THE PARETO COPULA, AGGREGATION OF RISKS AND THE EMPEROR'S SOCKS

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ABSTRACT. The copula of a multivariate distribution is the distribution transformed to have uniform one dimensional marginals. We review a transformation of the marginals of a multivariate distribution to a standard Pareto and the resulting distribution we call the Pareto copula. Use of the Pareto copula has a certain claim to naturalness when considering asymptotic limit distributions for sums, maxima and empirical processes. We discuss implications for aggregation of risk and offer some examples.

1. INTRODUCTION

Religious Copularians take as basic orthodoxy the desirability of transforming a multivariate distribution to have uniform marginals. Despite the shortcomings pointed out by the skeptic Mikosch (Mikosch, 2005, 2006), this practice has become a fairly standard procedure. We argue that when ones objective is the study of limit distributions and asymptotic approximations, if ones religion requires transformation of marginal distributions, one would do better to transform marginals to the standard Pareto distribution. The resulting transformed distribution, which we call the Pareto copula, has natural interpretations for limit theory and heavy tail analysis. This point of view will also show that several results attributed to be properties of special copulas, are in fact, examples of more general properties of distributions.

Our transformation to Pareto marginals is not new and has been used in the study of multivariate domains of attraction to characterize these domains by means of multivariate regular variation. The method consists of transforming a domain of attraction condition to *standard* regular variation in which all components of the transformed vector are normalized by the same linear function. The technique dates at least to de Haan and Resnick (1977) and has been explained in de Haan and Ferreira (2006), Resnick (1987, 2006).

Section 2 outlines the definition and basic properties of the Pareto copula in the context of a triangular array of random vectors $\{\boldsymbol{X}_{n,j}; j \geq 1, n \geq 1\}$, where rows consist of iid d dimensional random vectors. We discuss the role of the Pareto copula in the study of asymptotic properties of empirical measures, extremes, and sums of entries in the nth row of the array as $n \to \infty$.

Key words and phrases. Regular variation, risk, maximal domain of attraction, copula, Pareto.

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Then in Section 3, we specialize the triangular array setup to regular variation where $\boldsymbol{X}_{n,j} = \boldsymbol{X}_j/b(n)$ for suitable scaling function $b(t)$ and iid random vectors $\{\boldsymbol{X}_j\}$.

We also consider cases where the distribution of $\{\boldsymbol{X}_j\}$ is in a maximal domain of attraction and study aggregation of risks: the asymptotic properties of the distribution of the sum of the components of $\{\boldsymbol{X}_j\}$. We do this when the vector's distribution is multivariate regularly varying and also when the distribution of $\boldsymbol{X}_1$ is in a maximal domain of attraction with equal one dimensional marginals in a Gumbel domain and the distribution does not possess asymptotic independence. For this case, we obtain without further assumptions, a reasonably explicit expression for the tail probabilities of the sum of the components.

1.1. **Vector notation.** Vectors are denoted by bold letters, capitals for random vectors and lower case for non-random vectors. For example: $\boldsymbol{x} = (x^{(1)}, \dots, x^{(d)}) \in \mathbb{R}^d$. Operations between vectors should always be interpreted componentwise, so that for two vectors $\boldsymbol{x}$ and $\boldsymbol{z}$, $\boldsymbol{x} < \boldsymbol{z}$ means $x^{(i)} < z^{(i)}$ for $i = 1, \dots, d$, with analogous notations for $\boldsymbol{x} \le \boldsymbol{z}$ and $\boldsymbol{x} = \boldsymbol{z}$. If $\boldsymbol{x}_j$ for $j = 1, \dots, n$ are vectors, $\bigvee_{j=1}^n \boldsymbol{x}_j = (\bigvee_{j=1}^n x_j^{(i)}, i = 1, \dots, d)$. Also, if $\boldsymbol{\alpha} = (\alpha^{(1)}, \dots, \alpha^{(d)}) \ge \boldsymbol{0}$, we write $\boldsymbol{x}^{\boldsymbol{\alpha}} = \big((x^{(1)})^{\alpha^{(1)}}, \dots, (x^{(d)})^{\alpha^{(d)}}\big)$ for $\boldsymbol{x} \ge \boldsymbol{0}$. Further, we define $\boldsymbol{0} = (0, \dots, 0)$, $\boldsymbol{1} = (1, \dots, 1)$ and $\boldsymbol{\infty} = (\infty, \dots, \infty)$. For a real number c, we write as usual $c\boldsymbol{x} = (cx^{(1)}, \dots, cx^{(d)})$. We denote the rectangles (or the higher dimensional intervals) by $[\boldsymbol{a}, \boldsymbol{b}] = \{\boldsymbol{x} \in \mathbb{R}^d : \boldsymbol{a} \le \boldsymbol{x} \le \boldsymbol{b}\}$ with analogous notation for rectangles with one or both endpoints open.

To fix ideas, suppose for now that $\mathbb{E} = [\boldsymbol{0}, \boldsymbol{\infty}] \setminus \{\boldsymbol{0}\}$. Complements are taken with respect to $\mathbb{E}$, so that for $\boldsymbol{x} > \boldsymbol{0}$,

$$[\boldsymbol{0}, \boldsymbol{x}]^c = \mathbb{E} \setminus [\boldsymbol{0}, \boldsymbol{x}] = \{\boldsymbol{y} \in \mathbb{E} : \bigvee_{i=1}^d \frac{y^{(i)}}{x^{(i)}} > 1\}.$$

1.2. **Symbol and concept list.** Here is a glossary of miscellaneous symbols and nomenclature used throughout the paper.

RV_ρ	The class of regularly varying functions on $[0, \infty)$ with index $\rho \in \mathbb{R}$.
$f^{\leftarrow}$	The left continuous inverse of a monotone function f defined by $f^{\leftarrow}(x) = \inf\{y : f(y) \ge x\}$.
$b(t)$	Usually the quantile function of a distribution function $F(x)$, defined by $b(t) = F^{\leftarrow}(1 - \frac{1}{t})$ but usage can vary somewhat by context.
$\xrightarrow{v}$	Vague convergence of measures.
$\Rightarrow$	Convergence in distribution.
ϵ_x	The probability measure consisting of all mass at x.
$M_+(\mathbb{E})$	The space of non-negative Radon measures on $\mathbb{E}$.
$M_p(\mathbb{E})$	The space of Radon point measures on $\mathbb{E}$.
$\mathrm{PRM}(\mu)$	Poisson random measure on $\mathbb{E}$ with mean measure μ.

2. The Pareto copula

2.1. **Basics.** Consider a triangular array of random vectors $\{\boldsymbol{X}_{n,j}, n \geq 1, j \geq 1\}$ in which rows are iid. The distribution of $\boldsymbol{X}_{n,1}$ is F_n. We suppose random vectors are $\mathbb{R}^d$-valued and, for simplicity, assume the one dimensional marginal distributions $F_n^{(i)}$ are continuous. If $\boldsymbol{X}_{n,j} = (X_{n,j}^{(i)}; i = 1, \dots, d)$, we indicate the one dimensional marginal distributions by

$$F_n^{(i)}(x) = P\{X_{n,1}^{(i)} \leq x\}.$$

Let K be a closed, compact cone contained in $[-\boldsymbol{\infty}, \boldsymbol{\infty}]$ centered at the origin and for some $\boldsymbol{a} \in [-\boldsymbol{\infty}, \boldsymbol{\infty})$ set

$$\mathbb{E} = K \setminus \{\boldsymbol{a}\}$$

so that $\mathbb{E}$ is a one-point uncompactification of K (see Resnick (2006, page 170).) The cases of most interest are

- $\mathbb{E} = [\boldsymbol{0}, \boldsymbol{\infty}] \setminus \{\boldsymbol{0}\}$,
- $\mathbb{E} = [-\boldsymbol{\infty}, \boldsymbol{\infty}] \setminus \{-\boldsymbol{\infty}\}$,
- $\mathbb{E} = [-\boldsymbol{\infty}, \boldsymbol{\infty}] \setminus \{\boldsymbol{0}\}$.

Assume temporarily, for illustration, that $\mathbb{E} = [-\boldsymbol{\infty}, \boldsymbol{\infty}] \setminus \{-\boldsymbol{\infty}\}$. Our basic assumption is that there exists a Radon measure ν on Borel subsets of $\mathbb{E}$ such that

$$nF_n(\cdot) = nP\{\boldsymbol{X}_{n,1} \in \cdot\} \xrightarrow{v} \nu(\cdot) \tag{2.1}$$

in $M_+(\mathbb{E})$. This entails

$$n\bar{F}_n^{(i)}(x) = nP\{X_{n,1}^{(i)} > x\} \xrightarrow{v} \nu^{(i)}(x, \infty], \quad i = 1, \dots, d, \tag{2.2}$$

in, say, $M_+(-\infty, \infty]$ where, for instance,

$$\nu^{(1)}(x, \infty] = \nu\big((x, \infty] \times [-\infty, \infty]^{d-1}\big).$$

Define the random vectors

$$\boldsymbol{\mathcal{P}}_{n,j} = (\mathcal{P}_{n,j}^{(1)}, \dots, \mathcal{P}_{n,j}^{(d)}) = \Big(\frac{1}{1 - F_n^{(i)}(X_{n,j}^{(i)})}, i = 1, \dots, d\Big), \tag{2.3}$$

and note that $\mathcal{P}_{n,j}^{(i)}$ is standard Pareto distributed; for $i = 1, \dots, d$:

$$P\{\mathcal{P}_{n,j}^{(i)} > x\} = x^{-1}, \quad x \geq 1.$$

Definition 2.1 (Pareto copula)**.** *Suppose $\boldsymbol{X}_{n,1}$ has distribution F_n with continuous marginals. Define $\boldsymbol{\mathcal{P}}_{n,j}$ as in* (2.3)*. Then we call the distribution ψ_n of $\boldsymbol{\mathcal{P}}_{n,j}$ a* Pareto copula*.*

A variant of (2.2) obtained by taking reciprocals is

$$\frac{1}{n(1 - F_n^{(i)}(x))} \to \frac{1}{\nu^{(i)}(x, \infty]}, \quad i = 1, \dots, d \tag{2.4}$$

and inverting yields

$$\Big(\frac{1}{1 - F_n^{(i)}}\Big)^{\leftarrow}(ny) \to \Big(\frac{1}{\nu^{(i)}(\cdot, \infty]}\Big)^{\leftarrow}(y), \quad i = 1, \dots, d; \; y > 0. \tag{2.5}$$

To save writing, we define the non-decreasing functions

$$V^{(i)}(y) = \Big(\frac{1}{\nu^{(i)}(\cdot,\infty]}\Big)^{\leftarrow}(y), \quad i = 1,\dots,d;\ y > 0. \tag{2.6}$$

We summarize some properties of a Pareto copula; cf. de Haan and Resnick (1977), Resnick (1987, pages 265, 277) or Resnick (2006, page 204).

Proposition 2.2. *Let $\boldsymbol{X}_{n,1}$ be a random vector with distribution F_n such that (2.1) holds. Let ψ_n be its Pareto copula. Then the following holds.*
(a) *There exists a Radon measure ψ_∞ on the Borel subsets of $[\mathbf{0},\infty]\setminus\{\mathbf{0}\}$ such that*

$$n\psi_n(\cdot) \stackrel{v}{\to} \psi_\infty(\cdot) \tag{2.7}$$

in $M_+([\mathbf{0},\infty]\setminus\{\mathbf{0}\})$.
(b) *For $i = 1,\dots,d$,*

$$\psi_\infty^{(i)}(x,\infty] = \psi_\infty\big([0,\infty]^{i-1}\times(x,\infty]\times[0,\infty]^{d-i}\big) = x^{-1}, \quad x > 0. \tag{2.8}$$

(c) *ψ_∞ is a Lévy measure on $\mathbb{R}^d$.*

Proof. (a) From Lemma 6.1 in Resnick (2006, page 174), it is enough to consider regions $[\mathbf{0},\boldsymbol{x}]^c$ for $\boldsymbol{x} \ge \mathbf{0}$. Then

$$\begin{aligned} n\psi_n\big([\mathbf{0},n\boldsymbol{x}]^c\big) =& nP\{[\boldsymbol{\mathcal{P}}_{n,1} \le n\boldsymbol{x}]^c\} \\ =& nP\{[X_{n,1}^{(i)} \le \Big(\frac{1}{1-F_n^{(i)}}\Big)^{\leftarrow}(nx^{(i)}); i = 1,\dots,d]^c\} \\ =& nF_n\big([-\infty, \big(\frac{1}{1-F_n^{(i)}}\big)^{\leftarrow}(nx^{(i)});\ i = 1,\dots,d]^c\big) \end{aligned} \tag{2.9}$$

and from (2.1) and (2.5), this converges to

$$\nu\left(\Big[-\boldsymbol{\infty}, (V^{(i)}(x^{(i)}); i = 1,\dots,d)\Big]^c\right) =: \psi_\infty([\mathbf{0},\boldsymbol{x}]^c), \tag{2.10}$$

(b) This follows from $\boldsymbol{\mathcal{P}}_{n,1}$ having Pareto marginal distributions.
(c) Suppose for simplicity that $d = 2$. With $\|\boldsymbol{x}\| = |x^{(1)}| \vee |x^{(2)}|$ we have

$$\begin{aligned} \int_{\{\|\boldsymbol{x}\|\le 1\}} \|\boldsymbol{x}\|^2 \psi_\infty(d\boldsymbol{x}) =& \iint_{0\le x^{(1)}<x^{(2)}\le 1} (x^{(2)})^2\psi_\infty(d\boldsymbol{x}) + \iint_{0\le x^{(2)}\le x^{(1)}\le 1} (x^{(1)})^2\psi_\infty(d\boldsymbol{x}) \\ \le& 2\int_0^1 s^2 s^{-2}ds = 2 < \infty. \end{aligned}$$

□

2.2. **Point process interpretation.** Continue to suppose for illustration that $\mathbb{E} = [-\boldsymbol{\infty}, \boldsymbol{\infty}] \setminus \{-\boldsymbol{\infty}\}$. Condition (2.1) is equivalent to (cf. Resnick (2006, page 179, 180) or Resnick (1987)) point process convergence:

$$\sum_{j=1}^{n} \epsilon_{\boldsymbol{X}_{n,j}} \Rightarrow \mathrm{PRM}(\nu) = \sum_{k} \epsilon_{\boldsymbol{J}_k} \quad \text{in } M_p(\mathbb{E}), \tag{2.11}$$

or

$$\sum_{j=1}^{n} \epsilon_{\left(j/n, \boldsymbol{X}_{n,j}\right)} \Rightarrow \mathrm{PRM}(Leb \times \nu) = \sum_{k} \epsilon_{\left(t_k, \boldsymbol{J}_k\right)} \quad \text{in } M_p([0,\infty) \times \mathbb{E}), \tag{2.12}$$

where recall PRM(ν) means Poisson random measure with mean measure ν and *Leb* stands for Lebesgue measure. Similarly, (2.7) is equivalent to

$$\sum_{j=1}^{n} \epsilon_{\boldsymbol{\mathcal{P}}_{n,j}/n} \Rightarrow \mathrm{PRM}(\psi_\infty) = \sum_{k} \epsilon_{\boldsymbol{j}_k} \quad \text{in } M_+([\mathbf{0}, \boldsymbol{\infty}] \setminus \{\mathbf{0}\}), \tag{2.13}$$

or

$$\sum_{j=1}^{n} \epsilon_{\left(j/n, \boldsymbol{\mathcal{P}}_{n,j}/n\right)} \Rightarrow \mathrm{PRM}(Leb \times \psi_\infty) = \sum_{k} \epsilon_{\left(t_k, \boldsymbol{j}_k\right)} \quad \text{in } M_+([0,\infty) \times [\mathbf{0}, \boldsymbol{\infty}] \setminus \{\mathbf{0}\}), \tag{2.14}$$

From (2.3), (2.5), (2.11) and (2.13), we obtain the following result, which also explains the transformation of the points $\boldsymbol{j}_k$ to $\boldsymbol{J}_k$.

Proposition 2.3. *When* $\mathbb{E} = [-\infty, \infty] \setminus \{0\}$ *and* (2.1) *holds,*

$$\begin{aligned} \sum_{j=1}^{n} \epsilon_{\boldsymbol{X}_{n,j}} &\stackrel{d}{=} \sum_{j=1}^{n} \epsilon_{\left(\left(\frac{1}{1-F_n^{(i)}}\right)^{\leftarrow}(n\mathcal{P}_{n,j}^{(i)}/n); i=1,\dots,d\right)} \\ &\Rightarrow \quad \sum_{k} \epsilon_{(V^{(i)}(j_k^{(i)}); i=1,\dots,d)} \stackrel{d}{=} \sum_{k} \epsilon_{\boldsymbol{J}_k} = \mathrm{PRM}(\nu). \end{aligned} \tag{2.15}$$

An analogous result holds when a time component is included.

2.3. **Partial sum convergence.** As usual we denote by $D([0,\infty), \mathbb{R}^d)$, the space of $\mathbb{R}^d$-valued càdlàg functions on $[0,\infty)$. Since $\mathcal{P}_{n,1}^{(i)}$ has a standard Pareto distribution for $i = 1, \dots, d$, it follows that $\mathcal{P}_{n,1}^{(i)} \geq 1$. Therefore,

$$\lim_{\epsilon \downarrow 0} \limsup_{n\to\infty} nE\left((\mathcal{P}_{n,1}^{(i)})^2 1_{[|\mathcal{P}_{n,1}^{(i)}| \leq \epsilon]} \right) = 0. \tag{2.16}$$

Thus, by a standard result reviewed in Resnick (2006, page 214), we get from (2.13) or (2.14) the following.

Proposition 2.4. *Let* $\{\boldsymbol{\mathcal{P}}_{n,j}, n \geq 1, j \geq 1\}$ *be a triangular array of random vectors with standard Pareto marginals, in which rows are iid. Then*

$$\sum_{j \leq nt} \left(\boldsymbol{\mathcal{P}}_{n,j} - [nt] E\left(\frac{\boldsymbol{\mathcal{P}}_{n,1}}{n} 1_{[\|\boldsymbol{\mathcal{P}}_{n,1}\|]/n \leq 1]} \right) \right) \Rightarrow \boldsymbol{X}_1(t) \tag{2.17}$$

in $D([0,\infty), \mathbb{R}^d)$*, and where* $\boldsymbol{X}_1(\cdot)$ *is a Lévy process with Lévy measure* ψ_∞*.*

Definition 2.5 (Pareto Lévy copula). *Let* $\{\boldsymbol{X}_1(t), t \geq 0\}$ *be the limit process in* (2.17). *Then we call its Lévy measure* ψ_∞ *a* Pareto Lévy copula.

Be aware that others have attached meaning to the phrase *Lévy copula* to indicate Lebesgue marginals. See Barndorff-Nielsen and Lindner (2006), Böcker and Klüppelberg (2007), Bregman and Klüppelberg (2005), Cont and Tankov (2004), Kallsen and Tankov (2006). Our Pareto Lévy copula was also considered in Barndorff-Nielsen and Lindner (2006).

Remark 2.6. *Marginally, for* $i = 1, \dots, d$, $\{X_1^{(i)}(t), t \geq 0\}$ *is a* 1*-stable process with only positive jumps. However, the multivariate process* $\{\boldsymbol{X}_1(t), t \geq 0\}$ *is not stable unless* ψ_∞ *has the homogeneity property* $\psi_\infty(t\cdot) = t^{-1}\psi_\infty(\cdot)$.

Now suppose (2.1) holds with $\mathbb{E} = [\mathbf{0}, \boldsymbol{\infty}] \setminus \{\mathbf{0}\}$. We restrict attention to the first quadrant for the convenience of having only one multivariate tail specifying probabilities near $\boldsymbol{\infty}$. The full case of partial sum convergence for vectors in $\mathbb{R}^d$ and associated transformations to Pareto copulas can be considered in $[-\boldsymbol{\infty}, \boldsymbol{\infty}] \setminus \{\mathbf{0}\}$ but we would have to specify 2^d quadrants corresponding to the neighborhoods of the 2^d vertices of $[-\boldsymbol{\infty}, \boldsymbol{\infty}]$ which could be labelled $\{\boldsymbol{a} \cdot \infty : \boldsymbol{a} \in \{-1, 1\}^d\}$. (See the comments in Section 6.5.5 of Resnick (2006, page 201).) The following is a consequence of Section 7.2.1, Resnick (2006, page 214).

Proposition 2.7. *With* $\boldsymbol{X}_{n,j} \geq \mathbf{0}$ *and* $\mathbb{E} = [\mathbf{0}, \boldsymbol{\infty}] \setminus \{\mathbf{0}\}$ *suppose* (2.1) *holds in* $M_+(\mathbb{E})$ *and also that*

$$\lim_{\epsilon \downarrow 0} \limsup_{n \to \infty} nE(X_{n,1}^{(i)})^2 1_{[|X_{n,1}^{(i)}| \leq \epsilon]} = 0. \tag{2.18}$$

Then

$$\sum_{j \leq nt} \Big(\boldsymbol{X}_{n,j} - E\big(\boldsymbol{X}_{n,1} 1_{[\|\boldsymbol{X}_{n,1}\| \leq 1]}\big) \Big) \Rightarrow \boldsymbol{X}_2(t) \tag{2.19}$$

in $D([0, \infty), \mathbb{R}^d)$, *where* $\boldsymbol{X}_2(\cdot)$ *is a Lévy process with Lévy measure* ν.

The following result links the processes $\boldsymbol{X}_1(\cdot)$ and $\boldsymbol{X}_2(\cdot)$. It is a consequence of Proposition 2.3

Theorem 2.8. *If the Itô representation of* $\boldsymbol{X}_1(\cdot)$ *in* (2.17) *is*

$$\boldsymbol{X}_1(t) = \sum_{t_k \leq t} \boldsymbol{j}_k 1_{[\|\boldsymbol{j}_k\| > 1]} + \lim_{\epsilon \downarrow 0} \Big[\sum_{t_k \leq t} \boldsymbol{j}_k 1_{[\epsilon < \|\boldsymbol{j}_k\| \leq 1]} - t \int_{\{\epsilon < \|\boldsymbol{x}\| \leq 1\}} \boldsymbol{x} \psi_\infty(d\boldsymbol{x}) \Big] \tag{2.20}$$

which is consistent with the notation used in (2.14), *then the Itô representation for* $\boldsymbol{X}_2(\cdot)$ *is given by*

$$\boldsymbol{X}_2(t) = \sum_{t_k \leq t} \boldsymbol{J}_k 1_{[\|\boldsymbol{J}_k\| > 1]} + \lim_{\epsilon \downarrow 0} \Big[\sum_{t_k \leq t} \boldsymbol{J}_k 1_{[\epsilon < \|\boldsymbol{J}_k\| \leq 1]} - t \int_{\{\epsilon < \|\boldsymbol{x}\| \leq 1\}} \boldsymbol{x} \nu(d\boldsymbol{x}) \Big] \tag{2.21}$$

where

$$\boldsymbol{J}_k = (V^{(1)}(j_k^{(1)}), \dots, V^{(d)}(j_k^{(d)})).$$

2.4. **Extremes.** Assume again for simplicity that $\mathbb{E} = [-\infty, \infty] \setminus \{-\infty\}$. From (2.14) it is immediate that (Resnick, 1987)

$$\boldsymbol{Y}_n(t) := \bigvee_{\frac{j}{n} \le t} \boldsymbol{\mathcal{P}}_{n,j} \Rightarrow \bigvee_{t_k \le t} \boldsymbol{j}_k =: \boldsymbol{Y}(t), \tag{2.22}$$

in $D([0,\infty), \mathbb{R}^d)$, where $\boldsymbol{Y}$ is the multivariate extremal process associated with the limit in (2.14).

Proposition 2.9. *Set*

$$\boldsymbol{Z}_n(t) = \bigvee_{\frac{j}{n} \le t} \boldsymbol{X}_{n,j}$$

and assume that (2.1) *holds in* $M_+\big([-\infty, \infty] \setminus \{-\infty\}\big)$*. Then*

$$\begin{aligned} \boldsymbol{Z}_n(t) &\stackrel{d}{=} \left(\Big(\frac{1}{1 - F_n^{(i)}} \Big)^{\leftarrow} \Big(\bigvee_{j \le nt} \mathcal{P}_{n,j}^{(i)} \Big); \, i = 1, \dots, d \right) \\ &\Rightarrow \Big(V^{(i)} \big(Y^{(i)}(t) \big); \, i = 1, \dots, d \Big) =: \boldsymbol{Z}(t) \end{aligned}$$

in $D([0,\infty), \mathbb{R}^d)$*, where* $\boldsymbol{Z}$ *is the multivariate extremal process associated with the limit in* (2.12)*.*

3. Regular variation

Suppose $\boldsymbol{X} \ge \boldsymbol{0}$ is a random vector in $\mathbb{R}_+^d$ with distribution F and one dimensional marginal distributions $F^{(i)}$, $i = 1, \dots, d$. Define

$$\boldsymbol{\mathcal{P}} = \Big(\frac{1}{1 - F^{(i)}(X^{(i)})}; \, i = 1, \dots, d \Big). \tag{3.1}$$

Set

$$b_i(t) := \Big(\frac{1}{1 - F^{(i)}} \Big)^{\leftarrow}(t), \quad i = 1, \dots, d, \tag{3.2}$$

so

$$\boldsymbol{X} = \big(b_i(\mathcal{P}^{(i)}); \, i = 1, \dots, d \big). \tag{3.3}$$

Theorem 3.1. *Suppose that the distribution of* $\boldsymbol{X}$ *is regularly varying (Resnick, 2006, page 204), i.e., for* $i = 1, \dots, d$*, there exist functions* $b_i(t) \to \infty$ *as* $t \to \infty$ *such that*

$$tP\Big\{ \Big(\frac{X^{(i)}}{b_i(t)}; \, i = 1, \dots, d \Big) \in \cdot \Big\} \xrightarrow{v} \nu(\cdot) \tag{3.4}$$

in $M_+([\boldsymbol{0}, \boldsymbol{\infty}] \setminus \{\boldsymbol{0}\})$*, where* ν *is a Radon measure. This implies marginal distributions* $F^{(i)}$ *have regularly varying tails* $1 - F^{(i)} \in RV_{-\alpha^{(i)}}$ *and we assume* $0 < \alpha^{(i)} < \infty$*, for* $i = 1, \dots, d$*. Consider the Pareto copula* ψ *of* $\boldsymbol{X}$*. Then* ψ *is standard regularly varying,*

$$t\psi(t\cdot) = tP\Big\{ \frac{\boldsymbol{\mathcal{P}}}{t} \in \cdot \Big\} \xrightarrow{v} \psi_\infty(\cdot) \tag{3.5}$$

in $M_+([\mathbf{0},\boldsymbol{\infty}]\setminus\{\mathbf{0}\})$*, with*

$$\psi_\infty(t\cdot)=t^{-1}\psi_\infty(\cdot), \tag{3.6}$$

and

$$\nu([\mathbf{0},\boldsymbol{x}^{\mathbf{1}/\boldsymbol{\alpha}}]^c)=\psi_\infty([\mathbf{0},\boldsymbol{x}]^c),\quad \boldsymbol{x}>\mathbf{0}.$$

Conversely, suppose $\boldsymbol{X}$ *is a random vector in* $\mathbb{R}_+^d$ *with Pareto copula* ψ*. If* ψ *is standard regularly varying; i.e.* (3.5) *holds and additionally we have marginal regular variation*

$$1-F^{(i)}\in RV_{-\alpha^{(i)}},\quad \infty>\alpha^{(i)}>0,\ i=1,\dots,d,$$

then $\boldsymbol{X}$ *is regularly varying and* (3.4) *holds.*

Proof. The functions $b_i(\cdot)\in RV_{1/\alpha^{(i)}}$ where $\alpha^{(i)}>0$, $i=1,\dots,d$ and for $\boldsymbol{x}>\mathbf{0}$,

$$\begin{aligned} tP\{[\boldsymbol{\mathcal{P}}\le t\boldsymbol{x}]^c\} =&t\psi([\mathbf{0},t\boldsymbol{x}]^c)=tP\{[X^{(i)}\le b_i(tx^{(i)});\ i=1,\dots,d]^c\}\\ =&tP\{[\frac{X^{(i)}}{b_i(t)}\le\frac{b_i(tx^{(i)})}{b_i(t)};\ i=1,\dots,d]^c\}\\ \to&\nu([\mathbf{0},\big((x^{(i)})^{1/\alpha^{(i)}};\ i=1,\dots,d\big)]^c)=\nu([\mathbf{0},\boldsymbol{x}^{\mathbf{1}/\boldsymbol{\alpha}}]^c)=\psi_\infty([\mathbf{0},\boldsymbol{x}]^c) \end{aligned} \tag{3.7}$$

by (2.10). □

Suppose $\{\boldsymbol{X},\boldsymbol{X}_n,n\ge1\}$ is iid with the regularly varying distribution F on $\mathbb{R}_+^d$. To link with the notation of Section 2, set

$$F_n(\boldsymbol{x})=F(b_1(n)x^{(1)},\dots,b_d(n)x^{(d)})\quad \text{in } M_+([\mathbf{0},\boldsymbol{\infty}]\setminus\{\mathbf{0}\})$$

and

$$F_n^{(i)}(x)=F^{(i)}(b_i(n)x),\quad i=1,\dots,d.$$

In the notation of Section 2,

$$nF_n(\cdot)\xrightarrow{v}\nu(\cdot)$$

is equivalent to (3.4). Furthermore,

$$\mathcal{P}_{n,j}^{(i)}=\frac{1}{1-F^{(i)}(b_i(n)X_j^{(i)}/b_i(n))}=\frac{1}{1-F^{(i)}(X_j^{(i)})}=\mathcal{P}_j^{(i)},\quad i=1,\dots,d,$$

independent of n. The one dimensional variables are standard Pareto distributed.

This allows us to rephrase Proposition 2.4 and Theorem 2.8 for the case of regular variation.

Corollary 3.2. *Suppose* $\{\boldsymbol{X}_j,j\ge1\}$ *is iid on* $\mathbb{R}_+^d$ *with equal continuous univariate marginal distributions* $F^{(1)}$*. Set*

$$\mathcal{P}_j^{(i)}=\Big(\frac{1}{1-F^{(1)}}\Big)^{\leftarrow}(X_j^{(i)}),\quad b(t)=\Big(\frac{1}{1-F^{(1)}}\Big)^{\leftarrow}(t).$$

The following are equivalent:

(a) *With* $\boldsymbol{X}_{n,j}=\boldsymbol{X}_j/b(n)$, (2.19) *holds where* $\boldsymbol{X}_2(\cdot)$ *is* α*-stable Lévy motion* $(0<\alpha<2)$ *with Lévy measure* ν *satisfying* $\nu(t\cdot)=t^{-\alpha}\nu(\cdot)$*.*

(b) $\boldsymbol{X}_1$ *has a multivariate regularly varying distribution on* $\mathbb{R}_+^d$ *with index* $\alpha\in(0,2)$*.*

(c) $\bar F^{(1)}\in RV_{-\alpha}$, $0<\alpha<2$ *and with* $\boldsymbol{\mathcal{P}}_{n,j}=\boldsymbol{\mathcal{P}}_j/n$, (2.17) *holds with* $\boldsymbol{X}_1(\cdot)$ *1-stable Lévy motion and Lévy measure* ψ_∞*. The Pareto Lévy copula of* $\boldsymbol{X}_2(\cdot)$ *in (a) is* ψ_∞*.*

(d) $\bar{F}^{(1)} \in RV_{-\alpha}$, $0 < \alpha < 2$ *and the Pareto copula of the random vector* $\boldsymbol{X}_1$ *is standard regularly varying.*

The equivalence of (a) and (b) is discussed in Resnick (2006, page 214) and the rest follows from previous discussion.

3.1. **Aggregation of risks.** Assume $\mathbb{E} = [\mathbf{0}, \boldsymbol{\infty}] \setminus \{\mathbf{0}\}$. When the regular variation (3.4) holds, we get as $t \to \infty$, (Resnick, 2006, Section 7.3.1, page 227)

$$tP\Big\{\sum_{i=1}^{d} \frac{X^{(i)}}{b_i(t)} > x\Big\} \to \nu\Big\{\boldsymbol{x} \in \mathbb{E} : \sum_{i=1}^{d} x^{(i)} > x\Big\}. \tag{3.8}$$

If in (3.4)

$$b_i(t) = b(t) \in RV_{1/\alpha}, \quad i = 1, \ldots, d,\ \alpha > 0,$$

then

$$\nu(t\cdot) = t^{-\alpha}\nu(\cdot) \tag{3.9}$$

and therefore from (3.8) we have

$$tP\Big\{\sum_{i=1}^{d} X^{(i)} > b(t)y\Big\} \to \nu\Big\{\boldsymbol{x} \in \mathbb{E} : \sum_{i=1}^{d} x^{(i)} > y\Big\}$$

and by (3.9) this limit is

$$y^{-\alpha}\nu\Big\{\boldsymbol{x} \in \mathbb{E} : \sum_{i=1}^{d} x^{(i)} > 1\Big\}.$$

Thus

$$tP\Big\{\sum_{i=1}^{d} X^{(i)} > b(t)\Big\} \to \nu\Big\{\boldsymbol{x} \in \mathbb{E} : \sum_{i=1}^{d} x^{(i)} > 1\Big\}$$

and

$$\frac{P\{\sum_{i=1}^{d} X^{(i)} > t\}}{P\{X^{(1)} > t\}} \to \frac{\nu\{\boldsymbol{x} \in \mathbb{E} : \sum_{i=1}^{d} x^{(i)} > 1\}}{\nu\{\boldsymbol{x} \in \mathbb{E} :\ x^{(1)} > 1\}}. \tag{3.10}$$

The evaluation of the limit depends on the specific form of ν.

3.2. **An interesting special case.** An interesting case of the regular variation result in the previous section is discussed from the copula point of view by Alink et al. (2004) and reviewed in Albrecher et al. (2006). Suppose $d = 2$ and $\boldsymbol{X} = (X^{(1)}, X^{(2)})$ where $X^{(1)} \stackrel{d}{=} X^{(2)}$ so $F^{(1)} = F^{(2)}$. Write $\mathcal{P}$ in the following way:

$$U^{(i)} = F^{(1)}(X^{(i)});\ i = 1, 2 \quad \text{and} \quad \mathcal{P}^{(i)} = \frac{1}{1 - U^{(i)}};\ i = 1, 2.$$

Then for $\boldsymbol{x} > \mathbf{0}$.

$$\begin{aligned}\psi([\mathbf{0}, \boldsymbol{x}]) =& P\{\boldsymbol{\mathcal{P}} \le \boldsymbol{x}\} = P\{\frac{1}{1 - U^{(i)}} \le x^{(i)};\ i = 1, 2\}\\ =& P\{1 - U^{(i)} \ge (x^{(i)})^{-1};\ i = 1, 2\}\end{aligned}$$

$$
\begin{aligned}
&=1 - P\{[1-U^{(i)} \geq (x^{(i)})^{-1};\ i=1,2]^c\} \\
&=1 - P\{[1-U^{(1)} \leq (x^{(1)})^{-1}] \cup [1-U^{(2)} \leq (x^{(2)})^{-1}]\} \\
&=1 - \Big((x^{(1)})^{-1} + (x^{(2)})^{-1} - \hat{C}\big((x^{(1)})^{-1}, (x^{(1)})^{-1}\big)\Big).
\end{aligned}
$$

Thus, to summarize,

$$
\psi([\mathbf{0}, \boldsymbol{x}]^c) = (x^{(1)})^{-1} + (x^{(2)})^{-1} - \hat{C}\big((x^{(1)})^{-1}, (x^{(2)})^{-1}\big) \tag{3.11}
$$

where $\hat{C}$ is the copula

$$
\hat{C}\big(x^{(1)}, x^{(2)}\big) = P\{1 - U^{((i)} \leq x^{(i)};\ i = 1, 2\}.
$$

Now suppose the copula $\hat{C}$ is Archimedean so that

$$
\hat{C}\big(x^{(1)}, x^{(2)}\big) = \hat{\phi}^{-1}\big(\hat{\phi}(x^{(1)}) + \hat{\phi}(x^{(2)})\big)
$$

where $\hat{\phi}$ is the proper generator of the copula so that $\hat{\phi}$ is continuous, convex and strictly decreasing from $[0,1] \mapsto [0,\infty]$ such that $\hat{\phi}(1) = 0$. (See Albrecher et al. (2006), Alink et al. (2004).) Suppose additionally that $\hat{\phi}$ is regularly varying at 0 with index $-\xi$ for $\xi > 0$. Then

$$
R(t) := \hat{\phi}(\frac{1}{t}) \in RV_\xi
$$

at ∞ and

$$
R^{\leftarrow}(x) = \inf\{s : \hat{\phi}(\frac{1}{s}) \geq x\} = \frac{1}{\hat{\phi}^{-1}(x)} \in RV_{1/\xi}
$$

at ∞. Therefore, with this assumption we get from (3.11)

$$
n\psi([\mathbf{0}, n\boldsymbol{x}]^c) = (x^{(1)})^{-1} + (x^{(2)})^{-1} - n\hat{\phi}^{-1}\big(\hat{\phi}((nx^{(1)})^{-1}) + \hat{\phi}((nx^{(2)})^{-1})\big).
$$

The last term is

$$
\begin{aligned}
\frac{n}{R^{\leftarrow}\big(R(nx^{(1)}) + R(nx^{(2)})\big)} &= \left(\frac{R^{\leftarrow}\big(R(n)\big(\frac{R(nx^{(1)})}{R(n)} + \frac{R(nx^{(2)})}{R(n)}\big)\big)}{R^{\leftarrow}(R(n))}\right)^{-1} \\
&\sim \left(\lim_{n\to\infty}\big(\frac{R(nx^{(1)})}{R(n)} + \frac{R(nx^{(2)})}{R(n)}\big)\right)^{-1/\xi} \\
&= \Big((x^{(1)})^{\xi} + (x^{(2)})^{\xi}\Big)^{-1/\xi}.
\end{aligned}
$$

Thus

$$
\lim_{n\to\infty} n\psi([\mathbf{0}, n\boldsymbol{x}]^c) = (x^{(1)})^{-1} + (x^{(2)})^{-1} - \Big((x^{(1)})^{\xi} + (x^{(2)})^{\xi}\Big)^{-1/\xi} = \psi_\infty([\mathbf{0}, \boldsymbol{x}]^c). \tag{3.12}
$$

Note that in this model $\boldsymbol{\mathcal{P}}$ does *not* possess asymptotic independence (Resnick, 2006, page 192) since

$$
\lambda := \lim_{t\to\infty} P\{\mathcal{P}^{(2)} > t | \mathcal{P}^{(1)} > t\} = \lim_{t\to\infty} tP\{\boldsymbol{\mathcal{P}} > t(1,1)\} = \psi_\infty((\mathbf{1}, \boldsymbol{\infty}])
$$

where $\mathbf{1} = (1,1)$. Observe

$$
\psi_\infty((\boldsymbol{x}, \boldsymbol{\infty}]) = (x^{(1)})^{-1} + (x^{(2)})^{-1} - \psi_\infty([\mathbf{0}, \boldsymbol{x}]^c)
$$

and so we get from (3.12)

$$\lambda := \psi_\infty(\mathbf{1},\boldsymbol{\infty}]) = \Big(1+1-\big(1+1-(1^\xi+1^\xi)^{-1/\xi}\big)\Big) = 2^{-1/\xi}.$$

The measure ψ_∞ has a density $\psi_\infty'(u,v)$ which after differentiating $\psi_\infty([\mathbf{0},(u,v)]^c)$ is seen to be

$$\begin{aligned} \psi_\infty'(u,v) =&(1+\xi)\big(u^\xi+v^\xi\big)^{-1/\xi-2}(uv)^{\xi-1} && (3.13)\\ =&(1+\xi)u^{-2-\xi}\Big(1+(\frac{u}{v})^\xi\Big)^{-1/\xi-2}v^{\xi-1}, && (3.14) \end{aligned}$$

for $u>0, v>0$. From the formula for $\psi_\infty([\mathbf{0},(u,v)]^c)$ we can readily check that the denominator in (3.10) is

$$\psi_\infty\{(u,v)\in\mathbb{E}: v>1\} = \psi_\infty((1,\infty]\times[0,\infty]) = 1.$$

Calculating the numerator in the limit in (3.10), we get

$$\begin{aligned} \psi_\infty\{(u,v)\in\mathbb{E}: u+v>1\} =& \int_{u=1}^\infty\int_{v=0}^\infty \psi_\infty'(u,v)du\,dv + \int_{u=0}^1\int_{v=1-u}^\infty \psi_\infty'(u,v)du\,dv\\ =&\psi_\infty((1,\infty]\times[0,\infty]) + \int_{u=0}^1 (1+\xi)u^{-2-\xi}\left(\int_{v=1-u}^\infty\Big(1+(\frac{v}{u})^\xi\Big)^{-1/\xi-2}v^{\xi-1}dv\right)du \end{aligned}$$

and after some changes of variables this reduces to

$$=1+\int_0^\infty (1+v^\xi)^{-1/\xi-1}dv = 1+1 = 2,$$

since the integrand in the second term is a probability density (Alink et al., 2004, Lemma 2.4). This is an interesting limit because although this model does not possess asymptotic independence, the limit in (3.10) is the one predicted by asymptotic independence.

Next set $\boldsymbol{\alpha}=(\alpha,\alpha)$, $\alpha>0$, and following (3.7) we suppose

$$\nu([\mathbf{0},\boldsymbol{x}]^c) = \psi_\infty([\mathbf{0},\boldsymbol{x}^{\boldsymbol{\alpha}}]^c)$$

so that with $\boldsymbol{x}=(u,v)$ we have

$$\nu([\mathbf{0},(u,v)]^c) = u^{-\alpha}+v^{-\alpha}-\big(u^{\alpha\xi}+v^{\alpha\xi}\big)^{-1/\xi}.$$

Observe

$$\nu((1,\infty]\times[0,\infty]) = 1.$$

Furthermore, ν has a density $\nu'(u,v)$ given by

$$\begin{aligned} \nu'(u,v) =&\alpha^2(1+\xi)\big(u^{\alpha\xi}+v^{\alpha\xi}\big)^{-1/\xi-2}(uv)^{\alpha\xi-1}\\ =&\alpha^2(1+\xi)u^{-\alpha(1+\xi)-1}\Big(1+(\frac{v}{u})^{\alpha\xi}\Big)^{-1/\xi-2}v^{\alpha\xi-1}, \end{aligned}$$

for $u\geq 0, v\geq 0$.

We may now compute the limit in (3.10) for this model. We have the limit

$$\nu\{\boldsymbol{x}\in\mathbb{E}: x^{(1)}+x^{(2)}>1\} = \int_{u=1}^\infty\int_{v=0}^\infty \nu'(u,v)dudv + \int_{u=0}^1\int_{v=1-u}^\infty \nu'(u,v)dvdu$$

$$=1+\int_0^1 \alpha^2(1+\xi)u^{-\alpha(1+\xi)-1}\left(\int_{v=1-u}^{\infty}\Big(1+\big(\frac{v}{u}\big)^{\alpha\xi}\Big)^{-1/\xi-2}v^{\alpha\xi-1}\right)du$$

and after changes of variables this is

$$=1+\int_0^{\infty}\alpha\big(1+v^{\alpha\xi}\big)^{-1/\xi-1}\big(1+v\big)^{\alpha-1}dv$$
$$=1+\int_0^{\infty}\big(1+s^{\xi}\big)^{-1/\xi-1}\big(1+s^{-1/\alpha}\big)^{\alpha-1}ds,$$

If Y_ξ has the probability density $\big(1+s^{-1/\xi}\big)^{\alpha-1}$, $s>0$, this can be expressed as (Alink et al., 2004)

$$=1+E\big(1+Y_\xi^{-1/\alpha}\big)^{\alpha-1}.$$

Thus, for $d=2$ with equal marginals, whenever $F\in RV_{-\alpha}$ for $\alpha>0$, and ψ_∞ is given by (3.12), we have

$$\lim_{t\to\infty}\frac{P\{X^{(1)}+X^{(2)}>t\}}{P\{X^{(1)}>t\}}=1+E\big(1+Y_\xi^{-1/\alpha}\big)^{\alpha-1}.$$

4. The Pareto copula and distributions in the multivariate maximal domain of attraction

Suppose $\{\boldsymbol{X},\boldsymbol{X}_n,n\geq 1\}$ are iid random vectors with common distribution F. Then $\boldsymbol{X}$ or F is in a multivariate maximal domain of attraction if there exist

$$\boldsymbol{b}(t)=(b^{(1)}(t),\dots,b^{(d)}(t))\in\mathbb{R}^d,\quad \boldsymbol{a}(t)=(a^{(1)}(t),\dots,a^{(d)}(t))\in\mathbb{R}_+^d,$$

such that
(4.1)

$$P^n\Big[\frac{\boldsymbol{X}-\boldsymbol{b}(n)}{\boldsymbol{a}(n)}\leq\boldsymbol{x}\Big]=F^n\big(\boldsymbol{a}(n)\boldsymbol{x}+\boldsymbol{b}(n)\big)=\Big(P\Big[\frac{X^{(i)}-b^{(i)}(n)}{a^{(i)}(n)}\leq x^{(i)};\ i=1,\dots,d\Big]\Big)^n\to G(\boldsymbol{x}),$$

where G is a non-degenerate distribution called a max-stable or extreme value distribution. The marginal distributions $G^{(i)}_{\gamma^{(i)}}$, $i=1,\dots,d$ of G are one dimensional extreme value distributions of the form

$$G^{(i)}_{\gamma^{(i)}}=\exp\Big\{-\big(1+\gamma^{(i)}x^{(i)}\big)^{-1/\gamma^{(i)}}\Big\},\quad 1+\gamma^{(i)}x^{(i)}>0,$$

and $G^{(i)}$ concentrates on $\{u\in\mathbb{R}:1+\gamma^{(i)}u>0\}$. See, for example, de Haan and Ferreira (2006), Embrechts et al. (1997), Resnick (1987).

In the notation of Section 2, we may write

$$F_n(\cdot)=P\Big[\frac{\boldsymbol{X}-\boldsymbol{b}(n)}{\boldsymbol{a}(n)}\in\cdot\Big]=F(\boldsymbol{a}(n)(\cdot)+\boldsymbol{b}(n))$$

and then after the customary logarithmic transformation, it is seen that (4.1) is equivalent to (2.1). Further using the matchup with the notation of Section 2 we set

$$\boldsymbol{X}_{n,j} = \frac{\boldsymbol{X}_j - \boldsymbol{b}(n)}{\boldsymbol{a}(n)}.$$

The transformation given in (2.3) becomes

$$\begin{aligned}\mathcal{P}^{(i)}_{n,j} =& \frac{1}{1 - F^{(i)}_n(X^{(i)}_{n,j})} = \frac{1}{1 - F^{(i)}(a^{(i)}(n)X^{(i)}_{n,j} + b^{(i)}(n))}\\ =& \frac{1}{1 - F^{(i)}\Big(a^{(i)}(n)\big(\frac{X^{(i)}_j - b^{(i)}(n)}{a^{(i)}_n(n)}\big) + b^{(i)}(n)\Big)}\\ =& \frac{1}{1 - F^{(i)}(X^{(i)}_j)}\end{aligned}$$

independent of n.

As in Section 3, write for $\boldsymbol{x} > \boldsymbol{0}$,

$$\psi([\boldsymbol{0}, \boldsymbol{x}]) = P\{\boldsymbol{\mathcal{P}}_{n,1} \le \boldsymbol{x}\} = P\{\frac{1}{1 - F^{(i)}(X^{(i)}_1)} \le x^{(i)}; i = 1\dots, d\}.$$

Then (4.1) is equivalent to ψ being standard regularly varying

$$n\psi(n\cdot) \xrightarrow{v} \psi_\infty(\cdot) \quad \text{in } M_+([\boldsymbol{0}, \boldsymbol{\infty}] \setminus \{\boldsymbol{0}\}),$$

as $n \to \infty$ with $\psi_\infty(t\cdot) = t^{-1}\psi_\infty(\cdot)$ for $t > 0$ and for every $i = 1, \dots, d$ the random variable $X^{(i)}_1$ is in a one dimensional maximal domain of attraction of a univariate extreme value distribution $G_{\gamma^{(i)}}$. See de Haan and Resnick (1977), Resnick (1987, Chapter 5), de Haan and Ferreira (2006, Chapter 6).

4.1. **Aggregation of risks when marginals are in the maximal domain of attraction of the Gumbel.** We now discuss aggregation of risks when (4.1) holds with $\gamma^{(i)} = 0$ for $i = 1, \dots, d$ so that each marginal is in the domain of attraction of the Gumbel distribution. This is equivalent to supposing for $i = 1, \dots, d$ that there exists a self-neglecting function $f^{(i)}(t)$ with derivative converging to 0 such that

$$\frac{\bar{F}^{(i)}\big(t + xf^{(i)}(t)\big)}{\bar{F}^{(i)}(t)} \to e^{-x}, \quad x \in \mathbb{R}, \tag{4.2}$$

as t converges to the right endpoint of $F^{(i)}$ (de Haan (1970), de Haan and Ferreira (2006), Embrechts et al. (1997), Resnick (1987)). An acceptable choice of $f^{(i)}$ is the mean excess function (Bingham et al., 1987, de Haan, 1970, Geluk and de Haan, 1987). Then we may take

$$b^{(i)}(t) = \Big(\frac{1}{1 - F^{(i)}(\cdot)}\Big)^{\leftarrow}(t), \quad a^{(i)}(t) = f^{(i)}(b^{(i)}(t)), \quad i = 1, \dots, d. \tag{4.3}$$

To get attractive formulae, it is necessary to assume all marginals of F are the same so we proceed under the assumption

$$F^{(i)}(\cdot) = F^{(1)}(\cdot), \quad i = 1, \dots, d. \tag{4.4}$$

Formulae for aggregation of risks may be readily obtained when F does not possess asymptotic independence.

4.1.1. *Asymptotic independence is absent.* Special cases of this result have been given in Maulik et al. (2002, Proposition 3.1), Albrecher et al. (2006), Alink et al. (2004). We assume condition (4.4) of equal marginal distributions and write $\boldsymbol{b}(t) = b(t)\mathbf{1}$ and $a^{(1)}(t) = a(t)$.

Set $\mathbb{E} = [-\boldsymbol{\infty}, \boldsymbol{\infty}] \setminus \{-\boldsymbol{\infty}\}$. When the marginal distributions of F are in the maximal domain of attraction of a Gumbel distribution, (4.1) is equivalent to (see, for example, Resnick (2006, page 138))

$$\sum_{j=1}^{n} \epsilon_{a(n)^{-1}(\boldsymbol{X}_j - b(n)\mathbf{1})} \Rightarrow \epsilon_{\boldsymbol{j}_k} \tag{4.5}$$

in $M_p(\mathbb{E})$. Pick a large M. The restriction map $\mathbb{E} \mapsto \mathbb{E}^M := (-M\mathbf{1}, \boldsymbol{\infty}]$ is almost surely continuous so we get from (4.5) the same convergence restricted to $M_p(\mathbb{E}^M)$. Define the addition map $T : \mathbb{E}^M := (-\boldsymbol{\infty}, \boldsymbol{\infty}] \mapsto (-\infty, \infty]$ by

$$T\boldsymbol{x} = \sum_{i=1}^{d} x^{(i)}.$$

The map T is almost surely continuous from $\mathbb{E}^M \mapsto (-\infty, \infty]$ and applying it to the restricted version of (4.5) we get

$$N_n^M := \sum_{j=1}^{n} 1_{[(\boldsymbol{X}_j - b(n)\mathbf{1})/a(n) \geq -M\mathbf{1}]} \epsilon_{(\sum_{i=1}^{d} X_j^{(i)} - db(n))/a(n)} \Rightarrow N_\infty^M := \sum_k 1_{[\boldsymbol{j}_k > -M\mathbf{1}]} \epsilon_{\sum_{i=1}^{d} j_k^{(i)}}. \tag{4.6}$$

Note that asymptotic independence would require all points of the limit Poisson process to be on the lines through $-\boldsymbol{\infty}$ which would render the limit in (4.6) identically zero and hence useless; but this has been excluded.

We now proceed with a converging together argument (cf. Resnick (2006, Theorem 3.5, page 56) or Billingsley (1999)). Define

$$N_n := \sum_{j=1}^{n} \epsilon_{(\sum_{i=1}^{d} X_j^{(i)} - db(n))/a(n)}. \tag{4.7}$$

We make two claims. First we have, as $M \to \infty$,

$$N_\infty^M \Rightarrow N_\infty := \sum_k 1_{[\boldsymbol{j}_k > -\boldsymbol{\infty}]} \epsilon_{\sum_{i=1}^{d} j_k^{(i)}}, \tag{4.8}$$

considered as convergence in $M_p(-\infty, \infty]$. Second, we claim that if $d(\cdot, \cdot)$ is the vague metric on $M_p(-\infty, \infty]$, then for any $\eta > 0$,

$$\lim_{M \to \infty} \limsup_{n \to \infty} P\{d(N_n^M, N_n) > \eta\} = 0. \tag{4.9}$$

We are now in the position to state the following Proposition.

Proposition 4.1. *Suppose* (4.1) *holds where all marginals of* $F(\boldsymbol{x})$ *are equal and all marginals of* $G(\boldsymbol{x})$ *are Gumbel and* (4.2) *and* (4.4) *hold. Suppose* F *does NOT possess asymptotic independence and define* $\nu(\cdot)$ *by*

$$\nu\big([-\boldsymbol{\infty},\boldsymbol{x}]^c\big) = -\log G(\boldsymbol{x}), \quad \boldsymbol{x}\neq -\boldsymbol{\infty}. \tag{4.10}$$

Then

$$N_n := \sum_{j=1}^{n} \epsilon_{(\sum_{i=1}^{d} X_j^{(i)} - db(n))/a(n)} \Rightarrow \sum_{k} 1_{[\boldsymbol{j}_k > -\boldsymbol{\infty}]}\epsilon_{\sum_{i=1}^{d} j_k^{(i)}}, \tag{4.11}$$

in $M_p(-\infty,\infty]$, *where the limit* N_∞ *is Poisson random measure with mean measure*

$$\nu\{\boldsymbol{x}\in(-\boldsymbol{\infty},\boldsymbol{\infty}] : \sum_{i=1}^{d} x^{(i)} \in \cdot\}. \tag{4.12}$$

Therefore from Resnick (2006, page 138), as $n\to\infty$,

$$nP\{\frac{\sum_{i=1}^{d} X^{(i)} - db(n)}{a(n)} > y\} \to \nu\{\boldsymbol{x}\in(-\boldsymbol{\infty},\boldsymbol{\infty}] : \sum_{i=1}^{d} x^{(i)} > y\}. \tag{4.13}$$

Corollary 4.2. *Under the conditions of Proposition 4.1, we have from* (4.13) *that*

$$\lim_{t\to\infty} \frac{P\{\sum_{i=1}^{d} X^{(i)} > dt\}}{P\{X^{(1)} > t\}} = \nu\{\boldsymbol{x}\in(-\boldsymbol{\infty},\boldsymbol{\infty}] : \sum_{i=1}^{d} x^{(i)} > 0\}. \tag{4.14}$$

To verify (4.14), set $y=0$ in (4.13) and note from (4.3) that $P\{X^{(1)} > b(t)\} \sim t^{-1}$ as $t\to\infty$.

We now give the proof of Proposition 4.1.

Proof. The convergence in (4.8) is clear as it occurs almost surely. To prove (4.9), it suffices to take an arbitrary test function $f(\cdot)$ which is continous with compact support in $(-\infty,\infty]$ and show for any $\eta>0$,

$$\lim_{M\to\infty}\limsup_{n\to\infty} P\{|N_n(f) - N_n^M(f)| > \eta\} = 0,$$

which resolves to showing

$$\lim_{M\to\infty}\limsup_{n\to\infty} P\{\sum_{j=1}^{n} 1_{[a^{-1}(n)\left(\wedge_{i=1}^{d} X_j^{(i)} - b(n)\right) \le -M]}\epsilon_{a^{-1}(n)(\sum_{i=1}^{d} X_j^{(i)} - db(n))}(f) > \eta\} = 0. \tag{4.15}$$

Suppose the compact support of f is contained in $[-K,\infty]$ for some fixed K. Then the probability on the left side of (4.15) is bounded by

$$nP\{a^{-1}(n)\big(\wedge_{i=1}^{d} X_1^{(i)} - b(n)\big) \le -M, a^{-1}(n)(\sum_{l=1}^{d} X_1^{(l)} - db(n)) > -K\}.$$

We drop the subscript "1" for typographical simplicity. For the minimum to be less than $-M$, at least one of the terms must be less than $-M$, so the previous probability is bounded by

$$\leq \sum_{i=1}^{d} nP\{a^{-1}(n)\big(X^{(i)}-b(n)\big) \leq -M,\ a^{-1}(n)(\sum_{l=1}^{d} X^{(l)} - db(n)) > -K\}.$$

For the sum in the line above to be big when $a^{-1}(n)(X^{(i)}-b(n))$ is small requires the sum of the $d-1$ other terms with $l \neq i$ to be big which yields the next upper bound,

$$\leq \sum_{i=1}^{d} nP\{a^{-1}(n)\big(X^{(i)}-b(n)\big) \leq -M,\ a^{-1}(n)(\sum_{l\neq i} X^{(l)} - db(n)) > -K+M\},$$

and for the sum of $d-1$ terms to be bigger than (-K+M), at least one summand must be bigger than $(-K+M)/(d-1)$ and this leads to the bound.

$$\leq \sum_{i=1}^{d}\sum_{l\neq i} nP\{a^{-1}(n)\big(X^{(i)}-b(n)\big) \leq -M,\ a^{-1}(n)X^{(l)} - db(n)) > \frac{(-K+M)}{d-1}\}$$

and as $n \to \infty$, this converges to

$$\to \sum_{i=1}^{d}\sum_{l\neq i} \nu\{\boldsymbol{x} \in [-\boldsymbol{\infty},\boldsymbol{\infty}] \setminus \{-\boldsymbol{\infty}\} : x^{(i)} \leq -M,\ x^{(l} > \frac{-K+M}{d-1}\}$$
$$\leq \sum_{i=1}^{d}\sum_{l\neq i} \nu\{\boldsymbol{x} \in [-\boldsymbol{\infty},\boldsymbol{\infty}] \setminus \{-\boldsymbol{\infty}\} : x^{(i)} \leq -1,\ x^{(l} > \frac{-K+M}{d-1}\}$$

As $M \to \infty$, this converges to 0 since all bivariate marginals of the limit distribution $G(\boldsymbol{x})$ in (4.1) being proper precludes the limit from being positive. □

4.2. **Back to our interesting special case.** Consider again the example in Subsection 3.2, where the standard $\psi_\infty(\cdot)$ is given in (3.12). Since G has Gumbel marginals, we have $\nu^{(i)}(x,\infty] = \exp\{-x\}$ which makes $V^{(i)}(x) = \log x$ for $x > 0$. From the analogue of (2.10) with $-\boldsymbol{\infty}$ replacing $\mathbf{0}$ we have

$$\nu(-\boldsymbol{\infty}, (V^{(i)}(x^{(i)};\ i=1,2]^c) = \psi_\infty([\mathbf{0},\boldsymbol{x}]^c)$$

and thus

$$\nu([-\boldsymbol{\infty},\boldsymbol{x}]^c) = \psi_\infty([\mathbf{0}, e^{\boldsymbol{x}}]^c),$$

for $\boldsymbol{x} \neq -\boldsymbol{\infty}$ and where $e^{\boldsymbol{x}} = (e^{x^{(1)}}, e^{x^{(2)}})$. So ν has a density, which we call $\nu'(u,v)$, and from (3.13),

$$\nu'(u,v) = \psi'_\infty(e^u,e^v)e^u e^v = (1+\xi)\Big(e^{\xi u}+e^{\xi v}\Big)^{-1/\xi-2} e^{\xi u}e^{\xi v},$$

for $(u,v)\in\mathbb{R}^2$. For this example, the limit in (4.14) is

$$\nu\{(u,v)\in\mathbb{R}^2 : u+v>0\} = \iint_{\{(u,v)\in\mathbb{R}^2:u+v>0\}} \nu'(u,v)du\,dv$$

which we may evaluate as follows: Write $s=e^{\xi u}$, $t=e^{\xi v}$ and the integral becomes

$$= \iint_{\{(s,t)\in\mathbb{R}_+^2, st>1\}} \Big(\frac{1+\xi}{\xi^2}\Big)\big(s+t\big)^{-1/\xi-2} ds\,dt$$

and writing the double integral as $\int_{s=0}^\infty \int_{t>1/s}$ and doing the inner integral we get

$$= \int_0^\infty \frac{1}{\xi}\big(s+\frac{1}{s}\big)^{-1/\xi-1} ds.$$

Factor out $1/s$ from $s+1/s$ to get

$$= \int_0^\infty \frac{1}{\xi} s^{1/\xi+1}\big(1+s^2\big)^{-1/\xi-1} ds.$$

With the intent to convert this to a beta integral, we now substitute $y=1/(1+s^2)\in(0,1)$ to get

$$= \int_0^1 \frac{1}{2\xi} y^{1/(2\xi)-1}\big(1-y\big)^{1/(2\xi)+1-1} dy = \frac{1}{2\xi} B(\frac{1}{2\xi}, \frac{1}{2\xi}+1)$$

where $B(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}dt$. Expressing this in terms of the Gamma function yields

$$= \frac{\frac{1}{2\xi}\Gamma(\frac{1}{2\xi})\Gamma(\frac{1}{2\xi}+1)}{\Gamma(\frac{1}{\xi}+1)} = \frac{\Gamma(\frac{1}{2\xi}+1)^2}{\Gamma(\frac{1}{\xi}+1)}.$$

To summarize this example: Suppose $d=2$ and F is in a maximal domain of attraction as in (4.1) with the limit G having Gumbel marginals. Suppose further ψ_∞ has the form given in (3.12). Then Corollary 4.2 gives

$$\lim_{t\to\infty} \frac{P\{X^{(1)}+X^{(2)}>2t\}}{P\{X^{(1)}>t\}} = \frac{\Gamma(\frac{1}{2\xi}+1)^2}{\Gamma(\frac{1}{\xi}+1)}.$$

5. Concluding remarks

Religious Copularians have unshakable faith in the value of transforming a multivariate distribution to its copula. For the skeptics who believe the Emperor wears no clothes (Mikosch, 2006), perhaps use of the Pareto copula convinces some of them that the Emperor at least wears socks.

Constructing Lévy measures by transforming to the case of Lebesgue marginals seems, to us, uncritical transferrence of the copula philosophy to the domain of Lévy processes and it seems to us that our transformation of random vectors to those having Pareto marginals has much stronger probabilistic interpretation.

Adding dependent random variables in the domain of attraction of the Gumbel distribution as discussed in Proposition 4.1 produces a specific tail behavior when asymptotic independence is absent. When the random variables are independent, the result requires the concept of subexponentiality. We are actively thinking about the case of asymptotic independence specifically ruled out by Proposition 4.1 and Corollary 4.2.

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