The Painlevé Handbook

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Les cas où l'on peut intégrer une équation différentielle sont extrêmement rares, et doivent être regardés comme des exceptions; mais on peut considérer une équation différentielle comme définissant une fonction, et se proposer d'étudier les propriétés de cette fonction sur l'équation différentielle elle-même.

Charles Briot et Jean-Claude Bouquet, 1859.

Preface

Nonlinear differential or difference equations are encountered not only in mathematics, but also in many areas of physics (evolution equations, propagation of a signal in an optical fiber), chemistry (reaction-diffusion systems) and biology (competition of species).

The purpose of this book is to introduce the reader to nonperturbative methods allowing one to build *explicit* solutions to these equations. A prerequisite task is to investigate whether the chances of success are high or low, and this can be achieved without any *a priori* knowledge of the solutions, with a powerful algorithm called the Painlevé test. If the equation under study passes the Painlevé test, the equation is presumed *integrable* in some sense, and one can try to build the explicit information displaying this integrability:

- for an ordinary differential equation, the closed form expression of the general solution;
- for a partial differential equation, the nonlinear superposition formula to build soliton solutions;

and similar elements in the discrete situation. If on the contrary the test fails, the system is nonintegrable or even chaotic, but it may still be possible to find solutions. Indeed, the methods developed for the integrable case still apply and may in principle produce all the available pieces of integrability, such as the *solitary waves* of evolution equations, or solutions describing the collision of solitary waves, or the first integrals of dynamical systems, etc.

The examples chosen to illustrate these methods are mostly taken from physics. These include on the integrable side the nonlinear Schrödinger equation (continuous and discrete), the Korteweg–de Vries equation, the Boussinesq equation, the Hénon–Heiles Hamiltonians, and on the nonintegrable side the complex Ginzburg–Landau equation (encountered in optical fibers, turbulence, etc), the Kuramoto–Sivashinsky equation (phase turbulence), the reaction-diffusion model of Kolmogorov–Petrovski–Piskunov (KPP), the Lorenz model of atmospheric circulation and the Bianchi IX cosmological model which are both chaotic.

Written at a graduate level, the book contains tutorial text as well as detailed examples and describes the state of the art in some current areas of research.

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Outline

In Chap. 1, we insist that a nonlinear equation should *not* be considered as the perturbation of a linear equation. We illustrate using two simple examples the importance of taking account of the singularity structure in the complex plane to determine the general solution of nonlinear equations. We then present the point of view of the Painlevé school to *define new functions* from nonlinear ordinary differential equations (ODEs) possessing a general solution which can be made single valued in its domain of definition (*Painlevé property*, PP).

In Chap. 2, we present a local analysis, called the *Painlevé test*, in order to investigate the nature of the movable singularities (i.e. whose location depends on the initial conditions) of the general solution of a nonlinear differential equation. The simplest of the methods involved in this test was historically introduced by Sophie Kowalevski [257] and later turned into an algorithm by Bertrand Gambier [163]. For equations possessing the Painlevé property, the test is by construction satisfied, therefore we concentrate on equations which generically fail the test, in order to extract some constructive information on cases of partial integrability. We first choose four examples describing physical phenomena, for which the test selects cases which may admit closed form particular solutions[1] or first integrals.

This procedure is illustrated in several examples.

In the first example, the *Lorenz model* of atmospheric circulation [284]

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \sigma(y-x),\ \frac{\mathrm{d}y}{\mathrm{d}t} = rx - y - xz,\ \frac{\mathrm{d}z}{\mathrm{d}t} = xy - bz,$$

the test isolates four sets of values of the parameters (b, σ, r).

We next consider the *Kuramoto–Sivashinsky equation* (KS),

$$u_t + \nu u_{xxxx} + b u_{xxx} + \mu u_{xx} + u u_x = 0,\ \nu \neq 0,$$

[1] By definition, a solution is called particular if it can be obtained from the general solution by setting some constants of integration to numerical values.

an equation which describes the propagation of flames on a vertical wall, and we analyze for simplicity the ODE for its stationary flow. The test first detects the presence of movable multivaluedness[2] in the general solution whatever the parameters (ν, b, μ), then it displays the possible existence of particular solutions without movable branching.

We then analyze the one-dimensional cubic *complex Ginzburg–Landau equation* (CGL3),

$$iA_t + pA_{xx} + q|A|^2 A - i\gamma A = 0,\ pq\gamma \neq 0,\ (A, p, q) \in \mathscr{C},\ \gamma \in \mathscr{R}.$$

This is a generic equation which describes many physical phenomena, such as the propagation of a signal in an optical fiber [10], or spatiotemporal intermittency in spatially extended dissipative systems [296]. The test first uncovers the generic non-integrable nature of this PDE, then it selects as values of the parameters (p, q, γ) those $(q/p \in \mathscr{R}, \gamma = 0)$ of the *nonlinear Schrödinger equation* (NLS), an equation which is integrable in many acceptations. Finally it shows the possible existence of particular single valued solutions in the CGL3 case $\mathrm{Im}(q/p) \neq 0$.

The next example is the *Duffing–van der Pol oscillator*

$$E(u) \equiv u'' + (au^2 + b)u' - cu + \beta u^3 = 0.$$

It is chosen to illustrate a weaker form of the test (weak Painlevé test) in which the general solution is allowed to possess more than one determination around a movable singularity, but only a finite number (weak Painlevé property), like the square root function.

The last example is the two-degree of freedom Hamiltonian system

$$\begin{aligned} H &= \frac{1}{2}(p_1^2 + p_2^2 + \omega_1 q_1^2 + \omega_2 q_2^2) + \alpha q_1 q_2^2 - \frac{1}{3}\beta q_1^3 + \frac{c_3}{2q_2^2},\ \alpha \neq 0 \\ & q_1'' + \omega_1 q_1 - \beta q_1^2 + \alpha q_2^2 = 0, \\ & q_2'' + \omega_2 q_2 + 2\alpha q_1 q_2 - c_3 q_2^{-3} = 0, \end{aligned}$$

in which $\alpha, \beta, \omega_1, \omega_2, c_3$ are constants. In the case $c_3 = 0, \beta/\alpha = 1$, it was introduced by Hénon and Heiles to describe the chaotic motion of a star in the axisymmetric potential of a galaxy [198]. It is now known as the *cubic Hénon–Heiles Hamiltonian* system (HH3). The test selects only three values $\beta/\alpha = -1, -6, -16$.

The last two sections (2.2 and 2.3) deal with two fairly common situations when the test, as initiated by Sophie Kowalevski, is inconclusive, because of the insufficient number of arbitrary constants in the local representation of the general solution.

Chapter 3 is devoted to the explicit integration of nonlinear ODEs by methods based on singularities, mainly taking the examples of the previous chapter. We pro-

[2] A point where multivaluedness occurs is classically called a *critical point* or *ramification point* or *branch point*.

cess successively the integrable (Sect. 3.1) and partially integrable (Sect. 3.2) situations.

In Sect. 3.1.1, in the four cases when the Lorenz model passes the Painlevé test, we give a systematic method to compute the polynomial first integrals and we perform the full integration in terms of elliptic or Painlevé functions.

In Sects. 3.1.2 and 3.1.3, one looks for the traveling waves[3] of two important evolution equations, respectively the *Korteweg–de Vries equation* (KdV), which governs the propagation of waves in shallow water [39, 256],

$$bu_t + u_{xxx} - \frac{6}{a} u u_x = 0, \ (a,b) \text{ constant},$$

and the *nonlinear Schrödinger equation* (NLS),

$$iA_t + pA_{xx} + q|A|^2 A = 0, \ pq \neq 0, \ A \in \mathscr{C}, \ (p,q) \in \mathscr{R}.$$

This is an easy task because the ODEs have the Painlevé property and, from their general traveling wave, which is an elliptic function, one defines the various physically relevant particular solutions (pulses, fronts).

In Sect. 3.2, the partially integrable situation is mainly illustrated through the two examples of the equations for the traveling waves of the KS equation and the CGL3 equation, which have been seen to fail the Painlevé test.

In Sect. 3.2.1.2, we introduce the concept of *general analytic solution* of a nonintegrable ODE, defined as the closed form particular single valued solution which depends on the maximum possible number of integration constants, and we count precisely this number. We then look for two classes of solutions which are not too difficult to obtain and which have a great physical interest, the doubly periodic ones (elliptic) and the simply periodic ones (trigonometric).

Those particular solutions which are doubly periodic (elliptic) are easy to find because of necessary conditions arising from a nice property of elliptic functions. These conditions and the associated solutions are established in Sect. 3.2.2.

Among the particular solutions which are simply periodic (trigonometric), some are also easy to find by representing the possible solution as a polynomial in one elementary variable τ or two elementary variables (σ, τ) which obey fundamental nonlinear first order ODEs. These *truncation methods* are described in Sects. 3.2.3 (for KS) and 3.2.4 (for CGL3).

In Sect. 3.2.5, in order to overcome the limitations of the truncation methods, by implementing an old theorem of Briot and Bouquet (1856), we introduce a method able to find *all* the doubly periodic or simply periodic solutions of a given ODE, while any truncation method can only find *some* of these. Instead of searching an expression for the solution, it builds an intermediate, equivalent information, namely the *first order* autonomous ODE satisfied by the unknown solution. For KS and CGL3, it provides no new result, this fact will be explained in Sect. 3.2.8 as an application of the Nevanlinna theory.

[3] A **traveling wave** of a given PDE $E(u,x,t) = 0$ is any solution of the reduction $\xi = x - ct$ if it exists.

Section 3.2.6 deals with the *Duffing–van der Pol* oscillator when it passes the weak Painlevé test. In a particular case when a first integral exists, the resulting ODE can be mapped by a special transformation (hodograph) to another equation possessing the Painlevé property.

In Sect. 3.2.7, we display an example (the *Bianchi IX cosmological model* in vacuum), in which the necessary conditions to pass the test are used in a constructive, unusual way, in order to isolate all possible single valued solutions. The perturbative method of Sect. 2.2 shows the probable existence of one additional solution to the known ones.

In Sect. 3.2.8, we briefly present additional results on the KS equation which are obtained by the Nevanlinna theory. This theory, which is not based on singularity analysis, gives a complementary insight on the analytic structure of the solutions.

Chapter 4 deals with the extension to nonlinear *partial differential equations* (PDEs) of the Painlevé property and Painlevé test previously introduced for ODEs. In Sect. 4.1, we mention solutions of a PDE which are also solutions of some ODEs, i.e. what is called a *reduction*. In Sect. 4.2, we introduce the quite important class of *soliton equations*, together with their main properties: existence of an N-soliton solution and of a remarkable transformation called the Bäcklund transformation (BT). In Sect. 4.3, we extrapolate to PDEs the notion of integrability and the definition of the Painlevé property. After defining in Sect. 4.4.1 the expansion variable χ which minimizes the computation of the Laurent series representing the local solution, we present in Sect. 4.4.2 the successive steps of the Painlevé test, on the example of the KdV equation in order to establish necessary conditions for the Painlevé property. Finally, in Sect. 4.4.3, we apply the test to the equation of *Kolmogorov–Petrovski–Piskunov* (KPP) [255, 383] to generate necessary conditions for the existence of closed form particular solutions.

The subject of Chap. 5 is the "integration" of nonlinear PDEs. Constructive algorithms must be devised to establish the Painlevé property and ultimately to find explicit solutions. Known as the *singular manifold method* (SMM), these algorithms are the natural extension of the truncation methods already encountered in Chap. 3.

In Sect. 5.1, we first extract from the numerous results of the Painlevé test some global information about the analytic property of the solutions. In Sect. 5.2, we recall the two main approaches to build the so called N-soliton solution and briefly introduce the main integrability tools of the soliton equations: Lax pair, Darboux transformation, Bäcklund transformation, nonlinear superposition formula and the Crum transformation. The precise definitions are then given in Sect. 5.3, with application to two physically important equations, the KdV and Boussinesq equations [39], which are integrable by the *inverse spectral transformation* method (IST) [1].

In order to establish the Painlevé property of the PDEs under consideration, the challenge is to derive these integrability items by using methods based only on the singularity structure of the equations.

In Sect. 5.5.1 we present the basic ideas of this singular manifold method mainly consisting in converting the local information provided by the Painlevé test into the above mentioned (global) integrability items. The next two sections are respectively devoted to the SMM in the case of equations possessing the PP (Sect. 5.6) and in the case of partially integrable equations, i.e. equations which fail the Painlevé test but nevertheless admit particular singlevalued solutions (Sect. 5.7).

More precisely, in Sect. 5.6.1 we process the Korteweg-de Vries and Boussinesq equations, which possess only one family of movable singularities. Their two nonlinear superposition formulae are found to be the same, the reason being that the KdV and Boussinesq equations are two different reductions of a 2+1-dimensional IST-integrable equation, the Kadomtsev–Petviashvili (KP) equation [243]. However, the two reductions induce two different solitonic behaviors: KdV only describes the overtaking interaction of solitary waves, while Boussinesq may also describe the head-on collision of solitary waves.

In Sect. 5.6.2, the SMM is applied to two IST-integrable equations (sine-Gordon, modified KdV) which possess two families of movable singularities, and again obtain for both equations the same form of the NLSF.

In Sect. 5.6.3, we apply the SMM to two other integrable PDEs which have a third order Lax pair, the Sawada–Kotera (SK) [387] and Kaup–Kupershmidt (KK) [246, 148] equations. The key ingredient is to consider, in the list of Gambier [163] of second order first degree nonlinear ODEs possessing the PP, the very few equations which are linearizable into a third order ODE, yielding simultaneously the Darboux transformation and the x-part of the Lax pair. In addition to the auto-Bäcklund transformation and the NLSF in each case, the SMM provides a BT between SK and KK.

We next apply the SMM to partially integrable PDEs. In Sect. 5.7.1, we handle the *Fisher equation* [140], which models the evolution of mutant genes or the propagation of flames. In this one-family equation, by finding a particular solution of the necessary conditions generated by the Painlevé test, one obtains two elliptic solutions [8].

In Sect. 5.7.2, we handle the KPP reaction-diffusion equation, possessing two opposite families. The output is two one-soliton solutions (one tanh and one sech), and a degenerate two-soliton without coupling factor.

In the last section (5.8), we examine what these integrability items become when an integrable PDE reduces to an ODE:

- ○ Lax pair → isomonodromic deformation
- ○ Bäcklund transformation → birational transformation
- ○ nonlinear superposition formula → contiguity relation.

In Chap. 6, we give an illustration on the various ways to "integrate" a Hamiltonian system using two examples of Hamiltonian systems with two degrees of freedom: the cubic HH Hamiltonian introduced in Sect. 2.1.5, three cases of which pass the Painlevé test, and the quartic HH Hamiltonian (HH4),

$$\begin{aligned} H = &\frac{1}{2}(P_1^2 + P_2^2 + \Omega_1 Q_1^2 + \Omega_2 Q_2^2) + CQ_1^4 + BQ_1^2 Q_2^2 + AQ_2^4 \\ &+\frac{1}{2}\left(\frac{\alpha}{Q_1^2} + \frac{\beta}{Q_2^2}\right) + \gamma Q_1,\ B \neq 0, \end{aligned}$$

$$Q_1'' + \Omega_1 Q_1 + 4CQ_1^3 + 2BQ_1Q_2^2 - \alpha Q_1^{-3} + \gamma = 0,$$
$$Q_2'' + \Omega_2 Q_2 + 4AQ_2^3 + 2BQ_2Q_1^2 - \beta Q_2^{-3} = 0,$$

in which $A, B, C, \alpha, \beta, \gamma, \Omega_1, \Omega_2$ are constants. The Painlevé test selects four sets of values of these constants, $A : B : C =$ 1:2:1, 1:6:1, 1:6:8, 1:12:16 (the notation $A : B : C = p : q : r$ stands for $A/p = B/q = C/r =$ arbitrary).

These various ways to integrate are

- ◦ (Liouville integrability) to find a second invariant in involution with the Hamiltonian, which is however insufficient to perform a global integration; we recall the seven first integrals which establish this integrability for both HH3 (Sect. 6.2.1) and HH4 (Sect. 6.3.1);
- ◦ (Arnol'd–Liouville integrability) to find the variables which separate the Hamilton–Jacobi equation, thus leading to a global integration; this has been done for HH3 (Sect. 6.2.2), and nearly finished for HH4 (Sect. 6.3.2);
- ◦ (Painlevé property) to find an explicit closed form single valued expression for the general solution $q_j(t), Q_j(t)$; this has been done in all seven cases (Sects. 6.2.3 and 6.3.3), *via* birational transformations to fourth order ODEs isolated and integrated by Cosgrove.

Chapter 7 deals with discrete nonlinear equations. After some generalities, in Sect. 7.1 we consider the logistic map of Verhulst,

$$u_n = a u_{n-1}(1 - u_{n-1}),$$

a paradigm of chaotic behavior [405, 139], which admits a continuum limit to the Riccati equation. From the point of view of integrability, the logistic map is a "bad" discretization of the Riccati equation, because it cannot be linearized, and it must be replaced by a "good" discrete equation, i.e. one which preserves the property of linearizability. More generally, the goal is to extend the Painlevé property to the discrete world.

Section 7.2 presents an outlook of the difficulty to give an undisputed definition for the *discrete Painlevé property*.

In Sects. 7.3.1, 7.3.2 and 7.3.3, we present the three main methods of the discrete Painlevé test: the *singularity confinement method* [184], the *criterium of polynomial growth* [206], and the *perturbation of the continuum limit* [88].

In order to prove the discrete Painlevé property, one can either linearize the discrete equation, or explicitly integrate or, as admitted by most researchers, exhibit a discrete Lax pair.

In Sect. 7.4, we return to the question of finding a "good" discretization of the Riccati equation; this results in the homographic map

$$u_n = \frac{a_1 u_{n-1} + a_2}{a_3 u_{n-1} + a_4}.$$

The notion of discrete Lax pair is introduced in Sect. 7.5.

We then describe two examples of exact discretizations, i.e. for which the analytic expression of the general solution is the same for the continuous and discrete equations.

In Sect. 7.6.1 we consider the question of discretizing the nonlinear ODE for the modulus $v = |\psi|$ of the linear Schrödinger equation, namely [133, 305, 358],

$$v'' + fv + c^2 v^{-3} = 0,$$

usually called the *Ermakov–Pinney equation*. Again, the property to be preserved is the linearizability, since the starting equation is linear.

In Sect. 7.6.2 we recall the remark by Baxter and Potts that the addition formula of the Weierstrass function $\wp$ can be identified to an exact discretization of the Weierstrass equation. This is the foundation for a family of special two-component rational maps [367, 368] which, like its continuous counterpart, is a starting point to isolate discrete equations which may possess the discrete PP.

In Sect. 7.7, we briefly review two related problems. The first problem, still open but of a very high physical interest in optical fibers, is to find exact solitary waves (dark and bright) for the *nonintegrable discrete nonlinear Schrödinger equation*,

$$iu_t + p\frac{u(x+h,t) + u(x-h,t) - 2u(x)}{h^2} + q|u|^2 u = 0,\ i^2 = -1,\ pq \neq 0.$$

In the context of optical fibers or Bose–Einstein condensation [10], this equation is not obtained as a discretization of NLS but it arises by a direct construction. The second one is to isolate discrete versions of the nonlinear Schrödinger equation which might possess the discrete Painlevé property, and one such equation is the Ablowitz and Ladik [4] discrete equation.

Finally, in Sect. 7.8, after setting up the natural problem to extend to the discrete world the six transcendents of Painlevé, we introduce the two methods which have been devised to handle it. In the analytic method (Sect. 7.8.1), the procedure starts from the addition formula of the elliptic function, takes some inspiration from the method of Painlevé and Gambier and produces a rather long list of discrete Pn equations, but no proof exists that the list is exhaustive. The geometric method (Sect. 7.8.2) first displays the importance of two groups describing the continuous Pn, then uses the theory of rational surfaces to build an object which admits the largest of the just mentioned groups, object interpreted as the master discrete Painlevé equation e – P6, whose coefficients have an elliptic dependence on the independent variable. The main properties of all these d – Pn are then summarized in Sect. 7.8.3.

After an FAQ chapter, a few appendices collect material too technical to be presented in the main text.

Contents

Acronyms

AKNS	Ablowitz, Kaup, Newell and Segur
BT	Bäcklund transformation
CGL3	Cubic complex Ginzburg–Landau equation
CGL5	Quintic complex Ginzburg–Landau equation
DT	Darboux transformation
FAQ	Frequently asked question
Gn	Gambier equation number *n*
HH3	Cubic Hénon–Heiles Hamiltonian
HH4	Quartic Hénon–Heiles Hamiltonian
IST	Inverse spectral transform
KdV	Korteweg and de Vries equation
KK	Kaup-Kupershmidt equation
KP	Kadomtsev and Petviashvili equation
KPP	Kolmogorov–Petrovski–Piskunov equation
KS	Kuramoto and Sivashinsky equation
NLS	Nonlinear Schrödinger equation
NLSF	Nonlinear superposition formula
ODE	Ordinary differential equation
PDE	Partial differential equation
Pn	Painlevé equation number *n*
PP	Painlevé property
SG	sine-Gordon equation
SH	Swift and Hohenberg equation
SK	Sawada–Kotera equation
SME	Singular manifold equation
SMM	Singular manifold method
WTC	Weiss, Tabor and Carnevale

Chapter 1
Introduction

Abstract A nonlinear equation should *not* be considered as the perturbation of a linear equation. We illustrate using two simple examples the importance of taking account of the singularity structure in the complex plane to determine the general solution of nonlinear equations. We then present the point of view of the Painlevé school to *define new functions* from nonlinear ordinary differential equations possessing a general solution which can be made single valued in its domain of definition (the Painlevé property).

1.1 Singularities in the Complex Plane

Given some nonlinear differential equation, an intuitive approach to find a solution is to split the equation into the sum of a so-called *linear part* and a *perturbation*. Let us explain, using two examples, why this should *not* be done.

Consider the following elementary nonlinear equations

$$u' = k(1-u^2),\ ' = \frac{\mathrm{d}}{\mathrm{d}x}, \tag{1.1}$$

$$v'^2 = k^2 v^2 (1-v^2), \tag{1.2}$$

with the aim of finding their general solution,

$$u = \tanh k(x-x_0), \tag{1.3}$$

$$v = \frac{1}{\cosh k(x-x_0)} = \operatorname{sech} k(x-x_0). \tag{1.4}$$

The arbitrary complex constant x_0 is linked to the initial condition $u(x_i) = u_i, v(x_i) = v_i$ by the relation

$$u_i = \tanh k(x_i - x_0),\ v_i = \operatorname{sech} k(x_i - x_0). \tag{1.5}$$

On the real axis, u and v have no singularities but, in the complex plane, the singularities of u and v are a countable number of simple poles, located at $x = x_0 + (2n+1)i\pi/(2k), n \in \mathscr{Z}$. Such singularities are by definition said to be *movable*, as opposed to *fixed*, because their location depends on the initial conditions, i.e. on the constants of integration. The general solution of any linear differential equation has no movable singularity because it depends linearly on the constants of integration.

1.1.1 Perturbative Method

In the *perturbative* method [199], one first determines the stationary points, which leads to $u_0 = \pm 1$ and $v_0 = 0, \pm 1$, then one perturbs the solution in the neighborhood of a stationary point by expanding it in series of a small parameter ε. Under this perturbation

$$u = \sum_{n=0}^{\infty} \varepsilon^n u_n,\ u_0 = 1, \tag{1.6}$$

Equation (1.1) splits accordingly into

$$\sum_{n=0}^{\infty} \varepsilon^n E_n = 0,\ E_0 \equiv 0. \tag{1.7}$$

$$E_1 \equiv -u_1' - 2ku_0u_1 = 0, \tag{1.8}$$

$$E_2 \equiv -u_2' - 2ku_0u_2 - ku_1^2 = 0, \cdots \tag{1.9}$$

Choosing $u_1 = c_1 e^{-2kx}$ with c_1 arbitrary, this infinite set of linear equations with the same homogeneous part admits the particular solution

$$u_n = 2^{-n+1} c_1^n e^{-2knx},\ n \geq 1, \tag{1.10}$$

which defines a geometric series, and its sum provides the general solution of (1.1)

$$u = 1 + \frac{\varepsilon c_1 e^{-2kx}}{1 - (\varepsilon c_1/2)e^{-2kx}} = \tanh k(x - x_0),\ x_0 = \frac{1}{2k} \log\left(-\frac{\varepsilon c_1}{2}\right) \cdot \tag{1.11}$$

Equation (1.2) is handled slightly differently because of its nonlinearity in the highest derivative. One first takes its derivative,

$$v'' = k^2 v(1 - 2v^2), \tag{1.12}$$

to make the first perturbed equation $E_1 = 0$ linear in the highest derivative. Then the computation is quite similar: the expansion $v = \sum_{n=0}^{\infty} \varepsilon^n v_n$ around $v_0 = 0$ generates an infinite set of linear equations with the same homogeneous part. Choosing $v_1 = c_1 e^{-kx}$, with c_1 arbitrary, the particular solutions are

$$v_{2n+1} = (-4)^{-n} \left(c_1 e^{-kx}\right)^{2n+1}, \; v_{2n} = 0, \; n = 1, 2, \cdots \tag{1.13}$$

Therefore the series for v is geometric and it sums into:

$$v = \frac{\varepsilon c_1 e^{-kx}}{1 + (\varepsilon c_1/2)^2 e^{-2kx}} \equiv \operatorname{sech} k(x - x_0), \; x_0 = \frac{1}{k} \log \frac{\varepsilon c_1}{2}, \tag{1.14}$$

which represents the general solution of (1.2).

Why is this perturbative method not efficient for equations more complicated than (1.1)–(1.2)? There are several reasons for this:

1. In order to obtain the general term u_n, one must solve a recurrence relation, a difficult task even for a linear recurrence relation.
2. The resummation must be performed in closed form[1] and this is generically impossible; indeed, any solution which is not in closed form is what Painlevé calls "illusoire", in a sense to be developed soon.
3. After performing the resummation, one must check whether the closed form expression is valid everywhere except at a few points, called *singularities*; the location of these singularities cannot be restricted to the real axis but must be extended to the whole complex plane $\mathscr{C}$; in the above example, the reason for the finite value of the radius of convergence is the presence of a simple pole on the imaginary axis at $x = x_0 \pm i\pi/(2k)$.

To summarize, the main reason for the generic inapplicability of this perturbative method is that the singularity structure has not been taken into account: the movable singularity which is present in the exact solution is absent at all orders of the perturbation.

1.1.2 Nonperturbative Method

Let us now present a *nonperturbative* method, which yields the same result in a *finite* number of steps because it takes the singularity structure into account from the beginning.

Since nonlinear ODEs generically possess movable singularities, let us first establish the behavior of the general solution of (1.1) near such a movable singularity $x = x_0$. Assuming this behavior to be algebraic, this amounts to computing the possible values of the *leading power* p and the *leading coefficient* u_0 defined by

$$u \underset{\chi \to 0}{\sim} u_0 \chi^p, \; u_0 \neq 0, \; \chi = x - x_0, \tag{1.15}$$

with p not a positive integer. Then

[1] This will be defined precisely later. For the moment, it is sufficient to know that an example of such a closed form is $u = \psi'/\psi$, with ψ the solution of any linear equation.

$$u' \sim pu_0\chi^{p-1} + u_0'\chi^p \sim pu_0\chi^{p-1}, \tag{1.16}$$

so u_0 can be assumed constant when determining the leading behavior. The various terms of (1.1) then contribute as

term	$-u'$	$+k$	$-ku^2$
leading power	$p-1$	0	$2p$
leading coefficient	$-pu_0$	k	$-ku_0^2$

and the l.h.s. of the ODE, which must vanish, evaluates to

$$\begin{aligned} E(u) &\equiv -u' + k(1-u^2) \\ &= \left(-pu_0\chi^{p-1} + O(\chi^p)\right) + k\chi^0 - k\left(u_0^2\chi^{2p} + O(\chi^{2p+1})\right) && (1.17) \\ &= E_0\chi^q + O(\chi^{q+1}) = 0. && (1.18) \end{aligned}$$

The condition $u_0 \neq 0$ implies the equality of at least two of the three leading powers, the two equal powers being lower than or equal to the third one. As to the condition $E_0 = 0$, it expresses the vanishing of the sum of the two corresponding leading coefficients. Out of the three possibilities

$$(q = p-1 = 0 \leq 2p) \text{ and } (-pu_0 + k = 0), \tag{1.19}$$
$$(q = 0 = 2p \leq p-1) \text{ and } (k - ku_0^2 = 0), \tag{1.20}$$
$$(q = p-1 = 2p \leq 0) \text{ and } (-pu_0 - ku_0^2 = 0), \tag{1.21}$$

only the third one defines a solution,

$$p = -1,\ q = -2,\ u_0 = 1/k. \tag{1.22}$$

To summarize, the local behavior of u is that of a simple pole,

$$u \underset{\chi\to 0}{\sim} k^{-1}\chi^{-1},\ \chi = x - x_0. \tag{1.23}$$

In order to turn this local information into a global one, one then establishes a parallel with a well known generator of simple poles, namely the *logarithmic derivative* operator. If some function $\psi(x)$ has an algebraic behavior $\psi \sim \psi_0(x - x_0)^p$ near x_0 (with ψ_0 and p any complex numbers), under action of the logarithmic derivative operator,

$$\mathscr{D} = \frac{\mathrm{d}}{\mathrm{d}x}\log, \tag{1.24}$$

this behavior (whatever it is, regular or singular, multivalued or singlevalued) becomes that of a simple pole of residue p,

$$\frac{\mathrm{d}}{\mathrm{d}x}\log\psi \sim \frac{p}{x - x_0}. \tag{1.25}$$

The crucial point is then to match (1.23) and (1.25), by introducing the transformation from u to ψ *defined* by

$$u = k^{-1} \frac{\mathrm{d}}{\mathrm{d}x} \log \psi. \tag{1.26}$$

This transformation, called the *singular part transformation*, maps the Riccati ODE (1.1) to the second order ODE

$$\psi'' - k^2 \psi = 0, \tag{1.27}$$

which has no more movable singularities since it is linear. Therefore its general solution is known,

$$\psi = c \cosh k(x - x_1),\ (c, x_1) \text{ arbitrary}, \tag{1.28}$$

and this provides the closed form single valued expression (1.3) for the general solution of the Riccati ODE (1.1).

With our second example (1.2), one similarly obtains the two local behaviors

$$v \underset{\chi \to 0}{\sim} \pm ik^{-1} \chi^{-1},\ \chi = x - x_0. \tag{1.29}$$

This complex value $\pm ik^{-1}$ for the residue should be no surprise, since it is the root of an algebraic equation with real coefficients. The map from v to ψ must now involve two functions ψ_1, ψ_2, and indeed, if one defines the singular part transformation as

$$v = ik^{-1} (\log \psi_1)' - ik^{-1} (\log \psi_2)', \tag{1.30}$$

in which ψ_1 and ψ_2 are two different solutions of the same second order linear equation

$$\psi'' - \frac{k^2}{4} \psi = 0, \tag{1.31}$$

which can be chosen as

$$\psi_1 = c_1 \cosh \frac{k}{2}(x - x_1),\ \psi_2 = c_2 \cosh \frac{k}{2}(x - x_2), \tag{1.32}$$

the expression (1.30) satisfies the ODE (1.2), provided x_1, x_2, k obey the relation

$$k(x_1 - x_2) = i\pi + 2mi\pi,\ m \in \mathscr{Z}, \tag{1.33}$$

with the correspondence of notation $x_0 = (x_1 + x_2)/2$.

Therefore, the fact of taking account of the singularity structure (one family of simple poles, two families of simple poles with opposite residues, etc) allows one to establish an explicit closed form link towards another ODE (in our examples a linear ODE) which has no movable singularities, *ipso facto* performing the explicit integration of the nonlinear ODE.

The purpose of this book is to explain how to *explicitly* build analytic solutions of nonlinear differential equations, whether ordinary or partial, by nonperturbative methods such as the simple one presented above.

Since all the exact solutions one can derive by any method necessarily obey the singularity structure of the equation in the complex plane, it is therefore a prerequisite to study these singularities. For instance, the solutions (1.3)–(1.4) have respectively one family and two families of movable simple poles, therefore one must be able to detect, directly on their ODEs without knowing the solutions in advance, respectively one family and two families of movable simple poles.

1.2 Painlevé Property and the Six Transcendents

How can this be generalized? This is the whole problem of the explicit integration of ODEs. *To integrate an ODE*, according to a definition attributed to Poincaré, is to express its *general solution* as a finitely many term explicit expression, possibly multivalued, built from elementary objects called functions. A *function* in turn is defined as a map which can be made singlevalued in its whole domain of definition. Any linear ODE defines a function because its general solution can be made singlevalued, by classical uniformization procedures such as cuts in the complex plane. Typical examples are all the "special functions" of mathematical physics defined by some linear equation (exponential and trigonometric functions, functions of Bessel, Hermite, Legendre, Gauss, ...). Therefore, with the above definition, a large class of ODEs are considered as integrated[2]: linear ODEs, linearizable ODEs, ODEs whose general solution is rational in the solution of a linear equation, ...

In order to extend the class of available functions, L. Fuchs and Poincaré stated the problem of defining new functions from algebraic nonlinear differential equations. One such function had already been discovered by Jacobi when he solved the motion of the pendulum. In this Hamiltonian system

$$H = \frac{1}{2}ml^2\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2 + mgl(1-\cos\theta), \tag{1.34}$$

the problem is to find the position (characterized by the angle θ of the pendulum of length l and mass m with the vertical axis) as a function of the time t. After equating H to its constant value E, one obtains a first order second degree equation (the degree is by definition the polynomial degree in the highest derivative) which is often "integrated by separation of variables" as

[2] For instance, the stationary Schrödinger equation of quantum mechanics, called the Sturm–Liouville equation by mathematicians, is considered as integrated. To solve the spectral problem is outside the scope of this volume.

$$u = \tan\frac{\theta}{2},\ t = t_0 + \int_{u_0}^{u} \frac{\sqrt{2ml^2}\mathrm{d}u}{\sqrt{(1+u^2)(E(1+u^2)-2mglu^2)}}, \tag{1.35}$$

which expresses the time t as an *elliptic integral* of the position u. However, this does not answer the question, which was to express the position as a function of time. Indeed, the above answer is as bad as would be a multivalued expression like

$$t = t_0 + \int_{u_0}^{u} \frac{\mathrm{d}u}{1+u^2} = t_0 + \operatorname{Arctan} u - \operatorname{Arctan} u_0, \tag{1.36}$$

instead of the singlevalued answer

$$u = \frac{u_0 + \tan(t-t_0)}{1 - u_0 \tan(t-t_0)}. \tag{1.37}$$

This classical problem, called inversion of the elliptic integral, was solved by Abel and Jacobi, who proved that, for the pendulum, the coordinates $(l\cos\theta, l\sin\theta)$ of the position are singlevalued expressions of the time,

$$\sin\frac{\theta}{2} = k\,\mathrm{sn}\left(\sqrt{\frac{g}{l}}(t-t_0), k\right),\ k = \sqrt{\frac{E}{mgl}}, \tag{1.38}$$

$$\cos\frac{\theta}{2} = \mathrm{dn}\left(\sqrt{\frac{g}{l}}(t-t_0), k\right). \tag{1.39}$$

The symbols $\mathrm{sn}(x,k)$ and $\mathrm{dn}(x,k)$, in which k is a constant, denote two of the twelve Jacobi *elliptic functions* (Appendix C), which all satisfy equations of the type

$$\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2 - P(u) = 0, \tag{1.40}$$

with P a polynomial independent of x of degree four with complex coefficients. The general solution of (1.40) is singlevalued not only on the real x axis but in the whole complex plane. Considering the complex plane is mandatory to unveil the beautiful property of this function, which is to be a doubly periodic meromorphic function, a characteristic property of elliptic functions. This equation is form invariant under a transformation which plays a fundamental role in the present theories, the *homographic transformation* or homography,

$$u \mapsto \frac{\alpha u + \beta}{\gamma u + \delta},\ (\alpha, \beta, \gamma, \delta) \text{ complex constants},\ \alpha\delta - \beta\gamma \neq 0. \tag{1.41}$$

The canonical representative in this equivalence class is the *Weierstrass equation*

$$u'^2 = 4u^3 - g_2 u - g_3 = 4(u-e_1)(u-e_2)(u-e_3), \tag{1.42}$$

in which g_2, g_3, e_1, e_2, e_3 are complex constants and one zero of the polynomial P has been moved to infinity by choosing $-\delta/\gamma$ equal to the affix of that zero. The peculiarity of the homographic group is to be the *unique* bijection of the complex plane (to which one has added the point at infinity) to itself, this is why this group does not alter the singularity structure of the elliptic equation.

Another characteristic property of the elliptic equation, much more important in our context than the previous one, is to be the unique first order algebraic ODE able to define a "new" function in the above sense, i.e. from a nonlinear ODE.

This question (of defining new functions) has been investigated at higher orders (up to six for special classes) by the Painlevé school (Painlevé, Gambier, Chazy, Garnier) and its followers (Bureau, Exton, Martynov, Cosgrove). Its mathematical formulation [349, p. 2]

Déterminer toutes les équations différentielles algébriques du premier ordre, puis du second ordre, puis du troisième ordre, etc., dont l'intégrale a ses points critiques fixes.[3]

naturally leads to the definition of a property of differential equations.

Definition 1.1. If the general solution of an ODE can be made singlevalued, one says that such an ODE possesses the **Painlevé property** (PP).

A class of transformations which leaves invariant the singularity structure of u and therefore the PP of the ODE for u is the *homographic group* (also called Möbius group and denoted $\mathrm{PSL}(2,\mathscr{C})$)

$$\begin{aligned} &(u,x) \mapsto (U,X),\ u(x) = \frac{\alpha(x)U(X)+\beta(x)}{\gamma(x)U(X)+\delta(x)},\ X = \xi(x), \\ &(\alpha,\beta,\gamma,\delta,\xi) \text{ functions},\ \alpha\delta-\beta\gamma \neq 0, \end{aligned} \tag{1.43}$$

which depends on four arbitrary functions and generalizes the group (1.41).

At present time, only second order nonlinear equations have defined additional functions, the six ones discovered by Painlevé and Gambier, called *Painlevé transcendents* $\mathrm{P}n, n = 1, \dots, 6$[4]

$$\begin{aligned} \mathrm{P1} &: u'' = 6u^2 + x, \\ \mathrm{P2} &: u'' = \delta(2u^3 + xu) + \alpha, \\ \mathrm{P3} &: u'' = \frac{u'^2}{u} - \frac{u'}{x} + \frac{\alpha u^2 + \gamma u^3}{4x^2} + \frac{\beta}{4x} + \frac{\delta}{4u}, \\ \mathrm{P4} &: u'' = \frac{u'^2}{2u} + \gamma\left(\frac{3}{2}u^3 + 4xu^2 + 2x^2u\right) - 2\alpha u + \frac{\beta}{u}, \\ \mathrm{P5} &: u'' = \left[\frac{1}{2u} + \frac{1}{u-1}\right]u'^2 - \frac{u'}{x} + \frac{(u-1)^2}{x^2}\left[\alpha u + \frac{\beta}{u}\right] + \gamma\frac{u}{x} + \delta\frac{u(u+1)}{u-1}, \end{aligned}$$

[3] To determine all the algebraic differential equations of first order, then second order, then third order, etc., whose general solution has no movable critical points.

[4] We adopt for P3 the choice made by Painlevé in 1906 [350] to replace his original choice of 1900 [348].