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THE PATH-PARTITION PROBLEM IN BIPARTITE DISTANCE-HEREDITARY GRAPHS

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Abstract. A path partition of a graph is a collection of vertex-disjoint paths that cover all vertices of the graph. The path-partition problem is to find a path partition of minimum size. This paper gives a linear-time algorithm for the path-partition problem in bipartite distance-hereditary graphs.

1. Introduction

A *path partition* of a graph is a collection of vertex-disjoint paths that cover all vertices of the graph. The *path-partition problem* is finding the *path-partition number* $p(G)$ that is the minimum size of a path partition of G. Note that G has a Hamiltonian path if and only if $p(G) = 1$. Since the Hamiltonian path problem is $\mathcal{NP}$-complete for planar graphs [7], bipartite graphs [8], chordal graphs [8], chordal bipartite graphs [12], and strongly chordal graphs [12], so is the path-partition problem. On the other hand, the path-partition problem is polynomially solvable for trees [11, 14], interval graphs [1, 3], cographs [4, 5], and block graphs [15, 16]. In this paper we present a linear-time algorithm for the path-partition problem in bipartite distance-hereditary graphs. For technical reasons, we consider the following generalization of the path-partition problem. For a set S of vertices in a graph $G = (V, E)$, an *S-path partition* is a path partition $\mathcal{P}$ in which every vertex of S is an endpoint of a path in $\mathcal{P}$. The *S-path-partition problem* is to determine the *S-path-partition number* $p(G, S)$ that is the minimum size of an S-path partition of G. Note that the path-partition problem is a special case of the S-path-partition problem, since $p(G) = p(G, \emptyset)$.

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We now review distance-hereditary graphs. Suppose A and B are two sets of vertices in a graph $G = (V, E)$. $G[A]$ denotes the *subgraph of G induced* by A. The *deletion* of A from G, denoted by $G - A$, is the graph $G[V - A]$. The *neighborhood* $N_A(B)$ of B in A is the set of vertices in A that are adjacent to some vertex in B. The *closed neighborhood* $N_A[B]$ of B in A is $N_A(B) \bigcup B$. For simplicity, $N_A(v)$, $N_A[v]$, $N(B)$, and $N[B]$ stand for $N_A(\{v\})$, $N_A[\{v\}]$, $N_V(B)$, and $N_V[B]$, respectively. The *degree* of a vertex v is $\deg(v) = |N(v)|$. A vertex x is called a *leaf* of G if $\deg(x) = 1$. The *distance* $d_G(x, y)$ between two vertices x and y in G is the minimum length of an x-y path in G. The *hanging* h_u of a connected graph $G = (V, E)$ at a vertex $u \in V$ is the collection of sets $L_0(u),\ L_1(u),\ \ldots,\ L_t(u)$ (or $L_0,\ L_1,\ \ldots, L_t$ if there is no ambiguity), where $t = \max_{v \in V}\ d_G(u, v)$ and $L_i(u) = \{v \in V : d_G(u, v) = i\}$ for $0 \le i \le t$. For any $1 \le i \le t$ and any $v \in L_i$, let $N'(v) = N(v) \bigcap L_{i-1}$. Note that the notion $N'(v)$ depends on the hanging h_u. A vertex $v \in L_i$ with $1 \le i \le t$ has a *minimal neighborhood* in L_{i-1} if $N'(v) \subseteq N'(w)$ or $N'(v) \cap N'(w) = \emptyset$ for any $w \in L_i$.

A graph is *distance-hereditary* if every two vertices in a connected induced subgraph have the same distance as in the original graph. Distance-hereditary graphs were introduced by Howorka [10]. Characterizations and recognition of distance-hereditary graphs were studied in [2, 6, 9]. The following theorem contains some useful properties used in this paper.

Theorem 1. ([2, 9]) *Suppose $h_u = (L_0, L_1, \ldots, L_t)$ is a hanging of a connected distance-hereditary graph G at u. For each $1 \le i \le t$ and any two vertices $x, y \in L_i$, $N'(x) \cap N'(y) = \emptyset$ or $N'(x) \subseteq N'(y)$ or $N'(y) \subseteq N'(x)$. Consequently, for each $1 \le i \le t$, L_i contains a vertex v having a minimal neighborhood in L_{i-1}. In addition, for such a vertex v, we have $N_{V-N'(v)}(x) = N_{V-N'(v)}(y)$ for every pair of vertices x and y in $N'(v)$.*

Note that for any bipartite distance-hereditary graph G with a hanging $h_u = (L_0, L_1, \ldots, L_t)$, each $G[L_i]$ contains no edges. Consequently, $N(x) = N'(x)$ for any $x \in L_t$. We shall frequently use this fact in Sections 2 and 3.

In this paper, we use the following notation. For a graph G and vertices w, x, y, we use $G-x$ for $G-\{x\}$, $G-x-y$ for $(G-\{x\})-\{y\} \cong G-\{x, y\}$, and $G-w-x-y$ for $G-\{w, x, y\}$ ... etc. For a set A and elements x and y, we use $A-x$ for $A-\{x\}$, $A+x$ for $A \cup \{x\}$, $A-x-y$ for $(A-\{x\})-\{y\} = A-\{x, y\}$, $A-x+y$ for $(A-\{x\}) \cup \{y\}$ ... etc.

2. Path Partition in Bipartite Distance-hereditary Graphs

To give a linear-time algorithm for the path-partition problem in bipartite distance-hereditary graphs, we first establish three basic lemmas that are used later.

Lemma 2. *If x is a leaf of G, then $p(G,S) = p(G,S+x)$.*

In the following two lemmas, suppose $G = (V,E)$ is a connected bipartite distance-hereditary graph with a hanging $h_u = (L_0,\ldots,L_t)$ at u and $t \geq 1$. According to Lemma 2, for each vertex x in G, we may assume that either $x \in S$ or $x \notin S$ with $|N(x)| \geq 2$.

Lemma 3. *Suppose $x \in L_t$ has a minimal neighborhood in L_{t-1} and $N(x) \subseteq S$.*

(1) If $x \in S$, then $p(G,S) = p(G-x-y, S-x-y)+1$ for any $y \in N(x)$.

(2) If $x \notin S$ and $|N(x)| \geq 2$, then $p(G,S) = p(G-w-x-y, S-w-y)+1$ for any two distinct vertices w, $y \in N(x)$.

Proof. (1) Since an $(S-x-y)$-path partition of $G-x-y$, together with the path xy, forms an S-path partition of G, $p(G,S) \leq p(G-x-y,S-x-y)+1$. On the other hand, suppose $\mathcal{P}$ is an optimal S-path partition of G. Since $N[x] \subseteq S$, for any optimal S-path partition $\mathcal{P}$ of G either $x \in \mathcal{P}$ or $xy' \in \mathcal{P}$ for some $y' \in N(x)$. For the case in which $x \in \mathcal{P}$, let y be an endpoint of some $P \in \mathcal{P}$. Then, $\mathcal{P}' = \mathcal{P}-x-P+xy+(P-y)$ is another optimal S-path partition of G. So, in any case, we may assume that $xy' \in \mathcal{P}$ for some $y' \in N(x)$. Since x has a minimal neighborhood in L_{t-1}, by Theorem 1, $N(y') = N(y)$ and thus we may interchange the roles of y' and y to assume that $xy \in \mathcal{P}$. Hence, $\mathcal{P} - xy$ is an $(S-x-y)$-path partition of $G-x-y$. Thus, $p(G,S)-1 \geq p(G-x-y,S-x-y)$. Therefore, $p(G,S) = p(G-x-y,S-x-y)+1$.

(2) Since an $(S-w-y)$-path partition of $G-w-x-y$, together with the path wxy, forms an S-path partition of G, $p(G,S) \leq p(G-w-x-y,S-w-y)+1$. On the other hand, suppose $\mathcal{P}$ is an optimal S-path partition of G. Let P be the path of $\mathcal{P}$ that contains x. By $N(x) \subseteq S$, $|N(x)| \geq 2$, and $x \notin S$, we have that P is x or xy' or $w'xy'$. By an argument similar to that for (1), we may assume that $wxy \in \mathcal{P}$. Hence, $\mathcal{P} - wxy$ is an $(S-w-y)$-path partition of $G-w-x-y$. Thus, $p(G,S)-1 \geq p(G-w-x-y,S-w-y)$. Therefore, $p(G,S) = p(G-w-x-y,S-w-y)+1$. ∎

Lemma 4. *Suppose $x \in L_t$ has a minimal neighborhood in L_{t-1} and $N(x) \not\subseteq S$.*

(1) If $x \in S$, then $p(G,S) = p(G-x,S-x+y)$ for any $y \in N(x)-S$.

(2) If $x \notin S$ and $|N(x)| \geq 2$, then $p(G,S) = p(G-x-y,S)$ for any $y \in N(x)-S$.

Proof. (1) Suppose $\mathcal{P}$ is an optimal $(S-x+y)$-path partition of $G-x$ such that y is an endpoint of some path $P \in \mathcal{P}$. Then, $\mathcal{P}-P+Px$ is an S-path partition of G and so, $p(G,S) \le p(G-x, S-x+y)$. On the other hand, suppose $\mathcal{P}$ is an optimal S-path partition of G. Suppose the path P in $\mathcal{P}$ containing x is $xv_1v_2\ldots v_r$, where $r \ge 0$. For the case of $r=0$, let P_1yP_2 be the path of $\mathcal{P}$ that contains y. Then $\mathcal{P}-x-P_1yP_2+P_1y+P_2$ is an $(S-x+y)$-path partition of $G-x$. For the case of $r \ge 1$, we have $y, v_1 \in N(x)$. Since x has a minimal neighborhood in L_{t-1}, by Theorem 1, $N(y)=N(v_1)$. Thus, we may interchange the roles of y and v_1 and assume that $xyv_2\ldots v_r \in \mathcal{P}$. Then, $\mathcal{P}-P+yv_2\ldots v_r$ is an $(S-x+y)$-path partition of $G-x$. In any case, $p(G,S) \ge p(G-x,S-x+y)$. Therefore, $p(G,S)=p(G-x,S-x+y)$.

(2) Suppose $\mathcal{P}$ is an optimal S-path partition of $G-x-y$. Since $|N_G(x)| \ge 2$, without loss of generality, we may assume that $\mathcal{P}$ has a path $P=v_0v_1\ldots v_iv_{i+1}\ldots v_k$ such that $v_i \in N_G(x)$. Since x has a minimal neighborhood in L_{t-1}, by Theorem 1, $N(v_i)=N(y)$. Thus, $P'=v_0v_1\ldots v_ixyv_{i+1}\ldots v_k$ is a path of G. Therefore, $\mathcal{P}-P+P'$ is an S-path partition of G and so $p(G,S) \le p(G-x-y,S)$. On the other hand, suppose $\mathcal{P}$ is an optimal S-path partition of G. Consider first the case in which x and y lie on path

$$P = v_0v_1\ldots v_ixv_{i+1}\ldots v_jyv_{j+1}\ldots v_k \in \mathcal{P}.$$

By Theorem 1, v_i is adjacent to v_j and v_{i+1} is adjacent to v_{j+1}. Hence,

$$P' = v_0v_1\ldots v_{i-1}v_iv_jv_{j-1}v_{j-2}\ldots v_{i+2}v_{i+1}v_{j+1}v_{j+2}\ldots v_k$$

is a path in $G-x-y$ containing all vertices of P except x and y. Therefore, $\mathcal{P}-P+P'$ is an S-path partition of $G-x-y$. Next consider the case in which x and y lie on two distinct paths

$$P_1 = v_0v_1\ldots v_ixv_{i+1}\ldots v_{k-1}v_k \in \mathcal{P} \text{ and } P_2 = u_0u_1\ldots u_jyu_{j+1}\ldots u_{k'-1}u_{k'} \in \mathcal{P}.$$

By Theorem 1, v_i is adjacent to u_{j+1} and u_j is adjacent to v_{i+1}. Hence,

$$P_1' = v_0v_1\ldots v_{i-1}v_iu_{j+1}u_{j+2}\ldots u_{k'-1}u_{k'}$$

and

$$P_2' = u_0u_1\ldots u_{j-1}u_jv_{i+1}v_{i+2}\ldots v_{k-1}v_k$$

are paths in $G-x-y$ containing all vertices of P_1 and P_2 except x and y. Therefore, $\mathcal{P}-P_1-P_2+P_1'+P_2'$ is an S-path partition of $G-x-y$. In any case, we have that $p(G,S) \ge p(G-x-y,S)$. Therefore, $p(G,S)=p(G-x-y,S)$. ∎

Based on Lemmas 2 to 4, we have the following algorithm for the S-path partition problem in bipartite distance-hereditary graphs.

Algorithm PP-dh. Find the S-path partition number of a connected bipartite distance-hereditary graph.

Input: A connected bipartite distance-hereditary graph $G = (V, E)$ and $S \subseteq V$.
Output: The S-path partition number $p(G, S)$.
Method:

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P(G,S) ⟵ 0;
determine the hanging h_u = (L_0, L_1, ..., L_t) of G at a vertex u;
for  i = t to 1 step −1 do
{     let L_i = {x_1, x_2, ..., x_j};
      sort L_i such that |N'(x_{i_1})| ≤ |N'(x_{i_2})| ≤ ... ≤ |N'(x_{i_j})|;
      for  k = 1 to  j do
      {     x ⟵ x_{i_k};
            if deg(x) = 1 then S ⟵ S + x;
            if N(x) ⊆ S
            then {P(G,S) ⟵ P(G,S) + 1;
                   if x ∈ S  then {pick y ∈ N(x);
                                    G ⟵ G − x − y;
                                    S ⟵ S − x − y;
                                    L_{i−1} ⟵ L_{i−1} − y;}
                             else  {pick w, y ∈ N(x);
                                    G ⟵ G − w − x − y;
                                    S ⟵ S − w − y;
                                    L_{i−1} ⟵ L_{i−1} − w − y;}
                  }
            else  {pick y ∈ N(x) − S;
                   if x ∈ S  then {G ⟵ G − x;
                                    S ⟵ S − x + y;}
                             else  {G ⟵ G − x − y;
                                    L_{i−1} ⟵ L_{i−1} − y;}
                  }
      }
}
```

Theorem 5. *Algorithm PP-dh finds the S-path-partition number of a bipartite distance-hereditary graph $G = (V, E)$ with $S \subseteq V$ in linear time.*

Proof. The correctness of the theorem follows from Lemmas 2 to 4. In order to make the running time linear, we can use a bucket-sort to sort L_i. ∎

3. Discussion

This paper gives a linear-time algorithm for the path-partition problem in bipartite distance-hereditary graphs by using the concepts of hanging and a vertex with a minimal neighbor. The same idea also works for the Hamiltonian-cycle problem in bipartite distance-hereditary graphs.

Lemma 6. *Suppose $G = (V, E)$ is a connected bipartite distance-hereditary graph with a hanging $h_u = (L_0, L_1, \ldots, L_t)$ at u such that $t \geq 2$ and $|V| \geq 5$. If $x \in L_t$ has a minimal neighborhood in L_{t-1} and $\deg(x) \geq 2$, then for every $y \in N(x)$, G has a Hamiltonian cycle if and only if $G-x-y$ has a Hamiltonian cycle.*

Proof. Suppose G has a Hamiltonian cycle $C = v_1v_2v_3 \ldots v_nv_1$ with $x = v_1$. We first consider the case in which $y = v_i$ with $3 \leq i \leq n-1$. Since x has a minimal neighbor in L_{t-1}, by Theorem 1, $N(v_2) = N(v_i)$. Therefore, we may interchange the roles of v_2 and v_i and assume that $v_1v_iv_3v_4 \ldots v_{i-1}v_2v_{i+1} \ldots v_nv_1$ is a Hamiltonian cycle of G. So, without loss of generality, we may assume that $v_2 = y$ in C. Now consider the Hamiltonian cycle C of G. Since $v_n, v_2 \in N(x)$, by Theorem 1, $N(v_n) = N(v_2)$ and so v_n is adjacent to v_3. Therefore, $G-\{x, y\}$ has a Hamiltonian cycle $v_3v_4v_5 \ldots v_nv_3$.

Conversely, suppose $G - x - y$ has a Hamiltonian cycle $v_1v_2v_3 \ldots v_{n-2}v_1$. Since $\deg(x) \geq 2$, we may assume $v_1 \in N(x)$. Since $y, v_1 \in N(x)$, by Theorem 1, $N(y) = N(v_1)$ and so y is adjacent to v_2 in G. Therefore, G has a Hamiltonian cycle $v_1xyv_2v_3 \ldots v_{n-1}v_1$. ∎

Based on Lemma 6, we have the following algorithm for the Hamiltonian cycle problem in bipartite distance-hereditary graphs.

Algorithm HC-dh. Determine whether or not a connected bipartite distance-hereditary graph has a Hamiltonian cycle.

Input: A connected bipartite distance-hereditary graph $G = (V, E)$.
Output: "G has a Hamiltonian cycle" or "G has no Hamiltonian cycle."
Method:

determine the hanging $h_u = (L_0, L_1, \ldots, L_t)$ of G at a vertex u;

for $i = t$ **to** 1 **step** -1 **do**
{ let $L_i = \{x_1, x_2, \ldots, x_j\}$;
sort L_i such that $|N'(x_{i_1})| \leq |N'(x_{i_2})| \leq \ldots \leq |N'(x_{i_j})|$;
for $k = 1$ **to** j **do**
{ **if** $|V(G)| \leq 4$ **then if** $G \cong C_4$ **then goto** (y) **else goto** (n);
if $\deg(x_{i_k}) \leq 1$ **then goto** (n);
choose $y \in N(x_{i_k})$;
$G \longleftarrow G - x_{i_k} - y$;
$L_{i-1} \longleftarrow L_{i-1} - y$;
}
}
(y) print "G has a Hamiltonian cycle"; **stop**;
(n) print "G has no Hamiltonian cycle";

Theorem 7. *Algorithm HC-dh determines whether or not a connected bipartite distance-hereditary graph has a Hamiltonian cycle in linear time.*

Proof. The correctness of the algorithm follows from Lemma 6. In order to make the running time of the algorithm linear, we can use a bucket-sort to sort L_i. ∎

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