phases depending on the ageing temperature and the nature of the phases in equilibrium.

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The particle aspect of meson theory

BY N. KEMMER

Imperial College of Science and Technology, London, S.W. 7

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1. INTRODUCTION

For many years a central problem of theoretical physics has been to set up a satisfactory relativistic theory of elementary particles. This problem is yet far from solution, the notorious occurrence of infinite self-energies and similar divergencies having hitherto frustrated all attempts at complete formulation. Nevertheless, definite advances towards the understanding of the general problem have recently been made, not so much by improvement of the theory as by a more detailed study of all its possible types and variants and the resulting clarification of the essential underlying principles.

The picture of an elementary relativistic quantum-mechanical "particle" can now be roughly outlined as follows: The "exact theory" is one of quantized waves, the particle characteristics appearing as consequences of the non-commutation of the wave amplitudes. There exist possible theories with any given integral or half-integral value s of the "spin" of the particle, the number of independent states of polarization of the corresponding waves then being $2s+1 = N$ (Fierz 1939; Fierz and Pauli 1939).

For integral s, i.e. odd N, the particles must be taken to obey Bose statistics, for half-integral s, i.e. even N, only the assumption of Fermi statistics leads to physically permissible results.

In the case of particles with an electric charge (or even a magnetic moment, e.g. the neutron) the theory necessarily includes particles of either sign of the charge (moment), and only the charge density, not the particle density, is strictly an observable. However, if transitions involving the annihilation of two opposite charges are excluded, as is rigorously correct in the non-relativistic limit, the density can be interpreted as a true particle density. Therefore the transition to the limiting theory of a classical *particle* is possible.

On the other hand, for any uncharged particle the introduction of an antiparticle can be avoided by taking the wave functions to be real as in the well-known case of the photon (Majorana 1937; Kemmer 1938*b*; Møller 1938). However, one then finds that either the density (in the Bose case) or the energy density (in the Fermi case), if considered as c-numbers, vanish identically. Further, the non-relativistic Schrödinger equation is not satisfied by such wave functions in the limiting case. Therefore a limiting classical particle theory does *not* exist.

Conversely, one finds that, at least in the Bose case, a complete correspondence of the theory of uncharged particles with a classical *wave* theory exists, whereas the correspondence appears to *fail* for charged particles (Bhabha 1939). The latter fact can well be understood, for a classical entity corresponding to the quantum mechanical "charged field" is hard to envisage.

For the special case of the *meson* we thus have the following peculiar situation: although it seems likely that both charged and uncharged mesons exist, and the quantum treatment of the two is well nigh identical (Kemmer 1938*b*), the uncharged one (neutretto) is classically a true field, the charged one, on the other hand, a particle.

There can be no doubt that to the experimental worker also the charged meson appears first and foremost as a particle observed in a cloud chamber or by means of some other of its effects as a point charge. It is therefore very surprising to find that theoretical work has laid stress on the wave aspect of the meson practically throughout, the similarity to Maxwell's equations having been extensively used. It is clear from the above that this is neither justified by experimental considerations nor by arguments of correspondence.

In the present paper an entirely different aspect of the meson equations will be put forward, in which the similarity to Dirac's equation of the electron is emphasized. Although it would not seem to be so from previous work, the similarity is found to be very striking and the new procedure yields more than a mere restatement of old results. In the former presentation of the theory a number of points concerning the particle aspect of the meson undoubtedly remained somewhat obscure. For instance, the correct transition to the non-relativistic wave mechanics of a particle of spin 1 has never been completely given, and it can hardly be claimed that in the meson case the use of the term "spin" as distinct from polarization has been fully justified. In addition, the relation—so important in non-relativistic wave mechanics and in Dirac's theory—which interprets the value of any observable as the average of the corresponding operator over the given probability distribution of the particle, hitherto appeared to possess no counterpart in meson theory. The object of the following is to remedy these omissions.

The meson equations will appear as equations of the Dirac type, but will involve matrices obeying a different scheme of commutation rules. These rules were first given by Duffin (1938).* A treatment of similar character has also been developed by Belinfante (1939), who, however, presents the matter in an entirely different form, namely in a generalized spinor notation. Further, the "theory of the photon" proposed by de Broglie (1934, 1936) is in all essentials equivalent to the free particle case of the following. However, the treatment here proposed may claim to be more general in so far as no use is made of any particular representation of the fundamental matrices. Duffin confines himself to stating the representations, which in fact prove to be the only non-trivial irreducible ones, Belinfante considers only one of them, and the representation used by de Broglie is actually reducible. The complete avoidance of spinor notation in the following may also claim to be of some practical advantage and promises to simplify the description of particles with higher spin values (Dirac 1936; Fierz 1939; Fierz and Pauli 1939; also Majorana 1932; Klein 1936; Wigner 1939).

* The short note by Duffin contains much of the matter here more fully developed. The writer was studying the same subject previously, but is glad to acknowledge that Duffin's statement of the commutation rules was new to him and has greatly influenced the detailed development of the work. Many thanks for exchange of information are due both to Dr Duffin and to Dr Belinfante.

The interconnexion of the various formalisms mentioned will be discussed in more detail in §§ 6 and 7.

For the sake of conciseness it will be assumed that the physical contents of meson theory (Yukawa, Sakata and Taketani 1938; Kemmer 1938*a*; Fröhlich, Heitler and Kemmer 1938; Bhabha 1938) are known from the original "wave theoretical" presentation. Dirac's theory of the electron will also be assumed to be known, and it will be used as a pattern throughout in the form given to it by Pauli (1933) in the *Handbuch der Physik*. For details of proof it will be often sufficient to refer to Pauli's article.

2. The wave equation of the free meson

It is proposed to develop the theory in a purely deductive manner, without establishing the connexion with previous presentations until § 6. The term *meson* will, by definition, be applied to a particle of mass m and charge $\pm e$, which is described by the wave equation

$$\partial_\mu \beta_\mu \psi + \kappa\psi = 0, \tag{1}$$

together with the following commutation rules for the operators β_μ:

$$\beta_\mu \beta_\nu \beta_\rho + \beta_\rho \beta_\nu \beta_\mu = \beta_\mu \delta_{\nu\rho} + \beta_\rho \delta_{\nu\mu}. \tag{2}$$

No further specification of the β_μ is made.

In (1) the abbreviations

$$\kappa = \frac{mc}{\hbar}, \quad \partial_\mu = \frac{\partial}{\partial x_\mu}, \quad x_4 = ict,$$

are used and the usual convention regarding the summation over double suffices is implied.

An immediate consequence of (2) is that

$$\eta_4 = 2\beta_4^2 - 1 \tag{3}$$

obeys the following algebraic relations:

$$\left.\begin{aligned} &\beta_4 = \eta_4\beta_4 = \beta_4\eta_4, \quad \eta_4^2 = 1, \\ &\eta_4\beta_k + \beta_k\eta_4 = 0 \quad (k = 1, 2, 3). \end{aligned}\right\} \tag{3'}$$

Hence, if $\psi^\dagger$ is defined by $\qquad \psi^\dagger = i\psi^*\eta_4, \qquad (4)$

it satisfies the equation $\qquad \partial_\mu \psi^\dagger \beta_\mu - \kappa\psi^\dagger = 0. \qquad (5)$

Multiplying (1) by $\partial_\rho \beta_\rho \beta_\nu$ one deduces that

$$\partial_\mu \psi = \partial_\nu \beta_\nu \beta_\mu \psi, \tag{6}$$

and similarly from (5),
$$\partial_\mu \psi^\dagger = \partial_\nu \psi^\dagger \beta_\mu \beta_\nu. \tag{7}$$

It is to be noted that the differential relations (6) and (7) appear as *consequences* of (1) and (5) and not as initial conditions to be imposed on the wave function. As will be shown in more detail later, this represents an essential difference between the formalism here proposed and the aforementioned formalisms of Dirac and of de Broglie.

From (1) and (6) the second order wave equation

$$\partial_\mu \partial_\mu \psi = \kappa^2 \psi \tag{8}$$

can immediately be deduced, and (5) and (7) similarly lead to

$$\partial_\mu \partial_\mu \psi^\dagger = \kappa^2 \psi^\dagger. \tag{9}$$

A further consequence of (1) and (5) is that

$$\partial_\mu s_\mu = 0, \tag{10}$$

if
$$s_\mu = \psi^\dagger \beta_\mu \psi, \tag{10'}$$

so that s_μ may be interpreted as the four vector of current and density. It must yet be proved that the scheme here proposed can be made a relativistically invariant one by suitably defining the transformation undergone by the ψ functions when the space-time frame is subjected to a Lorentz transformation. This proof will be furnished in § 4. Apart from that point, however, equations (8) and (10) alone suffice to show that this formalism is adequate to describe a quantum mechanical particle. As in the case of the electron, it is sufficient to insert the well-known zero order "*WKB*" approximation to the wave function

$$\psi = a \exp(iS/\hbar)$$

into (8) to show that, in the limit $\hbar = 0$, (8) describes a classical relativistic particle (see Pauli 1933, p. 240). In addition, the existence of current and density enable the more general quantum mechanical statistical particle picture to be maintained. The density $s_0 = s_4/i$ is, of course, not necessarily positive, but the discussion by Pauli and Weisskopf (1934) has proved that this is in fact not a necessary requirement in the relativistic region. With exactly as much justification as in the electron case one can attempt to connect the expectation value $\overline{\omega}$ of any observable with an operator ω by the following definition (cf. de Broglie 1936):

$$\overline{\omega} = \frac{1}{i}\int \psi^\dagger \overline{\overline{\beta_4 \omega}} \psi \, dV. \tag{11}$$

The form of the right-hand side of (11) allows the direct physical interpretation of $\overline{\omega}$ as the mean value of ω taken over the density distribution s_0. As, however, contrary to the electron case, there exists no transformation of the expression for s_0 into a form which does not contain β_4, the above definition contains some ambiguity as to the order of the factors ω and β_4. In general one might expect some symmetrical combination to be the suitable one, and the double bar in (11) has been inserted to denote this symmetrization. It will be found, however, that for all operators of practical significance there can be no doubt whatever as to the correct order.

The operator of the density itself is, of course, unity; the three space components of s_μ on the other hand can be represented by the operators

$$s'_k = i(\beta_4 \beta_k + \beta_k \beta_4), \tag{12}$$

for

$$\frac{1}{i}\overline{\overline{s'_k \beta_4}} = \frac{1}{i}(\beta_4 s'_k + s'_k \beta_4) = \beta_4^2 \beta_k + \beta_k \beta_4^2 = \beta_k, \tag{13}$$

and therefore

$$\overline{s'_k} = \int s_k \, dV. \tag{14}$$

The most important application of (11) is to the case of the energy-momentum vector. If the analogy to the electron case is to be maintained, the operators ω should here be the differential operators $\frac{\hbar}{i}\frac{\partial}{\partial x_\mu}$:

$$\overline{p_\mu} = \frac{1}{i}\int \psi^\dagger \beta_4 \frac{\hbar}{i} \partial_\mu \psi \, dV. \tag{15}$$

Such a definition of energy and momentum was not known in former meson theory, and the expression usually given seems to be very different from (15). That in fact the two definitions are equivalent, will nevertheless be proved immediately. First, however, it should be noted that (15) is equivalent to the postulate that the tensor of energy and momentum density is

$$T_{\mu\nu} = \frac{c}{2}\left[\psi^\dagger \beta_\nu \frac{\hbar}{i} \partial_\mu \psi - \left(\frac{\hbar}{i} \partial_\mu \psi^\dagger\right) \beta_\nu \psi\right], \tag{16}$$

the second term being added in order to make $T_{\mu\nu}$ real. Its integral is equal to that of the first term. It follows immediately from (1) and (2) that

$$\frac{\partial T_{\mu\nu}}{\partial x_\nu} = 0, \tag{17}$$

so that the $\overline{p_\mu}$ are constants as required for a free particle. The tensor (16) is, however, not symmetrical; it can be replaced by a symmetrical one by a method similar to the one given by Tetrode for the Dirac electron. From (1) and (2) it follows that

$$\begin{aligned}\psi^\dagger\beta_\nu\partial_\mu\psi &= \psi^\dagger\beta_\nu\delta_{\rho\mu}\partial_\rho\psi \\ &= \psi^\dagger(\beta_\nu\beta_\mu\beta_\rho+\beta_\rho\beta_\mu\beta_\nu-\delta_{\mu\nu}\beta_\rho)\partial_\rho\psi,\end{aligned} \tag{18}$$

and a similar transformation can be performed with the second term in (16). Hence, putting

$$\Theta_{\mu\nu} = \frac{-mc^2}{i}[\psi^\dagger(\beta_\mu\beta_\nu+\beta_\nu\beta_\mu)\psi-\delta_{\mu\nu}\psi^\dagger\psi], \tag{19}$$

we obtain
$$T_{\mu\nu} = \Theta_{\mu\nu}+\frac{\hbar c}{2i}\frac{d}{dx_\rho}\psi^\dagger(\beta_\rho\beta_\mu\beta_\nu-\beta_\nu\beta_\mu\beta_\rho)\psi, \tag{20}$$

and
$$\frac{\partial\Theta_{\mu\nu}}{\partial x_\nu} = 0. \tag{21}$$

Now $\Theta_{\mu\nu}$ is symmetrical,‡ and moreover, by (4)

$$\Theta_{44} = -mc^2\psi^*\psi,$$

so that the energy density is essentially positive. Θ has in fact exactly the same properties as the energy-momentum tensor usually considered in meson theory. In § 6 it will be proved that for a suitable representation of the β_μ it is actually identical with that tensor. On the other hand, it follows from (20) that the expectation values of energy and momentum, as defined by (15) can equally well be given in terms of Θ, namely, by

$$\overline{p_\mu} = \frac{1}{ic}\int\Theta_{\mu4}\,dV. \tag{22}$$

It is thus clear that it is *not* necessary to abandon the connexion between the operators $\frac{\hbar}{i}\partial_\mu$ and the momenta, as seemed to be the case in previous presentations of meson theory.

It is a general theorem that the existence of a symmetrical energy-

‡ It should be noted that

$$\Theta_{\mu\nu}+\tfrac{1}{2}(T_{\mu\nu}+T_{\nu\mu}) = \Theta'_{\mu\nu}.$$

For the free meson the latter tensor can also be shown to satisfy a continuity equation $\frac{\partial\Theta'_{\mu\nu}}{\partial x_\nu} = 0$, but there is no way of generalizing this result to the case of electromagnetic interaction as will be done for $\Theta_{\mu\nu}$. Further, $-\Theta'_{44}$ is *not* necessarily positive. This alternative procedure of symmetrization may therefore be ignored.

momentum tensor automatically ensures the possibility of defining angular momentum. We have but to put

$$P_{ik} = -P_{ki} = \frac{1}{ic}\int (x_i \Theta_{k4} - x_k \Theta_{i4})\, dV, \tag{23}$$

and the P_{ik} are necessarily constant. Now by (20)

$$x_i \Theta_{k4} = x_i T_{k4} - \frac{\hbar c}{2i} x_i \frac{d}{dx_\rho} \psi^\dagger (\beta_\rho \beta_k \beta_4 - \beta_4 \beta_k \beta_\rho)\psi, \tag{24}$$

and hence
$$P_{ik} \doteq \frac{1}{i}\int \psi^\dagger \beta_4 \left(x_i \frac{\hbar}{i}\partial_k - x_k \frac{\hbar}{i}\partial_i\right)\psi\, dV + \frac{\hbar}{i}\int \psi^\dagger \beta_4 \frac{\beta_i\beta_k - \beta_k\beta_i}{i}\psi\, dV. \tag{25}$$

The first term of (25) is evidently to be interpreted as the orbital momentum, the second term as the spin of the meson, and by definition (11) the spin *operator* then is

$$S_{ik} = \frac{1}{i}(\beta_i\beta_k - \beta_k\beta_i). \tag{26}$$

The similarity to the electron case is striking; there the spin could be written as $\frac{1}{4i}(\gamma_i\gamma_k - \gamma_k\gamma_i)$. It is to be noted that S_{ik} commutes with β_4, so that no ambiguity of the kind discussed on p. 96 arises. It is further very noteworthy that

$$S_{ik}^3 = S_{ik}, \tag{27}$$

whence it follows that the eigenvalues of the spin in this theory can only be ± 1 and 0, as is to be expected if the formalism is indeed to be connected with the meson. It seems satisfactory that there is such a possibility of defining the spin as a *momentum*, independently of any consideration about the number of states of polarization.

3. The interaction with the electromagnetic field

In non-relativistic wave mechanics and for the Dirac electron the interaction with the electromagnetic field is introduced very simply, by the well-known substitutions

$$\partial_\mu \to \partial_\mu^- = \partial_\mu - \frac{ie}{\hbar c}\Phi_\mu, \tag{28}$$

when the differentiation applies to ψ, and

$$\partial_\mu \to \partial_\mu^+ = \partial_\mu + \frac{ie}{\hbar c}\Phi_\mu, \tag{28'}$$

when it applies to $\psi^\dagger$. It is not quite trivial that this mode of procedure can also be used here, in fact, in theories of similar type (Dirac 1936; Fierz and Pauli 1939) care must be taken in applying this rule, and it is inconsistent, if applied to *all* fundamental equations. However, here it is entirely correct to apply these substitutions to equations (1) and (6) which therefore now read as

$$\partial_\mu^- \beta_\mu \psi + \kappa\psi = 0 \tag{29}$$

and

$$\partial_\mu^+ \psi^\dagger \beta_\mu - \kappa\psi^\dagger = 0. \tag{30}$$

The considerations on relativistic invariance given in § 4 hold not only for (1) and (5) but also for these generalized equations, and as there are no "initial conditions" in the present formulation, there can be no inconsistency in this generalization.

The introduction of electromagnetic interaction thus results in very few changes to the developments of the previous section. As in the electron case the definition of s_μ, (10), remains completely unaltered, and here the same is also true of the symmetrical energy-momentum tensor $\Theta_{\mu\nu}$, as defined by (19). This can readily be seen to be so, in spite of the fact that the *un*symmetrical tensor now is

$$T_{\mu\nu} = \frac{c}{2}\left[\psi^\dagger \beta_\nu \frac{\hbar}{i} \partial_\mu^- \psi - \left(\frac{\hbar}{i} \partial_\mu^+ \psi^\dagger\right) \beta_\nu \psi\right]. \tag{31}$$

Instead of (17) and (21) one can now derive the equations

$$\frac{\partial T_{\mu\nu}}{\partial x_\nu} = \frac{\partial \Theta_{\mu\nu}}{\partial x_\nu} = eF_{\mu\nu} s_\nu, \tag{32}$$

where the $F_{\mu\nu}$ are the electric and magnetic field strengths.

Similarly, equation (25) for the angular momentum is only altered in so far as the operator ∂_μ in the orbital part is replaced by ∂_μ^-. The spin operator is entirely unaltered.

An important difference, however, occurs in equations (6) and (7). Using the commutation rules

$$\partial_\mu^\mp \partial_\nu^\mp - \partial_\nu^\mp \partial_\mu^\mp = \mp \frac{ie}{\hbar c} F_{\mu\nu}, \tag{33}$$

one now finds

$$\partial_\mu^- \psi = \partial_\nu^- \beta_\nu \beta_\mu \psi + \frac{ie}{2mc^2} F_{\nu\rho} (\beta_\rho \beta_\mu \beta_\nu - \delta_{\rho\mu} \beta_\nu) \psi \tag{34}$$

and

$$\partial_\mu^+ \psi^\dagger = \partial_\nu^+ \psi^\dagger \beta_\mu \beta_\nu + \frac{ie}{2mc^2} F_{\nu\rho} \psi^\dagger (\beta_\nu \beta_\mu \beta_\rho - \beta_\nu \delta_{\mu\rho}), \tag{34'}$$

and the second order wave equations are then readily seen to be

$$\partial_\mu^- \partial_\mu^- \psi = \kappa^2 \psi + \frac{ie}{\hbar c} F_{\mu\nu} \beta_\mu \beta_\nu \psi + \frac{ie}{2mc^2} \partial_\mu^- F_{\nu\rho} (\beta_\rho \beta_\mu \beta_\nu - \delta_{\rho\mu} \beta_\nu) \psi, \tag{35}$$

and
$$\partial_\mu^+ \partial_\mu^+ \psi^\dagger = \kappa^2 \psi^\dagger - \frac{ie}{\hbar c} F_{\mu\nu} \psi^\dagger \beta_\nu \beta_\mu + \frac{ie}{2mc^2} \partial_\mu^+ F_{\nu\rho} \psi^\dagger (\beta_\nu \beta_\mu \beta_\rho - \beta_\nu \delta_{\mu\rho}). \tag{36}$$

The first term on the right-hand side of (35) or (36) has its counterpart in the theory of the electron. It describes the interaction of the external field with the electric and magnetic moments of the particle. The second term, however, is peculiar to meson theory and cannot be so directly interpreted. To understand it better it is useful to define the magnetic moment in another way, namely, as

$$M_{ik} = \frac{e}{2} \int (x_i s_k - x_k s_i) \, dV. \tag{37}$$

In the case of the electron the expression corresponding to this can be split into two parts, the first giving an orbital moment, the second the moment due to the spin; the current can be split correspondingly. Here we can proceed analogously. By (29) and (30) we have

$$s_\mu = \psi^\dagger \beta_\mu \psi = \frac{1}{2\kappa} [\partial_\nu^+ \psi^\dagger \beta_\nu \beta_\mu \psi - \psi^\dagger \beta_\mu \beta_\nu \partial_\nu^- \psi], \tag{38}$$

and therefore by (34) and (34′)

$$s_\mu = \frac{1}{2\kappa} \left[(\partial_\mu^+ \psi^\dagger) \psi - \psi^\dagger \partial_\mu^- \psi + \frac{d}{dx_\nu} \psi^\dagger (\beta_\nu \beta_\mu - \beta_\mu \beta_\nu) \psi - \frac{ie}{mc^2} F_{\nu\rho} \psi^\dagger \beta_\nu \beta_\mu \beta_\rho \psi \right]. \tag{39}$$

Consequently

$$M_{ik} = \frac{e}{2mc} \left[\frac{1}{i} \int \psi^\dagger \left(x_i \frac{\hbar}{i} \partial_k^- - x_k \frac{\hbar}{i} \partial_i^- \right) \psi \, dV + \frac{\hbar}{i} \int \psi^\dagger \frac{\beta_i \beta_k - \beta_k \beta_i}{i} \psi \, dV + \frac{e\hbar}{2mc^2} \frac{1}{i} \int F_{\mu\nu} \psi^\dagger \beta_\mu (\beta_k x_i - \beta_i x_k) \beta_\nu \psi \, dV \right]. \tag{40}$$

The first two terms are, of course, the orbital and the spin moments of the meson respectively, and here, as for the electron, they are proportional to the corresponding terms of the mechanical moment, as given by (25), except for the operator β_4. A difference compared with the electron is, however, the absence of the anomalous gyromagnetic factor 2 in the spin term. Further, the additional third term is of a new type; it vanishes if there is no external field and might best be described as a polarizability of the meson. It is clear that it is connected with the appearance of the last terms in (35) and (36) already noted.

A circumstance peculiar to this theory is also the fact that the absence of β_4 in both orbital and spin magnetic moments makes the definition of operators in the sense of (11) impossible for these physical quantities.* It would appear that the latter do not possess a very direct meaning in the *particle* picture of the meson, but except in the non-relativistic limit are rather only measurable in a way not directly dependent on the probability distribution of the particle.

The division of the current into two parts, as given by (39), is the counterpart of the procedure given by Gordon for the electron. In both cases the second term or "polarization current" satisfies a continuity equation on its own. The first term has the same form as the current in non-relativistic theory, a fact of importance in the comparison with this limiting case.

These formulae may suffice to indicate the extent to which the matrix treatment of meson theory is successful. There can hardly be any doubt that it may serve to simplify practical calculations in cases where the meson appears primarily as a *particle*. It has not been here attempted to include the interaction of the meson with protons and neutrons, and it can readily be seen that in the scheme here used this would not be simple. It is, however, clear that in that interaction the meson enters primarily as a *field* (although a non-classical charged one), and it is then only natural to retain the old formulation.

4. Proof of relativistic invariance

The proof of the relativistic invariance of the formalism is so exactly analogous to the electron case that it will almost suffice to refer to the proof given for that case, for instance, by Pauli (1933). The behaviour of the ∂_μ under Lorentz transformations is known, and the requirement is to find a suitable linear transformation of the ψ among themselves by which (1) or (29) is brought back to its original form in the new co-ordinate system.

* [*Note added in proof:* It is however always possible to define such operators if explicit dependence upon the momentum operators is permitted. This is achieved by using the relation stated below as (69), and the procedure has proved essential in the applications of the transformations which have since been studied.]

It is sufficient to find such a transformation for the case of the general *infinitesimal* four-dimensional rotation. The latter can be given in the form

$$x'_\mu = x_\mu + \epsilon_{\mu\nu} x_\nu, \tag{41}$$

where $\epsilon_{\mu\nu} = -\epsilon_{\nu\mu}$, and the required transformation of the ψ may be expressed as

$$\psi' = S\psi, \quad (\psi^\dagger)' = \psi^\dagger S^{-1}, \tag{42}$$

where $$S = I + \tfrac{1}{2}\epsilon_{\mu\nu} t_{\mu\nu} \quad \text{and} \quad t_{\mu\nu} = -t_{\nu\mu}.$$

The wave equation can readily be seen to be invariant if the $t_{\mu\nu}$ are connected with the β_μ by the relations

$$\beta_\mu t_{\nu\rho} - t_{\nu\rho}\beta_\mu = \delta_{\mu\nu}\beta_\rho - \delta_{\mu\rho}\beta_\nu, \tag{43}$$

so that it remains to find a suitable set of $t_{\mu\nu}$. Equation (42), for μ and $\nu \neq 4$, however, also affords an independent definition of the spin operator, so that the results of previous sections directly indicate that

$$t_{\mu\nu} = \beta_\mu\beta_\nu - \beta_\nu\beta_\mu = iS_{\mu\nu}, \tag{44}$$

should be an adequate choice. This is indeed the case as can be proved directly by interchanging ν and ρ in equation (2) and subtracting from the original equation:

$$\beta_\mu(\beta_\nu\beta_\rho - \beta_\rho\beta_\nu) - (\beta_\nu\beta_\rho - \beta_\rho\beta_\nu)\beta_\mu = \delta_{\mu\nu}\beta_\rho - \delta_{\mu\rho}\beta_\nu. \tag{45}$$

This is precisely the relation required by (43). The invariance of the scheme is thus proved. By applying infinitesimal transformations similarly it would also be easy to show that the s_μ defined by (10) are indeed a vector, $T_{\mu\nu}$ a tensor and so on. The procedure is completely independent of the particular representation used for the β_μ. Once the theory is proved to be equivalent to the tensor form of meson theory used previously, the invariance is of course also evident directly.

5. Algebraic properties* of the β_μ

The previous development was entirely independent of the particular form of the β_μ matrices, the commutation rules (2) having been sufficient to define all the physical quantities that were of interest. In the case of the Dirac electron a similar development can of course be put forward, but it can then be shown that the four row matrices found by Dirac give the only irreducible representation of that particular algebra. The rigorous proof of this fact is a matter of abstract algebra and in the present case the corresponding procedure is, unfortunately, even more complicated than for

* The writer is greatly indebted to Professor W. Pauli for the suggestion of the detailed study of the algebra of the β_μ and for the introduction to the mathematical apparatus used.

the electron, owing to the fact that the β_μ matrices occurring here possess no reciprocals. The completely rigorous method has, however, actually been carried through, and it is merely for the sake of brevity that the proof is abridged. To find the possible independent irreducible representations of the algebra given by (2) one must first find the number of linearly independent quantities among the β_μ and their multiple products. In the electron case there are 16 such quantities. Here their number is considerably greater, but they may still be counted fairly easily if the auxiliary quantities

$$\eta_\mu = 2\beta_\mu^2 - 1 \tag{46}$$

are introduced, of which the fourth has already been used in § 2. There exist the following relations involving the β_μ and the η_μ:

$$\left.\begin{aligned} \beta_\mu^3 = \beta_\mu, \quad \eta_\mu^2 = 1, \\ \eta_\mu\eta_\nu - \eta_\nu\eta_\mu = 0, \quad \eta_\mu\beta_\nu + \beta_\nu\eta_\mu = 0 \quad (\mu \neq \nu), \\ \beta_\mu = \eta_\mu\beta_\mu = \beta_\mu\eta_\mu \quad \text{(no summation!)}. \end{aligned}\right\} \tag{47}$$

It follows from these that the following is a complete list of the linearly independent elements of the algebra:

Element	No. of elements of this type	Element	No. of elements of this type
I	1	$\eta_\mu\beta_\nu\beta_\rho\beta_\sigma$	12
β_μ	4	$\eta_\mu\eta_\nu$	6
$\beta_\mu\beta_\nu$	12	$\eta_\mu\eta_\nu\beta_\rho$	12
$\beta_\mu\beta_\nu\beta_\rho$	12	$\eta_\mu\eta_\nu\beta_\rho\beta_\sigma$	12
$\beta_\mu\beta_\nu\beta_\rho\beta_\sigma$	6	$\eta_\mu\eta_\nu\eta_\rho$	4
η_μ	4	$\eta_\mu\eta_\nu\eta_\rho\beta_\sigma$	4
$\eta_\mu\beta_\nu$	12	$\eta_\mu\eta_\nu\eta_\rho\eta_\sigma$	1
$\eta_\mu\beta_\nu\beta_\rho$	24	Total no.	126

(48)

There are thus 126 independent elements among the multiple products of the β_μ. It is next necessary to find the particular elements which commute with all the others. If one proceeds to construct these by successively postulating commutability with each of the quantities in the above list, one readily finds that the following expressions have the desired property:

$$I \text{ (the unit matrix)}, \quad M = \sum_\mu \eta_\mu - \sum_{\mu<\nu} \eta_\mu\eta_\nu$$

and

$$N = \eta_1\eta_2\eta_3\eta_4\Big(1 - \sum_\mu \eta_\mu\Big). \tag{49}$$

Any further expressions which might be shown to commute with all the 126 elements can be proved to be linear combinations of the above three. The following are instances of this fact:

$$\left.\begin{aligned} M^2 &= 10-6M+6N, \\ N^2 &= 5-2M, \\ MN &= 3M-4N. \end{aligned}\right\} \tag{50}$$

Provided an algebra satisfies a certain regularity condition,* which has been verified in the present case, the knowledge of the number of independent elements and of the number of elements commuting with all others is sufficient to determine the irreducible representations of the algebra. The latter number directly gives the number of inequivalent irreducible representations, whereas the total number of elements is known to be equal to the sum of the squares of the degrees of these representations. Thus here we must expect to find *three* inequivalent irreducible representations, say of degrees n_1, n_2 and n_3 respectively, and

$$n_1^2+n_2^2+n_3^2 = 126. \tag{51}$$

These three representations will be given in the next section. Their degrees are actually 10, 5, and 1 ($10^2+5^2+1^2 = 126$). We can be sure that there are no further inequivalent ones. Thus, in spite of the fact that the β_μ possess no reciprocals and are therefore initially more difficult to deal with than the Dirac γ_μ, it is comparatively simple to obtain a complete picture of their algebraic properties. Without going into any more details we may conclude this section by giving one more algebraic relation which should be of importance in further developments. This concerns the spurs of the 126 quantities listed in (48). It can be readily proved that, in any representation of the 126 matrices only 16 have non-vanishing spurs, namely the four η_μ with all their multiple products. This set of quantities incidentally

* This condition is that the algebra should be "halbeinfach" as for instance defined by v. d. Waerden (1931). Professor Pauli kindly furnished a proof—taken from lectures by Artin in 1927–8—that v. d. Waerden's definition of this condition can be replaced by the following postulate:

Let e_i be the matrices representing the independent elements of the algebra in the "regular representation", and let

$$g_{ik} = \mathrm{Spur}(e_i e_k).$$

The algebra is "halbeinfach" if

$$\mathrm{Det}\,\|g_{ik}\| \neq 0.$$

This determinant (which in the present case has 126^2 elements), proves comparatively easy to evaluate and does not vanish.

forms an abelian subgroup of the algebra. For the three irreducible representations one finds the following spurs:

Representation	10-row	5-row	1-row
spur I	10	5	1
spur η_μ	2	-1	-1
spur $\eta_\mu \eta_\nu$	-2	1	1
spur $\eta_\mu \eta_\nu \eta_\rho$	-2	3	-1
spur $\eta_\mu \eta_\nu \eta_\rho \eta_\sigma$	2	-3	1

(52)

and all other spurs vanish.

6. Representations of the β_μ and the connexion with alternative formulations

As stated above, there exist three inequivalent irreducible representations of the β_μ which can readily be given explicitly. They are

$$\beta_1=\begin{pmatrix}
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&-1\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&-1&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&-1&\cdot&\cdot&\cdot&\cdot&\cdot\\
-1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot
\end{pmatrix},\quad
\beta_2=\begin{pmatrix}
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&-1\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&-1&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&-1&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&-1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot
\end{pmatrix},$$

$$\beta_3=\begin{pmatrix}
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&-1\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&-1&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&-1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&-1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot
\end{pmatrix},\quad
\beta_4=\begin{pmatrix}
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&-i&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&-i&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&-i&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
i&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&i&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&i&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot
\end{pmatrix};$$

$$\beta_1=\begin{pmatrix}
\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&1\\
\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&1&\cdot&\cdot&\cdot
\end{pmatrix},\
\beta_2=\begin{pmatrix}
\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&1\\
\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&1&\cdot&\cdot
\end{pmatrix},\
\beta_3=\begin{pmatrix}
\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&1\\
\cdot&\cdot&\cdot&1&\cdot
\end{pmatrix},\
\beta_4=\begin{pmatrix}
\cdot&\cdot&\cdot&\cdot&-i\\
\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot\\
i&\cdot&\cdot&\cdot&\cdot
\end{pmatrix};$$

and

$$\beta_1=\beta_2=\beta_3=\beta_4=(\cdot). \tag{53}$$

(The dots in the above denote zeros.)

It can be seen that the matrices given are really irreducible, for one can readily find that any matrix of the same degree can be expressed as a linear combination of the four matrices β_μ and their multiple products. Of the three representations the third is a trivial one of little physical interest, but the other two both give possible meson theories. Neither of them, however, gives anything completely new; the ten-row representation simply leads to the usual theory based on Proca's (1936) equations, in which the wave function consists of four components forming a four-vector and six forming an antisymmetrical tensor; the five-row one to the Klein Gordon or so called "scalar" theory, in which the wave function consists of a scalar and its four-gradient. (From the point of view of spatial rotation only, the former wave function consists of three vectors and a scalar, the latter of one vector and two scalars. In (53) this has been marked by the dotted lines subdividing the matrices.) The reflexion character of the wave functions can still be determined arbitrarily, and therefore the above schemes may be equally well taken to describe the dual theories, i.e. the "pseudovector" and the "pseudoscalar" theory respectively (Kemmer 1938*a*). As no further possibilities are included in the formalism the fact is confirmed that the cases already known are the only possible theories for spin values 0 and 1.

It may appear surprising in this connexion that in the five-row or "scalar" theory there still exists a spin operator $S_{\mu\nu} = i(\beta_\mu \beta_\nu - \beta_\nu \beta_\mu)$. This, however, merely comes from the fact that the four-gradient is included as part of the wave function and, according to (44), the spin defines the infinitesimal rotations of *all* components of the wave function. In spite of this the quantity $\beta_4 S_{ik}$ which, according to (25), gives the *expectation value* of the mechanical moment is zero for this representation, so that it is still justifiable to say that the "scalar" meson has no spin. On the other hand, the second term in (40) does not vanish so that in the relativistic region a *magnetic* moment would exist even in scalar theory.

The results just given are already essentially contained in Duffin's (1938) note, the starting point of which is that Proca's and the scalar theory can be stated with the help of the matrices (53). The formulation due to Belinfante (1939) is also closely connected with the above; this can be seen as follows:

If one takes two sets of Dirac matrices that act on two separate suffices of a wave function, which would thus have to possess 16 components, and if one puts

$$\beta'_\mu = \tfrac{1}{2}(\gamma_\mu I' + \gamma'_\mu I), \tag{54}$$

these matrices satisfy Duffin's commutation rules (2). Thus a theory based on the equation

$$\frac{\partial}{\partial x_\mu}(\gamma_\mu I' + \gamma'_\mu I)\,\Psi + \kappa\Psi = 0 \tag{55}$$

is equivalent to the theory here presented, but uses a reducible set of β-matrices. A closer study of the reflexion character, as fixed by (54) and by the assumption that Ψ transforms like the product of two Dirac wave functions, shows that (55) gives the sum of the Proca theory and the *pseudo*scalar theory and in addition the trivial equation $\kappa\Psi_{16} = 0$ for a sixteenth, scalar component of Ψ. In other words, each of the three inequivalent representations of the β_μ is contained just once in the particular representation β'_μ.* Belinfante now specializes the representation by postulating that Ψ should behave like the *symmetrical* product of two Dirac functions. This procedure is equivalent to reduction of the β'_μ and restriction of one's considerations to the ten-row representation only. Belinfante thus studies a formulation of the Proca theory alone. In other respects his treatment is, however, more general than the present one in so far as he includes nuclear interaction which is here omitted altogether.

It is of considerable importance to note that the formulations hitherto described have the characteristic in common that they are analogous to the Dirac equation in the form

$$\partial_\mu \gamma_\mu \psi + \kappa\psi = 0, \tag{56}$$

and not to its alternative formulation

$$\frac{1}{c}\partial_t\psi + \partial_k \alpha_k \psi + i\kappa\gamma_4\psi = 0. \tag{57}$$

In the electron case (57) is for many purposes more useful than (56), and it is therefore of interest to see whether an equation analogous to it might also be used as the starting point in meson theory. For instance, one might attempt to put

$$A_\mu = \tfrac{1}{2}(\alpha_\mu I' + \alpha'_\mu I) \quad (\alpha_4 = \gamma_4), \tag{58}$$

and to take

$$\frac{1}{c}\partial_t\psi + \partial_k A_k \psi + i\kappa A_4 \psi = 0, \tag{59}$$

as the fundamental wave equation. This formalism has never actually been

* These facts are of interest in connexion with the suggestion put forward by Møller and Rosenfeld (1939) that a pseudoscalar meson should be included in the theory in order to obtain a more satisfactory description of nuclear interaction and β-decay.

used to describe the meson, but has been studied in various other connexions. In particular, the theory proposed by de Broglie (1934, 1936) for the photon and studied in detail in many papers by his pupils (e.g. Géhéniau 1938; Tonnelat 1939), is in fact based on (58) and (59). (An entirely different approach to the *same* formalism is afforded by the study of the interaction of two particles each obeying a Dirac equation; e.g. Kemmer 1937.)

For the description of a relativistic particle—meson or de Broglie photon—the theory of this paper appears to have some distinct advantages as compared with de Broglie's formulation. If (59) is postulated instead of (1) it becomes essential to introduce further initial conditions which the wave functions must satisfy, so that the second order wave equation (8) shall hold. There appear to be numerous different ways of stating these conditions; de Broglie's method involves the use of matrices not expressible by means of the β'_μ alone (namely, the matrices $\gamma_\mu I' - \gamma'_\mu I$), and two alternative methods will be stated in § 7. They are not algebraically equivalent to de Broglie's, but in his particular representation they give essentially the same equations. To avoid initial conditions altogether, the only method seems to be to postulate (1), as done here.

Relativistic wave equations of a most general kind have also been given by Dirac (1936). He presents the theory in two separate ways, in spinor notation and in a "Hamiltonian" form. A particular case of his theory are Proca's equations, but as can immediately be seen in his spinor formulation, Dirac's equations are again not algebraically equivalent to the above. This is partly due to the identification of a "self-dual" tensor with one symmetrical spinor, a connexion which is in any case inappropriate if complex tensors are to be considered, but even if his theory is reformulated with the introduction of a second symmetrical spinor (the *two* spinors representing the complex antisymmetrical tensor), it differs from ours in the handling of the additional conditions. In Dirac's form wave equation and initial conditions are so interwoven that the introduction of electromagnetic interaction by means of (28) is *not* a consistent procedure. On the other hand, Dirac's "Hamiltonian" formulation shows clearly that the differences of his theory compared with the present one are merely due to details of representation; in the following section a development of our theory will be given, which exactly parallels the Hamiltonian form. This, in fact, is shown to be essentially equivalent with the de Broglie form of the theory. We have preferred to leave this aspect to the end as it appears preferable to use (1) alone in the initial statement of the theory, but the alternative form is admittedly of some interest quite apart from its connexions with other authors' work. It is decidedly helpful in the study of the non-relativistic limit and may also prove to be of use in practical calculations.

7. The Hamiltonian formulation

Let equation (1) be multiplied by β_4:

$$\partial_4 \beta_4^2 \psi + \partial_k \beta_4 \beta_k \psi + \kappa \beta_4 \psi = 0, \tag{60}$$

and (6) be stated in the form

$$\partial_4 (1 - \beta_4^2)\psi - \partial_k \beta_k \beta_4 \psi = 0. \tag{61}$$

Then, adding the two equations, we obtain

$$\frac{1}{c}\partial_t \psi + \partial_k \frac{(\beta_k \beta_4 - \beta_4 \beta_k)}{i}\psi + i\kappa\beta_4 \psi = 0. \tag{62}$$

As this equation contains ∂_t multiplied into the unit matrix, it is the counterpart of the Dirac equation in the form (57), the hermitian matrices $\frac{1}{i}(\beta_k \beta_4 - \beta_4 \beta_k)$ corresponding to the α_k-matrices. We may properly call

$$H = \frac{c\hbar}{i}\partial_k \frac{(\beta_k \beta_4 - \beta_4 \beta_k)}{i} + mc^2 \beta_4, \tag{63}$$

the Hamiltonian—writing equation (62) as

$$\left(H + \frac{\hbar}{i}\frac{\partial}{\partial t}\right)\psi = 0, \tag{64}$$

for by (11) and (15) the expectation value of the energy is

$$E = \frac{1}{i}\int \psi^\dagger \left(-\frac{\hbar}{i}\frac{\partial}{\partial t}\right)\psi \, dV = \frac{1}{i}\int \psi^\dagger \beta_4 H \psi \, dV, \tag{65}$$

so that H is a possible form of the energy operator. The inclusion of the factor β_4 on the right-hand side of (65) is, of course, essential, the quantities

$$\frac{1}{i}\int \psi^\dagger H \psi \, dV \quad \text{or} \quad \int \psi^* H \psi \, dV,$$

having no connexion with the energy. It is important to note this in view of some criticism of Dirac's Hamiltonian conception which has recently been put forward, and is actually based on such an incorrect definition of the energy expectation value.

It follows from the above that if any physical quantity is represented by

$$\bar{Q} = \frac{1}{i}\int \psi^{\dagger} Q \psi \, dV, \tag{66}$$

its time derivative can be stated as

$$\frac{d\bar{Q}}{dt} = \bar{R} = \frac{1}{i}\int \psi^{\dagger} R \psi \, dV, \tag{67}$$

where

$$R = \frac{i}{\hbar}(HQ - QH). \tag{68}$$

The only fact about these relations, which makes them less simple than in electron theory is that Q and R are not the operators corresponding to $\bar{Q}$ and $\bar{R}$ by definition (11), so that great care regarding factors β_4 must be taken when using (68).

Equation (62), however, does not contain the whole of the original wave equation (1) as the multiplication by β_4 (equation (60)) obliterates the part of (1) which belongs to the eigenvalue 0 of β_4. This omitted part can readily be singled out by multiplying (1) by $1-\beta_4^2$. In this way one obtains

$$\partial_k \beta_k \beta_4^2 \psi + (1-\beta_4^2)\kappa\psi = 0, \tag{69}$$

an equation that does not contain the time, and can thus be regarded as an initial condition which the wave function must satisfy. By studying the particular β' representation used by de Broglie one finds that (69) contains exactly the same differential relations as postulated by de Broglie, although the algebraic form of his initial condition,

$$\partial_\mu(\gamma_4\gamma_\mu I' - \gamma_4'\gamma_\mu' I)\psi + \kappa(\gamma_4 I' - \gamma_4' I)\psi = 0, \tag{70}$$

has no connexion with (69). The form (69) certainly appears to be the most concise for stating these conditions. It can also be easily seen that (62) and (69) together are a complete substitute for (1), for the latter equation can be obtained from the other two by working backwards.

There is another alternative way of putting down the initial conditions, which consists in proceeding as in equations (60) to (62) for the three space co-ordinates as well as for the time. One then obtains the four equations

$$\partial_\mu \psi + \partial_\nu(\beta_\mu\beta_\nu - \beta_\nu\beta_\mu)\psi + \kappa\beta_\mu\psi = 0, \tag{71}$$

of which the fourth is the Hamiltonian wave equation and the other three are the initial conditions. This is the formulation of Dirac's paper. In spite of its symmetry it has considerable disadvantages because the wave equation (1) and the second order wave equation (8) are not deducible algebraically from (71) alone. The unsymmetrical statement of the initial condition as given by (69) is therefore to be preferred.

In the present section electromagnetic interaction was hitherto omitted. If we add it, it follows from (34) that the Hamiltonian equation will be

$$H = \frac{\hbar c}{i} \partial_{\bar{k}} \frac{\beta_k \beta_4 - \beta_4 \beta_k}{i} + mc^2 \beta_4 - \frac{ie}{\kappa} F_{\nu\rho}(\beta_\rho \beta_4 \beta_\nu - \delta_{\rho 4} \beta_\nu), \tag{72}$$

and the other equations of (71) will have a similar form. The condition (69), on the other hand, will simply become

$$\partial_{\bar{k}} \beta_k \beta_4^2 \psi + (1 - \beta_4^2) \kappa \psi = 0. \tag{73}$$

The final field-dependent term in H was not included in Dirac's statement of the Hamiltonian, but it must clearly be added to preserve the invariance of the scheme.

An interesting property of the Hamiltonian given in equation (63) is that for a plane wave solution with momentum p and energy $E = c(p^2 + m^2c^2)^{\frac{1}{2}}$ there exists the relation

$$H^3 \psi = E^2 H \psi. \tag{74}$$

Therefore the eigenvalues of H can only be 0 and $\pm E$. Now let us consider a plane wave solution of the equation (1). Its energy value can, according to (65), be given as

$$E = \frac{1}{i} \int \psi^\dagger \beta_4 H \psi \, dV. \tag{75}$$

Owing to the continuity equation (10)

$$n = \frac{1}{i} \int \psi^\dagger \beta_4 \psi \, dV \tag{76}$$

is a constant and can be so normalized that $n = \pm 1$. In the non-relativistic limit (§ 8) a wave function for which $n = +1$ will be a solution of the true one-body wave equation of a particle with positive charge, and if $n = -1$, the wave function belongs to a negatively charged state. Therefore we have

$$\frac{1}{i} \int \psi^\dagger \beta_4 (H - E) \psi \, dV = 0, \tag{77}$$

for "positive charge" states, and

$$\frac{1}{i}\int \psi^\dagger \beta_4 (H+E)\,\psi\,dV = 0, \tag{78}$$

for "negative charge" states. From this we see that the operators

$$D^+ = \frac{1}{2E}(H+E) \quad \text{and} \quad D^- = \frac{1}{2E}(H-E)$$

will have the same property of "annihilation operators" as in the electron theory, and a technique of calculation which merely uses spur conditions and annihilation operators appears to be possible here just as in the electron case. The necessary spur relations have already been given at the end of § 5.

8. The non-relativistic limit

A theory of the type here considered must go over into a classical relativistic particle theory, if $\hbar$ is put equal to 0, and must also contain non-relativistic quantum mechanics as the limiting case $c \to \infty$. The former limiting process has already been briefly mentioned on p. 95, and does not differ in the least from the corresponding limit in the electron case. It is therefore sufficient to refer to Pauli's article, p. 240, and no further discussion of this case is necessary. The non-relativistic limit $c \to \infty$, on the other hand, has some unexpected features when dealt with by means of our formalism, and will now be discussed. In the wave formulation this problem has been considered by Proca (1938).

It is not possible to give the theory in terms of abstract β_μ, but one can nevertheless state it in a general form that covers both the essential inequivalent representations of these matrices. The point of departure is the choice of a representation of the β_μ in which β_4 is diagonal. As $\beta_4^3 = \beta_4$, the eigenvalues of β_4 are $+1$, 0 and -1 only. For either of the two irreducible representations let the components of ψ be divided into three groups

$$\psi = (\psi^{\mathrm{I}}, \psi^{\mathrm{II}}, \psi^{\mathrm{III}}), \tag{79}$$

so that $$(\beta_4\psi)^{\mathrm{I}} = \psi^{\mathrm{I}}, \quad (\beta_4\psi)^{\mathrm{II}} = 0, \quad (\beta_4\psi)^{\mathrm{III}} = -\psi^{\mathrm{III}}. \tag{80}$$

It can be seen without much difficulty that the three other β_k can then be chosen in such a way that

$$\left.\begin{aligned} (\beta_k\psi)^{\mathrm{I}} &= \zeta_k\psi^{\mathrm{II}}, \\ (\beta_k\psi)^{\mathrm{II}} &= \zeta_k^\dagger\psi^{\mathrm{I}} + \tilde{\zeta}_k\psi^{\mathrm{III}}, \\ (\beta_k\psi)^{\mathrm{III}} &= \zeta_k^*\psi^{\mathrm{II}}. \end{aligned}\right\} \tag{81}$$

The ζ_k are rectangular matrices and the $\zeta_k^\dagger$, $\tilde{\zeta}_k$ and ζ_k^* their hermitian conjugates, transposed and complex conjugate matrices respectively. From (29) and (33) one then immediately obtains the equations

$$\left.\begin{aligned}(\partial_4^- + \kappa)\,\psi^{\mathrm{I}} + \partial_k^- \zeta_k \psi^{\mathrm{II}} &= 0,\\ (-\partial_4^- + \kappa)\,\psi^{\mathrm{III}} + \partial_k^- \zeta_k^* \psi^{\mathrm{II}} &= 0,\\ (\partial_4^- + \kappa)\,\psi^{\mathrm{II}} + 2\partial_k^- \tilde{\zeta}_k \psi^{\mathrm{III}} &= -\frac{ie}{2mc^2} F_{kl}(\zeta_k^\dagger \zeta_l - \tilde{\zeta}_k \zeta_l^*)\,\psi^{\mathrm{II}}\\ &\quad -\frac{ie}{mc^2} F_{k4}(\zeta_k^\dagger \psi^{\mathrm{I}} + \tilde{\zeta}_k \psi^{\mathrm{III}}),\\ \text{and}\qquad (-\partial_4^- + \kappa)\,\psi^{\mathrm{II}} + 2\partial_k^- \zeta_k \psi^{\mathrm{I}} &= +\frac{ie}{2mc^2} F_{kl}(\zeta_k^\dagger \zeta_l - \tilde{\zeta}_k \zeta_l^*)\,\psi^{\mathrm{II}}\\ &\quad +\frac{ie}{mc^2} F_{k4}(\zeta_k^\dagger \psi^{\mathrm{I}} + \tilde{\zeta}_k \psi^{\mathrm{III}}).\end{aligned}\right\} \tag{82}$$

This method is obviously a direct generalization of the procedure in the electron case, in which the Pauli spin matrices take the place of the ζ_k. Now, from (64) and (77) we see that for the wave functions corresponding to positive charge

$$\partial_4 \psi = -\frac{E}{\hbar c}\psi, \tag{83}$$

so that in the first approximation ($E \sim mc^2$):

$$\psi^{\mathrm{II}} = -\frac{1}{\kappa}\partial_k^- \zeta_k^\dagger \psi^{\mathrm{I}}, \tag{84}$$

and

$$\psi^{\mathrm{III}} = -\frac{1}{2\kappa}\partial_k^- \zeta_k^* \psi^{\mathrm{II}}. \tag{85}$$

Therefore ψ^{I} then becomes the "large" set of components, ψ^{II} being smaller by the factor v/c, ψ^{III} even by v^2/c^2. To the first approximation the wave equation for

$$\phi = \psi^{\mathrm{I}}/e^{-i\kappa ct} \tag{86}$$

then becomes

$$\frac{1}{ci}\partial_t \phi - \frac{1}{\kappa}\partial_k^- \partial_l^- \zeta_k \zeta_l^\dagger \phi = 0, \tag{87}$$

and further approximations can, of course, be found as for the electron (Pauli 1933). The actual form of the matrices as well as the number of components combined in the non-relativistic wave function naturally depends upon which of the two representations is used, and in practice it will be more convenient not to use the abstract equation (87) but to state the non-relativistic theory separately for the two representations. The above formulae then lead to the following:

(*a*) *Proca's equations:* ψ has the eigenvalue $+1$ three times, so that ϕ has three components which form a space vector. In the tensor notation of the usual meson theory (Kemmer 1938*a*) we find these three components to be the quantities $\frac{1}{\sqrt{2}}(\phi_i + i\chi_{0i})$.

The wave equation (87) then has the form

$$\frac{\hbar}{i}\partial_{\bar{t}}\,\phi_k - \frac{\hbar^2}{2m}\partial_{\bar{i}}\partial_{\bar{i}}\phi_k - \frac{e}{2mc}F_{kl}\phi_l = 0. \tag{88}$$

This is therefore the correct non-relativistic wave equation for a particle of spin 1.

(*b*) *Klein-Gordon equation:* There is but one eigenvalue $+1$ of ψ and ϕ has but one component. As is only natural to expect, the wave equation is

$$\frac{\hbar}{i}\partial_{\bar{t}}\,\phi - \frac{\hbar^2}{2m}\partial_{\bar{i}}\partial_{\bar{i}}\phi = 0, \tag{89}$$

i.e. simply the ordinary Schrödinger equation, but a detailed comparison with the above reveals that this ϕ is not the non-relativistic limit of the ψ of the Klein-Gordon equation but of $\frac{1}{\sqrt{2}}\left(\psi + \frac{i}{c\kappa}\frac{\partial\psi}{\partial t}\right)$.

Thus even in this seemingly well-known case the above treatment appears to bring some clarification of the physical interpretation.

9. Concluding remarks

The above must suffice to indicate what is meant by the "particle aspect" of meson theory. It has only been possible to go through the formalism very briefly and further interesting points may come to light when attempts are made to apply the above to practical calculations. However, it is felt that the main gaps hitherto left in the interpretation of meson theory have been filled at least in principle. As already stated earlier, no attempt has been made to tackle the problem of formulating nuclear interaction on this basis, because it seems clear that in the description of those effects the wave aspect must be the more fruitful. For a similar reason we have refrained from presenting the second quantization of the meson equations in this new form. In the quantized (q-number) theory the two aspects, particle and wave, are essentially inseparable and it is clear that the complete picture is already contained in the Pauli-Weisskopf (1934) theory and its generalization for spin 1. It would, of course, be possible to obtain the same operator equations by starting from the particle formalism given in this paper, but the final equations will be identical whichever way one proceeds. Let it therefore be sufficient to indicate the commutation rules as they would appear in the particle formulation. Particular care must be taken that those components of ψ are dealt with correctly, for which $\beta_4\psi = 0$, for in the case of these equation (1) does not determine the time derivative. In the quantized theory these components will therefore not occur as separate variables but must be considered as equal by definition to certain spatial derivatives of the other components. The defining equation is (69). The commutation rule must then contain only the other components. It can be readily seen that its correct form is

$$[(\psi^\dagger\beta_4^2)_\mu, (\beta_4\psi')_\nu] = [(\psi^\dagger\beta_4)_\mu, (\beta_4^2\psi')_\nu] = \frac{1}{i}\delta_{\mu\nu}\,\delta(x-x'). \tag{90}$$

The rest of the quantization is straightforward, and can in fact best be performed by translating the former wave theoretical formalism into the β language step by step.

A final question which immediately presents itself is whether similar formalisms can also be found for higher values of the spin, for which the theories have recently been formulated in spinor notation by Fierz and Pauli (1939). As kindly communicated to the author by Professor Pauli, a good deal of the foregoing formalism can be generalized for these cases, but it is yet an undecided question whether a completely satisfactory theory can be built up on these lines. Perhaps the opinion is justified that if such a formulation were to fail, the possibility of the existence of such higher particles would be, to say the least, doubtful. They would certainly not be particles in the full sense, i.e. as defined by the axioms of Dirac's electron theory, axioms which the meson has here been shown to obey.

In conclusion the writer wishes to express his sincerest thanks to Professor Pauli for the interest taken in the work and the help given in its algebraical part.

Summary

It is shown that a re-formulation of the meson equations is helpful in the interpretation of the meson as a localized particle. Instead of using the usual tensor form, the wave equations are stated as

$$\frac{\partial}{\partial x_\mu}\beta_\mu\psi + \frac{mc}{\hbar}\psi = 0,$$

where the β_μ are operators completely defined by a set of commutation rules first given by Duffin (1938). The theory can be developed in strikingly close correspondence to Dirac's electron theory, practically all the definitions of which find their exact counterpart, e.g. spin, magnetic moment, etc. The algebraic properties of the β_μ are studied in detail, a comparison with other similar formulations is given and the limiting non-relativistic theory is developed. The formalism proves simple to handle and is expected to be useful in all calculations primarily concerned with the particle aspect of the meson.

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