Commun. Math. Phys. 194, 61 – 70 (1998)

Communications in
Mathematical Physics

The Pair Correlation Function of Fractional Parts of Polynomials*

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Received: 22 July 1997 / Accepted: 24 September 1997

Abstract: We investigate the pair correlation function of the sequence of fractional parts of αn^d, $n = 1, 2, \ldots, N$, where $d \geq 2$ is an integer and α an irrational. We conjecture that for badly approximable α, the normalized spacings between elements of this sequence have Poisson statistics as $N \to \infty$.

We show that for almost all α (in the sense of measure theory), the pair correlation of this sequence is Poissonian.

In the quadratic case $d = 2$, this implies a similar result for the energy levels of the "boxed oscillator" in the high-energy limit. This is a simple integrable system in 2 degrees of freedom studied by Berry and Tabor as an example for their conjecture that the energy levels of generic completely integrable systems have Poisson spacing statistics.

1. Introduction

Hermann Weyl [11] proved that for an integer $d \geq 1$ and an irrational α, the sequence of fractional parts $\alpha n^d \mod 1$, $n = 1, 2, \ldots$ is equidistributed in the unit interval. A different aspect of the random behavior of the sequence has attracted attention recently: Are the spacings between members of the sequence distributed like those between members of a sequence of random numbers in the unit interval (or as some would say, do they have a "Poissonian" distribution)? This issue came up in the context of the distribution of spacings of the energy levels of integrable systems [1]. For the case $d = 1$ the spacings between the fractional parts of αn are essentially those of the energy levels of a two-dimensional harmonic oscillator [4, 2, 3]. For $d = 2$ the spacings are related to the spacings between the energy levels of the "boxed oscillator" [1], a particle in a

* Supported in part by grants from the U.S.-Israel Binational Science Foundation, the Israel Science Foundation, and the NSF.

2-dimensional potential well with hard walls in one direction and harmonic binding in the other. The spacings of $\alpha n^2 \mod 1$ were also investigated numerically in [5].

If $d = 1$ it is elementary that the consecutive spacings have at most 3 values [9, 10]. Hence the sequence is not random in this case. For $d \geq 2$ the picture is very different. To explain it, we recall a basic classification of real numbers with regards to their Diophantine approximation properties: We say α is *of type* κ if there is $c = c(\alpha) > 0$ so that

$$|\alpha - p/q| > c/q^\kappa$$

for all integers p, q. For rational α, $\kappa = 1$ and α is irrational if and only if $\kappa \geq 2$. It is well known that almost all α (in the sense of measure theory) are of type $\kappa = 2 + \epsilon$ for all $\epsilon > 0$. We will call such α "Diophantine". For instance, algebraic irrationals are of this type (Roth's theorem).

In [7] we establish some results towards the conjecture that $\alpha n^d \mod 1$ is Poissonian for any α of Diophantine type. In this note we examine the behavior for almost all α, which according to the above should be Poissonian. The statistic we examine is the *pair correlation*: The pair correlation density for a sequence of N numbers $\theta_1, \dots, \theta_N \in [0,1]$ which are equidistributed as $N \to \infty$, measures the distribution of spacings between the θ_j at distances of order of the mean spacing $1/N$. Precisely, if $\|x\| =$ distance$(x, \mathbf{Z})$ then for any interval $[-s, s]$ set

$$R_2([-s,s], \{\theta_n\}, N) = \frac{1}{N} \# \left\{ 1 \leq j \neq k \leq N : \|\theta_j - \theta_k\| \leq \frac{s}{N} \right\}. \tag{1.1}$$

For random numbers θ_j chosen uniformly and independently,

$$R_2([-s,s], \{\theta_n\}, N) \to 2s$$

with probability tending to 1 as $N \to \infty$. Our main result is that this holds for the sequence of fractional parts $\{\alpha n^d \mod 1\}$ for almost every α: Denoting by $R_2([-s,s], \alpha, N)$ the pair correlation sum (1.1) for this sequence, we show

Theorem 1. *For $d \geq 2$, there is a set $P \subset \mathbf{R}$ of full Lebesgue measure such that for any $\alpha \in P$, and any $s \geq 0$,*

$$R_2([-s,s], \alpha, N) \to 2s, \qquad N \to \infty.$$

Remark 1.1. The proof given below does not provide (and we do not know of) any specific α which is *provably* in P.

Remark 1.2. Already with the pair correlation we see the necessity of a condition on the type of α. For if there are arbitrarily large integers p, q so that

$$|\alpha - \frac{p}{q}| \leq \frac{1}{10q^{d+1}},$$

then $R_2([-s,s], \alpha, N) \not\to 2s$. Indeed if we choose $N = q$, then for $m \neq n \leq N$,

$$\left\| n^d\alpha - m^d\alpha \right\| = \left\| \frac{(n^d - m^d)p}{q} + \frac{t(n^d - m^d)}{10q^{d+1}} \right\|$$

with $|t| \leq 1$. Hence either $\left\| n^d\alpha - m^d\alpha \right\| \leq 1/10q = 1/10N$ if q divides $n^d - m^d$, or $\left\| n^d\alpha - m^d\alpha \right\| \geq 9/10q = 9/10N$ otherwise. Thus there are no normalized differences $N \left\| n^d\alpha - m^d\alpha \right\|$ in the interval $(1/10, 9/10)$ for this sequence of $N = q$.

The proof of the theorem follows the steps in [8] (where a similar assertion is proven for the values of binary quadratic forms). We first establish that as a function of $\alpha \in [0,1]$, $R_2([-s,s],\alpha,N) \to 2s$ in $L^2(0,1)$. This together with standard bounds on the Weyl sums $S(n,N) = \sum_{x\leq N} e(n\alpha x^d)$ allows us to pass to almost everywhere convergence. In Sect. 4 we briefly discuss higher correlations and show that they do not converge in L^2 to the expected value. Thus our approach does not lend itself directly to establishing almost everywhere convergence of the higher correlations.

2. Bounding the Variance

Let $f \in C_c^\infty(\mathbf{R})$ be a test function and set

$$R_2(f,\{\theta_n\},N) := \frac{1}{N} \sum_{1\leq j\neq k\leq N} F_N(\theta_j - \theta_k), \tag{2.1}$$

where

$$F_N(y) = \sum_{m\in\mathbf{Z}} f(N(y+m)). \tag{2.2}$$

The function $F_N(y)$ is periodic and has a Fourier expansion

$$F_N(y) = \frac{1}{N} \sum_{n\in\mathbf{Z}} \widehat{f}\left(\frac{n}{N}\right) e(ny). \tag{2.3}$$

Hence

$$R_2(f,\{\theta_n\},N) = \frac{1}{N^2} \sum_{n\in Z} \widehat{f}\left(\frac{n}{N}\right) \sum_{1\leq j\neq k\leq N} e(n(\theta_j-\theta_k)). \tag{2.4}$$

In particular, if $\theta_n = \alpha n^d \mod 1$, then the pair correlation function is given by

$$R_2(f,\alpha,N) = \frac{1}{N^2} \sum_{n\in Z} \widehat{f}\left(\frac{n}{N}\right) s_{off}(n,N), \tag{2.5}$$

where

$$s_{off}(n,N) := \sum_{1\leq x\neq y\leq N} e\left(n\alpha(x^d-y^d)\right). \tag{2.6}$$

As a function of α, $R_2(f,\alpha,N)$ is periodic and from (2.5) its Fourier expansion is

$$R_2(f,\alpha,N) = \sum_{l\in\mathbf{Z}} b_l(N) e(l\alpha), \tag{2.7}$$

where for $l \neq 0$,

$$b_l(N) = \frac{1}{N^2} \sum_{n\neq 0} \sum_{\substack{1\leq x\neq y\leq N \\ n(x^d-y^d)=l}} \widehat{f}\left(\frac{n}{N}\right). \tag{2.8}$$

The *mean* of $R_2(f,\alpha,N)$ over $\alpha \in [0,1]$ is

$$\langle R_2 \rangle = b_0(N) = \frac{1}{N^2} \sum_{1 \leq x \neq y \leq N} \widehat{f}(0) = \left(1 - \frac{1}{N}\right) \widehat{f}(0), \tag{2.9}$$

so that

$$\langle R_2 \rangle = \int_{-\infty}^{\infty} f(x)dx + O\left(\frac{1}{N}\right), \tag{2.10}$$

which is the expected value for a random sequence.

We next estimate the variance of $R_2(f, \alpha, N)$ as a function of α:

Proposition 2. *As a function of* $\alpha \in [0,1]$,

$$\left\| R_2(f, \alpha, N) - \widehat{f}(0) \right\|_2 \ll N^{-1/2+\epsilon} \tag{2.11}$$

for any $\epsilon > 0$, *the implied constants depending on* ϵ *and* f.

Proof. It is easy to see from (2.8) that since $f \in C_c^\infty(\mathbf{R})$, the Fourier coefficients $b_l(N)$ are negligable for $l \geq N^{d+1+\delta}$ for any fixed $\delta > 0$. Also from (2.8) we have for $l \neq 0$,

$$b_l(N) \ll \frac{\tau(|l|)^2}{N^2}, \tag{2.12}$$

where $\tau(|l|)$ is the numbers of divisors of $|l|$. This is because the factors of l determine n, x, y. We will use the well-known estimate

$$\tau(m) \ll m^\epsilon, \qquad \text{for any } \epsilon > 0. \tag{2.13}$$

Thus by Parseval

$$\begin{aligned}
\left\| R_2(f, \alpha, N) - \widehat{f}(0) \right\|_2^2 &= \left(\frac{\widehat{f}(0)}{N}\right)^2 + \sum_{l \neq 0} |b_l(N)|^2 \\
&\ll \sum_{l \neq 0} \frac{N^\epsilon}{N^2} |b_l(N)| \\
&= \sum_{0 \neq |l| \leq N^{d+1+\delta}} \frac{N^\epsilon}{N^2} |b_l(N)| + \text{ smaller order term} \\
&\ll \frac{N^\epsilon}{N^2} \sum_{l \neq 0} |b_l(N)| \\
&\ll \frac{N^\epsilon}{N^2} \sum_{\substack{1 \leq x \neq y \leq N \\ n \in \mathbf{Z}}} \frac{1}{N^2} \left| \widehat{f}\left(\frac{n}{N}\right) \right| \ll N^{-1+\epsilon}.
\end{aligned}$$

□

3. Almost-Everywhere Convergence

3.1. Overview of the argument for Theorem 1. In order to prove Theorem 1 from the decay of the variance of the pair correlation, we first show that for each $f \in C_c^\infty(\mathbf{R})$, there is a set of full measure $P(f) \subset \mathbf{R}$ so that for all $\alpha \in P(f)$,

$$R_2(f, \alpha, N_m) \to \widehat{f}(0) \tag{3.1}$$

for a subsequence N_m which grows faster than m. Indeed, fix $\delta > 0$, and let $\{N_m\}$ be a sequence of integers with

$$N_m \sim m^{1+\delta}.$$

Set

$$X_N(\alpha) = R_2(f, \alpha, N) - \widehat{f}(0).$$

By Proposition 2, $\|X_N\|_2^2 \ll_\epsilon N^{-1+\epsilon}$ for all $\epsilon > 0$ and so

$$\sum_{m=1}^{\infty} \int_0^1 |X_{N_m}(\alpha)^2| d\alpha < \infty.$$

Therefore (since $|X_{N_m}|^2 \geq 0$)

$$\int_0^1 \sum_m |X_{N_m}(\alpha)|^2 d\alpha = \sum_m \int_0^1 |X_{N_m}(\alpha)|^2 d\alpha < \infty,$$

and so $\sum_m |X_{N_m}|^2 \in L^1(0,1)$. Thus the sum is finite almost everywhere:

$$\sum_m |X_{N_m}(\alpha)|^2 < \infty, \qquad \text{for almost all } \alpha.$$

Therefore, $X_{N_m}(\alpha) \to 0$ as $m \to \infty$ for almost all α, that is we have (3.1) on a set $P(f)$ of α's which we may assume consists only of Diophantine numbers.

To go from almost everywhere convergence along a subsequence to almost everywhere convergence, we will show that as a function of N, $R_2(f, N, \alpha)$ does not oscillate much for Diophantine α. More precisely, there is some $\nu > 0$ so that if $N_m \leq n < N_{m+1}$ then for Diophantine α, there is $c(f, \alpha) > 0$ so that

$$|X_n(\alpha) - X_{N_m}(\alpha)| \ll c(f, \alpha) N_m^{-\nu}.$$

Because $0 \leq n - N_m \leq N_{m+1} - N_m \ll N_m^\delta$, this estimate in turn will follow from:

Proposition 3. *Let $0 < \delta < 1/2^{d-1}$. Then for all $f \in C_c(\mathbf{R})$ and all α of Diophantine type, there is some $c(f, \alpha) > 0$ so that for all $0 \leq k \leq N^\delta$,*

$$|X_{N+k}(\alpha) - X_N(\alpha)| \leq c(f, \alpha) N^{-\nu},$$

where $\nu < 1/2^{d-1} - \delta$.

Since $X_{N_m}(\alpha) \to 0$ for all $\alpha \in P(f)$, which by throwing out a measure-zero subset we assumed consisted only of Diophantine α's, Proposition 3 implies $X_n(\alpha) \to 0$ for all $\alpha \in P(f)$. We will prove this proposition after finishing the proof of Theorem 1. What remains to do is to find one subset $P \subset \mathbf{R}$ of full measure for which $R_2(f, \alpha, N) \to \int_{-\infty}^{\infty} f(x)dx$ for all $\alpha \in P$ and all f which are characteristic function of intervals $[-s, s]$ (or in $C_c^\infty(\mathbf{R})$). To do this, pick a (countable) sequence of positive $f_i \in C_c^\infty(\mathbf{R})$ so that for each $f \geq 0$ as above, there are subsequences $\{f_i^\pm\} \subset \{f_i\}$ which satisfy $f_i^- \leq f \leq f_i^+$ and $\int_{-\infty}^{\infty}(f_i^+ - f_i^-)(x)dx \to 0$. Take $P := \cap_i P(f_i)$ which is still of full measure. For every α we have

$$R_2(f_i^-, \alpha, N) \leq R_2(f, \alpha, N) \leq R_2(f_i^+, \alpha, N),$$

and in addition for $\alpha \in P$, we have $R_2(f_i^\pm, \alpha, N) \to \int_{-\infty}^{\infty} f_i^\pm$. Since $\int_{-\infty}^{\infty} f_i^\pm \to \int_{-\infty}^{\infty} f$, this shows that $R_2(f, \alpha, N) \to \int_{-\infty}^{\infty} f$ for $\alpha \in P$ and gives Theorem 1.

The proof of Proposition 3 will occupy the rest of this section.

3.2. Estimates for Weyl sums. We begin with some consequences of Weyl's estimates for the "Weyl sums" $S(n, N) = \sum_{x \leq N} e(n\alpha x^d)$ which we will need. Throughout the remainder of this section, we set $D = 2^{d-1}$.

Lemma 4. *For α Diophantine, and $M \geq 1$, we have*

$$\sum_{1 \leq n \leq M} |S(n, N)|^D \ll M^{1+\epsilon} N^{D-1+\epsilon}$$

for all $\epsilon > 0$ ($D = 2^{d-1}$).

Proof. This follows from proof of Weyl's inequality (see [6], Lemma 3). We will outline the steps. By repeated squaring, one finds that

$$|S(n, N)|^D \ll N^{D-1} + N^{D-d} \sum_{y_1, \ldots, y_{d-1}=1}^{N} \min\left\{N, \frac{1}{\|d! n\alpha y_1 \ldots y_{d-1}\|}\right\},$$

where $\|\cdot\|$ denotes the distance to the nearest integer. Now sum over $n \leq M$, collecting together terms with the product $d! n y_1 \ldots y_{d-1}$ having a given value m. The number of such terms is at most the divisor function $\tau(m) \ll m^\epsilon$. Since the maximal value of m is $d! M N^{d-1}$, we find

$$\sum_{1 \leq n \leq M} |S(n, N)|^D \ll M N^{D-1} + M^\epsilon N^{D-d+\epsilon} \sum_{m \leq d! M N^{d-1}} \min\left\{N, \frac{1}{\|m\alpha\|}\right\}. \tag{3.2}$$

Proceeding as in [6], we replace α by a rational approximation a/q with $|\alpha - a/q| \leq 1/q^2$, and divide the range of summation into consecutive blocks of length q. This will give

$$\sum_{m \leq d! M N^{d-1}} \min\left\{N, \frac{1}{\|m\alpha\|}\right\} \ll \left(\frac{M N^{d-1}}{q} + 1\right) \cdot (N + q \log q).$$

Inserting into (3.2) we get

$$\sum_{1\le n\le M} |S(n,N)|^D \ll MN^{D-1} + M^\epsilon N^{D-d+\epsilon}\left(\frac{MN^{d-1}}{q}+1\right)\cdot(N+q\log q). \tag{3.3}$$

Now choose $q \le MN^{d-1}$ with $|\alpha - a/q| \le 1/qMN^{d-1}$ (so certainly $|\alpha - a/q| \le 1/q^2$ so (3.3) holds). Since α is Diophantine, $|\alpha - a/q| \gg 1/q^{2+\epsilon}$ which gives $q \gg (MN^{d-1})^{1-\epsilon}$. Therefore

$$\left(\frac{MN^{d-1}}{q}+1\right)\cdot(N+q\log q) \ll (MN^{d-1})^{1+\epsilon},$$

and consequently

$$\sum_{1\le n\le M} |S(n,N)|^D \ll M^{1+\epsilon}N^{D-1+\epsilon}$$

as required. □

As an immediate consequence of this lemma, we get on repeatedly using the Cauchy-Schwarz inequality that

Corollary 5. *For α Diophantine, and $M \ge 1$,*

$$\sum_{1\le n\le M} |S(n,N)|^2 \ll M^{1+\epsilon}N^{2-2/D+\epsilon} \tag{3.4}$$

and

$$\sum_{1\le n\le M} |S(n,N)| \ll M^{1+\epsilon}N^{1-1/D+\epsilon}. \tag{3.5}$$

3.3. Proof of Proposition 3. We first show

$$X_{N+k}(\alpha) - X_N(\alpha) = \frac{1}{N^2}\sum_{0<|n|\le M} \widehat{f}\left(\frac{n}{N}\right)\{s_{off}(n,N+k) - s_{off}(n,N)\} + O\left(M^{2+\epsilon}N^{-2+\delta-2/D}\right). \tag{3.6}$$

We use the representation (2.5),

$$X_N(\alpha) = \frac{1}{N^2}\sum_{n\ne 0}\widehat{f}\left(\frac{n}{N}\right)s_{off}(n,N).$$

Since $f \in C_c^\infty(\mathbf{R})$, its Fourier transform $\widehat{f}$ is rapidly decreasing and so on using the trivial estimate $|s_{off}(n,N)| \le N^2$ we see that for any $b > 0$, $M = N^{1+b}$,

$$X_N(\alpha) = \frac{1}{N^2}\sum_{0\ne|n|\le M}\widehat{f}\left(\frac{n}{N}\right)s_{off}(n,N) + \text{ rapidly decaying term.}$$

Next we use $|s_{off}(n,N)| \le N + |S(n,N)|^2$ and Corollary 5 to deduce that

$$\sum_{0\ne|n|\le M} |s_{off}(n,N+k)| \le MN + \sum_{0\ne|n|\le M} |S(n,N+k)|^2 \ll M^{1+\epsilon}N^{2-2/D}. \tag{3.7}$$

Next we claim that

$$\frac{1}{(N+k)^2}\sum_{0\neq|n|\leq M}\widehat{f}\left(\frac{n}{N+k}\right)s_{off}(n,N+k)=\frac{1}{N^2}\sum_{0\neq|n|\leq M}\widehat{f}\left(\frac{n}{N}\right)s_{off}(n,N+k)$$
$$+O(M^{2+\epsilon}N^{-2+\delta-2/D}). \qquad (3.8)$$

Indeed, write

$$\frac{1}{(N+k)^2}=\frac{1}{N^2}+O\left(\frac{k}{N^3}\right)=\frac{1}{N^2}+O(N^{-3+\delta})$$

and

$$\frac{n}{N+k}=\frac{n}{N}+O(\frac{nk}{N^2})=\frac{n}{N}+O\left(\frac{M}{N^{2-\delta}}\right),$$

so that for $|n|\leq M$, $k<N^\delta$,

$$\widehat{f}\left(\frac{n}{N+k}\right)=\widehat{f}\left(\frac{n}{N}\right)+O\left(\frac{M}{N^{2-\delta}}\right).$$

Therefore

$$\begin{aligned}
&\frac{1}{(N+k)^2}\sum_{0\neq|n|\leq M}\widehat{f}\left(\frac{n}{N+k}\right)s_{off}(n,N+k)\\
&\qquad-\frac{1}{N^2}\sum_{0\neq|n|\leq M}\widehat{f}\left(\frac{n}{N}\right)s_{off}(n,N+k)\\
&=\left(\frac{1}{N^2}+O(\frac{1}{N^{3-\delta}})\right)\sum_{0\neq|n|\leq M}\left(\widehat{f}\left(\frac{n}{N}\right)+O\left(\frac{M}{N^{2-\delta}}\right)\right)s_{off}(n,N+k)\\
&\qquad-\frac{1}{N^2}\sum_{0\neq|n|\leq M}\widehat{f}\left(\frac{n}{N}\right)s_{off}(n,N+k)\\
&\ll\left(\frac{M}{N^{4-\delta}}+\frac{1}{N^{3-\delta}}\right)\sum_{0\neq|n|\leq M}|s_{off}(n,N+k)|\\
&\qquad\ll\frac{M}{N^{4-\delta}}M^{1+\epsilon}N^{2-2/D}=M^{2+\epsilon}N^{-2+\delta-2/D}\qquad\text{by (3.7)}
\end{aligned}$$

as required. This proves (3.6).

Next we express the difference $s_{off}(n,N+k)-s_{off}(n,N)$ as

$$s_{off}(n,N+k)-s_{off}(n,N)=2\,\mathrm{Re}\sum_{N+1\leq y\leq N+k}e(-n\alpha y^d)\sum_{1\leq x\leq N}e\left(n\alpha x^d\right)$$
$$+\sum_{N+1\leq x\neq y\leq N+k}e\left(n\alpha(x^d-y^d)\right).$$

We estimate the second term trivially by k^2:

$$|s_{off}(n,N+k)-s_{off}(n,N)|\leq k|S(n,N+k)|+k^2.$$

Then inserting this into (3.6) we get

$$\begin{aligned} X_{N+k}(\alpha) - X_N(\alpha) &\ll \frac{1}{N^2} \sum_{0<|n|\le M} \left(k|S(n, N+k)| + k^2 \right) + M^{2+\epsilon} N^{-2+\delta-2/D} \\ &\ll \frac{k}{N^2} \sum_{0<|n|\le M} |S(n,N)| + \frac{Mk^2}{N^2} + M^{2+\epsilon} N^{-2+\delta-2/D} \\ &\ll \frac{k}{N^2} M^{1+\epsilon} N^{1-1/D} + \frac{Mk^2}{N^2} + M^{2+\epsilon} N^{-2+\delta-2/D} \qquad \text{by (3.5)} \\ &\ll M^{1+\epsilon} k N^{-1-1/D} \ll N^{b+\delta-1/D+\epsilon}. \end{aligned}$$

Since $b > 0$ can be made arbitrarily small, this proves our proposition. □

4. Triple and Higher Correlations

The higher correlations run into some basic difficulties. For example, consider the triple correlation for $\alpha n^2 \mod 1$. For a test function $f \in C_c^\infty(\mathbf{R}^2)$, let

$$F_N(y_1, y_2) = \sum_{(m_1,m_2)\in \mathbf{Z}^2} f\left(N(y_1+m_1), N(y_2+m_2)\right). \tag{4.1}$$

This function is periodic and has a Fourier expansion

$$F_N(y) = \frac{1}{N^2} \sum_{k\in\mathbf{Z}^2} \widehat{f}\left(\frac{k}{N}\right) e(k\cdot y). \tag{4.2}$$

The triple correlation function of the sequence $\alpha n^2 \mod 1$ and for the test function f is

$$R_3(f,\alpha,N) = \frac{1}{N} \sideset{}{'}\sum_{1\le x,y,z\le N} F_N\left(\alpha(x^2-y^2), \alpha(y^2-z^2)\right), \tag{4.3}$$

where the sum $\sum'$ is over all triples of *distinct* integers x, y, z. The Fourier expansion of R_3 is

$$R_3(f,\alpha,N) = \sum_l c_l(N) e(l\alpha) \tag{4.4}$$

with

$$c_l(N) = \frac{1}{N^3} \sideset{}{'}\sum_{\substack{1,x,y,z\le N, \quad k_1,k_2 \\ k_1(x^2-y^2)+k_2(y^2-z^2)=l}} \widehat{f}\left(\frac{k_1}{N}, \frac{k_1}{N}\right). \tag{4.5}$$

There is no doubt that the mean $\langle R_3 \rangle = c_0(N) \to \widehat{f}(0,0)$, as $N \to \infty$, the expected answer for random sequence, and that more generally $c_l(N) \to 0$ if $l \neq 0$. That is to say that $R_3(f,\alpha,N) \to \widehat{f}(0,0)$ in the weak sense. This can probably be proven. However, a much greater difficulty appears and that is that if $f(0) \neq 0$ then

$$\left\| R_3(f,N) - \widehat{f}(0) \right\|_2^2 \gg N. \tag{4.6}$$

This shows that the L^2 approach to almost-everywhere convergence is problematic in this case. In fact, this feature of the L^2-norm being as large a manifestation of R_3 being very large at rational α's. For almost all α we still expect that $R_3(f,\alpha,N) \to \widehat{f}(0,0)$.

To prove (4.6) note that as $N \to \infty$,

$$\sum_l c_l(N) = \frac{1}{N^3} \sideset{}{'}\sum_{1\leq x,y,x\leq N} \sum_{k_1,k_2} \widehat{f}\left(\frac{k_1}{N}, \frac{k_1}{N}\right) \sim N^2 f(0).$$

Hence if $f(0) \neq 0$ then

$$N^2 \ll |\sum_l c_l(N)| \leq \left(\sum_l |c_l(N)|^2\right)^{1/2} \left(\sum_{l \ll N^3} 1\right)^{1/2}$$
$$= N^{3/2} \left(\sum_l |c_l(N)|^2\right)^{1/2},$$

which gives (4.6).

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Communicated by Ya. G. Sinai