi, \eta, \zeta)$ and parameters $(b, \sigma, r) \to (b, \sigma, \varepsilon)$ defined by

$$\xi = \varepsilon x,\ \eta = \varepsilon^2 \sigma y,\ \zeta = \varepsilon^2 \sigma z,\ \varepsilon^2 \sigma r = 1 \tag{5.37}$$

led Robbins [102] to believe to have found a new integrable case, defined by

$$\sigma \neq 0, \varepsilon = 0, r = \infty:\ \text{first integrals } \xi^2 - 2\zeta,\ -\xi^2 + \eta^2 + \zeta^2, \tag{5.38}$$

while in fact the new dynamical system is just the simplified of the original one, integrable by elliptic functions.

Example 2. The transformation $t \to t^2 \operatorname{Log} t$ with "$t \to 0$" from the Lorenz model to the system (24abc) of Ref. [81] is in fact the $\alpha-$transformation $(x, y, z, t) = (\varepsilon^{-1} X, \varepsilon^{-2} Y, \varepsilon^{-2} Z, t_0 + \varepsilon T)$ resulting in the system

$$\frac{\mathrm{d}X}{\mathrm{d}T} = \sigma(Y - X),\ \frac{\mathrm{d}Y}{\mathrm{d}T} = -Y - XZ,\ \frac{\mathrm{d}Z}{\mathrm{d}T} = XY, \tag{5.39}$$

whose general solution is elliptic.

5.5.1 The two examples

Example 1. "Complete (P1)" eq. (5.20) (BSMF p. 224 §15). The Cauchy form of the perturbed equation is

$$\alpha^{p-2}E \equiv -\frac{\mathrm{d}^2(u^{(0)}+\ldots)}{\mathrm{d}X^2}+6\alpha^{p+2}[u^{(0)}+\ldots]^2+\alpha^{2-p}[g(x_0)+\alpha X g'(x_0)+\ldots]=0. \tag{5.40}$$

First step. The holomorphy requirement $-2 \le p \le 2, p \in \mathcal{Z}$, selects five values for p, and the requirement of a nonlinear simplified equation only retains $p=-2$, i.e. $q=p-2=2p=-4$. The $g(x_0)$ term, which could vanish, does not contribute to the simplified equation

$$E^{(0)} \equiv -\frac{\mathrm{d}^2 u^{(0)}}{\mathrm{d}X^2}+6u^{(0)^2}=0. \tag{5.41}$$

Second step. The general solution of this particular Weierstrass equation is

$$u^{(0)}=\wp(X-c_0,0,g_3),\ (c_0,g_3) \text{ arbitrary}. \tag{5.42}$$

The auxiliary equation of the simplified equation is a Lamé equation

$$E'(X,u^{(0)})v \equiv -\frac{\mathrm{d}^2 v}{\mathrm{d}X^2}+12\wp(X-c_0,0,g_3)v=0 \tag{5.43}$$

whose general solution is a linear combination of $\partial\wp/\partial_{g_3}$ and $\partial\wp/\partial_{c_0}$ [5]

$$v=c_1(X\wp'+2\wp)+c_2\wp', \tag{5.44}$$

without any movable critical singularity.

Third step. The successive linear equations with their rhs are

$$\begin{aligned}
E'(X,u^{(0)})u^{(1)} &= 0 && (5.45)\\
E'(X,u^{(0)})u^{(2)} &= -6u^{(1)^2} && (5.46)\\
E'(X,u^{(0)})u^{(3)} &= -12u^{(1)}u^{(2)} && (5.47)\\
E'(X,u^{(0)})u^{(4)} &= -12u^{(1)}u^{(3)}-6u^{(2)^2}-g_0 && (5.48)\\
E'(X,u^{(0)})u^{(5)} &= -12(u^{(1)}u^{(4)}+u^{(2)}u^{(3)})-g_0'X && (5.49)\\
E'(X,u^{(0)})u^{(6)} &= -12(u^{(1)}u^{(5)}+u^{(2)}u^{(4)})-6u^{(3)^2}-\frac{1}{2}g_0''X^2 && (5.50)
\end{aligned}$$

with the particular solutions

$$\begin{aligned}
u^{(n)} &= 0,\ n=1,2,3 && (5.51)\\
u^{(4)} &= \frac{g_0}{24}\left[2X\wp\wp'+2\wp^2-\zeta\wp'\right] && (5.52)\\
u^{(5)} &= \frac{g_0'}{24}\left[2X^2\wp\wp'+2X\wp^2+(X\wp'+2\wp)\zeta\right] && (5.53)
\end{aligned}$$

$$
\begin{aligned}
u^{(6)} &= \frac{g_0''}{48}\big[(X\wp' + 2\wp)(X^2 + 2X\zeta - 2\,\mathrm{Log}\,\sigma) \\
&\qquad + (X^3\wp + X^2\zeta - 2X\,\mathrm{Log}\,\sigma + 2\int \mathrm{Log}\,\sigma dX)\wp'\big],
\end{aligned}
\tag{5.54}
$$

where the functions ζ and σ obey $\zeta' = -\wp, \sigma' = \zeta\sigma$. To prevent movable logarithms at $n = 6$ it is necessary that $g''(x_0) = 0$. Since x_0 is arbitrary, this condition is $\forall x :\ g''(x) = 0$, and Painlevé proved it to be sufficient, thus defining the (new in the sense of Section 3.2) function (P1) with the choice $g = x$.

Remarks.

- The reason why $u^{(1)}, u^{(2)}, u^{(3)}$ can be chosen zero is given on page 45. The reason given in ref. [75] p. 120 is not correct : even if the general solutions $u^{(1)}, u^{(2)}, u^{(3)}$ are meromorphic, they can in principle (this does not occur for the ODE under study) generate some multivaluedness further up in the computation. The theorem proven in *Leçons* p. 445 is quite profound : if a [second order in *Leçons*] ODE is stable, its general solution has a single valued dependence on the integration constants.

- Taking the particular solution $u^{(0)} = 1/(X - c_0)^2$ instead of the general one $\wp$ (see first remark section 5.5) makes all computations immediate $(\zeta = 1/(X - c_0), \sigma = X - c_0)$. For this particular equation, one would not miss the generation of the only necessary stability condition.

Example 2 (Chazy complete equation of class III (5.24)).

First step. For the canonical form of Cauchy of the perturbed equation

$$
\begin{aligned}
&-\frac{\mathrm{d}^2(u^{(0)} + \ldots)}{\mathrm{d}X^2} + 2\alpha^{p+1}u^{(0)}\frac{\mathrm{d}^2u^{(0)}}{\mathrm{d}X^2} - 3\alpha^{p+1}\left[\frac{\mathrm{d}^2u^{(0)}}{\mathrm{d}X^2}\right]^2 + \ldots \\
&+\alpha^{3-p}d_0(x_0) + \ldots = 0,
\end{aligned}
$$

the holomorphy condition is $-1 \le p \le 3, p \in \mathcal{Z}$, which the condition for a truly nonlinear simplified equation restricts to $p = -1, q = -4$. The value $p = -2$ [52, 31] of the method of pole-like expansions is therefore forbidden.

Second step. The simplified equation is that of Chazy (1.17), whose general solution $u^{(0)}$ is [17, 18] an algebraic transform (finite single valued expression) of the Hermite modular function $y(X)$

$$
u^{(0)} = \left[\mathrm{Log}(y_X^3 y^{-2}(y-1)^{-2})\right]_X, \tag{5.55}
$$

evaluated at the point $(c_1X + c_2)/(c_3X + c_4), c_1c_4 - c_2c_3 = 1$ and thus obviously depending on three arbitrary constants.

The auxiliary equation of Chazy's simplified equation

$$
E^{(0)'}v \equiv [-\partial_X^3 + 2u^{(0)}\partial_X^2 - 6u_X^{(0)}\partial_X + 2u_{XX}^{(0)}]v = 0 \tag{5.56}
$$

has the three independent solutions $\partial u^{(0)}/\partial_{c_i}, i=1,2,3$, all single valued.

Instead of the general solution $u^{(0)}$, which would make the computations rather involved, let us restrict to the two-parameter particular solution

$$u^{(0)} = -6\chi^{-1} + c\chi^{-2},\ \chi = X - x_0,\ (x_0, c) \text{ arbitrary constants,} \tag{5.57}$$

for which the auxiliary equation admits the general solution

$$v = k_2\chi^{-2} + k_3\chi^{-3} + k_4 v_4,\ v_4 = c^{-2}\chi^{-2}(e^{-2c/\chi} - 1 - 2c\chi^{-1}), \tag{5.58}$$

with (k_2, k_3, k_4) arbitrary constants.

Third step. The successive linear equations with their rhs are

$$-E^{(0)'}u^{(1)} = c_{1,0}u^{(0)}u^{(0)'} + d_{3,0}u^{(0)^3} \tag{5.59}$$

$$\begin{aligned} -E^{(0)'}u^{(2)} &= 2u^{(1)}u^{(1)''} - 3u^{(1)'^2} + c_{1,1}Xu^{(0)}u^{(0)'} + c_{1,0}u^{(0)'}(u^{(0)} + u^{(1)}) \\ &\quad + d_{3,1}Xu^{(0)^3} + 3d_{3,0}u^{(0)^2}u^{(1)} + c_{0,0}u^{(0)'}. \end{aligned} \tag{5.60}$$

A particular solution of the first one is provided by the method of variation of the constants $u^{(1)} = K_2(X)\chi^{-2} + K_3(X)\chi^{-3} + K_4(X)\chi^{-4}$

$$K_2'\chi^{-2} + K_3'\chi^{-3} + K_4'v_4 = 0 \tag{5.61}$$

$$-2K_2'\chi^{-3} - 3K_3'\chi^{-4} + K_4'v_4' = 0 \tag{5.62}$$

$$\begin{aligned} 6K_2'\chi^{-4} + 12K_3'\chi^{-5} + K_4'v_4'' &= c_{1,0}(-6\chi^{-1} + c\chi^{-2})(6\chi^{-2} - 2c\chi^{-3}) \\ &\quad + d_{3,0}(-6\chi^{-1} + c\chi^{-2})^3. \end{aligned} \tag{5.63}$$

To prevent a movable logarithm in K_2 (resp. K_3), it is necessary that, in the rhs of last equation, the coefficients of χ^{-5} and χ^{-6} vanish :

$$\forall (x_0, c): \ -2c^2 c_1(x_0) - 18c^3 d_3(x_0) = 0,\ c^3 d_3(x_0) = 0, \tag{5.64}$$

hence the two necessary stability conditions $\forall x\ d_3(x) = c_1(x) = 0$, obtained at the perturbation order $n = 1$. We leave it as an exercise to check that, after completion of $n = 4$, one has obtained all the conditions ($c_1 = c_0 = d_3 = d_1 = d_0 = 0$) which Chazy proved to be necessary and sufficient.

Theorem. For any family of the method of pole-like expansions, the value $i = -1$ is a Fuchs index.

Proof. Let $\mathbf{u} \sim \mathbf{u}_0\chi^{\mathbf{p}}$ be such a family and $\hat{\mathbf{E}}(x, \mathbf{u})$ be the dominant terms. The equation $\hat{\mathbf{E}}(x_0, \mathbf{u}) = 0$ admits as a particular solution the monomial $X \to \mathbf{u} = \mathbf{u}_0(x_0)(X - X_0)^{\mathbf{p}}$, therefore the linearized equation at the leading term (2.20) admits as a particular solution its derivative with respect to $X_0 : X \to$ const $\times\ \partial_{X_0}(X - X_0)^{\mathbf{p}}$. Since at least one component of $\mathbf{p}$ is negative, the associated component of $\partial_{X_0}(X - X_0)^{\mathbf{p}}$ is proportional to $(X - X_0)^{p-1}$, therefore $i = -1$ is a root of the indicial equation (2.36). □

5.5.2 General stability conditions (ODE of order m and degree 1)

Using his method, Painlevé could obtain quite general necessary stability conditions for algebraic ODEs of arbitrary order and degree, cf. BSMF p. 258, Acta p. 74, Chazy (Thèse). Consider the class, defined in the canonical form of Cauchy

$$u^{(m)} = R(u^{(m-1)}, u^{(m-2)}, \dots, u', u, x), \tag{5.65}$$

with R rational in u and its derivatives, analytic in x [for R algebraic, and for arbitrary order and degree, cf. Acta pp. 73, 77]. *Necessary stability conditions* are :

C1. As a rational fraction of $u^{(m-1)}$, R is a polynomial of degree at most two

$$u^{(m)} = A{u^{(m-1)}}^2 + Bu^{(m-1)} + C. \tag{5.66}$$

C2. As a rational fraction of $u^{(m-2)}$, A has only simple poles a_i with residues r_i equal to $1 - 1/n_i$, n_i nonzero integers possibly infinite

$$A = \sum_i \frac{1 - 1/n_i}{u^{(m-2)} - a_i}. \tag{5.67}$$

The above sum is finite.

For second order $m = 2$ the fraction A has at most four simple poles, and the set of their residues can only take the five values of Table 5.1

$$A = \sum_{i=1}^{4} \frac{r_i}{u - a_i},\ \sum_{i=1}^{4} r_i = 2,\ r_i = 1 - \frac{1}{n_i},\ n_i \in \mathcal{Z} \text{ or } n_i = \infty. \tag{5.68}$$

Table 5.1: Order two, degree one. Number of poles (nonzero r_i), list of their residues. The poles may be located at ∞ and may not be distinct. The type numbering convention is that of ([83] Table I p. 169). The least common multiplier (lcm) is shown for convenience.

Type	lcm(r_i)	r_1	r_2	r_3	r_4
I	$n \geq 1$	$1 + 1/n$	$1 - 1/n$	0	0
III	2	1/2	1/2	1/2	1/2
IV	3	2/3	2/3	2/3	0
V	4	3/4	3/4	1/2	0
VI	6	5/6	2/3	1/2	0

Note the one-to-one correspondence between Table 5.1 and the list of powers of the five Briot-Bouquet equations page 34.

Exercise 5.3 *For the six equations (Pn), determine the set* (a_i, r_i) *of simple poles with their residue.*

Solution.

$$
\begin{aligned}
&(P6) \quad && (\infty, 1/2),\ (0, 1/2),\ (1, 1/2),\ (x, 1/2), && (5.69)\\
&(P5) \quad && (\infty, 1/2),\ (0, 1/2),\ (1, 1), && (5.70)\\
&(P4) \quad && (\infty, 3/2),\ (0, 1/2), && (5.71)\\
&(P3) \quad && (\infty, 1),\ (0, 1), && (5.72)\\
&(P2) \quad && (\infty, 2), && (5.73)\\
&(P1) \quad && (\infty, 2). && (5.74)
\end{aligned}
$$

For instance, (P4) belongs to type I of Table 5.1, and it is also a confluent case of types III, V, VI. □

The similar finite lists of admissible values of A for any order m can be found in Painlevé ($m = 3$ Acta p. 68, Œuvres vol. III p. 254; $m \geq 4$ Acta p. 75, Œuvres vol. III p. 261).

C3. As rational fractions of $u^{(m-2)}$, B and C have no other poles than those of A, and these poles are all simple. Writing B, C as rational fractions of $u^{(m-2)}$ whose denominators are that of A, this implies the degrees limitations

$$\text{(order 2, degree 1)}: \ \text{deg num } B \leq 1, \ \text{deg num } C \leq 3. \qquad (5.75)$$

C4. (Chazy, Thèse). Every ODE $u^{(m-2)} - a_i = 0$ (a denominator of A) is stable.

C5. ([24]). All polynomial degrees in $u^{(k)}, k = 0, \dots, m-2$, (of the numerator and denominator of A, B, C written as irreducible fractions of u and its derivatives) are limited, except in the "Fuchsian" case $n_i = -2, r_i = 3/2$ (see Ref. [24] for details).

For additional conditions, see Ref. [91].

5.6 The method of Bureau

Firstly, this method exhibits a linear differential equation with a Fuchsian singularity which allows to interpret the indices i in the recursion relation of Kowalevski as Fuchs indices. Secondly, it brings rigor to the heuristic method of Kowalevski and Gambier. However, the generated no-log conditions are identical to those of the method of pole-like expansions.

Consider an $N^{\rm th}$ order ODE $E(x,u)=0$ (for simplicity, one assumes u and E unidimensional; the multidimensional case is handled in [19]) and a movable noncritical singular point x_0 where the general solution behaves like $u \sim u_0(x-x_0)^p$, with p a negative integer to be determined.

The integer p is computed by the method of pole-like expansions and the highest derivative is required to contribute (M. II p. 9) in order to be sure that one deals with the general, not a singular, solution.

One wants to apply the two fundamental theorems. Since the singularity x_0 violates the holomorphy assumption of theorem I, one defines an equivalent differential system (in fact two systems) for which x_0 is a point of holomorphy. These systems will depend on a perturbation parameter ε.

One first defines two new dependent variables (z,U) by the relations (Gambier Thèse p. 50, Bureau 1939)

$$u = s z^p,\ \frac{\mathrm{d}z}{\mathrm{d}x} = 1 + Uz,\ s \ne 0. \tag{5.76}$$

Elimination of u and the derivatives of z (M. II pp. 13, 77)

$$\begin{aligned}
z^{-p} u &= s && (5.77)\\
z^{-p+1} \frac{\mathrm{d}u}{\mathrm{d}x} &= ps + \left(\frac{\mathrm{d}s}{\mathrm{d}x} + psU\right) z && (5.78)\\
z^{-p+2} \frac{\mathrm{d}^2u}{\mathrm{d}x^2} &= p(p-1)s + \left(2p\frac{\mathrm{d}s}{\mathrm{d}x} + p(2p-1)sU\right) z\\
&\quad + \left(\frac{\mathrm{d}^2s}{\mathrm{d}x^2} + 2p\frac{\mathrm{d}s}{\mathrm{d}x}U + p^2 sU^2 + ps\frac{\mathrm{d}U}{\mathrm{d}x}\right) z^2 && (5.79)
\end{aligned}$$

etc, transforms E into

$$E \equiv E\big(x, U, \frac{\mathrm{d}U}{\mathrm{d}x}, \dots, \frac{\mathrm{d}^{(N-1)}U}{\mathrm{d}x^{N-1}}, s, \frac{\mathrm{d}s}{\mathrm{d}x}, \dots, \frac{\mathrm{d}^{(N)}s}{\mathrm{d}x^N}, z\big) = 0, \tag{5.80}$$

an equation for U of order $N-1$ polynomial in z. For the equivalent system (5.76), (5.80) made of two ODEs of orders one and $N-1$ in the unknowns (z,U), the point $z=0$ is still a point of meromorphy, see examples below.

To remove it, one introduces a dependence in a small nonzero parameter ε to obtain a perturbed system to which Theorem II can be applied. Two such perturbations have been defined (Bureau 1939).

First perturbation of Bureau

$$x = x_0 + \varepsilon X,\ z = \varepsilon Z,\ U \text{ unchanged}:\ E \equiv (\varepsilon Z)^q \sum_{n=0}^{+\infty} (\varepsilon Z)^n E^{(n)} = 0, \tag{5.81}$$

where the positive integer $-q$ is the singularity order of E. The coefficients must be expanded as Taylor series like in the $\alpha-$method

$$s(x) = s_0 + (\varepsilon X)s'_0 + \dots,\ s_0^{(k)} = \frac{d^{(k)}s}{\mathrm{d}x^k}(x_0),\ a(x) = a_0 + (\varepsilon X)a'_0 + \dots. \tag{5.82}$$

Expansions up to order one in ε for the above derivatives (5.77)–(5.79) are

$$z^{-p}u = s_0 + (s_0' X)\varepsilon + O(\varepsilon^2) \tag{5.83}$$

$$z^{-p+1}\frac{\mathrm{d}u}{\mathrm{d}x} = ps_0 + ((ps_0')X + (s_0' + ps_0 U)Z)\,\varepsilon + O(\varepsilon^2) \tag{5.84}$$

$$\begin{aligned} z^{-p+2}\frac{\mathrm{d}^2u}{\mathrm{d}x^2} &= p(p-1)s_0 \\ &+p\left((p-1)s_0' X + (2s_0' + (2p-1)s_0 U + s_0 Z\frac{\mathrm{d}U}{\mathrm{d}X})Z\right)\varepsilon + O(\varepsilon^2) \end{aligned} \tag{5.85}$$

etc, together with $\mathrm{d}Z/\mathrm{d}X = 1 + \varepsilon Z U = 1 + O(\varepsilon)$.

Order zero is an algebraic equation $E^{(0)}(x_0, s_0) = 0$ for the nonzero coefficient s_0.

Order one is subtle : it filters out all terms nonlinear in U and its derivatives $\mathrm{d}^{(k)}U/\mathrm{d}X^k$, and it extracts the contribution of $\mathrm{d}^{(k)}U/\mathrm{d}X^k$ from the term z^{k+1} in the expansions (5.77)–(5.79). This results in

$$E^{(1)} \equiv A\frac{X}{Z} + B + \sum_{k=0}^{N-1} c_k Z^k \frac{\mathrm{d}^{(k)}U}{\mathrm{d}X^k} = 0,\ (A, B, c_k) \text{ constant.} \tag{5.86}$$

Since $\mathrm{d}Z/\mathrm{d}X$ is unity at this order, this is a linear inhomogeneous ODE of order at most $N-1$ for U with constant coefficients, whose homogeneous part is by construction of Fuchsian type (exactly one singular point $Z = 0$, of the singular regular type) and even Eulerian type.

In order to be sure of dealing with the general solution of the original nonlinear ODE, the linear ODE (5.86) must have exactly the order $N-1$; a necessary stability condition for the nonlinear ODE is the single valuedness of the general solution of the linear ODE (5.86). Hence the necessary conditions, for each value of (p, s_0)

- the order of the linear ODE at perturbation order one is exactly $N-1$;
- its $N-1$ Fuchs indices are distinct integers;
- if 0 is an index, the rhs vanishes ($A = B = 0$ condition for the particular solution to contain no logarithm).

Since (5.86) is Eulerian, these conditions are sufficient for the general solution of the linear ODE (5.86) to be single valued, but only necessary for the stability of the nonlinear ODE.

Higher perturbation orders yield no information. The reasoning is then that any condition thus found at $x = x_0$, such as $s_0 = 1$, is valid at any x since x_0 is arbitrary.

Second perturbation of Bureau

$$x \text{ unchanged}, \ z = \varepsilon Z, \ U = \sum_{n=1}^{+\infty} (\varepsilon Z)^{n-1} U^{(n)} : \ E \equiv (\varepsilon Z)^q \sum_{n=0}^{+\infty} (\varepsilon Z)^n E^{(n)}. \tag{5.87}$$

Expansions for the above derivatives (5.77)–(5.79) are

$$\begin{aligned}
z^{-p} u &= s \\
z^{1-p} \frac{\mathrm{d}u}{\mathrm{d}x} &= ps + \left(psU^{(1)} + s'\right) \varepsilon Z + psU^{(2)} (\varepsilon Z)^2 + psU^{(3)} (\varepsilon Z)^3 + O(\varepsilon^4) \\
z^{2-p} \frac{\mathrm{d}^2 u}{\mathrm{d}x^2} &= p(p-1)s + p\left((2p-1)sU^{(1)} + 2s'\right) \varepsilon Z \\
& + \left(p^2 s(2U^{(2)} + U^{(1)^2}) + 2ps'U^{(1)} + s'' + ps\frac{\mathrm{d}U^{(1)}}{\mathrm{d}x}\right) (\varepsilon Z)^2 \\
& + O(\varepsilon^3)
\end{aligned}$$

etc, together with $\varepsilon \mathrm{d}Z/\mathrm{d}x = 1 + \varepsilon Z U^{(1)} + (\varepsilon Z)^2 U^{(2)} + O(\varepsilon^3)$.

Equation $E^{(0)}(x, s) = 0, s \ne 0$, is the same algebraic equation as above for the unknown $s(x)$, not $s(x_0)$. Each perturbation order $n \ge 1$ defines a linear algebraic equation

$$\forall n \ge 1 : \ P(n) U^{(n)} + Q_n(x, U^{(1)}, \dots, U^{(n-1)}) = 0, \tag{5.88}$$

where $P(n)$ is the indicial polynomial of Fuchsian equation (5.86), and Q_n depends on the previously computed coefficients. Necessary stability conditions $Q_i = 0$ arise at every value of i which is also one of the $N-1$ Fuchs indices. These conditions are identical to those of the method of pole-like expansions, as proven in section 5.6.1.

The successive steps and generated necessary conditions of the method of Bureau are

Step a. Determine all possible p like in the method of pole-like expansions (details M. I p. 256, M. II p. 9). For all p satisfying (**C0**, **C1**), perform step b.

C0. All p are integers.

C1. The linear ODE (order one of first perturbation) has exactly order $N-1$ [this holomorphy condition excludes for instance $p = -2$ in Chazy]. This implies the necessity for the highest derivation order to contribute to the dominant part during the computation of p.

Step b. Solve the algebraic equation for s_0 at order zero of first perturbation. For all nonzero s_0 perform steps c and d.

Step c. Solve the linear inhomogeneous Euler equation for $U(Z)$ at order one of first perturbation.

C2. Its $N-1$ Fuchs indices are distinct integers.

C3. If 0 is an index, the inhomogeneous part vanishes.

Step d. Solve the linear algebraic equation (5.88) (order n of second perturbation) from $n=1$ to the highest Fuchs index.

C4. Whenever the order n in step d is a Fuchs index i, Q_i is zero.

As compared with the $\alpha-$method, these stability conditions are directly taken at x, not at x_0. However, the method provides no conditions from the negative integer indices.

5.6.1 Bureau expansion *vs.* pole-like expansion

Let us first prove the existence of a one-to-one correspondence between the coefficients $U^{(n)}$ of Bureau (second perturbation) and those u_j of the method of pole-like expansions. The relations defining Bureau coefficients are

$$u = sz^p \tag{5.89}$$

$$\frac{\mathrm{d}z}{\mathrm{d}x} = 1 + U^{(1)}z + U^{(2)}z^2 + O(z^3) \tag{5.90}$$

and those defining the pole-like expansion are

$$u = \chi^p(u_0 + u_1\chi + u_2\chi^2 + O(\chi^3)) \tag{5.91}$$

$$\frac{\mathrm{d}\chi}{\mathrm{d}x} = 1. \tag{5.92}$$

The property $\chi_x = 1$ of χ first ensures $s = u_0$ [taking $\chi = x - x_0$ would just create useless complications]. The elimination of u between (5.89) and (5.91) yields

$$\begin{aligned} z &= \chi\left(1 + \frac{u_1}{u_0}\chi + \frac{u_2}{u_0}\chi^2 + O(\chi^3)\right)^{1/p} \\ &= \chi\left(1 + \frac{u_1}{pu_0}\chi + \frac{2pu_2 + (1-p)u_1^2}{2p^2u_0^2}\chi^2 + O(\chi^3)\right). \end{aligned} \tag{5.93}$$

Let us invert this Taylor series z of χ into a Taylor series χ of z

$$\chi = z\left(1 - \frac{u_1}{pu_0}z + \frac{-2pu_2 + (3+p)u_1^2}{2p^2u_0^2}z^2 + O(z^3)\right). \tag{5.94}$$

One finally substitutes this χ and $\mathrm{d}z/\mathrm{d}x$, both Taylor series in z, into eq. (5.92) to obtain

$$\begin{aligned}
\frac{\mathrm{d}\chi}{\mathrm{d}x} &= 1 \\
&= \left(1 - \frac{2u_1}{pu_0}z + 3\frac{-2pu_2 + (3+p)u_1^2}{2p^2u_0^2}z^2 + O(z^3)\right) \\
&\quad \times (1 + U^{(1)}z + U^{(2)}z^2 + O(z^3)) - \frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{u_1}{u_0}\right)z^2 + O(z^3). \quad (5.95)
\end{aligned}$$

The identification of the lhs and rhs as series in z provides the correspondence between the two sets of coefficients

$$s = u_0 \quad (5.96)$$

$$U^{(1)} = \frac{2u_1}{pu_0} \quad (5.97)$$

$$U^{(2)} = \frac{3u_2}{pu_0^2} + \left(\frac{2u_1}{pu_0}\right)^2 - (3p+1)\frac{u_1^2}{2p^2u_0^2} + \frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{u_1}{u_0}\right), \quad (5.98)$$

or

$$u_0 = s \quad (5.99)$$

$$u_1 = \frac{p}{2}sU^{(1)} \quad (5.100)$$

$$u_2 = \frac{p}{3}s^2U^{(2)} + p\frac{3p+1}{24}s^2U^{(1)^2} - \frac{p}{6}s^2\frac{\mathrm{d}U^{(1)}}{\mathrm{d}x}. \quad (5.101)$$

This bijection between the coefficients induces a bijection between the equations $E^{(n)} = 0$ of the expansion of Bureau (second perturbation) and the equations $E_j = 0$ of the method of pole-like expansions, hence a bijection between the no-log conditions.

This proves the equivalence between the method of pole-like expansions and the second perturbation of Bureau. As to the first perturbation of Bureau, it brings quite important information *not* obtaibable by the method of pole-like expansions.

5.6.2 The two examples

Example 1. “Complete (P1)” eq. (5.20). See Bureau M. I eq. (17.3), (21.2).

The unperturbed equivalent meromorphic system (5.76)–(5.80) in (z, U) is, in Cauchy form

$$\frac{\mathrm{d}z}{\mathrm{d}x} = 1 + Uz \quad (5.102)$$

$$\frac{\mathrm{d}U}{\mathrm{d}x} = 6s^2z^p - (p-1)z^{-2} - \left[2\frac{s'}{s} + (2p-1)U\right]z^{-1} - \frac{s'pU + s''}{ps} + \frac{g}{ps}z^{-p}.$$

Step a. The only value is $p = -2$, integer. The original ODE then reads, by increasing powers of z

$$E \equiv 6s(1-s)z^{-4} + 2s\left[5U - \frac{2}{s}\frac{\mathrm{d}s}{\mathrm{d}x}\right]z^{-3}$$
$$-2s\left[\frac{\mathrm{d}U}{\mathrm{d}x} + \frac{2}{s}\frac{\mathrm{d}s}{\mathrm{d}x}U - \frac{1}{2s}\frac{\mathrm{d}^2 s}{\mathrm{d}x^2} - 2U^2\right]z^{-2} - g(x) = 0 \quad (5.103)$$

Step b. Equation $E^{(0)} \equiv 6s_0(1-s_0) = 0$ has for only nonzero solution $s_0 = 1$.

Step c. At order one

$$\frac{E^{(1)}}{2s_0} \equiv -Z\frac{\mathrm{d}U}{\mathrm{d}X} + 5U - 2\frac{s_0'}{s_0} + 3\frac{s_0'}{s_0}\frac{X}{Z} = 0,\ \frac{\mathrm{d}Z}{\mathrm{d}X} = 1 + O(\varepsilon Z). \quad (5.104)$$

The only Fuchs index is $i = 5$. There is no condition on the rhs.

Step d. The computation presents no difficulty.

$$s = 1,\ U^{(1)} = U^{(2)} = U^{(3)} = 0,\ U^{(4)} = \frac{g}{4},\ U^{(5)} = \frac{g'(x)}{4},\ Q_6 \equiv -\frac{g''(x)}{2} = 0. \quad (5.105)$$

Remark. On the Cauchy form (5.102) with $p = -2, s = 1$, one sees easily how perturbations I and II remove the meromorphy.

Example 2 (Chazy complete equation of class III (5.24)).

Step a. The two solutions are $p = -1, p = -2$. For $p = -2$ the computation of the linear equation (5.86) yields a zero coefficient for $\mathrm{d}^2U/\mathrm{d}X^2$, thus violating condition **C1**.

For $p = -1$ the original ODE then reads, by increasing powers of z

$$E \equiv s(s+6)z^{-4} + s\left[6U - \frac{6}{s}\frac{\mathrm{d}s}{\mathrm{d}x} + 2\frac{\mathrm{d}s}{\mathrm{d}x} - c_1 s + d_3 s^2\right]z^{-3}$$
$$+s\Big[-2(s+2)\frac{\mathrm{d}U}{\mathrm{d}x} + ((2-\frac{9}{s})\frac{\mathrm{d}s}{\mathrm{d}x} - c_1 s)U - c_0$$
$$+c_1\frac{\mathrm{d}s}{\mathrm{d}x} - \frac{3}{s}(\frac{\mathrm{d}s}{\mathrm{d}x})^2 + (2+\frac{3}{s})\frac{\mathrm{d}^2 s}{\mathrm{d}x^2} + (7-s)U^2\Big]z^{-2}$$
$$+s\Big[\frac{\mathrm{d}^2U}{\mathrm{d}x^2} + 3\frac{\mathrm{d}s}{\mathrm{d}x}\frac{\mathrm{d}U}{\mathrm{d}x} + (\frac{3}{s}\frac{\mathrm{d}^2 s}{\mathrm{d}x^2} - c_0)U + d_1 + \frac{c_0}{s}\frac{\mathrm{d}s}{\mathrm{d}x}$$
$$-\frac{1}{s}\frac{\mathrm{d}^3 s}{\mathrm{d}x^3} - \frac{3}{s}\frac{\mathrm{d}s}{\mathrm{d}x}U^2 - 3U\frac{\mathrm{d}U}{\mathrm{d}x} + U^3\Big]z^{-1} + d_0. \quad (5.106)$$

Step b. Equation $E^{(0)} \equiv s_0(s_0+6) = 0$ has for only nonzero solution $s_0 = -6$.

Step c. At order one

$$\frac{E^{(1)}}{s_0} \equiv Z^2\frac{\mathrm{d}^2U}{\mathrm{d}X^2} - 2(s_0+2)Z\frac{\mathrm{d}U}{\mathrm{d}X} + 12U - s_0 c_{1,0} - s_0^2 d_{3,0}$$
$$+(2-\frac{6}{s_0})s_0' + (2+\frac{6}{s_0})s_0'\frac{X}{Z} = 0,\ \frac{\mathrm{d}Z}{\mathrm{d}X} = 1 + O(\varepsilon Z) \quad (5.107)$$

The Fuchs indices are $i = -4, -3$, there is no condition on the rhs, and the algorithm stops here, due to the absence of positive integer indices.

5.7 The Fuchsian perturbative method

It allows to extract the information contained in the negative indices [52], thus building infinitely many necessary conditions for the absence of movable critical singularities of the logarithmic type [31].

The perturbation which describes it is close to the identity

$$x \text{ unchanged},\ \mathbf{u} = \sum_{n=0}^{+\infty} \varepsilon^n \mathbf{u}^{(n)} :\ \mathbf{E} = \sum_{n=0}^{+\infty} \varepsilon^n \mathbf{E}^{(n)} = 0, \quad (5.108)$$

where, like for the $\alpha-$method, the small parameter ε is not in the original equation.

Then, the single equation (2.30) is equivalent to the infinite sequence

$$n=0:\ \mathbf{E}^{(0)} \equiv \mathbf{E}(x,\mathbf{u}^{(0)}) = 0 \quad (5.109)$$
$$\forall n \ge 1:\ \mathbf{E}^{(n)} \equiv \mathbf{E}'(x,\mathbf{u}^{(0)})\mathbf{u}^{(n)} + \mathbf{R}^{(n)}(x,\mathbf{u}^{(0)},\dots,\mathbf{u}^{(n-1)}) = 0 \quad (5.110)$$

with $\mathbf{R}^{(1)}$ identically zero. From Theorem II, necessary stability conditions are

- the general solution $\mathbf{u}^{(0)}$ of (5.109) has no movable critical points,
- the general solution $\mathbf{u}^{(1)}$ of (5.110) has no movable critical points,
- for every $n \ge 2$ there exists a particular solution of (5.110) without movable critical points.

Order zero is just the complete equation for the unknown $\mathbf{u}^{(0)}$, so, to get some information, one must apply Theorem II for a perturbation different from (5.108). Since Bureau has proven that the method of pole-like expansions, with more rigorous assumptions, can be casted into an application of the two basic theorems, one uses it at order zero, *only* to obtain the leading term $\mathbf{u}^{(0)} \sim \mathbf{u}_0^{(0)}\chi^{\mathbf{p}}$ of all the families of movable singularities.

First step. Determine all possible families $(\mathbf{p}, \mathbf{u}_0^{(0)})$

$$\mathbf{u}^{(0)} \sim \mathbf{u}_0^{(0)}\chi^{\mathbf{p}},\ \mathbf{E}^{(0)} \sim \mathbf{E}_0^{(0)}\chi^{\mathbf{q}},\ \mathbf{u}_0^{(0)} \ne \mathbf{0} \quad (5.111)$$

which do not describe a singular solution, by solving the algebraic equation

$$\mathbf{E}_0^{(0)} \equiv \lim_{\chi \to 0} \chi^{-\mathbf{q}} \hat{\mathbf{E}}(x, \mathbf{u}_0^{(0)} \chi^{\mathbf{p}}) = 0. \quad (5.112)$$

- **C0**. All components of $\mathbf{p}$ are integer.

If there exists no family which is truly singular (at least one component of $\mathbf{p}$ negative), the method stops without concluding.

Second step. For each family, compute the indicial equation (2.36) and require the necessary conditions :

- **C2**. Every zero of $\det \mathbf{P}$ (a Fuchs index) is integer.
- **C3**. Every zero i of $\det \mathbf{P}$ has a multiplicity equal to the dimension of the kernel of $\det \mathbf{P}(i)$

$$\forall \text{ index } i: \ (\text{multiplicity of } i) = \ \dim \text{Ker } \mathbf{P}(i). \quad (5.113)$$

Remark. There is no such condition as **C1** on page 14, i. e. the indicial polynomial may have a degree smaller than N. If the indicial equation has degree N, the conditions **C2** and **C3** (N distinct integers in the one-dimensional case) are slightly stronger than the conditions in Bureau ($N-1$ distinct integers).

The next step is easily computerizable [27, 47] if one represents $\mathbf{u}^{(0)}, \mathbf{u}^{(1)}, \ldots,$ as Laurent series bounded from below : $\mathbf{u}^{(0)}$ with powers in the range $(\mathbf{p} : +\infty)$, $\mathbf{u}^{(1)}$ with powers in the range $(\rho + \mathbf{p} : +\infty)$, where ρ denotes the smallest Fuchs index, an integer lower than or equal to -1, ...

Order $n = 0$ is identical to the method of pole-like expansions and the Laurent series for $\mathbf{u}^{(0)}$

$$\mathbf{u}^{(0)} = \sum_{j=0}^{+\infty} \mathbf{u}_j^{(0)} \chi^{j+\mathbf{p}}, \quad (5.114)$$

represents a particular solution containing a number of arbitrary coefficients equal to one (index -1) plus the number of positive Fuchs indices, counting their multiplicity.

Order $n = 1$ is identical to the "équation auxiliaire" of Darboux

$$\mathbf{E}^{(1)} \equiv \mathbf{E}'(x, \mathbf{u}^{(0)}) \mathbf{u}^{(1)} = 0, \quad (5.115)$$

and the Laurent series for $\mathbf{u}^{(1)}$

$$\mathbf{u}^{(1)} = \sum_{j=\rho}^{+\infty} \mathbf{u}_j^{(1)} \chi^{j+\mathbf{p}}, \quad (5.116)$$

represents a particular solution containing a number of arbitrary coefficients equal to the number of Fuchs indices, counting their multiplicity. If $\det \mathbf{P}(i)$ has degree N, it represents the general solution of (5.115).

Consequently the sum $\mathbf{u}^{(0)} + \varepsilon \mathbf{u}^{(1)}$ is already, in the neighborhood of $(\chi, \varepsilon) = (0,0)$, a *local representation of the greatest particular solution* of (2.30) available in this method (the general solution if $\det \mathbf{P}(i)$ has degree N), and this is a Laurent series with a strictly larger extension $(\rho + \mathbf{p} : +\infty)$ than that for the unperturbed expansion $(\mathbf{p} : +\infty)$.

At each order $n \geq 2$, the singularity order of the particular solution of the linear inhomogeneous equation (5.110) $\mathbf{E}^{(n)} = 0$ is dictated by the contribution $\mathbf{R}^{(n)}$ of the previously computed coefficients : it is increased by ρ at each order n and is equal to $n\rho + \mathbf{p}$

$$\forall n \geq 0 : \ \mathbf{u}^{(n)} = \sum_{j=n\rho}^{+\infty} \mathbf{u}_j^{(n)} \chi^{j+\mathbf{p}}. \tag{5.117}$$

Third step. Solve the recurrence relation for $\mathbf{u}_j^{(n)}$ for all values of $(n,j) \neq (0,0)$

$$\forall n \geq 0 \ \forall j \geq n\rho, \ (n,j) \neq (0,0) : \ \mathbf{E}_j^{(n)} \equiv \mathbf{P}(j)\mathbf{u}_j^{(n)} + \mathbf{Q}_j^{(n)}(x, \{\mathbf{u}_{j'}^{(n')}\}) = 0. \tag{5.118}$$

The generated necessary stability conditions are

- **C4**.

  $$\forall n \geq 0 \ \forall \text{ index } i, \ (n,i) \neq (0,0) : \ \mathbf{Q}_i^{(n)} \text{ orthogonal to Ker adj } \mathbf{P}(i). \tag{5.119}$$

  These orthogonality conditions must be satisfied whatever be the previously introduced arbitrary coefficients. For a single equation, the condition **C4** is simply $Q_i^{(n)} = 0$.

$\mathbf{Q}_j^{(n)}$ depends on all $\mathbf{u}_{j'}^{(n')}$ with $n' \leq n, j' - n'\rho \leq j - n\rho, (n', j') \neq (n,j)$, and this is the only ordering to be respected during the resolution. The costless ordering on (n,j) is the one which generates stability conditions the sooner, and it depends on the structure of indices of the DE under study.

In order to avoid introducing more arbitrary coefficients than N, the precise rule is :

- if $n = 0$ or ($n = 1$ and $i < 0$), assign arbitrary values to mult(i) components of $\mathbf{u}_i^{(n)}$ defining a basis of Ker $\mathbf{P}(i)$,
- if ($n = 1$ and $i \geq 0$) or $n \geq 2$, assign the value 0 to mult(i) components of $\mathbf{u}_i^{(n)}$ defining a basis of Ker $\mathbf{P}(i)$.

The resulting double expansion (Taylor in ε, Laurent in χ at each order in ε) can be rewritten as a *Laurent series in χ extending to both infinities*

$$\forall n \geq 0 : \ \mathbf{u}^{(n)} = \sum_{j=n\rho}^{+\infty} \mathbf{u}_j^{(n)} \chi^{j+\mathbf{p}}, \ \mathbf{E}^{(n)} = \sum_{j=n\rho}^{+\infty} \mathbf{E}_j^{(n)} \chi^{j+\mathbf{q}}, \tag{5.120}$$

$$\mathbf{u} = \sum_{n=0}^{+\infty} \varepsilon^n \left[\sum_{j=n\rho}^{+\infty} \mathbf{u}_j^{(n)} \chi^{j+\mathbf{p}} \right] = \sum_{j=-\infty}^{+\infty} \mathbf{u}_j \chi^{j+\mathbf{p}}. \quad (5.121)$$

Remarks.

1. The Fuchsian perturbative method (as well as the nonFuchsian one which will be seen section 5.8) is useful if and only if the zeroth order $n = 0$ fails to describe the general solution. This may happen for two reasons. The most common one is a negative Fuchs index in addition to -1 counted once, the second, less common one is a multiplicity higher than one for some family, as in the example of section 5.7.3.

2. We do not know of an upper bound for n, but there exists a lower bound. Indeed, in the linear inhomogeneous ODE (5.110), logarithms can arise only when some precise powers of χ, only depending on the homogeneous part, are present in the rhs Laurent series $R^{(n)}$. The lower bound n results from the condition that the lowest Fuchs index and the highest one, once forced to interfere by the nonlinear terms, start to contribute to such dangerous powers. An example of such a condition is given in section 5.7.5.

   Even if all Fuchs indices are positive (except -1 counted once), the lower bound on n may be greater than 0, as in the example of section 5.7.3.

3. *Remark on index* -1. In the case of a single equation, since indices must be distinct integers, the condition $Q_\rho^{(1)} = 0$ at the smallest Fuchs index $i = \rho$ is identically satisfied. Nevertheless, the frequently encountered statement "resonance -1 is always compatible" is erroneous, and numerous nonzero stability conditions $Q_{-1}^{(n)} = 0$ can be found in the examples of [31]. Indeed, even at first perturbation order, the stability condition at index -1 may not be satisfied : just like Fuchs index $i = \rho$ provides an identically satisfied stability condition, Painlevé "resonance" -1 has the same property if and only if $\rho = -1$, i.e. if -1 is the smallest integer index.

   If ρ is different from -1, Painlevé resonance -1 *seems to* be satisfied, but it is only because a Laurent series ranging from power p to $+\infty$ cannot represent the *general* solution, thus preventing the building of Painlevé stability condition $Q_{-1}^{(0)} = 0$.

5.7.1 Fuchs indices, Painlevé "resonances" or Kowalevski exponents?

Given a nonlinear algebraic DE of order N, one can define three sets of at most N numbers associated to it :

1. the Fuchs indices of the auxiliary equation of Darboux (section 5.7),

2. the "resonances" of the nonlinear equation (section 5.4),

3. the Kowalevski exponents, only defined if the nonlinear equation in invariant under a scaling transformation $(x, \mathbf{u}, \mathbf{E}) \to (kX, k^{\mathbf{p}}\mathbf{u}, k^{\mathbf{q}}\mathbf{E})$.

We have already seen the identity of the first two sets, defined for each family of movable singularities. Let us prove that the third notion is not distinct. The third set is defined as follows (for an introduction, see [10]). The invariance implies the particular solution ("scaling solution") $\mathbf{u}^{(0)} = \text{const}\ (x-x_0)^{\mathbf{p}}$, which is identical to a family of movable singularities. The Kowalevski exponents ρ are defined as the characteristic exponents of the linearized system near this solution, which proves the identity of the three notions.

Said differently, all these numbers are Fuchs indices, and this link to the theory of *linear* DEs proves the uselessness of the notions of Kowalevski exponents and Painlevé resonances.

5.7.2 Understanding negative Fuchs indices

The ODE with a meromorphic general solution [31]

$$E \equiv u_{xx} + 3uu_x + u^3 = 0,\ u = \frac{1}{x-a} + \frac{1}{x-b},\ a \text{ and } b \text{ arbitrary}, \tag{5.122}$$

has two families,

(F1) $p = -1, u_0^{(0)} = 1$, indices $(-1, 1)$,

(F2) $p = -1, u_0^{(0)} = 2$, indices $(-2, -1)$,

and this provides a clear comprehension of negative Fuchs indices, since the index -2 must coexist with the meromorphy. Indeed, the representation of the general solution (5.122) as a Laurent series of $x - x_0$ is the sum of two copies of an expansion of $1/(x-c)$, and there exist two expansions of $1/(x-c)$

$$(x-c)^{-1} = \sum_{j=-\infty}^{-1} (c-x_0)^{-1-j}(x-x_0)^j,\ |c-x_0| < |x-x_0| \tag{5.123}$$

$$= \sum_{j=0}^{+\infty} -(c-x_0)^{1-j}(x-x_0)^j,\ |x-x_0| < |c-x_0|. \tag{5.124}$$

The family (F1) corresponds to the sum (first expansion with $c = a = x_0$) plus (second expansion $c = b$), while the family (F2) corresponds to the sum (first expansion with $c = a$) plus (second expansion with $c = b$). This can be checked by a direct application of the algorithm, which for family (F2) gives [31]

$$u = 2\chi^{-1} + \varepsilon(A_1\chi^{-3} + B_1\chi^{-2}) + \varepsilon^2(\frac{A_1^2}{2}\chi^{-5} + \frac{3A_1B_1}{2}\chi^{-4})$$

$$+\varepsilon^3(\frac{A_1^3}{4}\chi^{-7}+\frac{5A_1^2B_1}{4}\chi^{-6}+A_1B_1^2\chi^{-5}-\frac{1}{2}B_1^3\chi^{-4})+O(\varepsilon^4) \quad (5.125)$$

$$= 2\chi^{-1}+\varepsilon B_1\chi^{-2}+\varepsilon A_1\chi^{-3}+(\frac{3}{2}\varepsilon^2 A_1B_1-\frac{1}{2}\varepsilon^3 B_1^3)\chi^{-4}+O(\chi^{-5}) \quad (5.126)$$

$$= \frac{2\chi-\varepsilon B_1}{\chi^2-\varepsilon B_1\chi+\frac{1}{2}(-\varepsilon A_1+\varepsilon^2 B_1^2)}, \quad (5.127)$$

where A_1 and B_1 are the arbitrary coefficients at order one. The simple pole $\chi = 0$ with residue 2 has been "unfolded" by the perturbation into two simple poles with residue 1, at the two arbitrary locations $\frac{1}{2}\left[\varepsilon B_1 \pm \sqrt{2\varepsilon A_1 - \varepsilon^2 B_1^2}\right]$, both close to 0.

For other examples, see [96] and conference proceedings referenced in [31].

5.7.3 The simplest constructive example

The equation

$$u'' + 4uu' + 2u^3 = 0 \quad (5.128)$$

is the simplest constructive example, because

1. there exists a movable logarithm, as shown by the $\alpha-$method (BSMF §13, p 221),
2. the method of pole-like expansions fails to find it,
3. the assumption of a "descending" Laurent series (5.34) fails to find it,
4. the Fuchsian perturbative method finds it after a very short computation, as we now show.

There exists a single family

$$p = -1,\ E_0^{(0)} = u_0^{(0)}(u_0^{(0)}-1)^2 = 0,\ \text{indices } (-1,0), \quad (5.129)$$

with the puzzling fact that $u_0^{(0)}$ should be at the same time equal to 1 according to the equation $E_0^{(0)} = 0$, and arbitrary according to the index 0. The application of the method provides

$$\begin{aligned}
u^{(0)} &= \chi^{-1} \text{ (the series terminates)} && (5.130)\\
E'(x,u^{(0)}) &= \partial_x^2 + 4\chi^{-1}\partial_x + 2\chi^{-2} && (5.131)\\
u^{(1)} &= u_0^{(1)}\chi^{-1},\ u_0^{(1)}) \text{ arbitrary}, && (5.132)\\
E^{(2)} &= E'(x,u^{(0)})u^{(2)} + 6u^{(0)}{u^{(1)}}^2 + 4u^{(1)}u^{(1)\prime} && \\
&= \chi^{-2}(\chi^2 u^{(2)})'' + 2{u^{(0)}}^2\chi^{-3} = 0 && (5.133)\\
u^{(2)} &= -2{u_0^{(1)}}^2(\chi^{-1}\operatorname{Log}\chi - \chi^{-1}). && (5.134)
\end{aligned}$$

The movable logarithmic branch point is therefore detected in a systematic way at order $n = 2$ and index $i = 0$.

The necessity to perform a perturbation arises from the multiple root of the equation for $u_0^{(0)}$, responsible for the insufficient number of arbitrary parameters in the zeroth order series $u^{(0)}$.

5.7.4 The two examples

Example 1. "Complete (P1)" eq. (5.20). The method is useless.

Example 2 (Chazy complete equation of class III (5.24)).

For the *second family* (in case $d_3 = 0$), the method is useless.

For the *first family*, since all indices are negative, one must start the perturbation process $n \geq 1$.

To obtain the stability conditions up to a given order $n \geq 1$, we only need to compute the first $3n$ coefficients of each element :

$$u^{(r)} = \sum_{j=-3r}^{-3r+3n-1} u_j^{(r)} \chi^{j-1}, r = 0, \dots, n, \tag{5.135}$$

i.e.

$$u_{0:3n-1}^{(0)},\ u_{-3:3n-4}^{(1)},\ \dots, u_{-3n+3:2}^{(n-1)},\ u_{-3n:-2}^{(n)}, \tag{5.136}$$

where $j_1 : j_2$ denotes a range of j values. The most efficient way to perform the double loop on (n, j) is to perform the outside loop in the variable $k = j - n\rho$, with $\rho = -3$, and the precise double loop is : for $k = 0$ to k_{max} do for $n =$ (if $k = 0$ then 1 else 0) to n_{max} do solve the linear algebraic equation (5.118) for $u_j^{(n)}, j = k + n\rho$.

Let us compute all the stability conditions at first and second order. The computer printout reads (full details are given in Ref. [31]) :

$$k = 0 : \quad Q_{-3}^{(1)} \equiv 0,\ u_{-3}^{(1)} \text{ arbitrary} \tag{5.137}$$

$$k = 1 : \quad Q_{-2}^{(1)} \equiv -6(5c_1 + 42d_3)u_{-3}^{(1)} = 0,\ u_{-2}^{(1)} \text{ arbitrary} \tag{5.138}$$

$$k = 2 : \quad Q_{-1}^{(1)} \equiv -12(c_1 + 9d_3)u_{-2}^{(1)} + 18(c_1 - 8d_3)u_{-3}^{(1)'} + \frac{6}{5}(2c_0 - 3c_1^2 - 117c_1 d_3 - 594d_3^2 + 18c_1' + 108d_3')u_{-3}^{(1)} = 0, \tag{5.139}$$

$$u_{-1}^{(1)} \text{ arbitrary}, \tag{5.140}$$

$$k = 3 : \quad Q_{-3}^{(2)} \equiv -\frac{66}{5} d_1 u_{-3}^{(1)^2} = 0, \tag{5.141}$$

$$k = 4 : \quad Q_{-2}^{(2)} \equiv \frac{1}{7}(8d_0 + 57d_1')u_{-3}^{(1)^2} - 12d_1 u_{-3}^{(1)} u_{-2}^{(1)} = 0 \tag{5.142}$$

$$k = 5 : \quad Q_{-1}^{(2)} \equiv -\frac{1}{35}(18d_0' + 99d_1'')u_{-3}^{(1)^2} - \frac{24}{5} d_1 u_{-2}^{(1)^2} + \frac{3}{35}(16d_0 + 72d_1')u_{-3}^{(1)} u_{-2}^{(1)} = 0. \tag{5.143}$$

Five conditions are obtained, three at order one, equivalent to $d_3 = c_1 = c_0 = 0$, and two at order two, equivalent to $d_1 = d_0 = 0$ [in order to simplify expressions, we have put the first order conditions in the above expressions for $k = j + 3n \geq 3$], after seventeen values of (n, j). These conditions were given without any detail by Chazy[22]. They restrict the complete ODE (5.24) to the simplified ODE (1.17), *modulo* (3.5).

Chazy proved the general solution of (1.17) to be single valued inside or outside a circle whose centre and radius depend on the choice of the three arbitrary constants; it is holomorphic in this domain, and the only singularity is a movable natural boundary ("coupure essentielle") defined by this circle. He also gave a parametric representation of the general solution $u(x)$ in terms of two solutions of the (linear) hypergeometric equation, but single valuedness is not at all apparent on this representation.

The direct explicit solution of Bureau [17, 18] is given section 5.5.1.

5.7.5 An example needing order seven to conclude

The following equation, isolated by Bureau ([15] p. 79),

$$u'''' + 3uu'' - 4u'^2 = 0 \tag{5.144}$$

possesses the two families

$$p = -2, u_0^{(0)} = -60, \text{ ind. } (-3, -2, -1, 20), \ \hat{K} = u'''' + 3uu'' - 4u'^2 \tag{5.145}$$

$$p = -3, u_0^{(0)} \text{ arbitrary, indices } (-1, 0), \ \hat{K} = 3uu'' - 4u'^2. \tag{5.146}$$

The *second family* has a Laurent series $(p : +\infty)$ which happens to terminate [31]

$$u^{(0)} = c(x - x_0)^{-3} - 60(x - x_0)^{-2}, \ (c, x_0) \text{ arbitrary.} \tag{5.147}$$

The Fuchsian perturbative method is useless, for the two arbitrary coefficients corresponding to the two Fuchs indices are already present at zeroth order.

The *first family* provides, at zeroth order, only a two-parameter expansion and, when one checks the existence of the perturbed solution

$$u = \sum_{n=0}^{+\infty} \varepsilon^n \left[\sum_{j=0}^{+\infty} u_j^{(n)} \chi^{j-2-3n} \right], \tag{5.148}$$

one finds that coefficients $u_{20}^{(0)}, u_{-3}^{(1)}, u_{-2}^{(1)}, u_{-1}^{(1)}$ can be chosen arbitrarily, and, at order $n = 7$, one finds two violations [31]

$$Q_{-1}^{(7)} \equiv u_{20}^{(0)} u_{-3}^{(1)^7} = 0, Q_{20}^{(7)} \equiv u_{20}^{(0)^2} u_{-3}^{(1)^6} u_{-2}^{(1)} = 0, \tag{5.149}$$

implying the existence of a movable logarithmic branch point.

Remark. The value $n = 7$ is the root of the linear equation $n(i_{\min} - p) + (i_{\max} - p) = -1$, with $p = -2, i_{\min} = -5, i_{\max} = 18$, linking the pole order p in the Fuchsian case $c = 0$, the smallest and the greatest Fuchs indices. It expresses the condition for the first occurence of a power χ^{-1}, leading by integration to a logarithm, in the r.h.s. $R^{(n)}$ of the linear inhomogeneous equation (5.110), r.h.s. created by the nonlinear terms $3uu'' - 4u'^2$.

5.7.6 Closed-form solutions of the Bianchi IX model

In this example, the no-log conditions are used in a constructive way, in order to isolate all possible single valued solutions.

The Bianchi IX cosmological model [78] is a system of three second order ODEs

$$(\mathrm{Log}\, A)'' = A^2 - (B - C)^2 \text{ and cyclically, } ' = \mathrm{d}/\mathrm{d}\tau, \tag{5.150}$$

or equivalently

$$(\mathrm{Log}\, \omega_1)'' = \omega_2^2 + \omega_3^2 - \omega_2^2\omega_3^2/\omega_1^2,\ A = \omega_2\omega_3/\omega_1,\ \omega_1^2 = BC \text{ and cyclically.} \tag{5.151}$$

One of the families [36, 79]

$$\begin{aligned} &A = \chi^{-1} + a_2\chi + O(\chi^3),\ \chi = \tau - \tau_2,\\ &B = \chi^{-1} + b_2\chi + O(\chi^3),\\ &C = \chi^{-1} + c_2\chi + O(\chi^3), \end{aligned} \tag{5.152}$$

has the Fuchs indices $(-1, -1, -1, 2, 2, 2)$. The Fuchsian perturbative method

$$A = \chi^{-1} \sum_{n=0}^{N} \varepsilon^n \sum_{j=-n}^{2+N-n} a_j^{(n)} \chi^j,\ \chi = \tau - \tau_2, \text{ and cyclically,} \tag{5.153}$$

then gives a failure of condition **C4** at $(n, i) = (3, -1)$ and $(5, -1)$ [79], and the satisfaction of these no-log conditions generates the three solutions :

$$(b_2^{(0)} = c_2^{(0)} \text{ and } b_{-1}^{(1)} = c_{-1}^{(1)}) \text{ or cyclically} \tag{5.154}$$

$$a_2^{(0)} = b_2^{(0)} = c_2^{(0)} = 0, \tag{5.155}$$

$$a_{-1}^{(1)} = b_{-1}^{(1)} = c_{-1}^{(1)}. \tag{5.156}$$

These are constraints which reduce the number of arbitrary coefficients to, respectively, four, three and four, thus defining particular solutions which may have no movable critical points. The question is : do they define additional solutions to what is known?

The only three closed-form solutions which are known are single valued, they are defined as the general solution of the following three subsystems.

1. The 4-dim axisymmetric case $B = C$ [108], whose general solution (2.60) is trigonometric.

2. The 3-dim *Darboux-Halphen system* [44, 61]

$$\omega_1' = \omega_2\omega_3 - \omega_1\omega_2 - \omega_1\omega_3, \quad \text{and cyclically.} \tag{5.157}$$

3. The 3-dim *Euler system* (1750) [8], describing the motion of a rigid body around its center of mass

$$\omega_1' = \omega_2\omega_3, \text{ and cyclically,} \tag{5.158}$$

whose general solution is elliptic [8], see (2.31) and (2.45).

The first constraint (5.154) implies the equality of two of the components (A, B, C) at every order and thus represents the four-parameter solution of Taub (2.60).

The second constraint (5.155) represents the three-parameter solution of the Darboux-Halphen system (5.157).

The third and last constraint (5.156) represents an extrapolation to four parameters of the three-parameter solution ot the Euler system described by $a_2^{(0)} + b_2^{(0)} + c_2^{(0)} = 0$. This would-be four-parameter, global, closed form, single valued exact solution has not yet been found.

5.8 The nonFuchsian perturbative method

Whenever the family under study has a number of Fuchs indices smaller than the differential order N, the Fuchsian perturbation method fails to build a representation of the general solution, thus possibly missing some stability conditions. Examples are (5.146) and the second family of (5.24) in the case $d_3 = 0$. The missing solutions of the auxiliary equation (5.115) are then nonFuchsian solutions, see section 5.2.2.

There is no difficulty to algorithmically compute the nonFuchsian expansions (5.11), but these are of no immediate help, due to their generic divergence.

There is one situation where some stability conditions can be generated *algorithmically* (indeed, we are not interested in computations adapted to a given equation, only in computerizable methods). It occurs when the two following conditions are met [85].

1. There exists a particular solution $\mathbf{u} = \mathbf{u}^{(0)}$ which is known globally, meromorphic and has at least one movable pole at a finite distance denoted x_0.

2. The only singular points of the linearized equation $\mathbf{E}^{(1)} = 0$ are $x = x_0$, nonFuchsian, and $x = \infty$, Fuchsian.

Then, the property that a fundamental set of solutions $\mathbf{u}^{(1)}$ be locally single valued near $\chi = x - x_0 = 0$ is equivalent to the same property near $\chi = \infty$. This is the global nature of $\mathbf{u}^{(0)}$ which allows the study of the point $\chi = \infty$, easy to perform with the Fuchsian perturbation method.

An important technical bonus is the lowering of the differential order N of equation $\mathbf{E}^{(1)} = 0$ by the number M of arbitrary parameters c which appear in $\mathbf{u}^{(0)}$. Indeed, again since $\mathbf{u}^{(0)}$ is closed form, its partial derivatives $\partial_c \mathbf{u}^{(0)}$ are closed form and are particular solutions of $\mathbf{E}^{(1)} = 0$, which allows this lowering of the order.

At each higher perturbation order $n \geq 2$, one similarly builds particular solutions $\mathbf{u}^{(n)}$ as expansions near $\chi = \infty$ and one requires the same properties.

5.8.1 An explanatory example : Chazy's class III ($N = 3, M = 2$)

The simplified equation (1.17), which possesses the PP [22] and therefore for which no $u^{(n)}$ is multivalued, is quite useful just to understand the method. This equation admits the global two-parameter solution (5.57) $u^{(0)} = c\chi^{-2} - 6\chi^{-1}$. The linearized equation

$$E^{(1)} \equiv E'(x, u^{(0)})u^{(1)} \equiv [\partial_x^3 - 2u^{(0)}\partial_x^2 + 6u_x^{(0)}\partial_x - 2u_{xx}^{(0)}]u^{(1)} = 0 \qquad (5.159)$$

possesses the two single valued global solutions $\partial_{x_0} u^{(0)}$ and $\partial_c u^{(0)}$, i.e. $u^{(1)} = \chi^{-3}, \chi^{-2}$, and it has only two singular points $\chi = 0$ (Fuchsian) and $\chi = \infty$ (nonFuchsian with Thomé rank two). The lowering by $M = 2$ units of the order of the linearized equation results from the change of function

$$u^{(1)} = \chi^{-3} v : \ E^{(1)} \equiv \chi^3 [\partial_x + 3\chi^{-1} - 2c\chi^{-2}]v'' = 0, \qquad (5.160)$$

and the study of the Fuchsian point $\chi = \infty$ yields an integer Fuchs index, which proves the *global* single valuedness of the general solution $u^{(1)}$.

Remarks.

- The local study of $\chi = 0$ provides a formal expansion (5.11) which happens to terminate, a nongeneric situation, thus providing the fundamental set of *global* solutions at perturbation order $n = 1$

  $$\forall \chi \ \forall c : \ u^{(1)} = \chi^{-2}, \ \chi^{-3}, \ (e^{-2c/\chi} - 1 + 2c\chi^{-1})\chi^{-2}/(2c^2). \qquad (5.161)$$

  This proves the existence of an essential singularity at $\chi = 0$ (ref. [66] chap. XVII).

- Going on with the formalism of Painlevé's lemma at higher orders constitutes the rigorous mathematical framework of the local representation of the general solution obtained by Joshi and Kruskal [70]

$$u = -6\chi^{-1} + c\chi^{-2}\left(1 + z - \frac{z^2}{8} + \frac{z^3}{144} - 7\frac{z^4}{13824} + O(\varepsilon^5)\right),\ z = \frac{\varepsilon}{c}e^{-2c/\chi}. \tag{5.162}$$

This representation reduces to the one given by Chazy (Taylor series in $1/\chi$) if one starts from the Fuchsian family $u \sim -6\chi^{-1}$.

5.8.2 The fourth order equation of Bureau ($N = 4, M = 2$)

In section 5.7.5, the fourth order equation (5.144) has been proven to be unstable after a computation practically untractable without a computer. Let us now prove this result without computation at all [85]. For the global two-parameter solution (5.147), the linearized equation

$$E^{(1)} = E'(x, u^{(0)})u^{(1)} \equiv [\partial_x^4 + 3u^{(0)}\partial_x^2 - 8u_x^{(0)}\partial_x + 3u_{xx}^{(0)}]u^{(1)} = 0 \tag{5.163}$$

has only two singular points $\chi = 0$ (nonFuchsian) and $\chi = \infty$ (Fuchsian), it admits the two global single valued solutions $\partial_{x_0}u^{(0)}$ and $\partial_c u^{(0)}$, i.e. $u^{(1)} = \chi^{-4}, \chi^{-3}$. The lowering by $M = 2$ units of the order of the linearized equation (5.163) is obtained with

$$u^{(1)} = \chi^{-4}v :\ [\partial_x^2 - 16\chi^{-1}\partial_x + 3c\chi^{-3} - 60\chi^{-2}]v'' = 0, \tag{5.164}$$

and the local study of $\chi = \infty$ is unnecessary, since one recognizes the confluent hypergeometric equation. The two other solutions in global form are

$$c \neq 0 :\ v_1'' = \chi^{-3}{}_0F_1(24; -3c/\chi) = \chi^{17/2}J_{23}(\sqrt{12c/\chi}), \tag{5.165}$$

$$v_2'' = \chi^{17/2}N_{23}(\sqrt{12c/\chi}), \tag{5.166}$$

where the hypergeometric fonction ${}_0F_1(24; -3c/\chi)$ is single valued and possesses an isolated essential singularity at $\chi = 0$, while the fonction N_{23} of Neumann is multivalued because of a $\operatorname{Log}\chi$ term.

Remark. The local study of (5.163) near $\chi = 0$ provides the formal expansions (5.11) for the two nonFuchsian solutions

$$\chi \to 0,\ c \neq 0 :\ u^{(1)} = e^{\pm\sqrt{-12c/\chi}}\chi^{31/4}(1 + O(\sqrt{\chi})), \tag{5.167}$$

detecting the presence in (5.163) of an essential singularity at $\chi = 0$, but the generically null radius of convergence of the formal series forbids to conclude to the multivaluedness of $u^{(1)}$. A nonobvious result is the existence, as seen above, of a linear combination of the two formal expansions (5.167) which is single valued.

5.8.3 An example in cosmology : Bianchi IX ($N = 6, M = 4$)

The Bianchi IX cosmological model in vacuum (5.150) does not possess the PP [37, 79]. Let us prove it rapidly [79, 85]. Taub [108] found the general solution of the axisymmetric case of two equal components, a meromorphic expression (2.60) depending on the four arbitrary parameters $(k_1, k_2, \tau_1, \tau_2)$. The linearized system generated by the perturbation

$$A = A^{(0)}(1 + \varepsilon A^{(1)} + O(\varepsilon^2)) \text{ and cyclically} \tag{5.168}$$

has the differential order $N = 6$, which is then lowered by $M = 4$ units by the change of function dictated by the symmetry of the system : $P^{(1)} = B^{(1)} + C^{(1)}, M^{(1)} = B^{(1)} - C^{(1)}$

$$A^{(1)''} - 2A^{(0)^2}A^{(1)} = 0, \tag{5.169}$$

$$P^{(1)''} - 2A^{(0)}B^{(0)}P^{(1)} = 4(A^{(0)}B^{(0)} - A^{(0)^2})A^{(1)}, \tag{5.170}$$

$$M^{(1)''} + 2(A^{(0)}B^{(0)} - 2B^{(0)^2})M^{(1)} = 0. \tag{5.171}$$

Indeed, the four single valued global solutions

$$(A^{(1)}, P^{(1)}) = \partial_c(\operatorname{Log} A^{(0)}, \operatorname{Log}(B^{(0)} + C^{(0)})),\ c = k_1, k_2, \tau_1, \tau_2, \tag{5.172}$$

are those of the equations (5.169)–(5.170),

$$M^{(1)} = 0, (A^{(1)}, P^{(1)} + 2A^{(1)}) = \begin{cases} ((\tau - \tau_1)\coth k_1(\tau - \tau_1) - 1/k_1, 0), \\ (0, (\tau - \tau_2)\coth k_2(\tau - \tau_2) - 1/k_2), \\ (\coth k_1(\tau - \tau_1), 0), \\ (0, \coth k_2(\tau - \tau_2)), \end{cases} \tag{5.173}$$

and there only remains to study the equation (5.171). It has a countable infinity of singular points : $\tau - \tau_2 = im\pi/k_2, m \in \mathcal{Z}$ (nonFuchsian, of Thomé rank two), accumulating at $\tau = \infty$. This uneasy situation can be overcome by taking the limit $k_1 = k_2 = 0$; it is indeed sufficient to exhibit a movable logarithm in this limit, for it will persist for $(k_1, k_2) \neq (0, 0)$. In this limit

$$k_1 = k_2 = 0 : \qquad \frac{\mathrm{d}^2 M^{(1)}}{\mathrm{d}t^2} + \left(\frac{2}{t^2} - \frac{4(t-1)^2}{t^4}\right) M^{(1)} = 0,\ t = \frac{\tau - \tau_2}{\tau_1 - \tau_2} \tag{5.174}$$

the only singular points are $t = 0$ (nonFuchsian) and $t = \infty$ (Fuchsian), the optimal situation. The Fuchs indices being -2 and 1, the computation of three terms is sufficient to exhibit a logarithm, and this proves the absence of the Painlevé property for the Bianchi IX model in vacuum.

Remarks

1. The two solutions are globally known [79] :

$$k_1 = k_2 = 0: \qquad M^{(1)} = e^{-2/t} t^{-1},\ e^{-2/t} t^{-1} \int^{1/t} z^{-4} e^{4z} \mathrm{d}z, \tag{5.175}$$

   which shows the presence of a logarithmic branch point at $t = 0$, or at $t = \infty$ as well.

2. The two formal non-Fuchsian solutions are

$$\tau - \tau_2 \to 0: \ M^{(1)} = e^{\alpha/(\tau-\tau_2)} \sum_{k=0}^{+\infty} \lambda_k (\tau - \tau_2)^{k+s},\ \lambda_0 \neq 0, \tag{5.176}$$

   with

$$\alpha = \pm 2 k_1^{-1} \sinh k_1(\tau_2 - \tau_1),\ s = 1 \mp 2 \cosh k_1(\tau_2 - \tau_1). \tag{5.177}$$

   The two generically irrational values for the Thomé exponents s allow to conclude only if the divergent series $\lambda_k(\tau - \tau_2)^k$ can be summed.

5.9 Miscellaneous perturbations

The differential complexity of the $\alpha-$method explains why it usually succeeds in case of failure of all the other methods, which only have an algebraic complexity. Consider the ODEs, none of which admits a power-law leading behaviour

$$-2uu'' + 3u'^2 + d_3 u^3 = 0,\ d_3 \neq 0, \tag{5.178}$$

$$u''' + uu'' - 2u'^2 = 0, \tag{5.179}$$

$$u'''' + 2uu'' - 3u'^2 = 0, \tag{5.180}$$

and let us prove that each of them has movable logarithms. The first one is extracted from Chazy's class III (5.24) by the perturbation $u = \varepsilon^{-1} U, x = x_0 + \varepsilon X$, and it represents its second family, see section 5.4.1. The second and third ones were considered by Chazy [23, 24] who had to establish a special theorem, using divergent series, to exhibit the movable logarithms. Having degree one, none of these ODEs admits singular solutions.

The first equation (5.178) is classically processed by the $\alpha-$method

$$u = \varepsilon^{-1} \sum_{n=0}^{+\infty} \varepsilon^n u^{(n)},\ E = \varepsilon^{-4} \sum_{n=0}^{+\infty} \varepsilon^n E^{(n)},\ x = x_0 + \varepsilon X, \tag{5.181}$$

resulting in

$$E^{(0)} \equiv -2u^{(0)} u^{(0)''} + 3u^{(0)'^2} = 0 \tag{5.182}$$

$$u^{(0)} = c(X - X_0)^{-2},\ (X_0, c) \text{ arbitrary}, \tag{5.183}$$

$$E^{(1)} \equiv c(X - X_0)^{-5}[-2((X - X_0)^3 u^{(1)})'' + c^2 d_3/(X - X_0)] = 0, \tag{5.184}$$

$$u^{(1)} = c^2 d_3 (X - X_0)^3 [(X - X_0) \operatorname{Log}(X - X_0) - (X - X_0)]/2. \tag{5.185}$$

and proving the instability at perturbation order one.

For equations (5.179) and (5.180), there exists no perturbation satisfying the assumptions of Theorem II page 45, there only exist singular perturbations, i. e. which discard the highest derivative. Since they however give the correct information, it would be desirable to extend Theorem II in that direction.

Equation (5.179) is handled by the singular perturbation

$$u = \varepsilon^{-1} \sum_{n=0}^{+\infty} \varepsilon^n u^{(n)},\ E = \varepsilon^{-2} \sum_{n=0}^{+\infty} \varepsilon^n E^{(n)}, \tag{5.186}$$

which excludes u''' from the simplified equation

$$E^{(0)} \equiv u^{(0)} u^{(0)''} - 2u^{(0)'^2} = 0 \tag{5.187}$$

$$u^{(0)} = c\chi^{-1},\ \chi = x - x_0,\ (x_0, c) \text{ arbitrary}, \tag{5.188}$$

$$E^{(1)} \equiv c(\chi^{-3}(\chi^2 u^{(1)})'' - 6\chi^{-4}) = 0, \tag{5.189}$$

$$u^{(1)} = 6\chi^{-1}(\mathrm{Log}\,\chi - 1). \tag{5.190}$$

This same perturbation (5.186) solves the case of the equation (5.180)

$$E^{(0)} \equiv 2u^{(0)} u^{(0)''} - 3u^{(0)'^2} = 0 \tag{5.191}$$

$$u^{(0)} = c\chi^{-2} \tag{5.192}$$

$$E^{(1)} \equiv c(2\chi^{-5}(\chi^3 u^{(1)})'' + 120\chi^{-6}) = 0 \tag{5.193}$$

$$u^{(1)} = -60\chi^{-2}(\mathrm{Log}\,\chi - 1). \tag{5.194}$$

5.10 The perturbation of the continuum limit of a discrete equation

Discrete equations can be considered as functional equations linking the values taken by some field variable u at a finite number $N+1$ of points, either arithmetically consecutive : $x+kh, k-k_0 = 0, 1, \dots, N$, or geometrically consecutive : $xq^k, k-k_0 = 0, 1, \dots, N$, where h or q is the lattice stepsize, assumed to lay in some neighborhood of, respectively, 0 or 1, and k_0 is just some convenient origin.

Definition [35]. A discrete equation is said to possess the *discrete Painlevé property* if and only if there exists a neighborhood of $h = 0$ at every point of which the general solution $x \to u(x, h)$ has no movable critical singularities.

Consider an arbitrary discrete equation (5.195),

$$\forall x\ \forall h\ :\ E(x, h, \{u(x+kh),\ k-k_0 = 0, \dots, N\}) = 0 \tag{5.195}$$

algebraic in the values of the field variable, with coefficients analytic in x, the stepsize and some parameters a. Let $(x, h, u, a) \to (X, H, U, A, \varepsilon)$ be an arbitrary perturbation admissible by the suitable extension of the theorem of

Poincaré to discrete systems. Two such perturbations are well known, the *autonomous limit*

$$x = x_0 + \varepsilon X,\ h = \varepsilon H,\ u = U,\ a = \text{ analytic } (A, \varepsilon), \tag{5.196}$$

and the *continuum limit*

$$x \text{ unchanged},\ h = \varepsilon,\ u = U,\ a = \text{ analytic } (A, \varepsilon). \tag{5.197}$$

The latter can be extended into a *perturbation of the continuum limit* [35]

$$x \text{ unchanged},\ h = \varepsilon,\ u = \sum_{n=0}^{+\infty} \varepsilon^n u^{(n)},\ a = \text{ analytic } (A, \varepsilon), \tag{5.198}$$

entirely analogous to the Fuchsian (section 5.7) or nonFuchsian (section 5.8) perturbative method of the continuous case.

It generates an infinite sequence of (continuous) differential equations $E^{(n)} = 0$ whose first one $n = 0$ is the continuum limit. The next ones $n \ge 1$, which are linear inhomogeneous, have the same homogeneous part $E^{(0)'} u^{(n)} = 0$ independent of n, defined by the derivative of the equation of the continuum limit, while their inhomogeneous part $R^{(n)}$ ("right-hand side") comes at the same time from the nonlinearities and the discretization.

Let us just handle the Euler scheme for the Bernoulli equation

$$E \equiv (\overline{u} - u)/h + u^2 = 0 \tag{5.199}$$

(notation is $u = u(x), \overline{u} = u(x+h)$), i.e. the logistic map of Verhulst, a paradigm of chaotic behaviour. Let us expand the terms of (5.199) according to the perturbation (5.198) up to an order in ε sufficient to build the first equation $E^{(1)} = 0$ beyond the continuum limit $E^{(0)} = 0$

$$\begin{aligned}
u &= u^{(0)} + u^{(1)}\varepsilon + O(\varepsilon^2) && (5.200)\\
u^2 &= u^{(0)^2} + 2u^{(0)}u^{(1)}\varepsilon + O(\varepsilon^2) && (5.201)\\
\overline{u} &= u + u'h + (1/2)u''h^2 + O(h^3) && (5.202)\\
\frac{\overline{u} - u}{h} &= u^{(0)'} + (u^{(1)'} + (1/2)u^{(0)''})\varepsilon + O(\varepsilon^2). && (5.203)
\end{aligned}$$

The equations of orders $n = 0$ et $n = 1$ are written as

$$\begin{aligned}
E^{(0)} &= u^{(0)'} + u^{(0)^2} = 0 && (5.204)\\
E^{(1)} &= E^{(0)'} u^{(1)} + (1/2)u^{(0)''} = 0,\ E^{(0)'} = \partial_x + 2u^{(0)}. && (5.205)
\end{aligned}$$

Their general solution is

$$\begin{aligned}
u^{(0)} &= \chi^{-1}, \chi = x - x_0,\ x_0 \text{ arbitrary} && (5.206)\\
u^{(1)} &= u^{(1)}_{-1}\chi^{-2} - \chi^{-2}\,\mathrm{Log}\,\psi,\ \psi = x - x_0,\ u^{(1)}_{-1} \text{ arbitrary}, && (5.207)
\end{aligned}$$

and the movable logarithm proves the instability as soon as order $n = 1$, at the Fuchs index $i = -1$.

5.11 The diophantine conditions

In fulfilling the systematic programme of Painlevé, one encounters the following kind of diophantine equations

$$\sum_{k=1}^{p} \frac{1}{n_k} = \frac{1}{n}, \tag{5.208}$$

with n and p given integers, whose unknowns (n_k) are Fuchs indices, which must therefore be either integer or infinite. They admit a finite set of solutions, which allows all cases to be further examined. Details can be found in Bureau 1964, M.I, M.II.

Such a diophantine condition always arises when there exists more than one family, as the constraint that, *simultaneously*, all Fuchs indices of all families be integer. Let us just give one example [31]. The Hamiltonian Hénon-Heiles system [62] in two coupled variables (q_1, q_2)

$$H \equiv (1/2)(q_{1,x}^2 + q_{2,x}^2 + c_1 q_1^2 + c_2 q_2^2) + \alpha q_1 q_2^2 - (1/3)\beta q_1^3 = E, \tag{5.209}$$

$$q_{1,xx} + c_1 q_1 - \beta q_1^2 + \alpha q_2^2 = 0 \tag{5.210}$$

$$q_{2,xx} + c_2 q_2 + 2\alpha q_1 q_2 = 0 \tag{5.211}$$

defines by elimination the fourth order ODE in $v = q_1$ [50]

$$\begin{aligned} &v_{xxxx} + (8\alpha - 2\beta) v v_{xx} - 2(\alpha + \beta) v_x^2 - (20/3)\alpha\beta v^3 \\ &+ (c_1 + 4c_2) v_{1,xx} + (6\alpha c_1 - 4\beta c_2) v^2 + 4 c_1 c_2 v + 4\alpha E = 0, \end{aligned} \tag{5.212}$$

with $(\alpha, \beta, c_1, c_2, E)$ constants. Let us restrict here to $c_1 = c_2 = 0$.

There exist two families

$$p = -1, v_0 = \frac{3}{\alpha}, \text{ indices } (-1, 10, r_1, r_2), \tag{5.213}$$

$$p = -1, v_0 = -\frac{6}{\beta,} \text{ indices } (-1, 5, s_1, s_2), \tag{5.214}$$

in which r_i and s_i satisfy the equations

$$r^2 - 5r + 12 + 6\gamma = 0, \; s^2 - 10s + 24 + 48\gamma^{-1} = 0, \; \gamma = \beta/\alpha. \tag{5.215}$$

The diophantine equations to be solved are

$$(r_1 - r_2)^2 = (2k-1)^2, (s_1 - s_2)^2 = (2l)^2, \; (r_i + 1)(r_i - 10)(s_i + 1)(s_i - 5) \neq 0, \tag{5.216}$$

with k and l two strictly positive integers. Making use of $(r_1 - r_2)^2 = (2r - 5)^2, (s_1 - s_2)^2 = 4(s-5)^2$, the elimination of γ between (5.215) yields

$$\gamma = \frac{48}{1 - l^2}, \; l^2 = 1 + \frac{1152}{23 + (2k-1)^2}, \tag{5.217}$$

and this provides sharp bounds for $l : 1 < l^2 \leq 49$. One thus obtains the four solutions for $\beta/\alpha, (k,l), (r_1,r_2), (s_1,s_2)$

$$-1 : \quad (1,7),\ (2,3),\ (-2,12)\ \ (\text{SK}), \tag{5.218}$$
$$-2 : \quad (3,5),\ (0,5),\ (0,10), \tag{5.219}$$
$$-6 : \quad (6,3),\ (-3,8),\ (2,8)\ \ (\text{KdV5}), \tag{5.220}$$
$$-16 : \quad (10,2),\ (-7,12),\ (3,7)\ \ (\text{KK}). \tag{5.221}$$

Three of them restrict the ODE to the stationary reduction of well-known soliton equations, thus proving the PP : Sawada-Kotera (SK [103]), higher-order Korteweg-de Vries (KdV5, [80]) and Kaup-Kupershmidt (KK, [71, 51]) equations.

The case $\beta = -2\alpha$ is similar to that of the ODE in section 5.7.3 : v_0 is a double root of its algebraic equation and is not arbitrary although 0 is an index. The results of the Fuchsian perturbative method are also similar; listed by increasing cost (number of needed values of (n,i)), the first stability conditions $Q_i^{(n)} = 0$ are

$$Q_0^{(1)} \equiv 0,\ \text{cost} = 2 \tag{5.222}$$
$$Q_0^{(2)} \equiv -40\alpha {u_0^{(1)}}^2 = 0,\ \text{cost} = 5 \tag{5.223}$$
$$Q_{10}^{(0)} \equiv -30\alpha^3 {u_5^{(0)}}^2 = 0,\ \text{cost} = 10 \tag{5.224}$$
$$Q_5^{(1)} \equiv -120\alpha u_5^{(0)} u_0^{(1)} = 0,\ \text{cost} = 12. \tag{5.225}$$

To detect the instability, the method of pole-like expansions is here sufficient but the Fuchsian perturbative method is much cheaper.

Chapter 6

Construction of necessary conditions. The Painlevé test

This chapter makes the synthesis of all the methods of chapter 5 in order to define a usable end product which makes obsolete the meromorphy test of section 2.3. This end product is widely known as the *Painlevé test*. Before detailing the steps of this algorithm in section 6.6, for ODEs as well as for PDEs, some prerequisite technical developments are needed : implementation of physicists' desiderata (section 6.1), technicalities to simplify the computations (section 6.2) and the quite important feature of the invariant Painlevé analysis (sections 6.3, 6.4 and 6.5).

6.1 Physical considerations

Some DEs encountered in physics are unstable, although integrable or partially integrable in some obvious physical sense. It is then extremely important not to discard them; this is achieved by relaxing some of the mathematical requirements.

Firstly, nonpolynomial DEs can be made polynomial by transformations on u like in section 3.3.3, necessarily outside the groups of invariance of the PP defined in sections 3.3.1 and 3.3.2.

Example 1 (sine-Gordon).

$$\text{(sine-Gordon) } u_{xt} = \sin u,\ e^{iu} = v,\ 2(v v_{xt} - v_x v_t) - v^3 + v = 0. \tag{6.1}$$

Example 2 (Benjamin-Ono). The nonlocal, nonpolynomial PDE

$$u_t + uu_x + \mathrm{H}(u_{xx}) = 0,\ \mathrm{H}(v) = \frac{1}{\pi}\mathrm{pp}\int_{-\infty}^{+\infty} \frac{v(x', t)}{x' - x}\mathrm{d}x', \tag{6.2}$$

in which H is the Hilbert transform, and pp the Cauchy principal value distribution, is equivalent [60, 101] to the local and polynomial system

$$u_t + uu_x + u_{xy},\ u_{xx} + u_{yy} = 0 \tag{6.3}$$

in one additional independent variable y.

Secondly, unstable polynomial DEs may be made stable and polynomial by transformations like (3.9).

Example 3 (parity invariance). The Ermakov-Pinney ODE [49, 97]

$$u_{xx} - \alpha^2 u + \beta^2 u^{-3} = 0. \tag{6.4}$$

is unstable (algebraic branch point $p = 1/2$) and invariant by parity on u : the transformation $u \to u^2$ or u^{-2} preserves its polynomial form and makes it stable.

6.2 Technicalities

A careful *choice of the dependent variables* can save many computations.

Example 1 (dynamical systems). These systems of first order ODEs sometimes possess an *equivalent* scalar ODE. This is the case of the Lorenz model (1.3), equivalent to [107]

$$xx''' - x'x'' + x^3x' + (b+\sigma+1)xx'' + (\sigma+1)(bxx' - x'^2) + \sigma x^4 + b(1-r)\sigma x^2 = 0 \tag{6.5}$$

and of the Hénon-Heiles Hamiltonian system in (q_1, q_2) (5.209) which implies the fourth order ODE in q_1 only (5.212). This offers two advantages. The first one is to reduce the matricial recurrence relation to a scalar one. The second one is much more interesting : the scalar ODE has a number of families lower than or equal to that of the DS, which saves a lot of useless cases to consider; thus, in the HH system, the leading powers for (q_1, q_2) are $(-2, -2), (-2, -1), (-2, 0)$, while the equivalent fourth order ODE for q_1 has only one leading power -2.

Choosing an *integrated dependent variable* for the computations saves a lot. The principle is that, if a DE for u is to be stable, this allows the presence of *one* movable logarithm in its primitive $v = \int u\ \mathrm{d}x$. If changing u to v_x allows the DE to be integrated once or more, expressions are shortened.

6.3 Equivalence of three fundamental ODEs

Let S be a given analytic function of a complex variable x, and let us consider the three differential equations in φ, χ, ψ :

$$
\begin{aligned}
\frac{\varphi_{xxx}}{\varphi_x} - \frac{3}{2}\left(\frac{\varphi_{xx}}{\varphi_x}\right)^2 &= S && (6.6)\\
\omega = \chi^{-1},\ -2\omega_x - 2\omega^2 &= S && (6.7)\\
-2\frac{\psi_{xx}}{\psi} &= S && (6.8)
\end{aligned}
$$

The first one is the Schwarz equation; if read backwards, it defines S as the Schwarzian $\{\varphi; x\}$ of φ. The second one is the Riccati equation in its normalized form (equations for ω or χ are equivalent and both of Riccati type). The third one is the second order linear Sturm-Liouville ODE in its normalized form.

Each of these three ODEs possesses a fundamental uniqueness property. As shown by S. Lie, the Schwarzian is the unique elementary homographic differential invariant of a function φ, i.e. the unique elementary function of the derivatives $D\varphi$ of φ, excluding φ itself, invariant under the 6-parameter group $\mathcal{H}$ (or Möbius group, or PSL(2,C)) of homographic transformations :

$$
\mathcal{H} : \varphi \to \frac{a\varphi + b}{c\varphi + d},\ (a, b, c, d) \text{ arbitrary complex constants, } ad - bc = 1. \tag{6.9}
$$

Among nonlinear first order ODEs in the class :

$$
u' = R(u, x) \tag{6.10}
$$

where R is rational in u and analytic in x, the Riccati equation is the unique one whose general integral has no movable critical points. As to equation (6.8), its uniqueness lies in its linear form.

It is a classical result due to Painlevé (1895, Leçons p. 230, Œuvres I) that the three ODEs (6.6), (6.7), (6.8) are equivalent : it is sufficient to integrate one in order to integrate the two others. Consequently, any ODE reducible to one of these three ODEs can be considered as *explicitly linearizable*. The six ODEs obtained by elimination of S between any two of the three ODEs have the general solution :

$$
\begin{aligned}
\omega(\varphi) &= \frac{c_1 \varphi_x}{c_1 \varphi + c_2} - \frac{\varphi_{xx}}{2\varphi_x} && (6.11)\\
\psi(\varphi) &= (c_1 \varphi + c_2) \varphi_x^{-\frac{1}{2}} && (6.12)\\
\varphi(\omega) &= \frac{c_1(\omega_2 - \omega_1) + c_2(\omega_3 - \omega_1)}{c_3(\omega_2 - \omega_1) + c_4(\omega_3 - \omega_1)},\ c_1 c_4 - c_2 c_3 = 1 && (6.13)\\
\psi(\omega) &= c_1 \psi_1 + c_2 \psi_2,\ \psi_1^2 = \frac{\omega_2 - \omega_3}{(\omega_2 - \omega_1)(\omega_3 - \omega_1)},\ \psi_2 = \psi_1 \frac{\omega_3 - \omega_1}{\omega_3 - \omega_2} && (6.14)\\
\varphi(\psi) &= \frac{c_1 \psi_1 + c_2 \psi_2}{c_3 \psi_1 + c_4 \psi_2},\ c_1 c_4 - c_2 c_3 = 1 && (6.15)\\
\omega(\psi) &= \frac{c_1 \psi_{1,x} + c_2 \psi_{2,x}}{c_1 \psi_1 + c_2 \psi_2} && (6.16)
\end{aligned}
$$

where the c_i 's are arbitrary constants, ω_i and ψ_i particular solutions of (6.7) and (6.8).

Only two of these six solutions, namely $\chi(\varphi)$ and $\psi(\varphi)$, eq. (6.11)–(6.12), are expressed with a single function; therefore, among the three equivalent functions, φ is the most elementary one, and we are going to see that the two others, $\chi(\varphi)$ and $\psi(\varphi)$, are the basic building blocks of the invariant Painlevé analysis of both PDEs and ODEs.

If the space of independent variables is multidimensional, for each additional independent variable t let us define a function $C(x, t, \ldots)$ by :

$$-\frac{\varphi_t}{\varphi_x} = C. \tag{6.17}$$

As seen from eq. (6.11)–(6.12), the t dependence of the three equivalent functions is then characterized by the three equivalent *linear* PDEs (two homogeneous, one inhomogeneous) :

$$\varphi_t + C\varphi_x = 0 \tag{6.18}$$

$$\omega = \chi^{-1},\ \omega_t + (C\omega - \frac{1}{2}C_x)_x = 0 \tag{6.19}$$

$$\psi_t + C\psi_x - \frac{1}{2}C_x\psi = 0. \tag{6.20}$$

The linearity of these PDEs, as well as the invariance of C under the change of function $\varphi \to F(\varphi)$, F arbitrary, show that all independent variables but one give rise to *linear* equations.

Systems (6.6)–(6.8) and (6.18)–(6.20) require the cross-derivative condition :

$$\varphi_x^{-1}((\varphi_{xxx})_t - (\varphi_t)_{xxx}) = 2((\chi^{-1})_t)_x - 2((\chi^{-1})_x)_t \tag{6.21}$$

$$= 2\psi^{-1}((\psi_t)_{xx} - (\psi_{xx})_t) = S_t + C_{xxx} + 2C_xS + CS_x = 0. \tag{6.22}$$

6.4 Optimal choice of the expansion variable

A PDE has movable singularities which are not isolated, on the contrary to an ODE, but which lay on a codimension one manifold

$$\varphi(x, t, \ldots) - \varphi_0 = 0, \tag{6.23}$$

in which φ is an arbitrary function of the independent variables and φ_0 an arbitrary movable constant. Even in the ODE case, the movable singularity can be defined as $\varphi(x) - \varphi_0$, since the implicit functions theorem allows this to be inverted to $x - x_0 = 0$; this provides a gaude freedom to be used later on in chapter 7.

The singular manifold and the expansion variable play two different roles, and there is no *a priori* reason to confuse them, so let us denote φ the function which defines the movable singular manifold $\varphi - \varphi_0 = 0$, and χ the expansion variable. The only requirement on χ is that it must vanish as $\varphi - \varphi_0$ and be a single valued function of $\varphi - \varphi_0$ and its derivatives.

The Laurent series for u and E are defined as

$$u = \sum_{j=0}^{+\infty} u_j \chi^{j+p}, \ -p \in \mathcal{N} \tag{6.24}$$

$$E = \sum_{j=0}^{+\infty} E_j \chi^{j+q}, \ -q \in \mathcal{N}^* \tag{6.25}$$

To illustrate our point, let us take as an example the Korteweg-de Vries equation

$$E \equiv -u_t + u_{xxx} + 6uu_x = 0 \tag{6.26}$$

(this is one of the very rare locations where this equation can be taken as an example; indeed, usually, things work so nicely for KdV that it is hazardous to draw general conclusions from its single study).

With the choice $\chi = \varphi - \varphi_0$ [111], the coefficients (u_j, E_j) are invariant under the two-parameter group of translations $\varphi \to \varphi + b, \ b$ arbitrary complex constant, and therefore they only depend on the differential invariant φ_x of this group and its derivatives :

$$u = -2\varphi_x^2\chi^{-2} + 2\varphi_{xx}\chi^{-1} + \frac{\varphi_t}{6\varphi_x} - \frac{2}{3}\frac{\varphi_{xxx}}{\varphi_x} + \frac{1}{2}\left[\frac{\varphi_{xx}}{\varphi_x}\right]^2 + O(\chi). \tag{6.27}$$

[The quantity $C = -\frac{\varphi_t}{\varphi_x}$ is invariant under $\varphi \to F(\varphi), \ F$ arbitrary function, and therefore is uninteresting for the moment.]

With the choice $\chi = (\varphi - \varphi_0)/\varphi_x$, always possible since the gradient of φ has at least one nonzero component, the invariance is extended to the four-parameter group of affine transformations $\varphi \to a\varphi + b, \ (a, b)$ arbitrary complex constants, with accordingly a dependence on the differential invariant φ_{xx}/φ_x and its derivatives :

$$u = -2\chi^{-2} + 2\frac{\varphi_{xx}}{\varphi_x}\chi^{-1} + \frac{\varphi_t}{6\varphi_x} - \frac{2}{3}\left[\frac{\varphi_{xx}}{\varphi_x}\right]_x - \frac{1}{6}\left[\frac{\varphi_{xx}}{\varphi_x}\right]^2 + O(\chi). \tag{6.28}$$

Let us extend this invariance to the six-parameter homographic group.

Eliminating φ_0 between χ and χ_x for each of the two choices of χ, one obtains the ODEs of order one for χ

$$\chi_x - \varphi_x = 0, \ \chi = \varphi - \varphi_0 \tag{6.29}$$

$$\frac{1 - \chi_x}{\chi} - \frac{\varphi_{xx}}{\varphi_x} = 0, \ \chi = \frac{\varphi - \varphi_0}{\varphi_x}, \tag{6.30}$$

whose coefficients only depend on the respective differential invariants. Now, one knows since S. Lie the differential invariant of the homographic group

$$S = \{\varphi; x\} = \left[\frac{\varphi_{xx}}{\varphi_x}\right]_x - \frac{1}{2}\left[\frac{\varphi_{xx}}{\varphi_x}\right]^2 \tag{6.31}$$

and *ipso facto* the associated ODE of order one. Its general solution leads, by taking the homographic transform which vanishes as $\varphi - \varphi_0$, to *the good expansion variable*

$$\chi = \frac{\varphi - \varphi_0}{\varphi_x - \frac{\varphi_{xx}}{2\varphi_x}(\varphi - \varphi_0)} = \left[\frac{\varphi_x}{\varphi - \varphi_0} - \frac{\varphi_{xx}}{2\varphi_x}\right]^{-1}. \tag{6.32}$$

Check : due to the homographic dependence of χ on φ, grad χ is a polynomial of degree two in χ with coefficients homographic invariants. Denoting t an arbitrary independent variable, possibly equal to x, one obtains

$$\chi_t = -C + C_x\chi - \frac{1}{2}(CS + C_{xx})\chi^2 \tag{6.33}$$

$$\chi_x = 1 + \frac{S}{2}\chi^2 \tag{6.34}$$

$$(\text{Log } \psi)_t = -C\chi^{-1} + \frac{1}{2}C_x = -C(\text{ Log } \psi)_x + \frac{1}{2}C_x \tag{6.35}$$

Again, eq. (6.33)–(6.34) are not different from eq. (6.7), (6.19).

The only price to pay for invariance is to privilege some coordinate x.

For our KdV example, the final Laurent series, to be compared with the initial one (6.27), is remarkably simple :

$$u = -2\chi^{-2} - \frac{C}{6} - \frac{2S}{3} + O(\chi). \tag{6.36}$$

The successive values of χ and the corresponding subgroup items are gathered in the following table.

Group	Invariant I	Riccati(χ)	Solution for χ
$\varphi + b$	φ_x	$\chi_x = I$	$\varphi - \varphi_0$
$a\varphi + b$	φ_{xx}/φ_x	$\chi_x = 1 - I\chi$	$(\varphi - \varphi_0)/\varphi_x$
$(a\varphi + b)/(c\varphi + d)$	$\{\varphi; x\}$	$\chi_x = 1 + (I/2)\chi^2$	$\frac{\varphi - \varphi_0}{\varphi_x - \frac{\varphi_{xx}}{2\varphi_x}(\varphi - \varphi_0)}$

Kruskal's choice

Kruskal [68] indicated the very simple choice $\chi = x - f(t, \ldots)$ of expansion variable to make the practical computations as short as possible. This choice is equivalent in our formalism to a choice of gauge, namely $S = 0, C_x = 0$, and φ is then an arbitrary homographic function of $x - f(t, \ldots)$ with constant coefficients. The choice of Kruskal is really a choice of the expansion variable, *not* of the singular manifold, i.e. $\chi_x = 1,$ not $\varphi_x = 1$.

Caution : this choice should only be used at the stage of building necessary conditions for the PP, and never at the stage of sufficiency because of the constraints put on (S, C).

6.5 Unified invariant Painlevé analysis (ODEs, PDEs)

This is a reference section containing all the items of that version of Painlevé analysis which is common to ODEs and PDEs and which generates the simplest possible expressions, due to its built-in invariance.

Consider a DE

$$\mathbf{E}(\mathbf{u}, \mathbf{x}) = 0 \tag{6.37}$$

polynomial in $\mathbf{u}$ and its derivatives, analytic in $\mathbf{x}$ ($\mathbf{E}, \mathbf{u}, \mathbf{x}$ multidimensional), and the Laurent series for $\mathbf{u}$ and $\mathbf{E}$ around the movable singular manifold $\varphi - \varphi_0 = 0$:

$$\mathbf{u} = \mathbf{u}_{-\mathbf{p},1} \operatorname{Log} \psi + \sum_{j=0}^{+\infty} \mathbf{u}_j \chi^{j+\mathbf{p}}, \ -\mathbf{p} \in \mathcal{Z} \tag{6.38}$$

$$\mathbf{E} = \sum_{j=0}^{+\infty} \mathbf{E}_j \chi^{j+\mathbf{q}}, \ -\mathbf{q} \in \mathcal{Z} \tag{6.39}$$

The coefficient $\mathbf{u}_{-\mathbf{p},1}$ can be nonzero only if $\mathbf{E}$ does not explicitly depend on $\mathbf{u}$. Let us denote x any independent variable such that $\varphi_x \neq 0$. In order to establish the most general formulae, we need two other independent variables, t and y. The gradient of expansion variables χ and ψ is (auxiliary notation is $\omega = \chi^{-1}$) :

$$\chi_x = 1 + \frac{S}{2}\chi^2 \tag{6.40}$$

$$\chi_t = -C + C_x\chi - \frac{1}{2}(CS + C_{xx})\chi^2 \tag{6.41}$$

$$\chi_y = -K + K_x\chi - \frac{1}{2}(KS + K_{xx})\chi^2 \tag{6.42}$$

$$(\operatorname{Log}\psi)_x = \chi^{-1} \tag{6.43}$$

$$(\operatorname{Log}\psi)_t = -C\chi^{-1} + \frac{1}{2}C_x = -C(\operatorname{Log}\psi)_x + \frac{1}{2}C_x \tag{6.44}$$

$$(\operatorname{Log}\psi)_y = -K\chi^{-1} + \frac{1}{2}K_x = -K(\operatorname{Log}\psi)_x + \frac{1}{2}K_x \tag{6.45}$$

$$\omega_x = -\omega^2 - \frac{S}{2} \tag{6.46}$$

$$\omega_t = C\omega^2 - C_x\omega + \frac{1}{2}(CS + C_{xx}) = (-C\omega + \frac{1}{2}C_x)_x \quad (6.47)$$
$$\omega_y = K\omega^2 - K_x\omega + \frac{1}{2}(KS + K_{xx}) = (-K\omega + \frac{1}{2}K_x)_x \quad (6.48)$$

(note that eq. (6.41) generates the eight others) where S, C, K are elementary homographic differential invariants linked by the cross-derivative conditions :

$$\varphi_x^{-1}\left((\varphi_{xxx})_t - (\varphi_t)_{xxx}\right) = S_t + C_{xxx} + 2C_xS + CS_x = 0 \quad (6.49)$$
$$\varphi_x^{-1}\left((\varphi_{xxx})_y - (\varphi_y)_{xxx}\right) = S_y + K_{xxx} + 2K_xS + KS_x = 0 \quad (6.50)$$
$$\varphi_x^{-1}\left((\varphi_y)_t - (\varphi_t)_y\right) = C_y - K_t + C_xK - CK_x = 0. \quad (6.51)$$

Kruskal's choice is implemented by putting $S = 0, C = f_t, K = f_y, \dots$ in eq. (6.40)–(6.48), thus reducing each rhs to one term and making eq. (6.49)–(6.51) useless.

The function $\varphi - \varphi_0$ never appears in the above formulae. Similarly, the explicit expressions of χ, ψ, S, C, K as functions of $\varphi - \varphi_0$ are *not* needed during the computations. We recall them here only for reference :

$$\chi = \left(\frac{\varphi_x}{\varphi - \varphi_0} - \frac{\varphi_{xx}}{2\varphi_x}\right)^{-1} \quad (6.52)$$
$$\psi = (\varphi - \varphi_0)\varphi_x^{-\frac{1}{2}} \quad (6.53)$$
$$S = \{\varphi; x\} = \frac{\varphi_{xxx}}{\varphi_x} - \frac{3}{2}\left(\frac{\varphi_{xx}}{\varphi_x}\right)^2 = \left(\frac{\varphi_{xx}}{\varphi_x}\right)_x - \frac{1}{2}\left(\frac{\varphi_{xx}}{\varphi_x}\right)^2 \quad (6.54)$$
$$= -2\left(\frac{\varphi_x}{\varphi - \varphi_0} - \frac{\varphi_{xx}}{2\varphi_x}\right)_x - 2\left(\frac{\varphi_x}{\varphi - \varphi_0} - \frac{\varphi_{xx}}{2\varphi_x}\right)^2, \quad (6.55)$$
$$C = -\frac{\varphi_t}{\varphi_x} \quad (6.56)$$
$$K = -\frac{\varphi_y}{\varphi_x}. \quad (6.57)$$

In some applications, it is necessary to choose for χ the most general homographic transform of (6.52) which vanishes as $\varphi - \varphi_0$

$$\operatorname{grad}\chi = \mathbf{X}_0 + \mathbf{X}_1\chi + \mathbf{X}_2\chi^2 , \quad (6.58)$$
$$\operatorname{grad}\omega = -\mathbf{X}_2 - \mathbf{X}_1\omega - \mathbf{X}_0\omega^2 \quad (6.59)$$
$$\operatorname{grad}\operatorname{Log}\psi = \mathbf{X}_0\chi^{-1} + \frac{1}{2}\mathbf{X}_1. \quad (6.60)$$

The vectorial coefficients $\mathbf{X}_i$ depend on $(S, C, K, \dots)$ and two additional arbitrary functions. The auxiliary expansion variable ψ is defined by its logarithmic gradient and by the condition that it should vanish as $\varphi - \varphi_0$.

6.6 The Painlevé test

The synthesis of the different methods to generate necessary conditions for the PP produces the following algorithm, called "Painlevé test".

Consider a DE (2.30) of order N, already transformed if necessary, see sections 6.1 and 6.2, so as to be polynomial in $\mathbf{u}$ and its derivatives, analytic in x. The Painlevé test is made of the following steps.

Step 0. Perform a transformation (3.5) in order to reduce the number of terms in the equation (details in section 5.3.1 and ref. M.I and, for PDEs, [39]).

Ex. : equation $u_x+u_t+u_{xxt}+u_x u_t = 0$, under the translation $u = U-x-t$, becomes $U_{xxt} + U_x U_t + 1 = 0$.

Step 1. Require the satisfaction of the very general necessary conditions obtained by Painlevé (details in section 5.5.2).

Ex. [74] : $-3u^2u'u''' + 5u^2u''^2 - uu'^2u'' - u'^4 = 0$. Unstable for 5/3 has not the required value $1 - 1/n$.

Ex. [25] : $(1+u^2)u_{xx} - 2uu_x^2 + u_t^2 = 0$. The ODE obtained by the reduction $(x,t) \to x-ct$ has an A with two simple poles $u = \pm i$ and residues $1 \pm ic^2/2$. The ODE is unstable, and so is the PDE.

Step 2. If the degree is greater than one, establish the ODE satisfied by the singular solutions (details in section 5.1).

Step 3. Put the DE under a canonical form of Cauchy; find all the exceptional points where the Cauchy theorem fails; for each such point, define a homographic transformation (3.5) allowing the Cauchy theorem to apply (details in sections 5.3 and 5.5). For each DE (the original one and all these homographic transforms), perform step 4.

Ex. : (P5) has the exceptional points $u = 1$ and $u = 0$ (poles of A, see section 5.5.2). These points are regular for the ODE in $(u-1)^{-1}$ and u^{-1}.

Ex. : the reduced three-wave interaction dynamical system

$$x' = -2y^2 + z + \gamma x + \delta y,\ y' = 2xy + \gamma y - \delta x,\ z' = -2xz - 2z$$

has the exceptional point $y = \delta/2$ [11], not so evident on the system itself but easily unveiled by considering the equivalent third order ODE for $y(t)$.

Step 4. Find all the families $\mathbf{u} \sim \mathbf{u}_0^{(0)} \chi^{\mathbf{p}}$ ($\mathbf{u}_0^{(0)} \neq \mathbf{0}$) (details in section 5.4). Discard those families which are also families of the ODE for singular solutions established at step 2. Require all components of remaining $\mathbf{p}$'s to be integer. Discard all families having all components of $\mathbf{p}$ positive. For each remaining family, perform step 5 and at least one of the steps 6 and 7.

Ex. : (P5) has six families of movable simple pole-like singularities

$$u \sim \pm(2\alpha)^{-\frac{1}{2}} x\chi^{-1},\ u^{-1} \sim \pm(-2\beta)^{-\frac{1}{2}} x\chi^{-1},\ (u-1)^{-1} \sim \pm(-2\delta)^{-\frac{1}{2}}\chi^{-1}. \tag{6.61}$$

Ex. : the reduced three-wave interaction has the families [11]

$$(x,y,z) \sim (-(1/2)\chi^{-1}, (i/2)\chi^{-1}, z_0),\ z_0 \text{ arbitrary},\ \text{indices } (-1,0,2)$$
$$(x,y,z) \sim (\chi^{-1}, \delta/2, -\chi^{-2}),\ \text{indices } (-1,2,2).$$

The first one will pass the test while the second will generate at index 2 the conditions $\gamma\delta = 0, \gamma(\gamma+1) = 0$.

In case the DE has too many terms, this step is worth being programmed on a computer, by fear of missing some families.

Warning. If one is unsure about some component u of $\mathbf{u}$ behaving like a positive integer power p of χ, it may be safer to switch to the DE for u^{-1}.

Step 5. From the auxiliary equation of the simplified equation, compute the linear operator $\mathbf{P}(i)$ eq. (2.36) and the indicial equation (2.36) $\det \mathbf{P}(i) = 0$ (details in section 5.4). Compute its zeroes (the Fuchs indices). Require each index to be integer (details in section 5.11) and to satisfy the rank condition (5.113).

Ex. ([31] example 5.B). These are two coupled PDEs with a single family whose linear operator $\mathbf{P}(i)$ is

$$\mathbf{P}(i) = \begin{pmatrix} -\frac{1}{3}(i+2)^2 & \frac{1}{3}(i+2) \\ -(i+2) & i^2 \end{pmatrix}. \tag{6.62}$$

The indices are the zeroes of its determinant $(-2,-2,-1,1)$. For the double index $i=-2$, the rank of $\mathbf{P}(i)$ is one, so the system of PDEs is unstable.

Ex. [22, 15, 52] : the equation $u_{xxx} - 7uu_{xx} + 11u_x^2 = 0$ has only one family with three indices : $p=-1, u_0^{(0)} = -2$, indices $(-6,-1,-1)$. The double index -1 immediately proves the instability.

Step 6. (NonFuchsian case). If the degree of the indicial polynomial is strictly lower than N, and if a particular solution is known in closed form, apply the NonFuchsian perturbative method (details in section 5.8).

Step 7. (Fuchsian case). Denote ρ the smallest integer Fuchs index, lower than or equal to -1. Define two positive integer upper bounds $k_{\max}$ and $n_{\max}$ representing the cost of the computation to come, see advice below. Solve the linear algebraic system (5.118) in the unknown $\mathbf{u}_j^{(n)}, (j,n) \neq (0,0)$, for the successive values $k=0$ to $k_{\max}, n=0$ to $n_{\max}$ with $j = k+n\rho$; whenever j is an index i of multiplicity $\mathrm{mult}(i)$,

- require the orthogonality condition (5.119) to be satisfied for any value of the previously introduced arbitrary coefficients,
- if $n = 0$ or ($n = 1$ and $i < 0$), assign arbitrary values to mult(i) components of $\mathbf{u}_i^{(n)}$ defining a basis of Ker $\mathbf{P}(i)$,
- if ($n = 1$ and $i \geq 0$) or $n \geq 2$, assign the value 0 to mult(i) components of $\mathbf{u}_i^{(n)}$ defining a basis of Ker $\mathbf{P}(i)$.

Details in section 5.7.

Advice for choosing $k_{\max}$ and $n_{\max}$: if the order $n = 0$ fails to describe the general solution, take at least $n_{\max} = 2$; take $k_{\max}$ so as to test the greatest Fuchs index for $n = n_{\max}$ (all details in the remarks at the end of section 5.7).

This ends the test. Step 6 has been put before step 7 because in all our examples it allows to conclude sooner.

Let us again stress that these sets of conditions may not be sufficient : Painlevé gave the counterexample of the second order ODE whose general solution is $\pm \operatorname{sn}[\lambda \operatorname{Log}(c_1 x + c_2); k)]$, with (c_1, c_2) arbitrary, for which no local test can generate the necessary and sufficient stability condition that $2\pi i\lambda$ be a period of the elliptic function sn. For advanced features, see section 5.9 and [41].

6.7 The partial Painlevé test

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In the search for the tiniest piece of integrability, the physicist, see section 3.7, will perform the above Painlevé test to its end, i. e. without stopping even in case of failure of some condition, so as to collect a bunch of necessary conditions.

Turning to sufficiency, these conditions will then be examined separately in the hope of finding some global element of integrability, most often a Darboux eigenvector.

For instance, the Lorenz model (1.3) has two families

$$x \sim 2i\chi^{-1},\ y \sim -(2i/\sigma)\chi^{-2},\ z \sim -(2/\sigma)\chi^{-2},\ i^2 = -1, \qquad (6.63)$$

with the same indices $(-1, 2, 4)$, which generate the no-log conditions [106, 32]

$$\begin{aligned}
&Q_2 \equiv (8/3)(b - 2\sigma)(b + 3\sigma - 1) = 0 \\
&Q_4 \equiv -4i(b - \sigma - 1)(b - 6\sigma + 2)x_2 + (8/3)(b - 1)(b - 3\sigma + 1)S \\
&-4b\sigma(b - 3\sigma + 5)r + f(b, \sigma) = 0,
\end{aligned}$$

in which x_2 is arbitrary, S is the Schwarzian of the invariant analysis, and f a polynomial irrelevant for what we want to emphasize. Performing a logical *or* operation on these conditions instead of the logical *and* of the mathematician, one obtains the condition on (b, σ)

$$(b-2\sigma)(b+3\sigma-1)(b-\sigma-1)(b-6\sigma+2)(b-1)(b-3\sigma+1) = 0 \quad (6.64)$$

What is remarkable is that *all* known analytic results on this model (first integrals [76], particular solutions [32], Darboux eigenvectors [48, 77]) belong to one of these six cases. Conversely, to each of the six factors there corresponds such a result, although sometimes only for a finite set of values of (b, σ).

Remarks.

1. With the restriction $S = 0$ one would miss two of the six factors.

2. First integrals $P(x, y, x)e^{\lambda t}$, with P polynomial and λ constant, should not be searched for with the assumption P the most general polynomial in three variables. Indeed, P must be an entire function of t i. e. have no singularities at a finite distance. The generating function of such polynomials is built from the singularity degrees of (x, y, z) [81]

   $$\frac{1}{(1-\alpha x)(1-\alpha^2 y)(1-\alpha^2 z)} \quad (6.65)$$

   and it provides the basis, ordered by singularity degrees

   $$(1),\ (x),\ (x^2, y, z),\ (x^3, xy, xz),\ (x^4, x^2 y, x^2 z, yz, z^2, y^2),\ . \quad (6.66)$$

   Thus, P_2 should be searched for as a linear combination of $(1, x, x^2, y, z)$. All known first integrals are found at the P_4 level [76].

3. The case $b = 1 - 3\sigma$ is on an equal footing with the case $b = 2\sigma$ which admits the first integral $(x^2 - 2\sigma z)e^{2\sigma t}$, but finding its first integral is still an open problem.

Chapter 7

Sufficiency : explicit integration methods

We review the algorithmic methods which *may* perform the explicit integration, with emphasis on ODEs. The PDE case is handled in another part of this volume [84].

We assume that the application of the Painlevé test (necessary conditions for the PP) has led either to no failure or to a minor failure, corresponding respectively to a presumption of integrability in the Painlevé sense or of partial integrability. If perturbative methods have been used, one has to decide to give up at some perturbation order n (remember the counterexample of Painlevé). The goal is then either to prove the sufficiency (integrability) or to build particular solutions (partial integrability).

If the DE belongs to one of the fully studied classes enumerated in chapter 4, the question is solved. Indeed, either it is possible, by some homographic transformation (3.5), to bring the DE back to a normalized ("classified") DE, in which case the integration is finished, or this is impossible, in which case the DE has not the PP.

For a DE which has not been classified, if one excludes the case where the DE is an ODE and defines a new function (a quite improbable event which has not occured since 1906), the explicit proof of sufficiency amounts to (the cases below are not mutually exclusive)

- either (ODE case) express the general solution as a finite expression of a finite number of elementary functions (solutions of linear equations, the Weierstrass $\wp$ function, the six Painlevé functions),
- or (PDE case) find a Lax pair.

In the partial integrability situation, one tries to obtain degeneracies of these results : a particular solution or a pair of linear operators able to generate a subclass of solutions.

The methods to handle both cases are the same, and they again only rely on the singularity structure. Their basic common idea is that the singular part of the Laurent expansions (of a *local* nature) contains all the information for a *global* knowledge of the solution.

The two existing methods are known as the *singular part transformation* and the *truncation method.* Before describing them, let us give a few definitions and explain how Painlevé proved the sufficiency for the six equations (P1)–(P6).

7.1 Sufficiency for the six Painlevé equations

Painlevé introduced the concept of "intégration parfaite" and used it to solve the question of sufficiency for the six equations discovered by himself and Gambier. The idea is to perform a finite (in the sense of Poincaré : finite expression) single valued transformation from (Pn) to another ODE which has no more movable singularities although it may still have fixed critical singularities. Such an ODE has qualitatively the same singularities than a linear ODE, and Painlevé says that its integration is then "parfaite" (achieved) (BSMF p. 205) : given any initial conditions, its solution can be computed with an arbitrary accuracy (by e.g. the sequence of coefficients of convergent Taylor series) since one knows in advance where the remaining (fixed) singularities are located. The movable singularities of the original ODE are then totally under control. The equations with fixed critical points therefore constitute a natural extension to the linear equations.

Painlevé defined such transformations (nowadays called "singular part transformations") for each of the six equations (P1)–(P6). These transformations, *via* logarithmic derivatives, transform (P1)–(P6) into equations for ψ without movable singularities ((P1) Acta p. 14, (P2) Acta p. 15, (P3) Acta p. 16, (P4,P5,P6) CRAS 1906, Œuvres III p. 120)

$$\text{(P1)} \qquad u = -\partial_x^2 \operatorname{Log} \psi \tag{7.1}$$
$$\text{(P2)} \qquad u = \partial_x \operatorname{Log} \psi_1 - \partial_x \operatorname{Log} \psi_2 \tag{7.2}$$
$$\text{(P3)} \qquad u = e^{-x}(\partial_x \operatorname{Log} \psi_1 - \partial_x \operatorname{Log} \psi_2) \tag{7.3}$$
$$\text{(P4)} \qquad u = \partial_x \operatorname{Log} \psi_1 - \partial_x \operatorname{Log} \psi_2 \tag{7.4}$$
$$\text{(P5)} \qquad u = xe^{-x}(2\alpha)^{-1/2}(\partial_x \operatorname{Log} \psi_1 - \partial_x \operatorname{Log} \psi_2). \tag{7.5}$$
$$\text{(P6)} \qquad u = x(x-1)e^{-x}(2\alpha)^{-1/2}(\partial_x \operatorname{Log} \psi_1 - \partial_x \operatorname{Log} \psi_2). \tag{7.6}$$

The Lax pairs of (P1)–(P6) can be found in Ref. [58] and [69].

The two methods developed in next sections (7.3) and (7.4) rely on this result.

The *logarithmic derivative* plays a privileged role, as generator of a movable simple pole with a residue generically unity. A prerequisite to the algorithmic derivation of a transformation from u to ψ such as (7.1) is the introduction of a free gauge function which we denote φ.

Such a gauge naturally arises if one thinks of an ODE as the canonical reduction of a PDE defined by suppressing the dependence upon all independent variables but x. This is the function φ used in the description of the movable singularities by (6.23) rather than $x - x_0 = 0$. Useless at the stage of building necessary conditions (the Painlevé test), this feature is the key to the algorithmic explicit integration methods.

7.2 The singular part(s)

Definition. The *singular part* of one of the families of movable singularities of a given DE is the finite sum of the Laurent series restricted to the nonpositive powers in the method of pole-like expansions

$$u_T = \sum_{j=0}^{-p} u_j \chi^{j+p}. \tag{7.7}$$

Synonyms are : truncation, truncated expansion.

Given φ, the singular part u_T is a one-parameter (φ_0) family of expressions $u_T(\varphi_0)$, and the two particular values $\varphi_0 = 0$ and $\varphi_0 = \infty$ are of special interest. For the example of KdV (6.26)

$$\begin{aligned} u_T(0) &= -2\left[\frac{\varphi_x}{\varphi} - \frac{\varphi_{xx}}{2\varphi_x}\right]^2 + \frac{\varphi_t}{6\varphi_x} - \frac{2}{3}\left(\frac{\varphi_{xxx}}{\varphi_x} - \frac{3}{2}\left[\frac{\varphi_{xx}}{\varphi_x}\right]^2\right) & (7.8) \\ u_T(\infty) &= -2\left[\quad - \frac{\varphi_{xx}}{2\varphi_x}\right]^2 + \frac{\varphi_t}{6\varphi_x} - \frac{2}{3}\left(\frac{\varphi_{xxx}}{\varphi_x} - \frac{3}{2}\left[\frac{\varphi_{xx}}{\varphi_x}\right]^2\right). & (7.9) \end{aligned}$$

Definition. The *singular part operator* $\mathcal{D}$ of a family is defined by

$$\operatorname{Log}\varphi \to \mathcal{D}\operatorname{Log}\varphi = u_T(0) - u_T(\infty). \tag{7.10}$$

Example 1 (KdV). The operator $\mathcal{D}$ is linear and equal to $2\partial_x^2$. This linearity is strongly linked with the Darboux transformation [84].

Example 2. The single family of (P1) and the two families of (P2) have the singular parts

$$\begin{aligned} (\text{P1}): \ u_T &= \chi^{-2} + \frac{S}{3},\ \mathcal{D} = -\partial_x^2 & (7.11) \\ (\text{P2}): \ u_T &= \pm\chi^{-1},\ \mathcal{D} = \pm\partial_x. & (7.12) \end{aligned}$$

7.3 Method of the singular part transformation

This is the method used by Painlevé and outlined in previous section 7.1. It consists of transforming the DE for u into a DE for φ by the nonlinear transformation

$$u = \mathcal{D} \operatorname{Log} \varphi, \tag{7.13}$$

where $\mathcal{D}$ is the singular part operator associated to one of the families of the equation for u.

If the transformed equation for φ can be integrated, so is the original equation.

Example 1 (linearization). The unique first order first degree ODE with the PP, namely the Riccati equation (1.1), has a $\mathcal{D}$ operator equal to $-a_2^{-1}\partial_x$, computable from the basic formulae (6.40), (6.52) and (7.10). The transformation $u = -a_2^{-1}\partial_x \operatorname{Log} \varphi$ from u to φ leads to the second order linear equation (1.2) for φ. It is then sufficient to know two particular solutions φ_1 and φ_2 (which *are* functions) of this linear equation to have a global knowledge of the general solution of the Riccati equation by the formula

$$u = -a_2^{-1}\partial_x \operatorname{Log}(c_1\varphi_1 + c_2\varphi_2). \tag{7.14}$$

Similarly, the transformation $\wp = -\partial_x^2 \operatorname{Log} \sigma$ associates to the Weierstrass elliptic function $\wp$ a function σ which is an entire function, solution of a nonlinear ODE.

Example 2 (simplified equation of one of the 50 stable ODEs (4.8)). The ODE

$$E \equiv u'' + uu' - u^3 = 0 \tag{7.15}$$

possesses two families of movable simple poles $u_0 = 1$ and $u_0 = -2$, with the one-parameter particular solutions $u_0/(x - x_0)$. The first family operator is $\mathcal{D} = \partial_x$ and it transforms it into

$$u = \partial_x \operatorname{Log} \varphi,\ E \equiv \varphi \left(\frac{\varphi''}{\varphi^2}\right)' = 0, \tag{7.16}$$

which integrates as $\varphi = a\wp(x - x_0, 0, g_3)$ with (a, x_0, g_3) arbitrary and provides the general solution. The two families of movable simple poles for u correspond to the movable simple zeroes of $\wp$ (residue $u_0 = 1$) and to the movable double poles of $\wp$ (residue $u_0 = -2$).

Example 3 (indirect linearization). The Ermakov-Pinney equation (6.4), after the transformation $u^{-2} = v$ removing its algebraic singularity

$$E \equiv -\frac{1}{2}vv_{xx} + \frac{3}{4}v_x^2 - \alpha^2 v^2 + \beta^2 v^4 = 0, \tag{7.17}$$

has two families $v \sim \pm(2\beta)^{-1}\chi^{-1}$, and the transformed ODE under $v = (2\beta)^{-1}\partial_x \operatorname{Log}\varphi$ [30]

$$\frac{\varphi_{xxx}}{\varphi_x} - \frac{3}{2}\left(\frac{\varphi_{xx}}{\varphi_x}\right)^2 = -2\alpha^2 \tag{7.18}$$

is a Schwarz ODE (6.6). This integrates the Ermakov-Pinney equation *via* a finite two-valued expression.

Well suited to DEs possessing only one family (Riccati, Weierstrass, (P1), KdV), this transformation must be adapted, following the Painlevé formulae for (P2)–(P6) in section 7.1, to suit DEs with more than one family (Jacobi elliptic equation, (P2) to (P6)). This is done in the course on PDEs [84].

7.4 Method of truncation (Darboux transformation)

Perfectly adapted to PDEs [84], this method is rather poor for ODEs, for an intrinsic reason which is the absence in this case of a Bäcklund transformation (link between two different solutions of the same DE introducing at each iteration at least one more arbitrary parameter in the solution). It nevertheless succeeds, at least partially, in many situations.

The idea [111] is to consider the singular part (7.7) of one family (or the sum $u = \sum_f \mathcal{D} \operatorname{Log}\psi_f$ of the singular parts of several families) as a *parametric representation* of a solution in terms of one function ψ linked to χ by $\chi^{-1} = \partial_x \operatorname{Log}\psi$ (or several functions ψ_f, one per family f). Every function ψ_f, which defines a singular manifold $\psi_f = 0$, is required to be an entire function, and for instance to satisfy the *same* linear system of two PDEs $L_1 = 0, L_2 = 0$ with some adjustable coefficients.

The method consists of identifying to zero the lhs $E(u_T)$ considered as a polynomial of ψ_f and its independent derivatives modulo the constraint that each ψ_f satisfies the linear system. This generates an overdetermined set of *determining equations* whose unknowns are the coefficients u_j of (7.7) and the coefficients of the linear system. The remarkable fact is that the determining equations are easy to solve.

The result is some class of exact solutions, and this class is easily interpreted. If the commutator $[L_1, L_2]$ is identically zero (which is always the case if the linear system has constant coefficients), the solutions are particular ones (PDE case) or any kind (particular or general) (ODE case). If this commutator is zero only when some coefficient of (L_1, L_2) satisfies some PDE, quite probably (L_1, L_2) define a Lax pair.

Again, the ODE case to which we restrict is much less rich than the PDE case [84] to which we refer the reader.

7.4.1 One-family truncation

This is the celebrated WTC truncation procedure [111]. Applicable to any DE with any number of families, it consists of selecting one of the families $\psi = 0$, in which ψ obeys the linear system of the invariant analysis

$$\psi_{xx} + \frac{S}{2}\psi = 0, \tag{7.19}$$

$$\psi_t + C\psi_x - \frac{C_x}{2}\psi = 0. \tag{7.20}$$

The functions S et C are adjustable fonctions, only constrained by the cross-derivative condition (6.49). Consider for instance the Ermakov-Pinney ODE [49, 97]

$$E \equiv -\frac{1}{2}vv_{xx} + \frac{3}{4}v_x^2 - \alpha^2 v^2 + \beta^2 v^4 = 0. \tag{7.21}$$

The infinite Laurent series is $v = (2\beta)^{-1}\chi^{-1} + v_1 + O(\chi)$ with v_1 arbitrary and β one of the two square roots of β^2. Thanks to the gauge φ, the coefficient v_1 is not a constant but a function.

The method consists of assuming that a solution v can be represented by the truncation

$$v = v_T = \frac{1}{2\beta}\chi^{-1} + v_1 = \mathcal{D}\,\mathrm{Log}\,\psi + v_1 \tag{7.22}$$

implying for the lhs E of the DE the similar truncated expansion

$$E \equiv \sum_{j=0}^{4} E_j \chi^{j-4}. \tag{7.23}$$

This generates, in this example, five equations $E_j = 0$ in the unknowns (v_1, S). Among them, E_0 is zero since the coefficient v_0 of the series for v is already the good one. E_1 is zero since 1 is a Fuchs index whose orthogonality condition is satisfied. Denoting $v_1 = -V_1/(2\beta)$, there remain the three equations

$$16\beta^2 E_2 \equiv -4\alpha^2 + S + 6V_1^2 + 6V_{1,x} = 0, \tag{7.24}$$

$$16\beta^2 E_3 \equiv 8\alpha^2 V_1 + 2SV_1 - 4V_1^3 + S_x + 2V_{1,xx} = 0, \tag{7.25}$$

$$16\beta^2 E_4 \equiv \frac{3}{4}S^2 - 4\alpha^2 V_1^2 + V_1^4 - V_1 S_x + 3SV_{1,x} + 3V_{1,x}^2 - 2V_1 V_{1,xx} = 0. \tag{7.26}$$

The algebraic elimination (i. e. without differentiation) of $V_{1,x}$ and $V_{1,xx}$ among these three equations yields $(S-s)^2 = 0$, with $s = -2\alpha^2$, then V_1 is found to satisfy the Riccati equation

$$-2V_{1,x} - 2V_1^2 = s. \tag{7.27}$$

Hence the particular solution

$$v = \frac{1}{2\beta}(\chi^{-1} - V_1), \tag{7.28}$$

in which each variable χ^{-1} and V_1 satisfies the same Riccati equation and depend on one arbitrary parameter. This is the general solution, which can be written as $v = (2\beta)^{-1}(\partial_x \operatorname{Log}\psi_1 - \partial_x \operatorname{Log}\psi_2)$ in agreement with the structure of singularities, cf. (7.2).

Remark. The class of particular solutions generically found by this method is the class of polynomials in tanh, which correspond to a constant value for S.

Another example is the (P2) equation, for which the one-family truncation $u = \chi^{-1} + u_1$ provides the one-parameter particular solution $u_1 = 0, S = x$ on the condition $\alpha = 1/2$, i. e. an algebraic transform of the Airy equation.

7.4.2 Two-family truncation

When a DE admits two families with opposite principal parts, such as (7.17), it is natural to seek particular solutions described by two singular manifolds [34]

$$v = \frac{1}{2\beta}[\partial_x \operatorname{Log}\psi_1 - \partial_x \operatorname{Log}\psi_2 + v_0], \tag{7.29}$$

in which (ψ_1, ψ_2) is a basis of the two-dimensional space of solutions of some ODE whose general solution is entire, e.g. the second order linear equation with constant coefficients

$$\psi_{xx} - \frac{k^2}{4}\psi = 0 \tag{7.30}$$

$$\Psi_2 = C_1 e^{\frac{k}{2}x} + C_2 e^{-\frac{k}{2}x} = C_0 \cosh\frac{k}{2}(x - x_0), \tag{7.31}$$

$$\psi_1(x) = \Psi_2(x + a),\ \psi_2(x) = \Psi_2(x - a),\ a \text{ arbitrary}, \tag{7.32}$$

Substituting (7.29) into (7.17) and eliminating any derivative of (ψ_1, ψ_2) of order higher than or equal to two in x results into a polynomial in the two variables $\psi_{1,x}/\psi_1, \psi_{2,x}/\psi_2$. Before identifying it to the null polynomial, one must take account of the first integral μ_0, the ratio of two constant Wronskians

$$\frac{\psi_{1,x}}{\psi_1}\frac{\psi_{2,x}}{\psi_2} = \frac{k^2}{4} - \mu_0\frac{k}{2}\left(\frac{\psi_{1,x}}{\psi_1} - \frac{\psi_{2,x}}{\psi_2}\right),\ \mu_0 = \operatorname{cotanh} ka, \tag{7.33}$$

which splits the polynomial of two variables into the sum of two polynomials in one variable :

$$16\beta^2 E \equiv (k^2 - 4\alpha^2 + 6v_0^2 + 6k\mu_0 v_0)(\left(\frac{\psi_{1,x}}{\psi_1}\right)^2 + \left(\frac{\psi_{2,x}}{\psi_2}\right)^2)$$

$$
\begin{aligned}
&+(k\mu_0(k^2-4\alpha^2+6v_0^2)+2(3k^2\mu_0^2-k^2-4\alpha^2+4v_0^2)v_0)\\
&\times(\frac{\psi_{1,x}}{\psi_1}-\frac{\psi_{2,x}}{\psi_2})\\
&+2\alpha^2k^2-\frac{k^4}{2}-3k^3\mu_0v_0-4\alpha^2v_0^2-3k^2v_0^2+v_0^4. \qquad (7.34)
\end{aligned}
$$

This defines three different algebraic equations in the unknowns (k, v_0, μ_0); their two solutions

$$k^2 = 4\alpha^2,\ v_0 = 0,\ \mu_0 \text{ arbitrary}, \qquad (7.35)$$

$$k^2 = 4\alpha^2,\ v_0 = 2\alpha,\ k\mu_0 = -2\alpha \qquad (7.36)$$

are just two different representations [34] of a solution of (7.17) depending on two arbitrary constants $(\mu_0, x_0):$ with this simple assumption, we have obtained the general solution

$$u^{-2} = v = \frac{1}{2\beta}\left[\frac{\psi_{1,x}}{\psi_1}-\frac{\psi_{2,x}}{\psi_2}\right] = \frac{\alpha}{\beta}\frac{\sinh ka}{\cosh k(x-x_0)+\cosh ka}. \qquad (7.37)$$

In particular, with $\mu_0 = 0$ one thus obtains immediately the class of solutions polynomial in the two variables tanh and sech [33], thus augmenting the class indicated at the end of previous section. Evidently, if the DE has only one family, no dependence on sech can be found.

Chapter 8

Conclusion

The solution of an ODE cannot escape the structure of singularities of the ODE. Such a structure can be studied on the equation itself, without any *a priori* knowledge of the solution, providing a deep insight on the possibility or not to perform the explicit integration.

Two levels of integrability have been defined : the Painlevé property (the most elementary level) and the integrability in the sense of Poincaré (the practical level).

A first series of methods (globally called "the Painlevé test") provide *necessary* conditions for a differential equation to have the Painlevé property, without any guarantee on the sufficiency. In case of a negative answer from these first methods, there exist other methods (*Leçons* 8, 9, 10, 13, 19), not developed here, to provide necessary conditions for the general solution to have only a finite amount of movable branching, which implies the integrability in the sense of Poincaré, a weaker property than the PP.

In case of a positive answer, the DE *may* have the PP, i.e. a general solution free from movable critical singularities. Then, a second series of methods are available to perhaps constructively prove the PP by explicitly building the general solution or some equivalent information (Lax pair). In case of failure of these second methods, the only remaining tool is human ability.

There exists another approach to DEs which is not based on the study of singularities, this is the method of infinitesimal symmetries [87, 88]. It provides reductions of PDEs to "smaller" PDEs or to ODEs, and it may provide first integrals of ODEs. However, the PDEs or ODEs left over after its completion still require to be integrated, and the only methods to do so are those based on singularities. For instance, with the ODE (P1), the method of symmetries cannot provide any information (existence or not of a first integral, single valuedness or multivaluedness).

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$$\begin{aligned}
\frac{\mathrm{d}x_1}{\mathrm{d}t} &= a_1x_2x_3 + a_2x_3x_1 + a_3x_1x_2 \\
\frac{\mathrm{d}x_2}{\mathrm{d}t} &= b_1x_2x_3 + b_2x_3x_1 + b_3x_1x_2 \\
\frac{\mathrm{d}x_3}{\mathrm{d}t} &= c_1x_2x_3 + c_2x_3x_1 + c_3x_1x_2
\end{aligned}$$

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