THE PARABOLIC UMBILIC CATASTROPHE AND ITS APPLICATION IN THE THEORY OF ELASTIC STABILITY*

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Abstract. The implications of the parabolic umbilic catastrophe in the theory of elastic stability are investigated. In particular, the influence of terms in the potential energy which are deemed necessary for a complete analysis and the isolation of primary critical surfaces are considered. The results are demonstrated for the example of the buckling and initial post-buckling of a spherical shell under the influence of a constant as well as a spatially variable pressure.

Introduction. Catastrophe theory has been hailed as the most important mathematical discovery in decades. It has also been termed the Emperor with no clothes! This paper makes no effort to address either of the above comments; rather, it is concerned with the application of a specific catastrophe to problems in the theory of elastic stability. Thus, the intended contribution is the evaluation of certain of the implications of catastrophe theory in the context of the theory of elastic stability.

Catastrophe theory [1] and the theory of elastic stability are extremely similar and it is fair to say that the latter is a special case of the former. This relationship has led to a number of papers [2, 3] which provided comparisons between actual physical problems and catastrophe theory as well as a description of areas in which catastrophe theory may be applied. It has become apparent that catastrophe theory has certain features which are of interest in the theory of elastic stability; however, it is equally obvious that the theory of elastic stability is not a trivial application of catastrophe theory. In fact, catastrophe theory does not even address some of the most difficult aspects of the prebuckling and buckling solutions for a given problem [4, 5]. Furthermore, the physical implications involved in a loss of stability play a predominant role in the analysis of physical systems. This has been demonstrated in [6] where it is shown that the least critical surface for a problem is not necessarily related to the initial loss of stability. The above points notwithstanding, it appears that the contribution which catastrophe theory has to offer is in the realm of classification for complex systems and of the determination of the correct number of loads, imperfections, etc. (control parameters) which should be involved in a stability analysis.

This paper investigates the class of elastic stability problems which are described by a potential energy expression that can be reduced to the parabolic umbilic form. This particular problem is of interest for two reasons: first, the potential energy expression used is

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apparently inconsistent from a perturbation point of view and, second, catastrophe theory dictates that this problem requires two independent load-type parameters in order to achieve a proper unfolding. Neither of these features would be considered necessary in a stability analysis and it is therefore appropriate to investigate their influence. Previous investigations of the critical surfaces of the parabolic umbilic model have been examined by Godwin [8] and Bröcker and Lander [9], but they were not oriented towards the theory of elastic stability. In particular, the aim of this paper is to analyze this model from a structural stability point of view while taking advantage of the classification scheme provided by Catastrophe theory.

The first portion of the paper is devoted to the evaluation and analysis of the critical surfaces of the parabolic umbilic catastrophe. In particular, two forms are identified and termed the parabolic umbilic types one and two. The particular critical surfaces which are of relevance in a stability analysis have been isolated and extensive parameter studies have been undertaken.

The above results are then demonstrated for the two-mode buckling problem of a shallow section of a spherical shell. The influence of the additional parameters which are implied by catastrophe theory are also evaluated and are demonstrated to be significant.

The parabolic umbilic catastrophe. The parabolic umbilic catastrophe arises in the analysis of systems which have two coincident least eigenvalues and for which cubic terms in the expansion of the potential energy function about the critical state are in general non-vanishing. That is, if the expansion of the potential energy V, about the ideal critical state, takes the form

$$V = Ax^3 + Bx^2y + Cxy^2 + Dy^3$$

then there are a number of possibilities which arise. If the cubic equation $V = 0$ has one real root and a pair of complex conjugate roots, then V leads to the hyperbolic umbilic catastrophe. If there are three real and unequal roots, then V is classed as an elliptic umbilic. These particular forms have been considered previously in [2, 6]. In addition, there are two singular cases which occur for the three-real-root situation. These are the parabolic umbilic, when there are two equal roots, and the symbolic umbilic, when there are three equal roots. As may be appreciated from the root structure, the parabolic and symbolic umbilics are the non-trivial transitions which exist between the hyperbolic and elliptic umbilics.

The problem of interest in the present investigation is the parabolic umbilic model which is described in standard form as

$$V = \pm x^4 + xy^2 + L_1x^2 + L_2 y^2 - \varepsilon_1 x - \varepsilon_2 y \tag{1}$$

where L_1, L_2, ε_1, ε_2 are the control parameters and x, y are the behavior variables. In a typical elastic stability analysis L_1 and L_2 would be related to some applied loads while ε_1 and ε_2 would be related to the amplitudes of certain geometric imperfections. This is, of course, not necessary as L_1, L_2, ε_1, ε_2 may represent loads, imperfections, material parameters, dimensions and so on. The behavior parameters x, y are related to the amplitudes of the critical modes of the problem. As mentioned previously, the quartic term $\pm x^4$ and the independence of L_1 and L_2 are not the norm in elastic stability analyses and are therefore of particular interest. These factors have been included by catastrophe theorists to provide a stable jet and a complete unfolding of the catastrophe, respectively. It should also be emphasized that the plus or minus possibility for the quartic term is extremely important as the change in sign leads to quite different results. In the present paper, the cases with the plus or minus sign are referred to as the parabolic umbilic type one or type two, respectively.

Critical sets. The critical sets are defined by the criterion that the first and second variations of the potential energy vanish simultaneously. This yields the equilibrium equations

$$\pm 4x^3 + y^2 + 2L_1 x = \varepsilon_1, \tag{2}$$

$$2y(x + L_2) = \varepsilon_2, \tag{3}$$

while critical states of equilibrium occur when the solutions of (2) and (3) also satisfy

$$(\pm 6x^2 + L_1)(x + L_2) = y^2. \tag{4}$$

The critical sets are the surfaces defined by the relationship between L_1, L_2, ε_1, ε_2 when x and y are eliminated from (2), (3) and (4). This elimination is not a trivial matter and it does not appear that an explicit relationship can be obtained in closed form. In the present circumstance it was accomplished numerically by first eliminating y from (2) and (3) by way of Eq. (4). This operation yields

$$\varepsilon_1 = \pm 10x^3 \pm 6L_2 x^2 + 3L_1 x + L_1 L_2 \tag{5}$$

and

$$\varepsilon_2 = \pm 2(x + L_2)\sqrt{(\pm 6x^2 + L_1)(x + L_2)}. \tag{6}$$

In the solution for the critical surface it is required that this surface correspond to real values of x, y, ε_1, ε_2, L_1 and L_2. Further, it may be noted that for real values of ε_1, L_1 and L_2 Eq. (5) has at least one real solution for x. In addition, for real values of ε_2, Eq. (6) implies that $(x + L_2)(\pm 6x^2 + L_1)$ must be positive. Also, from Eq. (3) it follows that if x and ε_2 are real then y is real. Therefore, the existence of real critical states is dependent only on the condition that the discriminant in Eq. (6) be positive.

In practical stability problems L_1 and L_2 would represent load parameters which are generally treated as the unknowns and which are functions of the imperfection parameters ε_1 and ε_2. There will be real solutions for L_1 and L_2 if the system is capable of buckling. Thus the ideal computation technique is to determine L_1 and L_2 given ε_1 and ε_2. This method, however, leads to excess complexity and therefore the following procedure was adopted in this study. Numerical values are provided for ε_1, L_1 and L_2 and the corresponding value of ε_2 was to be evaluated. Thus, ε_1, L_1 and L_2 were substituted into Eq. (5) which then yielded either one or three real roots for x. The real root(s) for x were then substituted into Eq. (6) in order to determine the appropriate value of ε_2. Of course only real values of ε_2 are acceptable and it can be seen that there may be one or three real values of ε_2 corresponding to each set of ε_1, L_1, L_2.

It is clear that a graphical presentation of the critical surfaces is difficult and they are four-dimensional. Thus the figures are presented as projections of the general surfaces onto a two-dimensional plane. This is usually accomplished by introducing the relation $L_2 = \kappa L_1$ where κ is a constant which is assigned a series of values. The critical surfaces are then evaluated on the L_2 versus ε_2 plane for different values of ε_1 and a series of figures are presented for each value of κ.

A particularly interesting aspect of this class of problem is the existence of more than one critical surface for a given combination of $\varepsilon_1, \varepsilon_2$. All of the critical surfaces so obtained are of interest; however, they are not all of physical relevance, as it is only the first critical load encountered on a particular load-deflection path which specifies the buckling load. Furthermore, it may not be the least critical load on a given load deflection path which is predominant and it is often difficult to assess which is the predominant critical load. It should also be pointed out that the critical sets do not allow this interpretation and it is only the combination of the equilibrium equations of the stability criterion that contains the complete information. In this paper, the locus of critical loads which are encountered first as the applied load is increased from zero will be termed the primary critical surface, whereas all other critical surfaces will be termed secondary surfaces.

Typical results of the numerical work are presented in Figs. 1–8, where Figs. 1–4 and Figs. 5–8 are devoted to the type-one and type-two parabolic umbilics respectively. Figs. 1 and 5 are of particular interest as they are representative of the calculations which must be considered in order to distinguish between primary and secondary critical surfaces. In addition, Figs. 2–4 and 6–8 demonstrate the influence of the independence of the "load" parameters L_1 and L_2. All of the figures are plotted with the ordinate being expressed as $\lambda = 1 - L_1$, which is the usual way of presenting load-imperfection curves in a stability analysis. These figures demonstrate quite dramatically the influence of the independence of the "load" parameters, even in the region of $L_1 = L_2 = 0$. The influence of the quartic term in the potential energy has been demonstrated in Fig. 9 where sample critical load-initial imperfection results are calculated for the case of a positive, zero and negative quartic term respectively. It may be appreciated that in certain cases the quartic term may be quite significant.

Example: buckling of a shallow segment of a spherical shell. The results obtained for the parabolic umbilic catastrophe are completely general within the context of the asymptotic nature of the analysis. It is therefore of interest to investigate the implications of the terms which are peculiar to catastrophe theory on a real problem.

The problem under consideration is that of the two-mode buckling of a shallow spherical shell under the influence of external pressure. It represents the extension of a portion of an analysis by Hutchinson [7] in 1967. Within the context of the assumptions presented in that paper, the two-mode buckling case reduces to the analysis of the potential energy expression

$$PE = (\text{const})\left\{\frac{1}{2}\left(1 - \frac{\lambda}{\lambda_{\text{cl}}}\right)\xi_1^2 + \frac{1}{4}\left(1 - \frac{\lambda}{\lambda_{\text{cl}}}\right)\xi_2^2 - \frac{9}{32}\sqrt{3(1 - v^2)}\xi_1\xi_2^2 - \frac{\lambda}{\lambda_{\text{cl}}}\xi_1\bar{\xi}_1 - \frac{1}{2}\frac{\lambda}{\lambda_{\text{cl}}}\xi_2\bar{\xi}_2\right\} \tag{7}$$

where λ is the load parameter corresponding to the magnitude of the external pressure, λ_{cl} is the classical critical load, v is Poisson's ratio, ξ_1 and ξ_2 are the amplitudes of the critical modes while $\bar{\xi}_1$ and $\bar{\xi}_2$ are the amplitudes of imperfections with mode shapes identical to those of ξ_1 and ξ_2 respectively. It should be noted that the above Eq. (7) is expressible as

$$V = (\lambda - \lambda_{\text{cl}})(x^2 + y^2) + xy^2 - \varepsilon_1 x - \varepsilon_2 y \tag{8}$$

which is of precisely the form which leads to the parabolic umbilic catastrophe. It should be

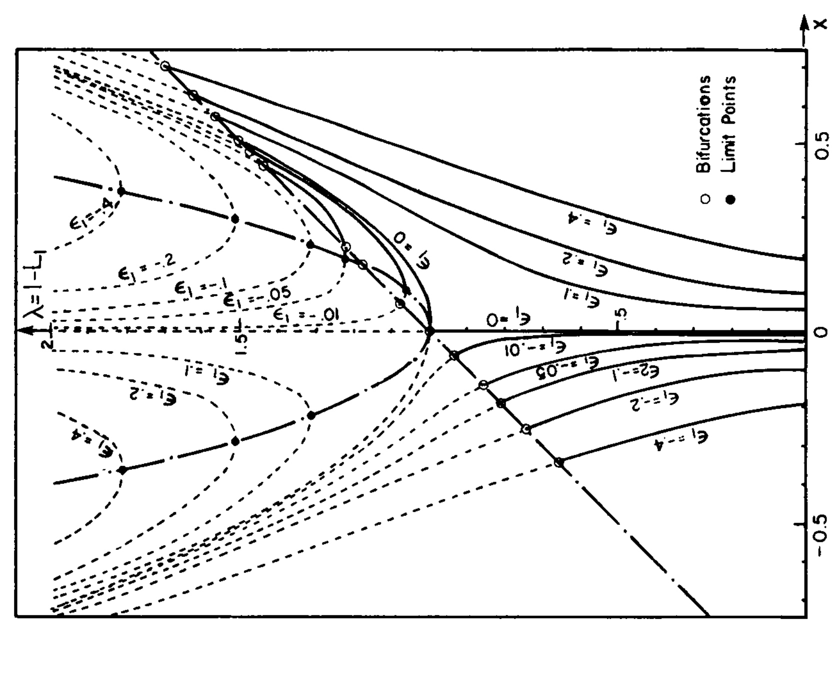

FIG. 1. Parabolic umbilic type-one equilibrium paths, $L_2 = L_1, \varepsilon_2 = y = 0(\varepsilon_1 = 4x^3 + 2L_1x)$.

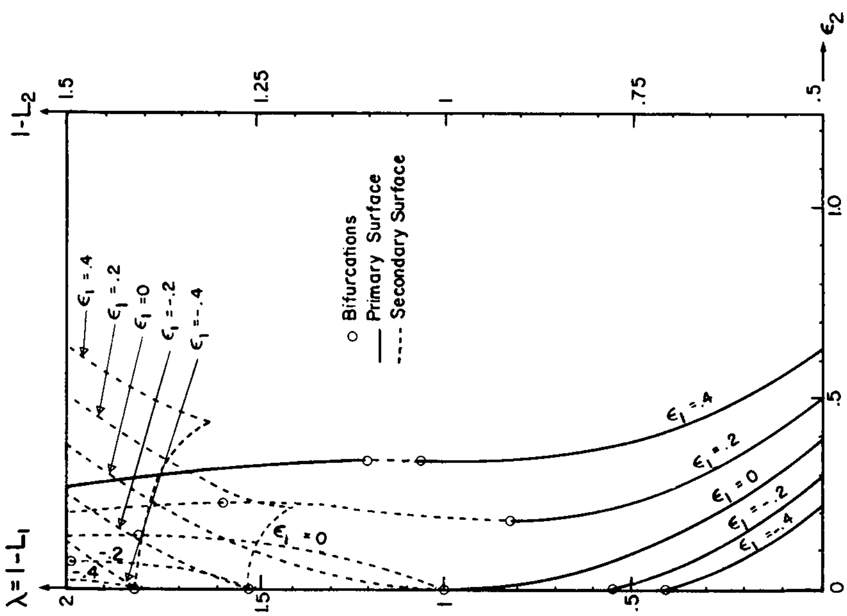

FIG. 2. Parabolic umbilic type-one critical surface, $L_2 = 0.5L_1$.

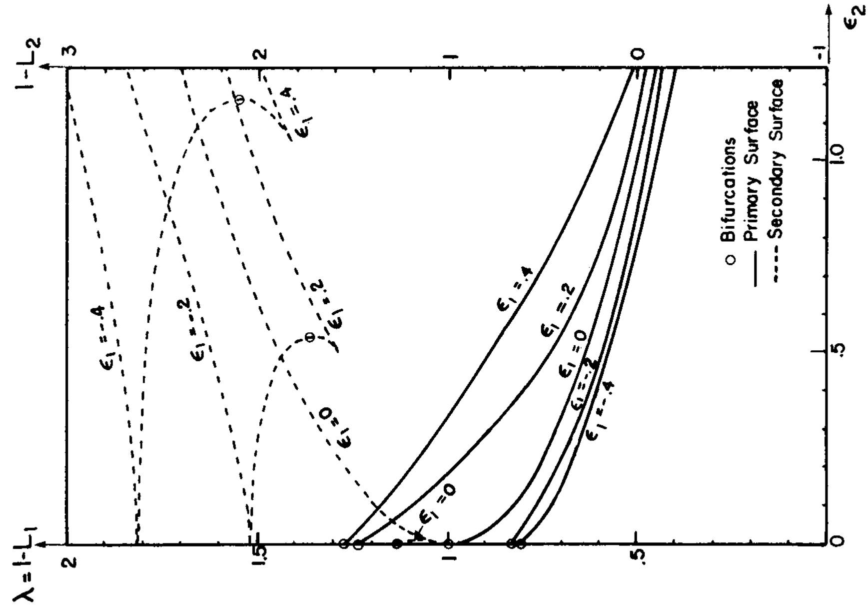

FIG. 4. Parabolic umbilic type-one critical surface, $L_2 = 2L_1$.

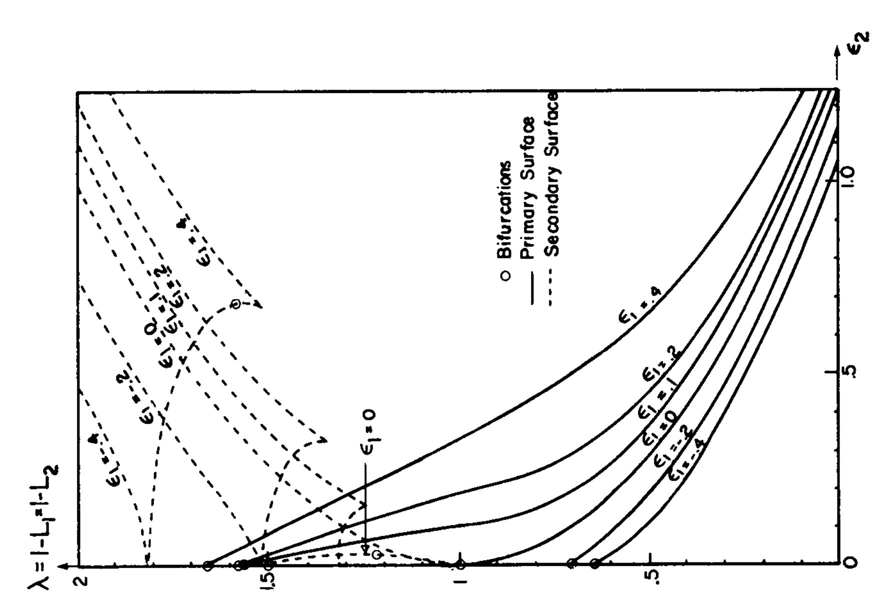

FIG. 3. Parabolic umbilic type-one critical surface, $L_2 = L_1$.

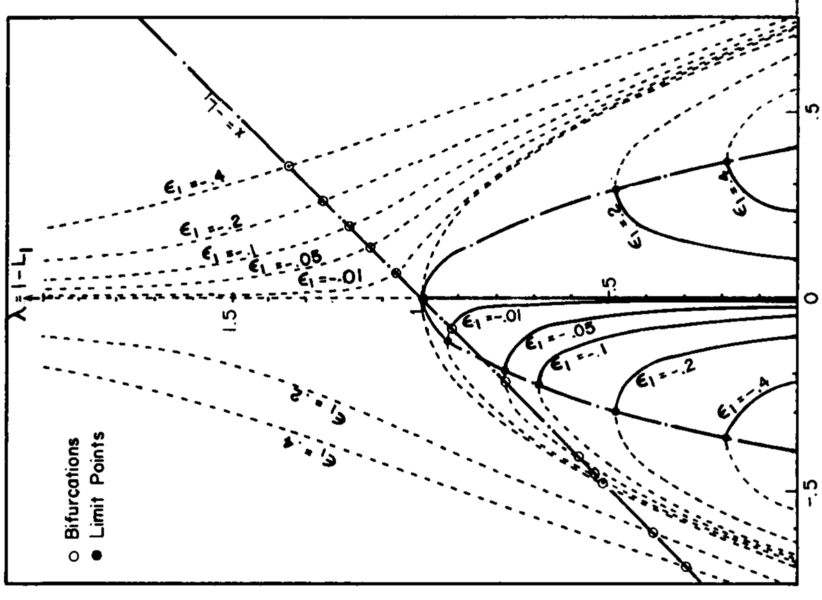

FIG. 5. Parabolic umbilic type-two equilibrium paths, $L_2 = L_1$, $\varepsilon_2 = y = O(\varepsilon_1 = -4x^3 + 2L_1 x)$.

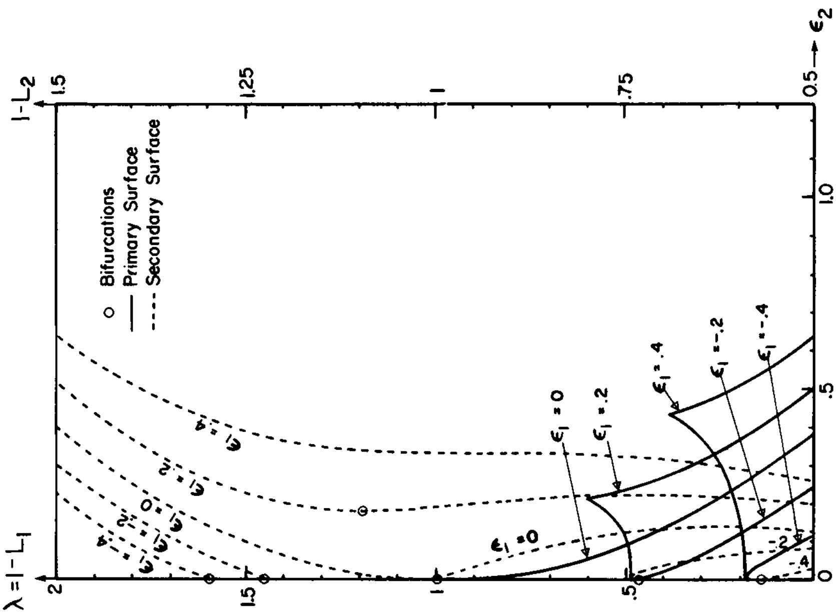

FIG. 6. Parabolic umbilic type-two critical surface $L_2 = 0.5L_1$.

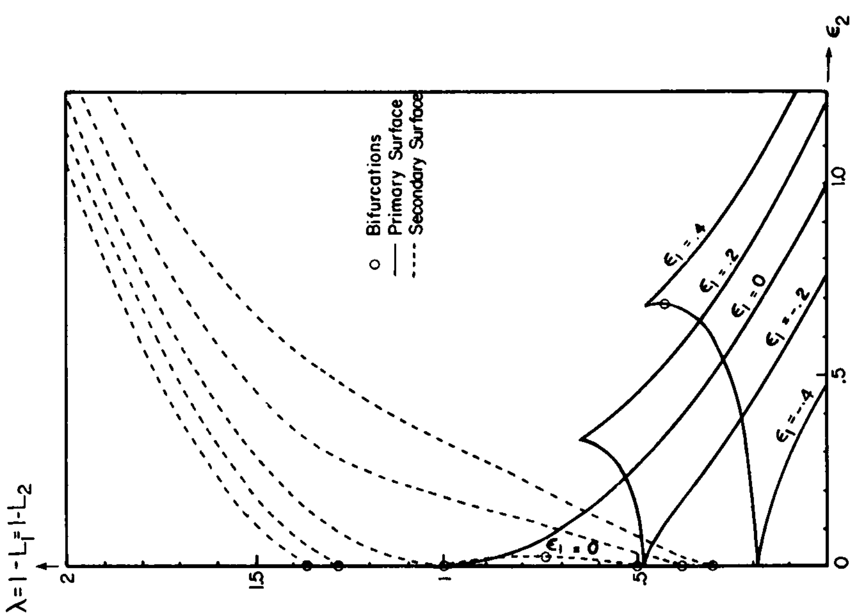

FIG. 7. Parabolic umbilic type-two critical surface, $L_2 = L_1$.

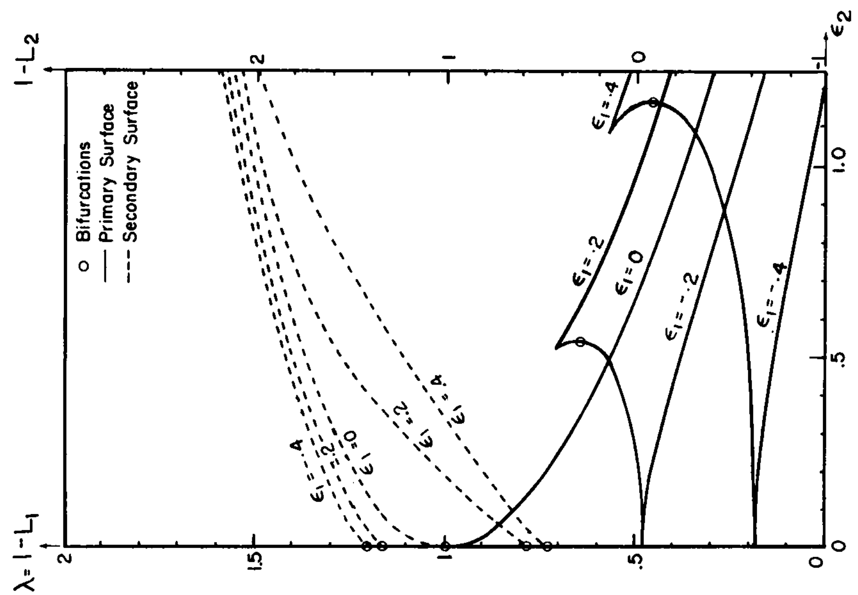

FIG. 8. Parabolic umbilic type-two critical surface, $L_2 = 2L_1$.

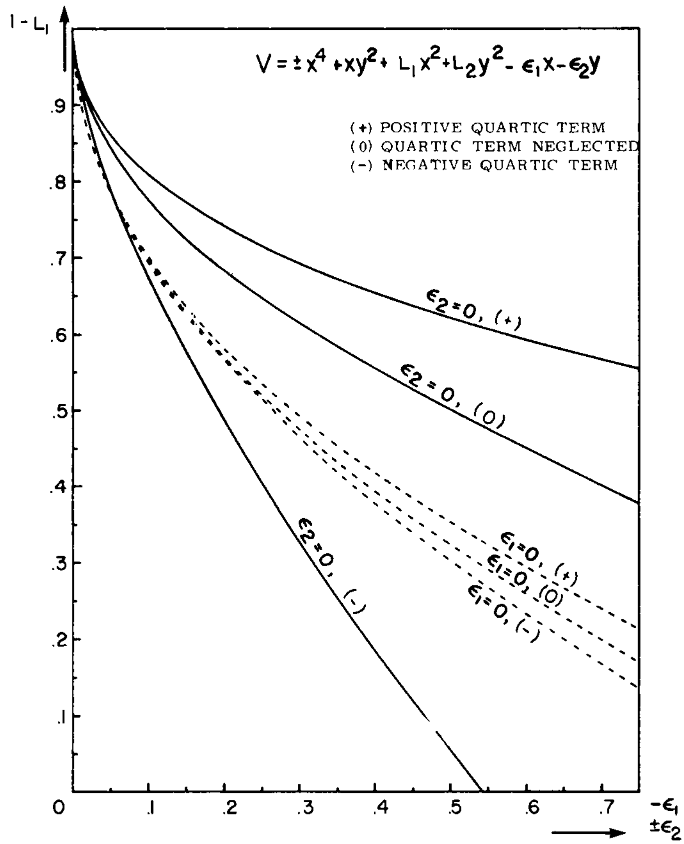

FIG. 9. Effect of the quartic term of the potential energy on the imperfection-sensitivity of the parabolic umbilic model, $L_2 = L_1$.

cautioned that there may be some debate as to whether this is in fact enough justification to assume that this problem is representable as a parabolic umbilic [3]. This is particularly true since there exists the possibility of coupling with other buckling modes. However, it is the opinion of the present authors that if this two-mode problem is going to be investigated then the least that must be done is to analyze it as if it were equivalent to the parabolic umbilic.

A comparison of the present potential energy expression, Eq. (8), with the form of the parabolic umbilic expression, Eq. (1), reveals that Eq. (8) must be supplemented by the addition of quartic terms and another independent load parameter. The first of these is accomplished by extending the analysis [7] to include fourth-order terms while the second is accomplished by assuming that the shell is acted upon by a constant and a spatially variable applied pressure.

For the analysis, the applied load parameter λ is taken in the form

$$\lambda = \lambda_u + \lambda_n \cos\frac{q_0 x}{R}\cos\frac{\sqrt{3}\,q_0 y}{R} \tag{9}$$

where $q_0^2 \equiv 2CR/t$. In the above expression, the first term represents applied uniform pressure and is independent of the coordinates, while λ_n is the amplitude of the non-uniform applied sinusoidal pressure. Thus, the problem considered results from the influence of perturbations in the uniform and non-uniform pressure about the critical uniform pressure state ($\lambda_u = \lambda_{u_{cl}}$ and $\lambda_n = 0$). With these modifications, the approximation to the potential energy becomes (as shown in the appendix)

$$\frac{PE}{\lambda_{cr_{cl}}} = \left[\frac{Et}{4}\left(\frac{tq_0}{R}\right)^2\right] \cdot \left\{\left(1 - \frac{\lambda_u}{\lambda_{u_{cl}}}\right)(\xi_1)^2 + \left[\frac{1}{2}\left(1 - \frac{\lambda_u}{\lambda_{u_{cl}}}\right) + \frac{1}{32}\frac{\lambda_n}{\lambda_{u_{cl}}}\right](\xi_2)^2 + \frac{9C}{16}\xi_1\xi_2^2 + \frac{3}{8}(1-\nu)\xi_1^4 - \left(\frac{\lambda_u}{\lambda_{u_{cl}}}\right)(2\xi_1\bar{\xi}_1) + \left(-\frac{\lambda_u}{\lambda_{u_{cl}}} + \frac{1}{16}\frac{\lambda_n}{\lambda_{u_{cl}}}\right)(\xi_2\bar{\xi}_2)\right\} S_0. \quad (10)$$

It can readily be shown that this problem transforms to a type-one parabolic umbilic

$$V = x^4 + xy^2 + L_1x^2 + L_2y^2 - \varepsilon_1 x - \varepsilon_2 y \quad (11)$$

and is therefore properly unfolded. When Eqs. (10) and (11) are compared, it may be noted that L_2 may be expressed as $L_2 = \frac{1}{2}L_1 + L_2'$. Thus, in Figs. 10, the influence of the L_2'

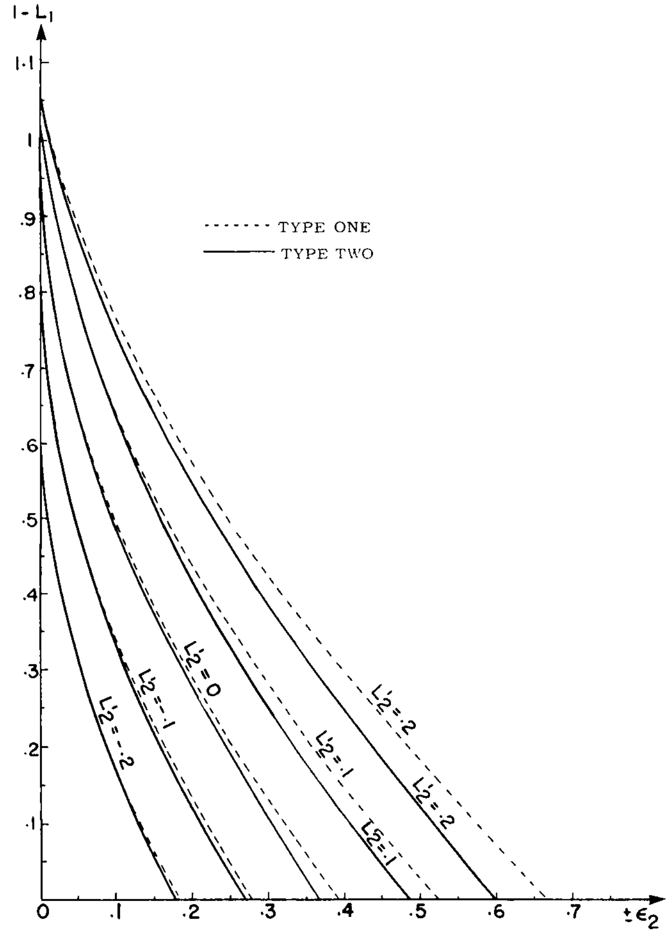

FIG. 10a. Influence of two independent applied loads on the imperfection sensitivity of the parabolic umbilic type-one and type-two models, $L_2 = 0.5L_1 + L_2'$, $\varepsilon_1 = 0$.

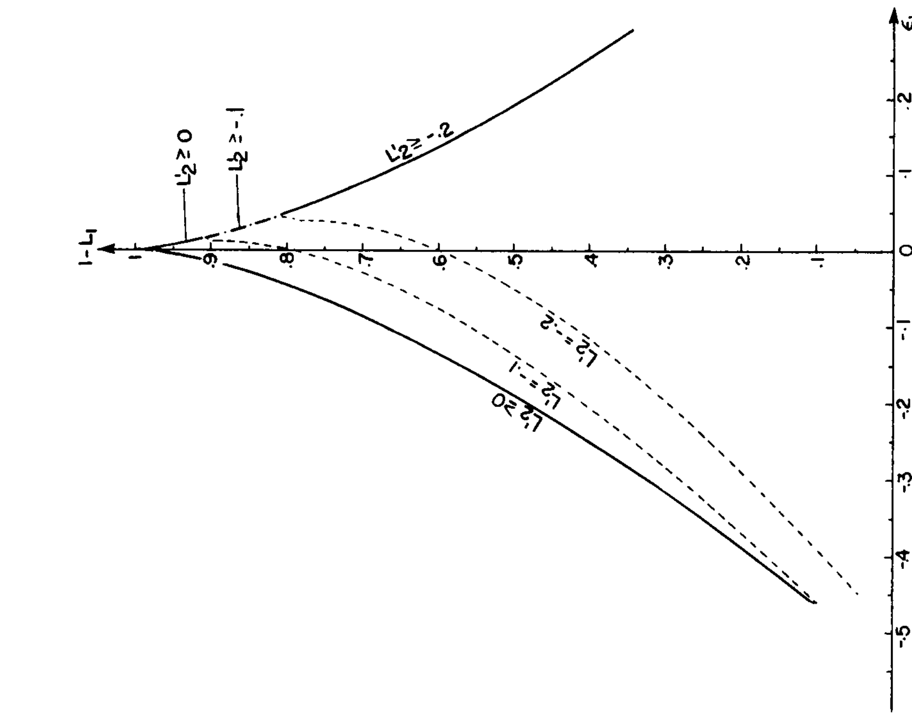

FIG. 10c. Influence of two independent applied loads on the imperfection sensitivity of the parabolic umbilic type-two model, $L_2 = 0.5L_1 + L'_2$, $\varepsilon_2 = 0$.

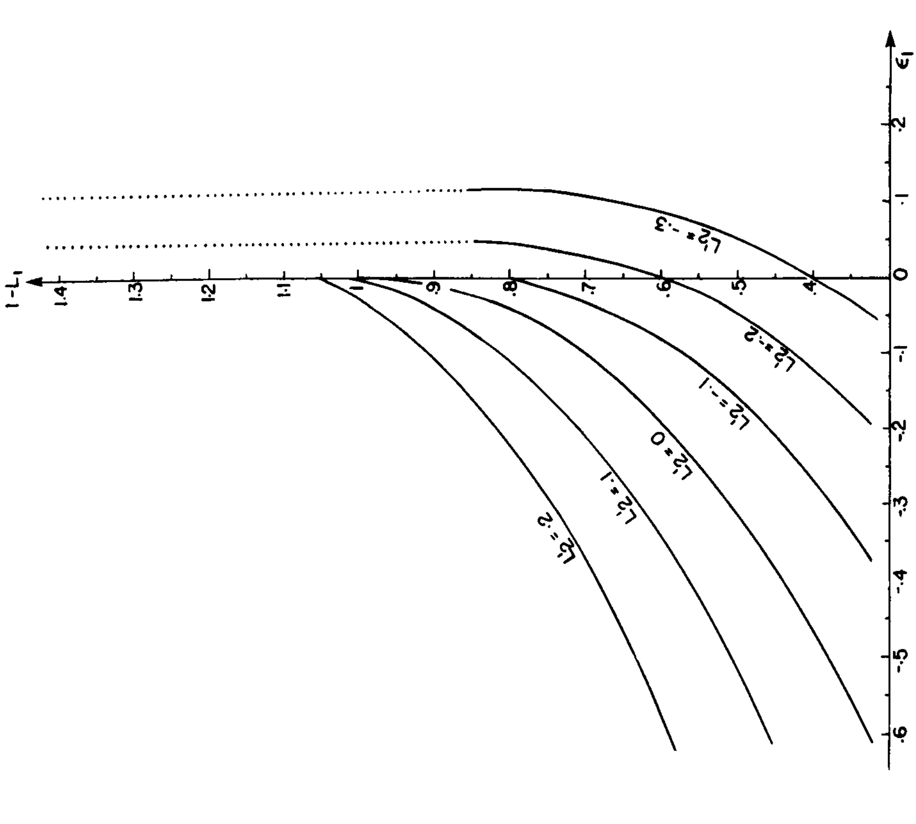

FIG. 10b. Influence of two independent applied loads on the imperfection sensitivity of the parabolic umbilic type-one model, $L_2 = 0.5L_1 + L'_2$, $\varepsilon_2 = 0$.

factor (or nonuniform pressure component) has been considered. In addition, both the type-one and type-two cases have been plotted as these represent the extremes which may be encountered in a critical load–initial imperfection plot. In Fig. 10a the type-one results are shown as dotted lines while the type-two results are given as a solid line. Although there are some differences, it is evident that the inclusion of the quartic term does not play a vital role. On the other hand, when the same cases are plotted for $\xi_2 = 0$ (Figs. 10b and 10c), it may be appreciated that the results are significantly different. This is because $\xi_2 = 0$ implies that the cubic term xy^2 has no influence and thus the quartic term x^4 predominates. Furthermore, from the equilibrium paths given in Figs. 1 and 5, it is clear that the type-two situation (Fig. 10c) is far more imperfection-sensitive than the type-one model (Fig. 10b). Also, since the effect of L'_2 is to raise or lower the linear stability boundaries in Figs. 1 and 5, the type-two model is relatively unaffected by changes in L'_2. In order to illustrate other situations, the case $L_2 = L_1 + L'_2$ has been investigated in Figs. 11, These results demonstrate the same trends as in the previous case. Thus, it may be concluded that for certain situations the retention of the quartic term is essential.

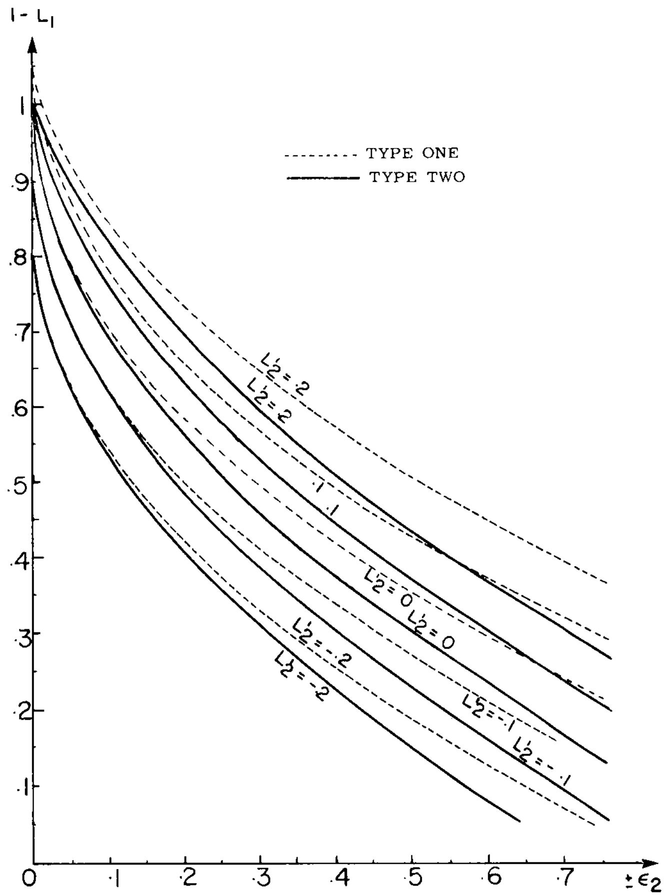

FIG. 11a. Influence of two independent applied loads on the imperfection sensitivity of the parabolic umbilic type-one and type-two models, $L_2 = L_1 + L'_2$, $\varepsilon_1 = 0$.

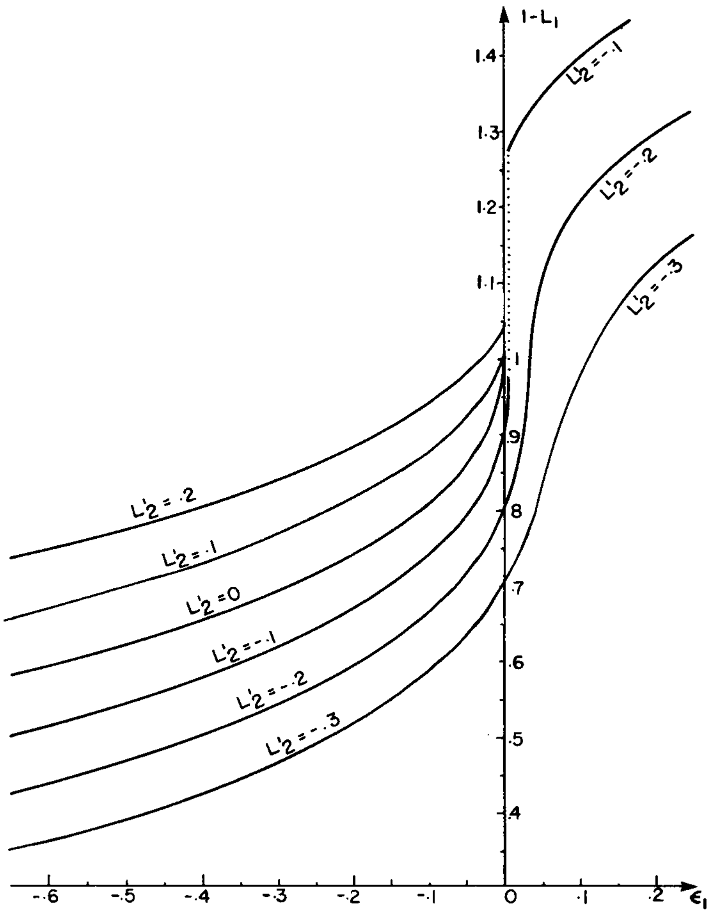

FIG. 11b. Influence of two independent applied loads on the imperfection sensitivity of the parabolic umbilic type-one model, $L_2 = L_1 + L'_2, \varepsilon_2 = 0$.

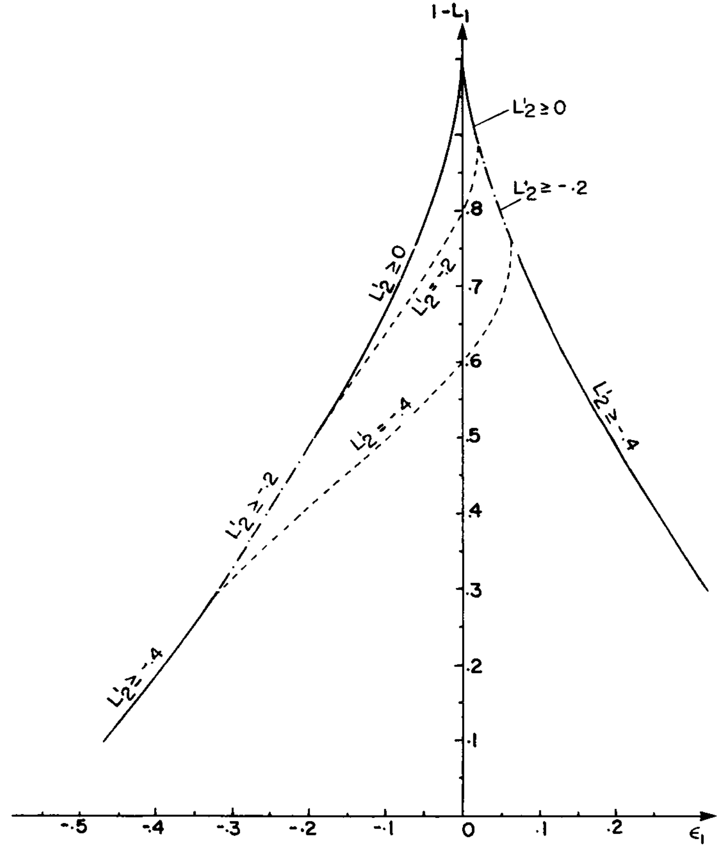

FIG. 11c. Influence of two independent applied loads on the imperfection sensitivity of the parabolic umbilic type-two model, $L_2 = L_1 + L'_2, \varepsilon_2 = 0$.

Finally, it should be noted that although the spherical shell buckling problem clearly displays the effects of the independence of the two applied loads, the effect of the quartic term is not particularly decisive because the coefficient of this term turns out to be positive, and thus the resulting potential energy expression is classified as parabolic umbilic type one. Nevertheless, the work involved in computing this term is not wasted, since there is no a priori way to predict whether it is positive or negative. Further, even though the quartic term found is positive, a close examination of its equilibrium paths (Fig. 1) shows that the buckling load for positive values of geometric imperfection ε_1 is always above 1.5, which is quite different from the prediction of 1.0 if the quartic term is omitted. The complementary (secondary) paths for the spherical shell problem, as shown in dotted lines in Fig. 1, would have been also totally different if the quartic term were omitted. Further applications of this model can be found in [10].

Summary. The parabolic umbilic catastrophe has been solved in its most general form and the critical surfaces have been determined. In addition, those critical surfaces which are encountered first as the load is increased from zero have been isolated and designated as the primary critical surfaces. The results have been demonstrated in the two-mode buckling problem of a pressured spherical shell. It has been demonstrated that the addition of terms which are dictated by catastrophe theory is indeed highly important and can alter the critical surfaces significantly.

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Appendix: expanded potential energy of a complete spherical shell under a uniform and a non-uniform applied pressure. The membrane strain-displacement relations are nonlinear of the form

$$\varepsilon_x = U_{,x} + W/R + \tfrac{1}{2}W_{,x}^2,$$

$$\varepsilon_y = V_{,y} + W/R + \tfrac{1}{2}W_{,y}^2,$$

$$\varepsilon_{xy} = \tfrac{1}{2}(U_{,y} + V_{,x}) + \tfrac{1}{2}W_{,x}W_{,y}, \tag{A.1}$$

while the bending strain-displacement relations are taken in the linear form as

$$\kappa_x = W_{,xx}, \qquad \kappa_y = W_{,yy}, \qquad \kappa_{xy} = W_{,xy}. \tag{A.2}$$

Furthermore, the stress-strain relationship is assumed so that,

$$N_x = \frac{Et}{1-\nu^2}(\varepsilon_x + \nu\varepsilon_y), \qquad M_x = D(\kappa_x + \nu\kappa_y),$$

$$N_y = \frac{Et}{1-\nu^2}(\varepsilon_y + \nu\varepsilon_x), \qquad M_y = D(\kappa_y + \nu\kappa_x), \qquad \text{(A.3, A.4)}$$

$$N_{xy} = \frac{Et}{1+\nu}\varepsilon_{xy}, \qquad M_{xy} = D(1-\nu)\kappa_{xy},$$

where E is Young's modulus and t is the thickness of the spherical shell, N_x, N_y and N_{xy} are the membrane stress resultants, M_x, M_y and M_{xy} are the bending stress resultants and D is the flexural rigidity which equals $Et^3/(4C^2)$ with $C = (3(1-\nu^2))^{1/2}$.

The potential energy is therefore expressible as

$$PE = U_m + U_b - \Omega \qquad \text{(A.5)}$$

where U_m is the membrane strain energy, U_b is the bending strain energy and Ω is the work term. These quantities are defined by

$$U_m = \frac{1}{2}\iint N_x\varepsilon_x + N_y\varepsilon_y + (N_{xy})(2\varepsilon_{xy})\,dx\,dy, \qquad \text{(A.6)}$$

$$U_b = \frac{1}{2}\iint M_x\kappa_x + M_y\kappa_y + (M_{xy})(2\kappa_{xy})\,dx\,dy, \qquad \text{(A.7)}$$

$$\Omega = \iint pW\,dx\,dy, \qquad \text{(A.8)}$$

where negative values of p indicate external pressure and negative values of W denote an inward displacement.

It is assumed that the pre-buckling state is linear so that there is a uniform contraction of the spherical shell with no in-plane displacements. The total displacement can therefore be expressed as

$$U = u, \qquad V = v, \qquad W = c_0 R + w, \qquad \text{(A.9)}$$

where u, v and w are incremental quantities which represent the change in displacements from the pre-buckling state to the equilibrium state under consideration. Also, c_0 is proportional to the applied pressure and is independent of the coordinates.

At this stage, the potential energy can be grouped according to the degree of the incremental displacements u, v and w into the form

$$PE = P'_1[u] + (P^0_2[u] + P'_2[u]) + (P^0_3[u] + P'_3[u]) + P^0_4[u] + \cdots \qquad \text{(A.10)}$$

where $P_i[u]$ contain terms which are of the ith degree in the displacements. The superscript "0" denotes that the functional is independent of the applied load while a prime denotes that the functional is a linear function of the applied load.

Substituting the total displacement into the potential energy expression and grouping the result according to the powers of u, v and w yields

$$P'_1[u] = \iint \left[\frac{Et}{1-\nu^2} c_0 \frac{2(1+\nu)}{R} - p \right] w \, dx \, dy, \tag{A.11}$$

$$\begin{aligned} P_2^0[u] + P'_2[u] = \frac{Et}{2(1-\nu^2)} \iint & \left(u_{,x} + \frac{w}{R}\right)^2 + \left(v_{,y} + \frac{w}{R}\right)^2 \\ & + (2\nu)\left(u_{,x} + \frac{w}{R}\right)\left(v_{,y} + \frac{w}{R}\right) + \frac{1}{2}(1-\nu)(u_{,y} + v_{,x})^2 \\ & + \frac{t^2}{12}[w_{,xx}^2 + w_{,yy}^2 + 2\nu w_{,xx} w_{,yy} + 2(1-\nu) w_{,xy}^2] \\ & + c_0(1+\nu)(w_{,x}^2 + w_{,y}^2) \, dx \, dy, \end{aligned} \tag{A.12}$$

$$\begin{aligned} P_3^0[u] = \frac{Et}{2(1-\nu^2)} \iint & \left(u_{,x} - \frac{1}{R} w\right)(w_{,x})^2 + \left(\nu v_{,y} + \frac{w}{R}\right)(w_{,x})^2 \\ & + \left(v_{,y} + \frac{w}{R}\right)(w_{,y}^2) + \nu\left(u_{,x} + \frac{w}{R}\right)(w_{,y})^2 \\ & + (1-\nu)(u_{,y} + v_{,x})(w_{,x} w_{,y}) \, dx \, dy, \end{aligned} \tag{A.13}$$

$$P_4^0[u] = \frac{Et}{2(1-\nu^2)} \frac{1}{4} \iint (w_{,x}^2 + w_{,y}^2)^2 \, dx \, dy. \tag{A.14}$$

By stipulating that the pre-buckling state is an equilibrium state, that is, setting $\delta P'_1[u]$ to zero, one obtains

$$c_0 = \frac{p(1-\nu)R}{2Et}. \tag{A.15}$$

Furthermore, the differential equations for classical buckling are obtained by setting the first variation of $P_2^0[u] + P'_2[u]$ to zero, so that

$$\begin{aligned} & \delta(P_2^0[u] + P'_2[u]) \\ & = \frac{Et}{2(1-\nu^2)} \iint \Bigg\{ (-2)\left[\left(u_{,xx} + \nu v_{,xy} + \frac{1+\nu}{R} w_{,x}\right) + \left(\frac{1-\nu}{2}\right)(u_{,yy} + v_{,xy})\right](\delta u) \\ & + (-2)\left[\left(v_{,yy} + \nu u_{,xy} + \frac{1+\nu}{R} w_{,y}\right) + \left(\frac{1-\nu}{2}\right)(u_{,xy} + v_{,xx})\right](\delta v) \\ & + (2)\left[\left(\frac{1+\nu}{R}\right)\left(u_{,x} + v_{,y} + \frac{2w}{R}\right) + \frac{t^2}{12} \nabla^4 w \right. \\ & \left. - c_0(1+\nu)(w_{,xx} + w_{,yy})\right](\delta w) \Bigg\} \, dx \, dy = 0. \end{aligned} \tag{A.16}$$

Note that by assuming that the buckling wavelength is small compared with the radius of the shell, all the forced and natural boundary conditions are replaced by periodicity requirements. The solution is of the form

$$\begin{aligned} u_c &= \sum A_{k_x k_y} \sin\left(\frac{k_x x}{R}\right) \cos\left(\frac{k_y y}{R}\right), \\ v_c &= \sum B_{k_x k_y} \cos\left(\frac{k_x x}{R}\right) \sin\left(\frac{k_y y}{R}\right), \\ w_c &= \sum C_{k_x k_y} \cos\left(\frac{k_x x}{R}\right) \cos\left(\frac{k_y y}{R}\right), \end{aligned} \tag{A.17}$$

where the summation is taken over all possible wave numbers k_x and k_y. Thus, the three equations for classical buckling can be expressed as

$$\begin{bmatrix} k_x^2 + \frac{1-\nu}{2} k_y^2 & \frac{1+\nu}{2} k_x k_y & k_x(1+\nu) \\ \left(\frac{1+\nu}{2}\right) k_x k_y & k_x^2\left(\frac{1-\nu}{2}\right) + k_y^2 & (1+\nu)k_y \\ (1+\nu)k_x & (1+\nu)k_y & 2(1+\nu) + \frac{t^2}{12R^2} K^2 \\ & & + c_0(1+\nu)K \end{bmatrix} \begin{bmatrix} \mathrm{A} \\ \mathrm{B} \\ \mathrm{C} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{A.18}$$

where K is defined to be $k_x^2 + k_y^2$. Also, since no confusion can arise, the subscripts on A, B and C have been dropped. By setting the determinant of the above homogeneous system to zero, the eigenvalue is found to be

$$\lambda \triangleq \frac{-c_0}{1-\nu} \equiv -p\,\frac{R}{2Et} = \frac{1}{K} + \frac{K}{q_0^4}. \tag{A,19}$$

Minimizing the eigenvalue λ with respect to K yields the condition for minimum eigenvalue

$$K = q_0^2 \tag{A.20}$$

which implies that

$$\lambda_{c1} = 2/q_0^2. \tag{A.21}$$

Setting the quantity C to be the thickness of the shell, the quantities A and B are found to be

$$A = \frac{-(1+\nu)k_x t}{K}, \qquad B = \frac{-(1+\nu)k_y t}{K}, \qquad C = t. \tag{A.22}$$

It was shown in Hutchinson's paper [7] that the two-mode case occurs for the following two sets of wave numbers:

$$k_x = 1 \qquad k_y = 0 \qquad \text{Set 1,} \qquad k_x = \tfrac{1}{2} \qquad k_y = \frac{\sqrt{3}}{2} \qquad \text{Set 2.} \tag{A.23}$$

The first set of wave numbers implies

$$\begin{bmatrix} u_c^{(1)} \\ v_c^{(1)} \\ w_c^{(1)} \end{bmatrix} = t\xi_1 \begin{bmatrix} \dfrac{-(1+\nu)}{q_0} \sin \dfrac{q_0 x}{R} \\ 0 \\ \cos \dfrac{q_0 x}{R} \end{bmatrix}, \tag{A.24}$$

while the remaining set implies

$$\begin{bmatrix} u_c^{(2)} \\ v_c^{(2)} \\ w_c^{(2)} \end{bmatrix} = t\xi_2 \begin{bmatrix} \dfrac{-(1+\nu)}{2q_0} & \cos \dfrac{q_0 x}{2R} \cos \dfrac{\sqrt{3}\, q_0 y}{2R} \\ \dfrac{-\sqrt{3}(1+\nu)}{2q_0} & \cos \dfrac{q_0 x}{2R} \sin \dfrac{\sqrt{3}\, q_0 y}{2R} \\ & \cos \dfrac{q_0 x}{2R} \cos \dfrac{\sqrt{3}\, q_0 y}{2R} \end{bmatrix} \tag{A.25}$$

For the analysis of the post-buckling behavior of the system it is necessary to include higher order terms in the potential energy. The cubic term of a two-mode system is found to be

$$P_3[^{u(1)}_{c} + u_c^{(2)}] = P_3[u_c^{(1)}] + P_{12}[^{(1)}_{c}, u_c^{(2)}] + P_{21}[u_c^{(1)}, u_c^{(2)}] + P_3[u_c^{(2)}] \tag{A.26}$$

where $P_{ij}[u, v]$ means that the functional is of the ith power in u and jth power in v. Substituting the first and second modes into the above and carrying out the integration, one obtains, after some manipulation,

$$P_{12}[u_c^{(1)}, u_c^{(2)}] = \frac{Et^3}{R^2} \frac{9C}{32} \xi_1 \xi_2^2 S_0 \tag{A.27}$$

where S_0 is the area of a section of the shell and $P_3[u_c^{(1)}] = P_2[u_c^{(1)}, u_c^{(2)}] = P_3[u_c^{(2)}] = 0$.

Since the non-vanishing cubic term is of the form xy^2 it is necessary to compute the quartic term. The desired coefficient of this term is

$$A_4 \xi_1^4 = P_4[u_c^{(1)}] - P_2[\phi] \tag{A.28}$$

where the differential equation for ϕ is given by $\delta P_2[\phi] = -\delta P_3[u_{cr}^{(1)}]$. Thus the appropriate equations are

$$-2\left[\phi^u_{,xx} + \frac{1+\nu}{R} \phi^w_{,x}\right](\delta\phi^u) = +\xi_1^2 \left[t^2 \left(\frac{q_0}{R}\right)^3 \sin \frac{2q_0 x}{R}\right](\delta\phi^u)$$

$$(\phi^u_{,xx})(\delta\phi^v) = 0,$$

$$2\left[\left(\frac{1+\nu}{R}\right)\left(\phi^u_{,x} + \frac{2}{R}\phi^w\right) + \frac{t^2}{12}\phi^w_{,xxxx} + \frac{2}{q_0^2}(1-\nu^2)\phi^w_{,xx}\right](\delta\phi^w)$$

$$= -\xi_1^2 \left[\frac{(1+\nu)}{R}\left(\frac{q_0}{R}\right)^2 \left(\frac{1}{2} - \frac{1}{2}\cos \frac{2q_0 x}{R}\right)\right] t^2(\delta\phi^w). \tag{A.29}$$

Since the buckling mode $u_c^{(1)}$ is axisymmetric, $-\delta P_3[u_c^{(1)}]$ will also be axisymmetric. Thus, in the above, only terms independent of the y coordinate are retained. Furthermore, we have replaced λ by λ_{cl} as a first approximation. Again, all boundary conditions are replaced by the periodicity requirements. Thus, the solution of these differential equations is found to be

$$\phi = \begin{bmatrix} \phi^u \\ \phi^v \\ \phi^w \end{bmatrix} = \xi_1^2 \begin{bmatrix} \frac{t^2 q_0}{8R} \sin \frac{2q_0 x}{R} \\ \gamma_1 x + \gamma_0 \\ -\frac{1}{8} \frac{q_0^2 t^2}{R} \end{bmatrix} \tag{A.30}$$

Since the modification term $(-1)P_2[\phi]$ is identical to $\frac{1}{2}P_{21}[u_c^{(1)}, \phi]$ [4], and the cubic term (upon neglecting all terms which involve derivatives with respect to y coordinates) is independent of ϕ^v, the constant γ_1 should be set to zero. By inspection, the constant γ_0 has no influence on the modification term.

Thus, the modification term is found to be

$$(-1)P_2[\phi] = \tfrac{1}{2}P_{21}[u_c^{(1)}, \phi] = \frac{(Et^3)}{R^2}\left(\frac{-3}{16}\right)(2+\nu)\xi_1^4 S_0 \tag{A.31}$$

and the desired quartic term is

$$A_4 \xi_1^4 = P_4[u_c^{(1)}] - P_2[\phi] = \frac{Et^3}{R^2}\frac{3}{16}(1-\nu)\xi_1^4 S_0 \tag{A.32}$$

For the case of a uniform and a non-uniform applied pressure, the eigenvalue parameter λ (which is a small positive nondimensional quantity) can be expressed as

$$\lambda = \lambda_u + \lambda_n \cos\frac{q_0 x}{R} \cos\frac{\sqrt{3}\, q_0 y}{R} \tag{A.33}$$

so that in a two-mode system,

$$P_2'[u] = \frac{Et}{2(1-\nu^2)} \iint -\left(\lambda_u + \lambda_n \cos\frac{q_0 x}{R}\cos\frac{\sqrt{3}\, q_0 y}{R}\right)(1-\nu^2) \cdot [(w_{c,x}^{(1)} + w_{c,x}^{(2)})^2 + (w_{c,y}^{(2)})^2]\, dx\, dy. \tag{A.34}$$

The Taylor's expanded potential energy at the point $\lambda_u = \lambda_{u_{cl}}$ and $\lambda_n = \lambda_{n_{cl}}$ is

$$PE = (\lambda_u - \lambda_{u_{cl}})\frac{d}{d\lambda_u}(P_2[u]) + (\lambda_n - \lambda_{n_{cl}})\frac{d}{d\lambda_n}(P_2[u]) + P_{12}[u_c^{(1)}, u_c^{(2)}] + (P_4[u_c^{(1)}] - P_2[\phi]) + Q_1[u] \tag{A.35}$$

where, from the classical buckling analysis, $\lambda_{u_{cl}} = 2/q_0$, $\lambda_{n_{cl}} = 0$. Furthermore, it is found that

$$(\lambda_u - \lambda_{u_{cl}})\frac{d}{d\lambda_u}(P_2[u]) = (\lambda_u - \lambda_{u_{cl}})\left(\frac{-Et^3}{R^2}\right)\left(\frac{q_0^2}{2}\right)\left[\frac{1}{2}\xi_1^2 + \frac{1}{4}\xi_2^2\right]S_0, \tag{A.36}$$

$$(\lambda_n - 0)\frac{d}{d\lambda_n}(P_2[u]) = (\lambda_n)\left(\frac{-Et^3}{R^2}\frac{q_0^2}{2}\right)\left(\frac{-1}{64}\xi_2^2 S_0\right). \tag{A.37}$$

From the above, the imperfection terms are easily found to be

$$\left(\frac{Et^3}{R^2}\right)\left(\frac{q_0^2}{2}\right)\left[-\lambda_u(\xi_1\overline{\xi_1}+\frac{1}{2}\,\xi_2\overline{\xi_2})+(\lambda_n)\left(\frac{1}{32}\,\xi_2\overline{\xi_2}\right)\right]S_0.$$

Thus the expanded potential energy is found to be

$$\begin{aligned}\frac{PE}{\lambda_{u_{cl}}}=\left[\left(\frac{Et^3}{R^2}\right)\left(\frac{q_0^2}{4}\right)\right]\cdot\Bigg\{&\left(1-\frac{\lambda_u}{\lambda_{u_{cl}}}\right)(\xi_1)^2\\&+\left[\frac{1}{2}\left(1-\frac{\lambda_u}{\lambda_{u_{cl}}}\right)+\frac{1}{32}\left(\frac{\lambda_n}{\lambda_{u_{cl}}}\right)\right](\xi_2)^2\\&+\frac{9C}{16}\,\xi_1\xi_2^2+\tfrac{3}{8}(1-\nu)\xi_1^4\\&-\left(\frac{\lambda_u}{\lambda_{u_{cl}}}\right)(2\xi_1\overline{\xi_1})+\left(-\frac{\lambda_u}{\lambda_{u_{cl}}}+\frac{1}{16}\frac{\lambda_n}{\lambda_{u_{cl}}}\right)(\xi_2\overline{\xi_2})\Bigg\}S_0.\end{aligned}\qquad\text{(A.38)}$$

Minimizing the above expanded potential energy expression with respect to the amplitudes of the buckling modes ξ_1 and ξ_2 yields the equilibrium equations of the two-mode system

$$\left(1-\frac{\lambda_u}{\lambda_{u_{cl}}}\right)(2\xi_1)+\left(\frac{9C}{16}\right)(\xi_2)^2+\tfrac{3}{2}(1-\nu)\xi_1^3=2\,\frac{\lambda_u}{\lambda_{u_{cl}}}\,\overline{\xi_1}$$

$$\cdot\left[\left(1-\frac{\lambda_u}{\lambda_{u_{cl}}}\right)+\left(\frac{1}{16}\frac{\lambda_n}{\lambda_{u_{cl}}}\right)\right](\xi_2)+\frac{9c}{8}\,\xi_1\xi_2=\left(\frac{\lambda_u}{\lambda_{u_{cl}}}-\frac{1}{16}\frac{\lambda_n}{\lambda_{u_{cl}}}\right)(\overline{\xi_2}).\qquad\text{(A.39)}$$

This is the appropriate set of equations which has been analyzed in general terms in the paper.