The pattern of multiple rings from morphogenesis in development

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Abstract

Under certain conditions the problem of morphogenesis in development and the problem of morphology in block copolymers may be reduced to one geometric problem. In two dimensions two new types of solutions are found. The first type of solution is a disconnected set of many components, each of which is close to a ring. The sizes and locations of the rings are precisely determined from the parameters and the domain shape of the problem. The solution of the second type has a coexistence pattern. Each component of the solution is either close to a ring or to a round disc. The first-type solutions are stable for certain parameter values but unstable for other values; the second-type solutions are always unstable. In both cases one establishes the equal area condition: the components in a solution all have asymptotically the same area.

Key words. ring, disc, nonlocal geometric problem, mean curvature, approximate solution, Lyapunov-Schmidt reduction.

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1 Introduction

Activator-inhibitor systems were studied by Turing [27]. They may be used to model animal coats and skin pigmentation. The Gierer-Meinhardt theory [6] for morphogenesis in cell development is an activator-inhibitor type reaction-diffusion system of two unknowns. The first unknown, denoted by u, describes the short-range autocatalytic substance, i.e., the activator, and the second unknown, denoted by v, is its long-range antagonist, i.e., the inhibitor. Both are functions of a space variable x in a domain $D \subset R^2$ and a time variable $t > 0$. They satisfy the equations

$$u_t = \epsilon^2 \Delta u - u + \frac{u^p}{(1+\kappa u^p)v^q}; \quad \iota v_t = d\Delta v - v + \frac{u^r}{v^s} \tag{1.1}$$

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with the Neumann boundary condition. The system (1.1) is known as the Gierer-Meinhardt system with saturation (the Gierer-Meinhardt system without saturation does not have the κu^p term). It is a minimal model that provides a theoretical bridge between observations on the one hand and the deduction of the underlying molecular-genetic mechanisms on the other hand.

In the case where ϵ is small and d is large like $d = \frac{d_0}{\epsilon}$, this reaction-diffusion system may be reduced, at least formally, to a nonlocal geometric problem.

To state this geometric problem, let D be an open, bounded and smooth domain in R^n and $a \in (0,1)$ and $\gamma > 0$ be two parameters. We seek a subset E of D and a number λ such that $|E| = a|D|$, where $|E|$ and $|D|$ are the n-dimensional Lebesgue measures of E and D respectively, and the equation

$$H(\partial_D E) + \gamma(-\Delta)^{-1}(\chi_E - a) = \lambda \tag{1.2}$$

holds on $\partial_D E$, the part of the boundary of E which is inside D. Here $H(\partial_D E)$ is the mean curvature of the hypersurface $\partial_D E$ viewed from E. The characteristic function of E is denoted by χ_E, i.e. $\chi_E(x) = 1$ if $x \in E$ and $\chi_E(x) = 0$ if $x \notin E$. The nonlocal operator $(-\Delta)^{-1}$ is defined by solving

$$-\Delta v = q \text{ in } D, \ \partial_\nu v = 0 \text{ on the boundary of } D, \overline{v} = 0$$

for $q \in L^2(D)$, $\overline{q} = 0$, i.e. $v = (-\Delta)^{-1}q$. We denote the outward normal derivative of v on ∂D by $\partial_\nu v$. A bar over a function, like $\overline{v}$, is the average of the function on D.

A formal deduction of (1.2), as a reduced steady-state problem, from (1.1) and the connection between the parameters in (1.1) and the ones in (1.2) may be found in [24]. In short, as $\epsilon \to 0$ in (1.1), one can formally show that the u variable of a steady state solution of (1.1) tends to a function which is a positive constant on a set E and vanishes outside E. The set E satisfies the geometric equation (1.2). The parameters a and γ in (1.2) are derived from κ, p, q, r, s, and d_0 of (1.1). There is a restriction on the range of κ in order to have $a \in (0,1)$.

In addition to (1.2) if $\partial_D E$ of a solution E meets the boundary of D, the two surfaces $\partial_D E$ and ∂D must meet orthogonally. However, such a situation is not considered in this paper.

The geometric problem (1.2) is also found in the study of block copolymer morphology. A diblock copolymer molecule is a linear chain of an A-monomer block grafted covalently to a B-monomer block. Because of the repulsion between the unlike monomers, the different type subchains tend to segregate, but as they are chemically bonded in chain molecules, segregation of subchains cannot lead to a macroscopic phase separation. Only a local microphase separation occurs: microdomains rich in A monomers and microdomains rich in B monomers emerge as a result. A pattern formed from microdomains is known as a morphology phase. Various phases, including the lamellar, the cylindrical, and the spherical phases, have been observed in experiments [2].

Ohta and Kawasaki [11] proposed a free energy for a diblock copolymer melt which takes the form

$$I(u) = \int_D \left[\frac{\epsilon^2}{2}|\nabla u|^2 + W(u) + \frac{\epsilon\gamma}{2}|(-\Delta)^{-1/2}(u-a)|^2\right] dx. \tag{1.3}$$

The diblock copolymer occupies the domain D. The relative density of the A monomers is u and the relative density of the B monomers is $1-u$. Here $(-\Delta)^{1/2}$ is the square root of $(-\Delta)^{-1}$ in the sense of operators. There is a constraint on the average of u: $\overline{u} = a \in (0,1)$. The function W is a balanced double-well potential such as $(1/4)u^2(1-u)^2$. A solution to the Euler-Lagrange equation of (1.3) is a pair of a function u with the Neumann boundary condition and a number λ such that

$$-\epsilon^2\Delta u + W'(u) + \epsilon\gamma(-\Delta)^{-1}(u-a) = \lambda \text{ in } D. \tag{1.4}$$

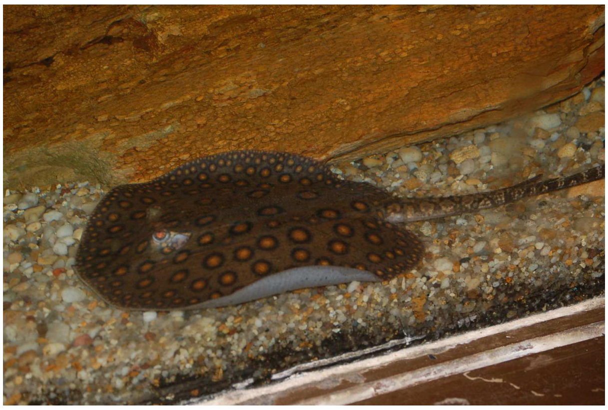

Figure 1: The ring pattern on a freshwater ray.

Nishiura and Ohnish [10] identified (1.2) as a formal singular limit of (1.4). Ren and Wei [13] noted that as $\epsilon \to 0$, I tends to a limit in the sense of Γ-convergence. The Γ-limit is a functional on subsets E of D whose sizes are fixed at $a|D|$. It is given by

$$J(E) = |D\chi_E|(D) + \frac{\gamma}{2}\int_D |(-\Delta)^{-1/2}(\chi_E - a)|^2\,dx. \tag{1.5}$$

Here χ_E is the characteristic function of E and $|D\chi_E|(D)$ is the perimeter of E in D, i.e. the area of $\partial_D E$. The geometric problem (1.2) is precisely the Euler-Lagrange equation of J. The Γ-convergence theory [4, 9, 8] gives a mathematically precise meaning of I converging to J.

The lamellar phase of diblock copolymers was studied extensively [5, 15, 14, 16, 18, 17, 3, 19, 20]. A perfect lamellar solution is a critical point of J in one dimension. All critical points in one dimension are found in [13]. They are all local minima of J. Depending on the parameters a and γ, either two or four of them are global minima. The cylindrical phase and the spherical phase of diblock copolymers are critical points of J in two and three dimensions respectively, when the parameter a is sufficiently small and γ appropriately large. Ren and Wei constructed such solutions as local minima of J through a type of Lyapunov-Schmidt reduction method [22, 21, 23]. Identification of the global minima of J in two and three dimensions appears to be a difficult problem. Partial results were obtained by Alberti, Choksi and Otto [1].

Two new types of solutions in two dimensions are found in this paper. The first has a ring pattern that is often observed on animal skins (Figure 1). A ring pattern solution consists of a number of components, each of which is close to a ring. By a ring we mean a set of the form $\{x \in R^2 : r_1 \leq |x - \xi| \leq r_2\}$, where ξ is the center, r_1 the inner radius and r_2 the outer radius. The rings in the solution all have approximately the same size, i.e., the same inner and outer radii. These radii are determined by the parameters a and γ, and the locations of the rings are determined by the shape of the domain D (Figure 2, Plot 1). We are not aware if the ring pattern has been observed in block copolymer systems.

The existence and the stability of ring pattern solutions depend on the parameters γ and a. The

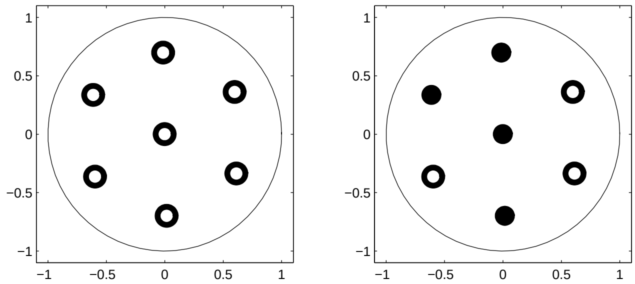

Figure 2: A ring solution with $K = 7$ and a ring/disc solution with $K_1 = 3$ and $K_2 = 4$ are found by numerically minimizing F.

first parameter a must be sufficiently small while the second parameter γ must be suitably large. See Theorem 2.1 for the precise parameter range and the properties of the ring solutions.

It is known from [22, 21] that there exist disc pattern solutions. A disc solution consists of a number of components each of which is close to a round disc.

In Theorem 2.2 we find yet another type of solution. Such a solution has a pattern of coexisting rings and discs (Figure 2, Plot 2). The rings and the discs have almost the same area. In numerical simulations of an evolving system governed by the Ohta-Kawasaki free energy I, (1.3), the ring/disc coexistence phenomenon has been observed [7]. Although always unstable, a coexistence pattern always manages to stay for a very long time. The reduced free energy J, (1.5), appears to have captured the slow dynamics of the full Ohta-Kawasaki system.

A special case of Theorem 2.1 was proved by the authors in [7] where the solution consists of only one ring. The many-component case studied here is significantly more complicated. The proof of Theorem 2.1 is divided into several steps. Step 1 is long and technical. It shows that the deviation of each component in a ring pattern solution from a perfect ring is negligible. It suffices to minimize the free energy J among the sets whose components are exact rings. Step 2 asserts that all the rings in a solution must have approximately the same area. Step 3 determines the inner and the outer radii of each ring solely in terms of a scaled version of γ. Step 4, the final step, finds the locations of the rings. In the proof of Theorem 2.2 the equal area condition in Step 2 remains valid: all the rings and discs in a solution have approximately the same area.

Our main results are stated in the next section. Theorem 2.1 is proved in Sections 3 through 7. Step 1 of the proof spans from Section 3 to Section 6. Steps 2, 3, and 4 are all contained in Section 7, the heart of this work. Theorem 2.2 is proved in Section 8. Since the proof of the second theorem proceeds along the same lines, we only explain the key differences between the two proofs.

2 Results

Let D be an open, smooth, and bounded domain in R^2. On D the Green's function G of $-\Delta$ with zero Neumann boundary condition is a sum of two parts:

$$G(x, y) = \frac{1}{2\pi} \log \frac{1}{|x-y|} + R(x, y), \tag{2.1}$$

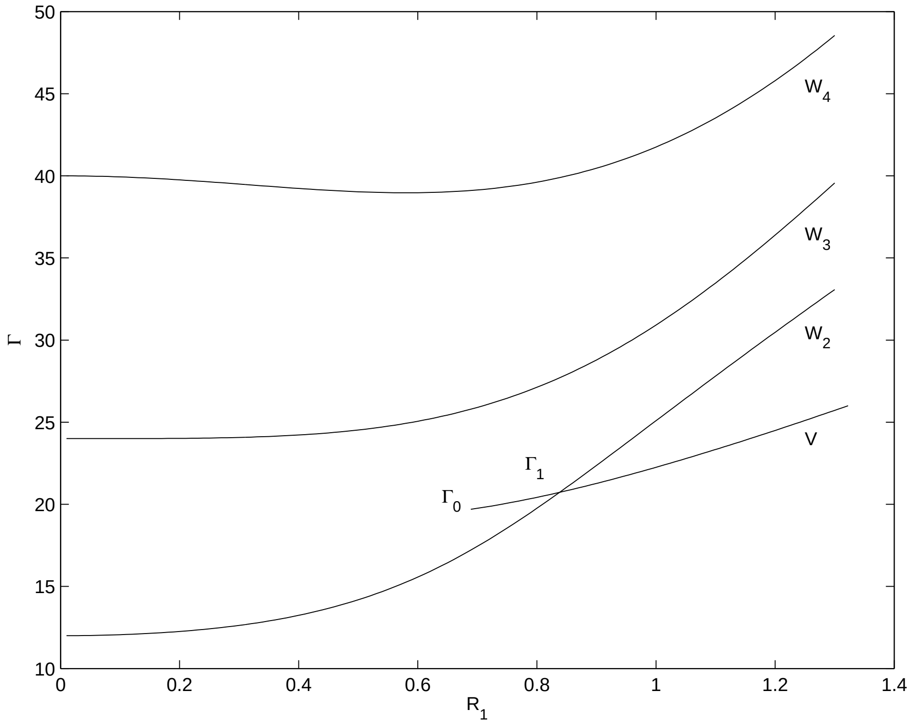

Figure 3: Γ_0 and Γ_1 are written next to the points whose heights are Γ_0 and Γ_1 respectively.

where $R(x, y)$ is the regular part. The Green's function satisfies the equation

$$-\Delta_x G(x,y) = \delta(x-y) - \frac{1}{|D|} \text{ in } D, \quad \partial_{\nu(x)} G(x,y) = 0 \text{ on } \partial D, \quad \overline{G(\cdot, y)} = 0 \ \forall y \in D. \tag{2.2}$$

Here Δ_x is the Laplacian with respect to the x-variable of G, $\nu(x)$ is the outward normal direction at $x \in \partial D$, and $\partial_{\nu(x)}$ is the normal derivative there with respect to the x-variable. We set

$$F(\xi_1, \xi_2, ..., \xi_K) := \sum_{k=1}^{K} R(\xi_k, \xi_k) + \sum_{k=1}^{K} \sum_{l=1, l\neq k}^{K} G(\xi_k, \xi_l), \tag{2.3}$$

for $\xi_k \in D$, and $\xi_k \neq \xi_l$ if $k \neq l$. Because $G(x, y) \to \infty$ if $|x - y| \to 0$ and $R(x, x) \to \infty$ if $x \to \partial D$, F admits at least one global minimum.

We recall some quantities first introduced in [7]. Let R_1, $R_2 > 0$ be such that

$$R_2^2 - R_1^2 = 1. \tag{2.4}$$

First for each integer $n \geq 2$ the quadratic equation

$$\frac{1 - (\frac{R_1}{R_2})^{2n} - n(1 - (\frac{R_1}{R_2})^2)}{16n^2}\Gamma^2 + \Big[\frac{n^2-1}{8nR_2^3} + \frac{n^2-1}{8R_1^3}\Big(\frac{1}{n} - 1 + (\frac{R_1}{R_2})^2\Big)\Big]\Gamma + \frac{(n^2-1)^2}{4R_1^3R_2^3} = 0$$

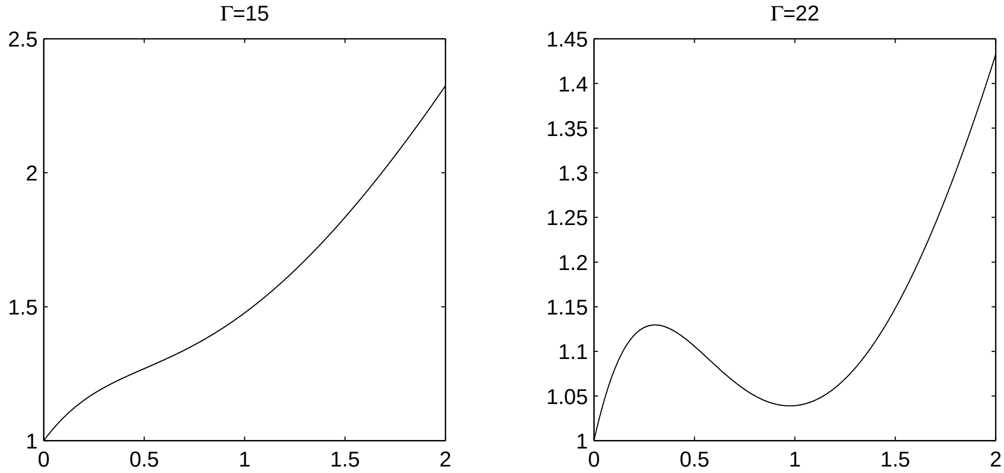

Figure 4: Two graphs of Q_Γ with $\Gamma = 15$ and $\Gamma = 22$. The first one does not have a local minimum but the second one does.

of Γ has one positive root and one negative root. Denote the positive root by $\tilde{\Gamma}_n(R_1)$ as a quantity that depends on R_1. Define curves W_n (Figure 3) in the first quadrant of the R_1-Γ plane by

$$W_n = \{(R_1, \tilde{\Gamma}_n(R_1)) : R_1 > 0\},\ n = 2, 3, ... \tag{2.5}$$

Next let Q_Γ be a function of $R_1 > 0$ given by

$$Q_\Gamma(R_1) = 2\pi(R_1 + R_2) + \frac{\Gamma\pi}{2}\Big[- \frac{R_2^4 \log R_2}{2} - \frac{R_1^4 \log R_1}{2} + R_1^2 R_2^2 \log R_2 - \frac{R_1^2}{4} + \frac{1}{8}\Big] \tag{2.6}$$

where $R_2 = \sqrt{1 + R_1^2}$. In the function Q_Γ, Γ is a positive parameter. The function Q_Γ admits a positive local minimum if Γ is sufficiently large (Figure 4). There is a constant $\Gamma_0 > 0$ such that if $\Gamma > \Gamma_0$, Q_Γ has a positive local minimum. For each $\Gamma > \Gamma_0$ we denote this positive local minimum by $T_1(\Gamma)$. The curve $V = \{(T_1(\Gamma), \Gamma) : \Gamma > \Gamma_0\}$ (also plotted in Figure 3) intersects the curve W_2 at one point, which we denote by $(T_1(\Gamma_1), \Gamma_1)$ where $\Gamma_1 > \Gamma_0$. It does not intersect with the other W_n's ($n = 3, 4, ...$). For each $\Gamma > \Gamma_0$, we write

$$T_1 = T_1(\Gamma) \text{ and } T_2 = \sqrt{1 + T_1^2}. \tag{2.7}$$

More discussions on Γ_0 and Γ_1 are found in the last section.

We can now state our first theorem on the ring pattern.

Theorem 2.1 *Let K be a positive integer. There exist two universal constants Γ_0 and Γ_1, with $0 < \Gamma_0 < \Gamma_1$, such that for each $\Gamma \in (\Gamma_0, \Gamma_1) \cup (\Gamma_1, \infty)$, there is a constant $a_0 > 0$ such that if*

$$a < a_0 \quad \textit{and} \quad \gamma = \Gamma(\frac{a|D|}{K\pi})^{-3/2}$$

there exists a solution of (1.2) with the following properties.

1. *The solution has K components, each of which has the shape of a ring.*
2. *All the components have approximately the same area.*
3. *In each ring the inner boundary is close to a circle of radius $(\frac{a|D|}{K\pi})^{1/2}T_1$ and the outer boundary is close to a circle of radius $(\frac{a|D|}{K\pi})^{1/2}T_2$ where T_1 and T_2, given in (2.7), depend on Γ only.*
4. *Let the centers of the rings be ζ_1, ζ_2,...,ζ_K. Then $(\zeta_1, \zeta_2, ..., \zeta_K)$ is close to a global minimum of F given in (2.3).*
5. *The solution is stable if $\Gamma > \Gamma_1$ and unstable if $\Gamma \in (\Gamma_0, \Gamma_1)$.*

The amount of deviation of the inner and outer boundaries of a component from exact circles is estimated in the proof of this theorem. The notion of stability is explained after we prove existence. When the domain D itself is a disc, the Green's function G is explicitly known. Numerical minimization of F can be done rather easily (see Figure 2).

Our second theorem is about the coexistence pattern.

Theorem 2.2 *Let K_1 and K_2 be positive integers and $K = K_1 + K_2$. For $\Gamma \in (\Gamma_0, \infty)\backslash\{\Gamma_1, 2n(n+1) : n = 2, 3, 4, ...\}$, there is a constant $a_0 > 0$ such that if*

$$a < a_0 \quad \text{and} \quad \gamma = \Gamma(\frac{a|D|}{K\pi})^{-3/2}$$

there exists an unstable solution of (1.2) with the following properties.

1. *The solution has K components. K_1 of them have the shapes of rings and K_2 of them have the shapes of discs.*
2. *All the components have approximately the same area.*
3. *The boundary of each disc is close to a circle of radius $(\frac{a|D|}{K\pi})^{1/2}$. The inner boundary of each ring is close to a circle of radius $(\frac{a|D|}{K\pi})^{1/2}T_1$ and the outer boundary of the ring is close to a circle of radius $(\frac{a|D|}{K\pi})^{1/2}T_1$.*
4. *Let the centers of the rings and the discs be ζ_1, ζ_2,...,ζ_K. Then $(\zeta_1, \zeta_2, ..., \zeta_K)$ is close to a global minimum of F.*

3 Approximate solutions

The proof of Theorem 2.1 starts with the construction of a family of approximate solutions.

Let $\xi_1, \xi_2, ..., \xi_K$ be K distinct points in D, and $r = (r_{1,1}, r_{1,2}, r_{2,1}, r_{2,2}, ..., r_{K,1}, r_{K,2})$ be the collection of inner and outer radii. First we must specify the regions where $\xi = (\xi_1, \xi_2, ..., \xi_K)$ and r are defined. The domain of F is the set $D(F) = \{\xi \in D^K : \ \xi_i \neq \xi_j \text{ if } i = j\}$. Since $R(x, x) \to \infty$ as $x \to \partial D$ and $G(x, y) \to \infty$ as $|x - y| \to 0$, we can find an open neighborhood $U_1 \subset D(F)$ of the set $\{\xi \in D(F) : \ F(\xi) = \min_{\eta \in D(F)} F(\eta)\}$ so that the closure of U_1 is also contained in $D(F)$. This closure $\overline{U}_1$ is the region for ξ. For r it is better to consider a scaled variable R whose components are $R_{k,\alpha}$ ($k = 1, 2, ..., K$ and $\alpha = 1, 2$), and

$$R_{k,\alpha} = (\frac{a|D|}{K\pi})^{-1/2} r_{k,\alpha}. \tag{3.1}$$

Recall T_1 and $T_2 = \sqrt{1 + T_1^2}$, two numbers defined in (2.7) that depend on Γ only. For each $\Gamma > \Gamma_0$ there is $\delta_1 > 0$ such that when restricted to $[T_1 - \delta_1, T_1 + \delta_1]$, Q_Γ is minimized at T_1. Also choose $\delta_2 > 0$ such that $T_1 + \delta_1 < T_2 - \delta_2$. Let

$$U_2 = \{R \in R^{2K} : \ R_{k,1} \in (T_1 - \delta_1, T_1 + \delta_1), \ R_{k,2} \in (T_2 - \delta_2, T_2 + \delta_2), \ \sum_{k=1}^{K} (R_{k,2}^2 - R_{k,1}^2) = K\}. \tag{3.2}$$

At this point the values of δ_1 and δ_2 are not completely determined yet. They will be made more specific in the proof of Lemma 5.1.

We let $(\xi, r) = (\xi, (\frac{a|D|}{K\pi})^{1/2} R)$ where

$$(\xi, R) \in \overline{U_1 \times U_2} \tag{3.3}$$

Denote the ring centered at ξ_k of outer radius $r_{k,2}$ and inner radius $r_{k,1}$ by P_k. The union of the P_k's is

$$E_0 = \bigcup_{k=1}^{K} P_k = \bigcup_{k=1}^{K} \{x \in R^2 : r_{k,1} \le |x - \xi_k| \le r_{k,2}\}. \tag{3.4}$$

When a is small, E_0 is a disconnected subset of D with K components. The constraint $\sum_{k=1}^{K} (R_{k,2}^2 - R_{k,1}^2) = K$ implies that

$$\sum_{k=1}^{K} \pi (r_{k,2}^2 - r_{k,1}^2) = a|D|, \tag{3.5}$$

so the area of E_0 is $a|D|$.

Lemma 3.1 *If $E = E_0$, the left side of (1.2) is, on the outer boundary of the k-th ring,*

$$\frac{1}{r_{k,2}} + \gamma \Big\{ - \frac{r_{k,2}^2 - r_{k,1}^2}{2} \log r_{k,2} + \pi (r_{k,2}^2 - r_{k,1}^2) R(\xi_k, \xi_k) + \sum_{l \ne k} \pi (r_{l,2}^2 - r_{l,1}^2) G(\xi_k, \xi_l) \Big\} + O(1);$$

and is, on the inner boundary of the same ring,

$$-\frac{1}{r_{k,1}} + \gamma \Big\{ - \frac{r_{k,2}^2 \log r_{k,2} - r_{k,1}^2 \log r_{k,1}}{2} + \frac{r_{k,2}^2 - r_{k,1}^2}{4}$$
$$+ \pi (r_{k,2}^2 - r_{k,1}^2) R(\xi_k, \xi_k) + \sum_{l \ne k} \pi (r_{l,2}^2 - r_{l,1}^2) G(\xi_k, \xi_l) \Big\} + O(1).$$

Proof. Write

$$P_k = B_k - \tilde{B}_k, \ \forall k = 1, 2, ..., K,$$

where

$$B_k = \{x \in D : |x - \xi_k| \leq r_{k,2}\}, \quad \tilde{B}_k = \{x \in D : |x - \xi_k| < r_{k,1}\}.$$

Set v_k to be the solution of

$$-\Delta v_k = \chi_{B_k} - \frac{\pi r_{k,2}^2}{|D|} \text{ in } D, \quad \partial_\nu v_k = 0 \text{ on } \partial D, \ \overline{v_k} = 0,$$

and $\tilde{v}_k$ to be the solution of

$$-\Delta \tilde{v}_k = \chi_{\tilde{B}_k} - \frac{\pi r_{k,1}^2}{|D|} \text{ in } D, \quad \partial_\nu \tilde{v}_k = 0 \text{ on } \partial D, \ \overline{\tilde{v}_k} = 0.$$

If $v = (-\Delta)^{-1}(\chi_{E_0} - a)$, then

$$v = \sum_{k=1}^{K} (v_k - \tilde{v}_k).$$

Define

$$V_k(x) = \begin{cases} -\frac{|x|^2}{4} + \frac{r_{k,2}^2}{4} - \frac{r_{k,2}^2}{2} \log r_{k,2}, & \text{if } |x| < r_{k,2} \\ \\ -\frac{r_{k,2}^2}{2} \log |x|, & \text{if } |x| \geq r_{k,2} \end{cases}.$$

Then $-\Delta V_k(\cdot - \xi_k) = \chi_{B_k}$. Write $v_k = V_k(\cdot - \xi_k) + W_k(\cdot, \xi_k)$. Clearly,

$$-\Delta W_k(x, \xi_k) = -\frac{\pi r_{k,2}^2}{|D|}, \ \partial_\nu W_k(x, \xi_k) = \partial_\nu \frac{r_{k,2}^2}{2} \log |x - \xi_k| \text{ on } \partial D, \ \overline{W_k(\cdot, \xi_k)} = -\overline{V_k(| \cdot - \xi_k|)}.$$

Here the Laplacian Δ and the outward normal derivative ∂_ν are taken with respect to x.

Note that from (2.2), $W_k(x, \xi_k)$ and $\pi r_{k,2}^2 R(x, \xi_k)$ satisfy the same equation and the same boundary condition. Therefore, they can differ only by a constant. This constant is $\overline{W_k(\cdot, \xi_k)} - \pi r_{k,2}^2 \overline{R(\cdot, \xi_k)}$. But $\overline{v_k} = \overline{G(\cdot, \xi_k)} = 0$ implies that this constant is also equal to

$$-\frac{r_{k,2}^2}{2} \overline{\log | \cdot - \xi_k|} - \overline{V_k(\cdot - \xi_k)} = \frac{\pi r_{k,2}^4}{8|D|}.$$

Hence

$$W_k(x, \xi_k) = \pi r_{k,2}^2 R(x, \xi_k) + \frac{\pi r_{k,2}^4}{8|D|}. \tag{3.6}$$

This gives us the exact expression of v_k. A similar formula holds for $\tilde{v}_k$.

On the outer boundary of the k-th ring P_k,

$$\begin{aligned} & \frac{1}{r_{k,2}} + \gamma v(\xi_k + r_{k,2} e^{i\theta_k}) \\ = \quad & \frac{1}{r_{k,2}} + \gamma \Big\{ - \frac{r_{k,2}^2 - r_{k,1}^2}{2} \log r_{k,2} \end{aligned}$$

$$+\pi(r_{k,2}^2 - r_{k,1}^2)R(\xi_k + r_{k,2}e^{i\theta_k}, \xi_k) + \sum_{l\neq k} \pi(r_{l,2}^2 - r_{l,1}^2)G(\xi_k + r_{k,2}e^{i\theta_k}, \xi_l)$$

$$+\sum_{l=1}^{K} \frac{\pi(r_{l,2}^4 - r_{l,1}^4)}{8|D|}\Big\};$$

and on the inner boundary of the same ring

$$\begin{aligned}
&-\frac{1}{r_{k,1}} + \gamma v(\xi_k + r_{k,1}e^{i\theta_k}) \\
= \quad & -\frac{1}{r_{k,1}} + \gamma\Big\{ -\frac{r_{k,2}^2 \log r_{k,2} - r_{k,1}^2 \log r_{k,1}}{2} + \frac{r_{k,2}^2 - r_{k,1}^2}{4} \\
& +\pi(r_{k,2}^2 - r_{k,1}^2)R(\xi_k + r_{k,1}e^{i\theta_k}, \xi_k) + \sum_{l\neq k} \pi(r_{l,2}^2 - r_{l,1}^2)G(\xi_k + r_{k,1}e^{i\theta_k}, \xi_l) \\
& +\sum_{l=1}^{K} \frac{\pi(r_{l,2}^4 - r_{l,1}^4)}{8|D|}\Big\}.
\end{aligned}$$

The lemma now follows. $\square$

Lemma 3.2 *The value of J at E_0 is*

$$\begin{aligned}
J(E_0) \;=\; & 2\pi \sum_{k=1}^{K} (r_{k,2} + r_{k,1}) \\
&+\frac{\gamma\pi^2}{2} \sum_{k=1}^{K} \Big[-\frac{r_{k,2}^4 \log r_{k,2}}{2\pi} - \frac{r_{k,1}^4 \log r_{k,1}}{2\pi} + \frac{r_{k,2}^2 r_{k,1}^2 \log r_{k,2}}{\pi} - \frac{(r_{k,2}^2 - r_{k,1}^2)r_{k,1}^2}{4\pi} + \frac{(r_{k,2}^2 - r_{k,1}^2)^2}{8\pi} \\
&+(r_{k,2}^2 - r_{k,1}^2)^2 R(\xi_k, \xi_k) + \frac{(r_{k,2}^2 - r_{k,1}^2)(r_{k,2}^4 - r_{k,1}^4)}{4|D|}\Big] \\
&+\frac{\gamma\pi^2}{2} \sum_{k=1}^{K} \sum_{l\neq k}^{K} \Big[(r_{k,2}^2 - r_{k,1}^2)(r_{l,2}^2 - r_{l,1}^2)G(\xi_k, \xi_l) + \frac{(r_{l,2}^2 - r_{l,1}^2)(r_{k,2}^4 - r_{k,1}^4)}{4|D|}\Big].
\end{aligned}$$

Proof. The first term of $J(E_0)$ is clearly $2\pi \sum_{k=1}^{K} (r_{k,2} + r_{k,1})$. To compute the second term, note that

$$\int_D |(-\Delta)^{-1/2}(\chi_{E_0} - a)|^2 \, dx = \int_D v(x)(\chi_{E_0} - a) \, dx = \int_{E_0} v(x) \, dx = \sum_l \sum_k \int_{P_l} (v_k(x) - \tilde{v}_k(x)) \, dx.$$

In this double sum we first consider the terms when $k = l$:

$$\int_{P_k} (v_k(x) - \tilde{v}_k(x)) \, dx = \int_{\tilde{B}_k} \tilde{v}_k + \int_{B_k} v_k - \int_{\tilde{B}_k} v_k - \int_{B_k} \tilde{v}_k.$$

From the definition of V_k one finds that

$$\int_{B_k} V_k(x) \, dx = -\frac{\pi r_{k,2}^4 \log r_{k,2}}{2} + \frac{\pi r_{k,2}^4}{8}. \tag{3.7}$$

For the integral of W_k, note that, since $\Delta W_k(\cdot,\xi_k) = \frac{\pi r_{k,2}^2}{|D|}$, $W_k(x,\xi_k) - \frac{\pi r_{k,2}^2}{4|D|}|x-\xi_k|^2$ is harmonic in x. By the mean value property of harmonic functions

$$\begin{aligned}\int_{B_k} W_k(x,\xi_k)\,dx &= \int_{B_k} (W_k(x,\xi_k) - \frac{\pi r_{k,2}^2}{4|D|}|x-\xi_k|^2)\,dx + \int_{B_k} \frac{\pi r_{k,2}^2}{4|D|}|x-\xi_k|^2\,dx \\ &= \pi r_{k,2}^2 W_k(\xi_k,\xi_k) + \frac{\pi^2 r_{k,2}^6}{8|D|} \;=\; \pi^2 r_{k,2}^4 R(\xi_k,\xi_k) + \frac{\pi^2 r_{k,2}^6}{4|D|}. \end{aligned} \tag{3.8}$$

Therefore, from (3.7) and (3.8),

$$\int_{B_k} v_k = -\frac{\pi r_{k,2}^4 \log r_{k,2}}{2} + \frac{\pi r_{k,2}^4}{8} + \pi^2 r_{k,2}^4 R(\xi_k,\xi_k) + \frac{\pi^2 r_{k,2}^6}{4|D|}, \tag{3.9}$$

and similarly,

$$\int_{\tilde B_k} \tilde v_k = -\frac{\pi r_{k,1}^4 \log r_{k,1}}{2} + \frac{\pi r_{k,1}^4}{8} + \pi^2 r_{k,1}^4 R(\xi_k,\xi_k) + \frac{\pi^2 r_{k,1}^6}{4|D|}. \tag{3.10}$$

Integration by parts shows that

$$-\int_{\tilde B_k} v_k\,dx = \int_D (\Delta \tilde v_k) v_k\,dx = -\int_D \nabla \tilde v_k \cdot \nabla v_k\,dx = \int_D \tilde v_k (\Delta v_k)\,dx = -\int_{B_k} \tilde v_k\,dx.$$

Therefore,

$$\begin{aligned} -\int_{\tilde B_k} v_k\,dx \;&=\; -\int_{B_k} \tilde v_k\,dx \;=\; -\int_{\tilde B_k} V_k\,dx - \int_{\tilde B_k} W_k\,dx \\ &= \frac{\pi r_{k,1}^4}{8} - \pi r_{k,1}^2 (\frac{r_{k,2}^2}{4} - \frac{r_{k,2}^2}{2}\log r_{k,2}) - \int_{\tilde B_k} (W_k(x,\xi_k) - \frac{\pi r_{k,2}^2}{4|D|}|x-\xi_k|^2)\,dx \\ &\quad - \int_{\tilde B_k} \frac{\pi r_{k,2}^2}{4|D|}|x-\xi_k|^2\,dx \\ &= \frac{\pi r_{k,1}^4}{8} - \pi r_{k,1}^2 (\frac{r_{k,2}^2}{4} - \frac{r_{k,2}^2}{2}\log r_{k,2}) - \pi r_{k,1}^2 W_k(\xi_k,\xi_k) - \frac{\pi^2 r_{k,2}^2 r_{k,1}^4}{8|D|} \\ &= \frac{\pi r_{k,1}^4}{8} - \pi r_{k,1}^2 (\frac{r_{k,2}^2}{4} - \frac{r_{k,2}^2}{2}\log r_{k,2}) - \pi^2 r_{k,1}^2 r_{k,2}^2 R(\xi_k,\xi_k) - \frac{\pi^2 r_{k,2}^2 r_{k,1}^4}{8|D|} - \frac{\pi^2 r_{k,1}^2 r_{k,2}^4}{8|D|}. \end{aligned}$$

Consequently,

$$\begin{aligned} &\int_{P_k} (v_k(x) - \tilde v_k(x))\,dx \\ &= -\frac{\pi r_{k,2}^4 \log r_{k,2}}{2} - \frac{\pi r_{k,1}^4 \log r_{k,1}}{2} + \pi r_{k,2}^2 r_{k,1}^2 \log r_{k,2} - \frac{\pi^2 (r_{k,2}^2 - r_{k,1}^2) r_{k,1}^2}{4} + \frac{\pi^2 (r_{k,2}^2 - r_{k,1}^2)^2}{8} \\ &\quad + \pi^2 (r_{k,2}^2 - r_{k,1}^2)^2 R(\xi_k,\xi_k) + \frac{\pi^2 (r_{k,2}^2 - r_{k,1}^2)(r_{k,2}^4 - r_{k,1}^4)}{4|D|}. \end{aligned}$$

Next we consider the terms with $k \neq l$:

$$\int_{P_l} (v_k(x) - \tilde v_k(x))\,dx = \int_{\tilde B_l} \tilde v_k + \int_{B_l} v_k - \int_{\tilde B_l} v_k - \int_{B_l} \tilde v_k.$$

In this case note that $V_k(x-\xi_k) = -\frac{r_{k,2}^2}{2}\log|x-\xi_k|$ is harmonic on B_l and $\tilde{B}_l$. Then by the mean value property,

$$\int_{B_l} V_k = -\frac{r_{k,2}^2 r_{l,2}^2}{2}\log|\xi_k-\xi_l|, \text{ and } \int_{\tilde{B}_l} V_k = -\frac{r_{k,2}^2 r_{l,1}^2}{2}\log|\xi_k-\xi_l|.$$

These imply, after some similar computation, that when $k \neq l$,

$$\int_{P_l}(v_k(x)-\tilde{v}_k(x))\,dx = \pi^2(r_{k,2}^2-r_{k,1}^2)(r_{l,2}^2-r_{l,1}^2)G(\xi_k,\xi_l) + \frac{\pi^2(r_{l,2}^2-r_{l,1}^2)(r_{k,2}^4-r_{k,1}^4)}{4|D|}$$

The lemma now follows. $\square$

4 Perturbed rings

A perturbed ring E_k from P_k is characterized by a pair of 2π-periodic functions $\phi_k(\theta_k) = (\phi_{k,1}(\theta_k), \phi_{k,2}(\theta_k))$ so that

$$E_k = \bigcup_{\theta_k\in[0,2\pi]}\Big\{\xi_k + te^{i\theta_k} : t \in \Big[\sqrt{r_{k,1}^2+\phi_{k,1}(\theta_k)}, \sqrt{r_{k,2}^2+\phi_{k,2}(\theta_k)}\Big]\Big\}, \tag{4.1}$$

and the boundaries of the perturbed ring E_k are two curves parametrized by θ_k:

$$\theta_k \to \xi_k + \sqrt{r_{k,1}^2+\phi_{k,1}(\theta_k)}e^{i\theta_k}, \tag{4.2}$$

which is the perturbed inner circle, and

$$\theta_k \to \xi_k + \sqrt{r_{k,2}^2+\phi_{k,2}(\theta_k)}e^{i\theta_k}, \tag{4.3}$$

the perturbed outer circle. We will restrict the size of $\phi_{k,1}$, $\phi_{k,2}$ so that $r_{k,1}^2+\phi_{k,1}$, $r_{k,2}^2+\phi_{k,2}$ are positive. Moreover it is always assumed that $\phi_{k,\alpha}$ satisfies

$$\sum_{k=1}^K \int_0^{2\pi}(-\phi_{k,1}(\theta_k)+\phi_{k,2}(\theta_k))\,d\theta_k = 0. \tag{4.4}$$

This ensures that the size of $\cup_{k=1}^K E_k$, which we denote by E_ϕ, stays equal to $a|D|$:

$$\begin{aligned}
|E_\phi| &= \sum_{k=1}^K\int_0^{2\pi}\int_{\sqrt{r_{k,1}^2+\phi_{k,1}(\theta_k)}}^{\sqrt{r_{k,2}^2+\phi_{k,2}(\theta_k)}} r\,drd\theta_k = \sum_{k=1}^K\int_0^{2\pi}\frac{r_{k,2}^2+\phi_{k,2}(\theta_k)-r_{k,1}^2-\phi_{k,1}(\theta_k)}{2}\,d\theta_k \\
&= \sum_{k=1}^K(\pi r_{k,2}^2-\pi r_{k,1}^2) = a|D|.
\end{aligned}$$

The arc length of $\partial_D E_\phi$ can be expressed as

$$\begin{aligned}
&|D\chi_{E_\phi}|(D) \\
&= \sum_{k=1}^K\int_0^{2\pi}\Big[\sqrt{r_{k,1}^2+\phi_{k,1}(\theta_k)+\frac{(\phi_{k,1}'(\theta_k))^2}{4(r_{k,1}^2+\phi_{k,1}(\theta_k))}}+\sqrt{r_{k,2}^2+\phi_{k,2}(\theta_k)+\frac{(\phi_{k,2}'(\theta_k))^2}{4(r_{k,2}^2+\phi_{k,2}(\theta_k))}}\Big]\,d\theta_k.
\end{aligned}$$

Calculating the variations of (4.5) we obtain $2K$ quasi-linear operators

$$\mathcal{H}_{k,\alpha}(\phi_{k,\alpha})(\theta_k) = \frac{r_{k,\alpha}^2 + \phi_{k,\alpha}(\theta_k) + \frac{3(\phi'_{k,\alpha}(\theta_k))^2}{4(r_{k,\alpha}^2+\phi_{k,\alpha}(\theta_k))} - \frac{\phi''_{k,\alpha}(\theta_k)}{2}}{2\Big(r_{k,\alpha}^2 + \phi_{k,\alpha}(\theta_k) + \frac{(\phi'_{k,\alpha}(\theta_k))^2}{4(r_{k,\alpha}^2+\phi_{k,\alpha}(\theta_k))}\Big)^{3/2}}, \ k = 1, 2, ..., K, \ \alpha = 1, 2. \tag{4.5}$$

Note that $\mathcal{H}_{k,2}$ gives half of the curvature of the perturbed outer boundary viewed from E_k. However, $\mathcal{H}_{k,1}$ is *negative* half of the curvature of the perturbed inner boundary viewed from E_k.

The nonlocal part of J in (1.5) may be written in terms of $\phi_{k,\alpha}$ as

$$\begin{aligned}
&\frac{\gamma}{2}\int_D |(-\Delta)^{-1/2}(\chi_{E_\phi} - a)|^2\, dx = \frac{\gamma}{2}\int_{E_\phi}\int_{E_\phi} G(x,y)\, dxdy \\
&= \frac{\gamma}{2}\sum_{k,l=1}^{K}\int_0^{2\pi} d\theta_k \int_{\sqrt{r_{k,1}^2+\phi_{k,1}(\theta_k)}}^{\sqrt{r_{k,2}^2+\phi_{k,2}(\theta_k)}} dr \int_0^{2\pi} d\omega_l \int_{\sqrt{r_{l,1}^2+\phi_{l,1}(\omega_l)}}^{\sqrt{r_{l,2}^2+\phi_{l,2}(\omega_l)}} dt\ G(\xi_k + re^{i\theta_k}, \xi_l + te^{i\omega_l})rt.
\end{aligned}$$

The variation of the nonlocal part of J with respect to $\phi_{k,1}$ is

$$-\frac{\gamma}{2}(-\Delta)^{-1}(\chi_{E_\phi} - a)(\xi_k + \sqrt{r_{k,1}^2 + \phi_{k,1}(\theta_k)}e^{i\theta_k}) = -\frac{\gamma}{2}\int_{E_\phi} G(\xi_k + \sqrt{r_{k,1}^2 + \phi_{k,1}(\theta_k)}e^{i\theta_k}, y)\, dy; \tag{4.6}$$

and the variation with respect to $\phi_{k,2}$ is

$$\frac{\gamma}{2}(-\Delta)^{-1}(\chi_{E_\phi} - a)(\xi_k + \sqrt{r_{k,2}^2 + \phi_{k,2}(\theta_k)}e^{i\theta_k}) = \frac{\gamma}{2}\int_{E_\phi} G(\xi_k + \sqrt{r_{k,2}^2 + \phi_{k,2}(\theta_k)}e^{i\theta_k}, y)\, dy. \tag{4.7}$$

Under the constraint (4.4) the Euler-Lagrange equations of J are

$$\mathcal{H}_{k,1}(\phi_{k,1})(\theta_k) - \frac{\gamma}{2}(-\Delta)^{-1}(\chi_{E_\phi} - a)(\xi_k + \sqrt{r_{k,1}^2 + \phi_{k,1}(\theta_k)}e^{i\theta_k}) = \lambda \tag{4.8}$$

$$\mathcal{H}_{k,2}(\phi_{k,2})(\theta_k) + \frac{\gamma}{2}(-\Delta)^{-1}(\chi_{E_\phi} - a)(\xi_k + \sqrt{r_{k,2}^2 + \phi_{k,2}(\theta_k)}e^{i\theta_k}) = -\lambda \tag{4.9}$$

in terms of $\phi_{k,1}$ and $\phi_{k,2}$.

Remark 4.1 *Note that (4.9) differs from (1.2) by a half while (4.8) differs from (1.2) by a* negative *half.*

Let us define

$$\mathcal{A}_{k,1}(\phi_k)(\theta_k) = \frac{\gamma}{4\pi}\int_{E_k-\xi_k} \log|\sqrt{r_{k,1}^2 + \phi_{k,1}(\theta_k)}e^{i\theta_k} - y|\, dy \tag{4.10}$$

$$\mathcal{A}_{k,2}(\phi_k)(\theta_k) = -\frac{\gamma}{4\pi}\int_{E_k-\xi_k} \log|\sqrt{r_{k,2}^2 + \phi_{k,2}(\theta_k)}e^{i\theta_k} - y|\, dy \tag{4.11}$$

$$\mathcal{B}_{k,1}(\phi_k)(\theta_k) = -\frac{\gamma}{2}\int_{E_k} R(\xi_k + \sqrt{r_{k,1}^2 + \phi_{k,1}(\theta_k)}e^{i\theta_k}, y)\, dy \tag{4.12}$$

$$\mathcal{B}_{k,2}(\phi_k)(\theta_k) = \frac{\gamma}{2}\int_{E_k} R(\xi_k + \sqrt{r_{k,2}^2 + \phi_{k,2}(\theta_k)}e^{i\theta_k}, y)\, dy \tag{4.13}$$

$$\mathcal{C}_{kl,1}(\phi_k,\phi_l)(\theta_k) = -\frac{\gamma}{2}\int_{E_l} G(\xi_k+\sqrt{r_{k,1}^2+\phi_{k,1}(\theta_k)}e^{i\theta_k},y)\,dy \tag{4.14}$$

$$\mathcal{C}_{kl,2}(\phi_k,\phi_l)(\theta_k) = \frac{\gamma}{2}\int_{E_l} G(\xi_k+\sqrt{r_{k,2}^2+\phi_{k,2}(\theta_k)}e^{i\theta_k},y)\,dy \tag{4.15}$$

so that (4.8) and (4.9) become

$$\begin{aligned}
\mathcal{H}_{k,1}(\phi_{k,1})+\mathcal{A}_{k,1}(\phi_k)+\mathcal{B}_{k,1}(\phi_k)+\sum_{l\neq k}\mathcal{C}_{kl,1}(\phi_k,\phi_l) &= \lambda,\\
\mathcal{H}_{k,2}(\phi_{k,2})+\mathcal{A}_{k,2}(\phi_k)+\mathcal{B}_{k,2}(\phi_k)+\sum_{l\neq k}\mathcal{C}_{kl,2}(\phi_k,\phi_l) &= -\lambda.
\end{aligned} \tag{4.16}$$

Let $\mathcal{S}=(\mathcal{S}_{1,1},\mathcal{S}_{1,2},\mathcal{S}_{2,1},\mathcal{S}_{2,2},...,\mathcal{S}_{K,1},\mathcal{S}_{K,2})$ be the operator that appears on the left side of (4.16) projected to $\{(-1,1,-1,1,...,-1,1)\}^{\perp}$, i.e.,

$$\mathcal{S}_{k,\alpha}(\phi)=\mathcal{H}_{k,\alpha}(\phi_{k,\alpha})+\mathcal{A}_{k,\alpha}(\phi_k)+\mathcal{B}_{k,\alpha}(\phi_k)+\sum_{l\neq k}\mathcal{C}_{kl}(\phi_k,\phi_l)+(-1)^{\alpha}\lambda(\phi) \tag{4.17}$$

for $k=1,2,...,K$, $\alpha=1,2$. Here $\lambda(\phi)$ is a number so chosen that $\mathcal{S}(\phi)\perp(-1,1,-1,1,...,-1,1)$, i.e.

$$\sum_{k=1}^{K}\int_0^{2\pi}(-\mathcal{S}_{k,1}(\phi)+\mathcal{S}_{k,2}(\phi))\,d\theta_k=0. \tag{4.18}$$

Note E_ϕ is a solution of (1.2) (and of course (4.16)) if and only if

$$\mathcal{S}(\phi)=0. \tag{4.19}$$

The operator $\mathcal{S}$ maps from a closed ball centered at the origin of the space

$$\mathcal{X}=\left\{\phi=\begin{bmatrix}\phi_{1,1}(\theta_1)\\ \phi_{1,2}(\theta_1)\\ \phi_{2,1}(\theta_2)\\ \phi_{2,2}(\theta_2)\\ ...\\ \phi_{K,1}(\theta_K)\\ \phi_{K,2}(\theta_K)\end{bmatrix}:\phi_{k,\alpha}\in H^2(S^1),\quad \phi\perp\begin{bmatrix}-1\\1\\-1\\1\\...\\-1\\1\end{bmatrix}\right\} \tag{4.20}$$

to

$$\mathcal{Y}=\left\{q=\begin{bmatrix}q_{1,1}(\theta_1)\\ q_{1,2}(\theta_1)\\ q_{2,1}(\theta_2)\\ q_{2,2}(\theta_2)\\ ...\\ q_{K,1}(\theta_K)\\ q_{K,2}(\theta_K)\end{bmatrix}:q_{k,\alpha}\in L^2(S^1),\quad q\perp\begin{bmatrix}-1\\1\\-1\\1\\...\\-1\\1\end{bmatrix}\right\}. \tag{4.21}$$

5 Linear analysis

Let $\mathcal{S}'(0)$ be the linearized operator of $\mathcal{S}$ at $\phi = 0$, i.e, at $E = E_0 = \cup_{k=1}^K P_k$. Calculations show that

$$
\begin{aligned}
\mathcal{H}'_{k,\alpha}(0)(u_{k,\alpha}) &= -\frac{1}{4r_{k,\alpha}^3}(u''_{k,\alpha} + u_{k,\alpha}), \\
\mathcal{A}'_{k,1}(0)(u_k) &= -\frac{\gamma}{8\pi}\int_0^{2\pi} u_{k,1}(\omega)\log|r_{k,1}e^{i\theta_k} - r_{k,1}e^{i\omega}|\,d\omega \qquad (5.1)\\
&\quad +\frac{\gamma}{8\pi}\int_0^{2\pi} u_{k,2}(\omega)\log|r_{k,1}e^{i\theta_k} - r_{k,2}e^{i\omega}|\,d\omega, \\
\mathcal{A}'_{k,2}(0)(u_k) &= \frac{\gamma}{8\pi}\int_0^{2\pi} u_{k,1}(\omega)\log|r_{k,2}e^{i\theta_k} - r_{k,1}e^{i\omega}|\,d\omega \qquad (5.2)\\
&\quad -\frac{\gamma}{8\pi}\int_0^{2\pi} u_{k,2}(\omega)\log|r_{k,2}e^{i\theta_k} - r_{k,2}e^{i\omega}|\,d\omega \\
&\quad -\frac{\gamma u_{k,2}(\theta_k)}{8}\left(1 - \frac{r_{k,1}^2}{r_{k,2}^2}\right), \\
\mathcal{B}'_{k,1}(0)(u_k) &= \frac{\gamma}{4}\int_0^{2\pi} u_{k,1}(\omega)R(\xi_k + r_{k,1}e^{i\theta_k}, \xi_k + r_{k,1}e^{i\omega})\,d\omega \\
&\quad -\frac{\gamma u_{k,1}(\theta_k)}{4r_{k,1}}\int_{P_k}\nabla R(\xi_k + r_{k,1}e^{i\theta_k}, y)\cdot e^{i\theta_k}\,dy \qquad (5.3)\\
&\quad -\frac{\gamma}{4}\int_0^{2\pi} u_{k,2}(\omega)R(\xi_k + r_{k,1}e^{i\theta_k}, \xi_k + r_{k,2}e^{i\omega})\,d\omega, \\
\mathcal{B}'_{k,2}(0)(u_k) &= -\frac{\gamma}{4}\int_0^{2\pi} u_{k,1}(\omega)R(\xi_k + r_{k,2}e^{i\theta_k}, \xi_k + r_{k,1}e^{i\omega})\,d\omega \qquad (5.4)\\
&\quad +\frac{\gamma}{4}\int_0^{2\pi} u_{k,2}(\omega)R(\xi_k + r_{k,2}e^{i\theta_k}, \xi_k + r_{k,2}e^{i\omega})\,d\omega \\
&\quad +\frac{\gamma u_{k,2}(\theta_k)}{4r_{k,2}}\int_{P_k}\nabla R(\xi_k + r_{k,2}e^{i\theta_k}, y)\cdot e^{i\theta_k}\,dy, \\
\mathcal{C}'_{kl,1}(0)(u_k, u_l) &= \frac{\gamma}{4}\int_0^{2\pi} u_{l,1}(\omega)G(\xi_k + r_{k,1}e^{i\theta_k}, \xi_l + r_{l,1}e^{i\omega})\,d\omega \\
&\quad -\frac{\gamma u_{k,1}(\theta_k)}{4r_{k,1}}\int_{P_l}\nabla G(\xi_k + r_{k,1}e^{i\theta_k}, y)\cdot e^{i\theta_k}\,dy \qquad (5.5)\\
&\quad -\frac{\gamma}{4}\int_0^{2\pi} u_{l,2}(\omega)G(\xi_k + r_{k,1}e^{i\theta_k}, \xi_l + r_{l,2}e^{i\omega})\,d\omega, \\
\mathcal{C}'_{kl,2}(0)(u_k, u_l) &= -\frac{\gamma}{4}\int_0^{2\pi} u_{l,1}(\omega)G(\xi_k + r_{k,2}e^{i\theta_k}, \xi_l + r_{l,1}e^{i\omega})\,d\omega \qquad (5.6)\\
&\quad +\frac{\gamma}{4}\int_0^{2\pi} u_{l,2}(\omega)G(\xi_k + r_{k,2}e^{i\theta_k}, \xi_l + r_{l,2}e^{i\omega})\,d\omega
\end{aligned}
$$

$$+\frac{\gamma u_{k,2}(\theta_k)}{4r_{k,2}}\int_{P_l}\nabla G(\xi_k+r_{k,2}e^{i\theta_k},y)\cdot e^{i\theta_k}\,dy. \tag{5.7}$$

Let us separate $\mathcal{S}'(0)$ into a dominant part $\mathcal{L}$ and a minor part $\mathcal{M}$: $\mathcal{S}'(0)=\mathcal{L}+\mathcal{M}$. We define $\mathcal{L}_{k,\alpha}$, the k,α component of $\mathcal{L}$, to be

$$\begin{aligned}
\mathcal{L}_{k,1}(u)(\theta_k) &= -\frac{1}{4r_{k,1}^3}(u''_{k,1}(\theta_k)+u_{k,1}(\theta_k)) \\
&\quad -\frac{\gamma}{8\pi}\int_0^{2\pi}u_{k,1}(\omega)\log|r_{k,1}e^{i\theta_k}-r_{k,1}e^{i\omega}|\,d\omega \\
&\quad +\frac{\gamma}{8\pi}\int_0^{2\pi}u_{k,2}(\omega)\log|r_{k,1}e^{i\theta_k}-r_{k,2}e^{i\omega}|\,d\omega \\
&\quad -l(u) \\
\mathcal{L}_{k,2}(u)(\theta_k) &= -\frac{1}{4r_{k,2}^3}(u''_{k,2}(\theta_k)+u_{k,2}(\theta_k)) \\
&\quad +\frac{\gamma}{8\pi}\int_0^{2\pi}u_{k,1}(\omega)\log|r_{k,2}e^{i\theta_k}-r_{k,1}e^{i\omega}|\,d\omega \\
&\quad -\frac{\gamma}{8\pi}\int_0^{2\pi}u_{k,2}(\omega)\log|r_{k,2}e^{i\theta_k}-r_{k,2}e^{i\omega}|\,d\omega \\
&\quad -\frac{\gamma u_{k,2}(\theta_k)}{8}\left(1-\frac{r_{k,1}^2}{r_{k,2}^2}\right) \\
&\quad +l(u).
\end{aligned}$$

The real-valued linear operator l is independent of k and α. It is so chosen that $\mathcal{L}$ maps from $\mathcal{X}$ to $\mathcal{Y}$. $\mathcal{M}$ is set to be $\mathcal{S}'(0)-\mathcal{L}$.

From now on we let

$$\rho_k=\frac{r_{k,1}}{r_{k,2}}<1. \tag{5.8}$$

We are more interested in the operators $\Pi\mathcal{S}'(0)$, $\Pi\mathcal{L}$, and $\Pi\mathcal{M}$ where Π is the orthogonal projection operator from $\mathcal{Y}$ to

$$\mathcal{Y}_*=\left\{q\in\mathcal{Y}:\begin{bmatrix}q_{k,1}\\ q_{k,2}\end{bmatrix}\perp\cos\theta_k\begin{bmatrix}\rho_k\\ 1\end{bmatrix},\ \begin{bmatrix}q_{k,1}\\ q_{k,2}\end{bmatrix}\perp\sin\theta_k\begin{bmatrix}\rho_k\\ 1\end{bmatrix},\ q_{k,\alpha}\perp 1,\ \forall k,\alpha\right\}. \tag{5.9}$$

The operator $\Pi\mathcal{L}$ is defined on

$$\mathcal{X}_*=\mathcal{X}\cap\mathcal{Y}_*. \tag{5.10}$$

Note that

$$l(u)=0,\ \text{ if } u\in\mathcal{X}_*. \tag{5.11}$$

Lemma 5.1 *Assume that the conditions of Theorem 2.1 hold.*

1. $\|u\|_{H^2}\le Ca^{3/2}\|\mathcal{L}(u)\|_{L^2}$ *for all* $u\in\mathcal{X}_*$.

2. If $\Gamma > \Gamma_1$, we have $\|u\|_{H^1}^2 \le Ca^{3/2}\langle \mathcal{L}(u), u\rangle$ for all $u \in \mathcal{X}_*$.

Proof. The spectrum of $\Pi\mathcal{L}$ can be computed explicitly using Fourier series. The Fourier space of $\mathcal{X}_*$ is

$$\begin{aligned}\widehat{\mathcal{X}_*} &= \{(\{l_{1,1}(n_1)\}, \{l_{1,2}(n_1)\}, \{l_{2,1}(n_2)\}, \{l_{2,2}(n_2)\}, ..., \{l_{K,1}(n_K)\}, \{l_{K,2}(n_K)\},) : \\ &\quad \sum_{n_k=-\infty}^{\infty} |l_{k,\alpha}(n_k)|^2 < \infty,\ \forall k, \alpha;\ l_{k,\alpha}(0) = 0,\ \forall k, \alpha; \\ &\quad (l_{k,1}(1), l_{k,2}(1)) \perp (\rho_k, 1),\ (l_{k,1}(-1), l_{k,2}(-1)) \perp (\rho_k, 1),\ \forall k\}. \end{aligned} \tag{5.12}$$

Let

$$\widehat{u_{k,\alpha}}(n) = \int_0^{2\pi} u_{k,\alpha}(\theta_k) e^{-in\theta_k}\, d\theta_k \tag{5.13}$$

be the n-th Fourier coefficient of $u_{k,\alpha}$. Then when $n \neq 0$

$$\widehat{\mathcal{L}_{k,1}(u)}(n) = (\frac{n^2-1}{4r_{k,1}^3} + \frac{\gamma}{8|n|})\widehat{u_{k,1}}(n) - \frac{\gamma\rho_k^{|n|}}{8|n|}\widehat{u_{k,2}}(n) \tag{5.14}$$

$$\widehat{\mathcal{L}_{k,2}(u)}(n) = -\frac{\gamma\rho_k^{|n|}}{8|n|}\widehat{u_{k,1}}(n) + [\frac{n^2-1}{4r_{k,2}^3} + \frac{\gamma}{8}(\frac{1}{|n|} - 1 + \rho_k^2)]\widehat{u_{k,2}}(n). \tag{5.15}$$

To derive (5.14) and (5.15) we have used the well-known formula

$$\log|1 - e^{i\theta_k}| = -\sum_{n=1}^{\infty} \frac{\cos n\theta_k}{n}. \tag{5.16}$$

See for instance Tolstov [26, Page 93]. We have also used the formula

$$\log|1 - re^{i\theta_k}| = -\sum_{n=1}^{\infty} \frac{r^n \cos n\theta_k}{n},\ r \in (0,1). \tag{5.17}$$

See [7, Appendix B] for a simple derivation of (5.17).

Note that $\Pi\mathcal{L} = \mathcal{L}$ on $\mathcal{X}_*$, which follows from the facts

$$\begin{bmatrix} \widehat{\mathcal{L}_{k,1}(u)}(1) \\ \widehat{\mathcal{L}_{k,2}(u)}(1) \end{bmatrix} \perp \begin{bmatrix} \rho_k \\ 1 \end{bmatrix},\ \begin{bmatrix} \widehat{\mathcal{L}_{k,1}(u)}(-1) \\ \widehat{\mathcal{L}_{k,2}(u)}(-1) \end{bmatrix} \perp \begin{bmatrix} \rho_k \\ 1 \end{bmatrix}. \tag{5.18}$$

By (5.14) and (5.15) we have effectively diagonalized the operator $\Pi\mathcal{L}$ into 2 by 2 blocks labeled by k and n, which we denote by

$$M_{k,n} = \begin{bmatrix} \frac{n^2-1}{4r_{k,1}^3} + \frac{\gamma}{8n} & -\frac{\gamma\rho_k^n}{8n} \\ -\frac{\gamma\rho_k^n}{8n} & \frac{n^2-1}{4r_{k,2}^3} + \frac{\gamma}{8}[\frac{1}{n} - 1 + \rho_k^2] \end{bmatrix} \tag{5.19}$$

for $n \ge 1$.

For $n=1$, each

$$M_{k,1} = \begin{bmatrix} \frac{\gamma}{8} & -\frac{\gamma\rho_k}{8} \\ \\ -\frac{\gamma\rho_k}{8} & \frac{\gamma\rho_k^2}{8} \end{bmatrix}$$

has two eigenvalues. One is $\tilde{\lambda}_{k,1} = 0$, with eigenvectors

$$\cos\theta_k \begin{bmatrix} \rho_k \\ 1 \end{bmatrix}, \quad \sin\theta_k \begin{bmatrix} \rho_k \\ 1 \end{bmatrix}.$$

However they do not meet the orthogonality conditions in the definition of $\mathcal{X}_*$ and are therefore discarded.

The second eigenvalue is $\lambda_{k,1} = \frac{\gamma(1+\rho_k^2)}{8} > Ca^{-3/2}$ for some $C > 0$ independent of a, with eigenvectors

$$\cos\theta_k \begin{bmatrix} 1 \\ -\rho_k \end{bmatrix}, \quad \sin\theta_k \begin{bmatrix} 1 \\ -\rho_k \end{bmatrix},$$

which satisfy the orthogonality conditions of $\mathcal{X}_*$.

For $n \geq 2$, denote the $(1,1)$ entry of $M_{k,n}$ by c_1, the $(2,2)$ entry by c_2, and the $(1,2)$ and $(2,1)$ entries by d. Then

$$\det(\lambda I - M_{k,n}) = \lambda^2 - (c_1+c_2)\lambda + c_1c_2 - d^2. \tag{5.20}$$

Let $\tilde{\lambda}_{k,n}, \lambda_{k,n}$ be the two eigenvalues of M_n; then we find that

$$\tilde{\lambda}_{k,n} = \frac{c_1 + c_2 + \sqrt{(c_1-c_2)^2 + 4d^2}}{2}, \tag{5.21}$$

$$\lambda_{k,n} = \frac{c_1 + c_2 - \sqrt{(c_1-c_2)^2 + 4d^2}}{2}. \tag{5.22}$$

It is obvious that $c_1 > c_2$; therefore,

$$\tilde{\lambda}_{k,n} > \frac{c_1 + c_2 + c_1 - c_2}{2} = c_1 = \frac{n^2-1}{4r_{k,1}^3} + \frac{\gamma}{8n} > Cn^2a^{-3/2} > 0$$

where $C > 0$ is independent of a.

It remains to study $\lambda_{k,n}$. Let us use scaled variables $R_{k,\alpha}$ and Γ, where

$$R_{k,\alpha} = (\frac{a|D|}{K\pi})^{-1/2} r_{k,\alpha}, \quad \Gamma = (\frac{a|D|}{K\pi})^{3/2}\gamma. \tag{5.23}$$

The matrices M_n can be written as

$$M_{k,n} = (\frac{a|D|}{K\pi})^{-3/2} \begin{bmatrix} \frac{n^2-1}{4R_{k,1}^3} + \frac{\Gamma}{8n} & -\frac{\Gamma}{8n}(\frac{R_{k,1}}{R_{k,2}})^n \\ \\ -\frac{\Gamma}{8n}(\frac{R_{k,1}}{R_{k,2}})^n & \frac{n^2-1}{4R_{k,2}^3} + \frac{\Gamma}{8}[\frac{1}{n} - 1 + (\frac{R_{k,1}}{R_{k,2}})^2] \end{bmatrix}, \quad n \geq 2.$$

It is easy to see that asymptotically for fixed $R_{k,1}$ and Γ

$$\lim_{n\to\infty} \frac{\tilde{\lambda}_{k,n}}{(\frac{a|D|}{K\pi})^{-3/2}(\frac{n^2-1}{4R_{k,1}^3})} = 1, \quad \lim_{n\to\infty} \frac{\lambda_{k,n}}{(\frac{a|D|}{K\pi})^{-3/2}(\frac{n^2-1}{4R_{k,2}^3})} = 1. \tag{5.24}$$

Note that the second eigenvalue $\lambda_{k,n}$ is not zero if $\det M_{k,n} \neq 0$, and it is positive if $\det M_{k,n} > 0$. The equation $\det M_{k,n} = 0$ is quadratic in Γ:

$$\frac{1-\rho_k^{2n} - n(1-\rho^2)}{64n^2}\Gamma^2 + [\frac{n^2-1}{32nR_{k,2}^3} + \frac{n^2-1}{32R_{k,1}^3}(\frac{1}{n} - 1 + \rho_k^2)]\Gamma + \frac{(n^2-1)^2}{16R_{k,1}^3R_{k,2}^3} = 0.$$

The graph of the left side, as a function of Γ, is a downward parabola. Its intersection with the vertical axis is $(0, \frac{(n^2-1)^2}{4R_{k,1}^3R_{k,2}^3})$. Therefore, one root for Γ is negative, and the other root is positive.

Since $\Gamma > 0$, we focus on the positive root which in Section 2 is denoted by $\tilde{\Gamma}_n(R_{k,1})$. Recall the curves W_n defined in (2.5) and $\overline{U_1 \times U_2}$ from (3.3). For $\Gamma \in (\Gamma_0, \Gamma_1) \cup (\Gamma_1, \infty)$, we have $R_{k,1} \in [T_1 - \delta_1, T_1 + \delta_1]$ and $R_{k,2} \in [T_2 - \delta_2, T_2 + \delta_2]$. Since (T_1, Γ) is not on the curves W_n, $n = 2, 3, ...$, we can choose δ_1 and δ_2 sufficiently small so that the determinants of the matrices

$$\begin{bmatrix} \frac{n^2-1}{4R_{k,1}^3} + \frac{\Gamma}{8n} & -\frac{\Gamma}{8n}(\frac{R_{k,1}}{R_{k,2}})^n \\ -\frac{\Gamma}{8n}(\frac{R_{k,1}}{R_{k,2}})^n & \frac{n^2-1}{4R_{k,2}^3} + \frac{\Gamma}{8}[\frac{1}{n} - 1 + (\frac{R_{k,1}}{R_{k,2}})^2] \end{bmatrix}, \ n = 2, 3, ...$$

are not zero for all $R_{k,1} \in [T_1 - \delta_1, T_1 + \delta_1]$ and $R_{k,2} \in [T_2 - \delta_2, T_2 + \delta_2]$. With the help of the asymptotic formulas (5.24), we find $C > 0$, independent of a, such that

$$\frac{|\lambda_{k,n}|}{n^2} > Ca^{-3/2}, k = 1, 2, ..., K, \ n = 2, 3, ... \tag{5.25}$$

This implies that

$$\|u\|_{H^2} \le Ca^{3/2}\|\mathcal{L}(u)\|_{L^2}, \tag{5.26}$$

for all $u \in \mathcal{X}_*$.

If we further assume that $\Gamma > \Gamma_1$, then (T_1, Γ) lies below all the W_n's and by making δ_1 and δ_2 small we ensure that $\det M_{k,n} > 0$ and there exists $C > 0$ such that

$$\frac{\lambda_{k,n}}{n^2} > Ca^{-3/2}, k = 1, 2, ..., K, \ n = 2, 3, ... \tag{5.27}$$

This implies that

$$\|u\|_{H^1}^2 \le Ca^{3/2}\langle \mathcal{L}(u), u\rangle. \tag{5.28}$$

In particular $\mathcal{L}$ is a positive operator on $\mathcal{X}_*$. This proves the lemma. □

The next lemma states that the second part $\mathcal{M}$ in $\mathcal{S}'(0)$ is minor. We omit its proof, which is similar to that of [7, Lemma 5.2].

Lemma 5.2 *There exists $C > 0$ independent of ξ_k and $R_{k,\alpha}$ such that $\|\mathcal{M}(u)\|_{L^2} \le Ca^{-1}\|u\|_{L^2}$ for all $u \in \mathcal{X}_*$.*

The properties of $\Pi\mathcal{S}'(0)$ may now be derived from the last two lemmas. See [7, Lemma 5.3] for a proof.

Lemma 5.3 *Assume that the conditions of Theorem 2.1 hold.*

1. *For $u \in \mathcal{X}_*$, $\|u\|_{H^2} \le Ca^{3/2}\|\Pi\mathcal{S}'(0)(u)\|_{L^2}$.*

2. If $\Gamma > \Gamma_1$, then $\|u\|^2_{H^1} \le Ca^{3/2}\langle \Pi\mathcal{S}'(0)(u), u\rangle$.

3. $\Pi\mathcal{S}'(0) : \mathcal{X}_* \to \mathcal{Y}_*$ is one-to-one and onto.

Finally in this section we state a bound on the second Fréchet derivative of $\mathcal{S}$.

Lemma 5.4 *Assume that $\|\phi\|_{H^2} \le ca$, where c is sufficiently small. Then $\|\mathcal{S}''(\phi)(u,v)\|_{L^2} \le Ca^{-5/2}\|u\|_{H^2}\|v\|_{H^2}$.*

Note that by taking c small, we keep $r^2_{k,\alpha} + \phi_{k,\alpha}$ positive, so E_k is a perturbed ring. For a proof of this lemma, see [22, Lemma 3.2] or [21, Lemma 6.1].

6 Reduction to finite dimensions

In this section we prove that for each given (ξ, r) with $(\xi, R) = (\xi, (\frac{a|D|}{K\pi})^{-1/2} r) \in \overline{U_1 \times U_2}$, there exists a function $\varphi(\cdot, \xi, r) \in \mathcal{X}_*$ such that

$$\mathcal{S}(\varphi(\cdot,\xi,r)) = \begin{bmatrix} A_1\cos\theta_1 \begin{bmatrix} \rho_1 \\ 1 \end{bmatrix} \\ A_2\cos\theta_2 \begin{bmatrix} \rho_2 \\ 1 \end{bmatrix} \\ \cdots \\ A_K\cos\theta_K \begin{bmatrix} \rho_K \\ 1 \end{bmatrix} \end{bmatrix} + \begin{bmatrix} B_1\sin\theta_1 \begin{bmatrix} \rho_1 \\ 1 \end{bmatrix} \\ B_2\sin\theta_2 \begin{bmatrix} \rho_2 \\ 1 \end{bmatrix} \\ \cdots \\ B_K\sin\theta_K \begin{bmatrix} \rho_K \\ 1 \end{bmatrix} \end{bmatrix} + \begin{bmatrix} \begin{bmatrix} C_{1,1} \\ C_{1,2} \end{bmatrix} \\ \begin{bmatrix} C_{2,1} \\ C_{2,2} \end{bmatrix} \\ \cdots \\ \begin{bmatrix} C_{K,1} \\ C_{K,2} \end{bmatrix} \end{bmatrix} \tag{6.1}$$

for some real numbers A_k, B_k, and $C_{k,\alpha}$. Note that φ is sought in $\mathcal{X}_*$.

The equation (6.1) may be written as

$$\Pi\mathcal{S}(\varphi(\cdot,\xi,r)) = 0 \tag{6.2}$$

where Π is the orthogonal projection operator from $\mathcal{Y}$ to $\mathcal{Y}_*$. In the next section we will find a particular (ξ, r), say (ζ, s), such that at $\xi = \zeta$ and $r = s$, $A_k = B_k = C_{k,\alpha} = 0$, i.e. $\mathcal{S}(\varphi(\cdot,\zeta,s)) = 0$. This means that by finding φ one reduces the original problem (1.2) to a problem of finding ζ and s in a finite-dimensional set.

Expand $\mathcal{S}(\phi)$ as

$$\mathcal{S}(\phi) = \mathcal{S}(0) + \mathcal{S}'(0)(\phi) + \mathcal{N}(\phi) \tag{6.3}$$

where $\mathcal{N}$ is a higher order term defined by (6.3). Rewrite (6.2) in a fixed point form:

$$\phi = -(\Pi\mathcal{S}'(0))^{-1}(\Pi\mathcal{S}(0) + \Pi\mathcal{N}(\phi)) \tag{6.4}$$

Lemma 6.1 *There is $\varphi = \varphi(\cdot,\xi,r)$ such that for every $(\xi, R) \in \overline{U_1 \times U_2}$, $\varphi(\cdot,\xi,r) \in \mathcal{X}_*$ solves (6.4) and $\|\varphi(\cdot,\xi,r)\|_{H^2} \le ca^{3/2}$, where c is a sufficiently large constant independent of a, ξ, and r.*

Proof. To use the contraction mapping principle in the fixed point setting (6.4), let

$$\mathcal{T}(\phi) = -(\Pi\mathcal{S}'(0))^{-1}(\Pi\mathcal{S}(0) + \Pi\mathcal{N}(\phi)) \tag{6.5}$$

be an operator defined on

$$D(\mathcal{T}) = \{\phi \in \mathcal{X}_* : \|\phi\|_{H^2} \le ca^{3/2}\} \tag{6.6}$$

where the constant c is sufficiently large and will be made more precise later.

We know from Lemma 3.1 that $\mathcal{S}(0)$ is a sum of a θ_k independent part and a quantity of order $O(1)$. After one applies Π, the θ_k independent part vanishes and we obtain

$$\|\Pi\mathcal{S}(0)\|_{L^2} = O(1). \tag{6.7}$$

From Lemma 5.3 we deduce that

$$\|(\Pi\mathcal{S}'(0))^{-1}\Pi\mathcal{S}(0)\|_{H^2} \le Ca^{3/2}. \tag{6.8}$$

Lemma 5.4 implies that

$$\|\mathcal{N}(\phi)\|_{L^2} \le Ca^{-5/2}\|\phi\|_{H^2}^2 \tag{6.9}$$

and consequently,

$$\|(\Pi\mathcal{S}'(0))^{-1}\Pi\mathcal{N}(\phi)\|_{H^2} \le Ca^{-1}\|\phi\|_{H^2}^2. \tag{6.10}$$

Using (6.5), (6.8), (6.6), and (6.10) we find that

$$\|\mathcal{T}(\phi)\|_{H^2} \le Ca^{3/2} + Ca^{-1}c^2a^3 \le ca^{3/2}$$

if c is sufficiently large and a sufficiently small. Therefore, $\mathcal{T}$ is a map from $D(\mathcal{T})$ into itself.

Next we show that $\mathcal{T}$ is a contraction. Let $\phi_1, \phi_2 \in D(\mathcal{T})$. First note that

$$\mathcal{T}(\phi_1) - \mathcal{T}(\phi_2) = (\Pi\mathcal{S}'(0))^{-1}(-\Pi)(\mathcal{N}(\phi_1) - \mathcal{N}(\phi_2)). \tag{6.11}$$

Because

$$\mathcal{N}(\phi_1) - \mathcal{N}(\phi_2) = \mathcal{S}(\phi_1) - \mathcal{S}(\phi_2) - \mathcal{S}'(0)(\phi_1 - \phi_2),$$

we deduce, with the help of Lemma 5.4 and (6.6), that

$$\begin{aligned}
\|\mathcal{N}(\phi_1) - \mathcal{N}(\phi_2)\|_{L^2} &\le \|\mathcal{S}'(\phi_2)(\phi_1 - \phi_2) - \mathcal{S}'(0)(\phi_1 - \phi_2)\|_{L^2} + Ca^{-5/2}\|\phi_1 - \phi_2\|_{H^2}^2 \\
&\le Ca^{-5/2}\|\phi_2\|_{H^2}\|\phi_1 - \phi_2\|_{H^2} + Ca^{-5/2}\|\phi_1 - \phi_2\|_{H^2}^2 \\
&\le Ca^{-5/2}(\|\phi_1\|_{H^2} + \|\phi_2\|_{H^2})\|\phi_1 - \phi_2\|_{H^2} \\
&\le Ca^{-1}\|\phi_1 - \phi_2\|_{H^2}.
\end{aligned}$$

Then Lemma 5.3 implies that

$$\|\mathcal{T}(\phi_1) - \mathcal{T}(\phi_2)\|_{H^2} \le Ca^{1/2}\|\phi_1 - \phi_2\|_{H^2}. \tag{6.12}$$

Therefore, $\mathcal{T}$ is a contraction mapping in $D(\mathcal{T})$, if a is sufficiently small. There is a fixed point φ, which we denote by $\varphi = \varphi(\theta, \xi, r)$. Being in $D(\mathcal{T})$, $\|\varphi\|_{H^2} = O(a^{3/2})$, which is smaller than the radii square of $r_{k,\alpha}^2$. Hence each E_k is a perturbed ring. $\square$

7 Minimization

We prove Theorem 2.1 in this section. From Lemma 6.1 we know that for every $(\xi, R) \in \overline{U_1 \times U_2}$ there exists $\varphi(\cdot, \xi, r) \in \mathcal{X}_*$ such that $\Pi\mathcal{S}(\varphi(\cdot, \xi, r)) = 0$, i.e. (6.1) holds. In this section we find particular ξ and r denoted by ζ and s such that $\mathcal{S}(\varphi(\cdot, \zeta, s)) = 0$.

Lemma 7.1 $J(E_{\varphi(\cdot,\xi,r)}) = J(E_0) + O(a^{3/2})$.

Proof. Expanding $J(E_\varphi)$ yields

$$J(E_\varphi) = J(E_0) + \sum_{k,\alpha} \int_0^{2\pi} \mathcal{S}_{k,\alpha}(0)\varphi_{k,\alpha}\, d\theta_k + \frac{1}{2}\sum_{k,\alpha} \int_0^{2\pi} \mathcal{S}'(0)_{k,\alpha}(\varphi)\varphi_{k,\alpha}\, d\theta_k + O(a^2). \tag{7.1}$$

The error term in (7.1) is obtained by Lemma 5.4. and the fact $\|\varphi\|_{H^2} = O(a^{3/2})$.

On the other hand $\Pi\mathcal{S}(\varphi) = 0$ implies that

$$\Pi(\mathcal{S}(0) + \mathcal{S}'(0)(\varphi) + \mathcal{N}(\varphi)) = 0$$

where $\mathcal{N}$ is given in (6.3). We multiply the last equation by φ and integrate to derive, again with the help of Lemma 5.4,

$$\sum_{k,\alpha} \int_0^{2\pi} \mathcal{S}_{k,\alpha}(0)\varphi_{k,\alpha}\, d\theta_k + \sum_{k,\alpha} \int_0^{2\pi} \mathcal{S}'(0)_{k,\alpha}(\varphi)\varphi_{k,\alpha}\, d\theta_k = O(a^2).$$

We can now rewrite (7.1) as

$$J(E_\varphi) = J(E_0) + \frac{1}{2}\sum_{k,\alpha} \int_0^{2\pi} \mathcal{S}_{k,\alpha}(0)\varphi_{k,\alpha}\, d\theta_k + O(a^2).$$

Lemma 3.1 and the fact that $\|\varphi\|_{H^2} = O(a^{3/2})$ imply that

$$J(E_\varphi) = J(E_0) + O(a^{3/2}) + O(a^2) = J(E_0) + O(a^{3/2}).$$

When we use Lemma 3.1, note that $\mathcal{S}(0)$ is a sum of a θ_k independent part and a quantity of order $O(1)$, and that $\varphi_{k,\alpha} \perp 1$. This proves the lemma. $\square$

Lemma 7.2 *As a function of (ξ, r), $J(E_{\varphi(\cdot,\xi,r)})$ is locally minimized at some (ζ, s), when a is small. As $a \to 0$,*

$$\zeta \to \zeta^*, \quad (\frac{a|D|}{\pi})^{-1/2} s \to S^*,$$

possibly along a subsequence, where $F(\zeta^*) = \min_{\xi\in D} F(\xi)$ and $S^* = (T_1, T_2, T_1, T_2, ..., T_1, T_2)$.

Proof. The functional $J(E_\varphi)$ can be viewed as a function of r and ξ. Let $J(E_\varphi)$ be minimized at (ζ, s). It is more convenient if we use the scaled variable R and Γ as in (5.23). Then (3.5) implies that

$$\sum_{k=1}^{K} (R_{k,2}^2 - R_{k,1}^2) = K. \tag{7.2}$$

Corresponding to s, we introduce S by

$$S_{k,\alpha} = (\frac{a|D|}{K\pi})^{-1/2} s_{k,\alpha}. \tag{7.3}$$

Note that $(\zeta, S) \in \overline{U_1 \times U_2}$.

As a function of the scaled variables R and Γ, by Lemmas 3.2 and 7.1, $J(E_\varphi)$ becomes

$$J(E_\varphi) = (\frac{a|D|}{K\pi})^{1/2} \log(\frac{K\pi}{a|D|})^{1/2} J_1(R) + (\frac{a|D|}{K\pi})^{1/2} J_2(R) + (\frac{a|D|}{K\pi})^{1/2} J_3(R,\xi) + O(a^{3/2}) \tag{7.4}$$

where J_1, J_2, and J_3 are given by

$$\begin{aligned} J_1(R) &= \sum_{k=1}^K \frac{\Gamma\pi}{4} (R_{k,2}^2 - R_{k,1}^2)^2 \\ J_2(R) &= \sum_{k=1}^K 2\pi (R_{k,1} + R_{k,2}) + \frac{\Gamma\pi^2}{2} \sum_{k=1}^K \Big[-\frac{R_{k,2}^4 \log R_{k,2}}{2\pi} - \frac{R_{k,1}^4 \log R_{k,1}}{2\pi} + \frac{R_{k,2}^2 R_{k,1}^2 \log R_{k,2}}{\pi} \\ &\quad - \frac{(R_{k,2}^2 - R_{k,1}^2) R_{k,1}^2}{4\pi} + \frac{(R_{k,2}^2 - R_{k,1}^2)^2}{8\pi} \Big] \\ J_3(\xi, R) &= \frac{\Gamma\pi^2}{2} \Big[\sum_{k=1}^K (R_{k,2}^2 - R_{k,1}^2)^2 R(\xi_k, \xi_k) + \sum_{k=1}^K \sum_{l \neq k} (R_{k,2}^2 - R_{k,1}^2)(R_{l,2}^2 - R_{l,1}^2) G(\xi_k, \xi_l) \Big] \end{aligned}$$

Assume

$$(\zeta, S) \to (\zeta', S') \text{ as } a \to 0, \text{ possibly along a subsequence.} \tag{7.5}$$

First we claim that

$$(S'_{k,2})^2 - (S'_{k,1})^2 = 1, \ \forall k = 1, 2, ..., K. \tag{7.6}$$

This means that asymptotically all the rings have the same area. Suppose that (7.6) is not true. Because of the constraint (7.2), $J_1(S') > J_1(S^*)$ by the convexity of the square function and Jessen's inequality. Compare the energy of (S, ζ) and that of (S^*, ζ) to find

$$\lim_{a\to 0} \frac{1}{(\frac{a|D|}{K\pi})^{1/2} \log(\frac{K\pi}{a|D|})^{1/2}} [J(\zeta, S) - J(\zeta, S^*)] = \lim_{q \to 0} [J_1(S) - J_1(S^*)] = J_1(S') - J_1(S^*) > 0,$$

a contradiction to the assumption that (ζ, S) is a minimum.

Next we claim that

$$S' = S^*. \tag{7.7}$$

Otherwise $S'_{k,1} \neq T_1$ for at least one k. We again compare the energy of (ζ, S) and that of (ζ, S^*). Recall Q_Γ and F from (2.6) and (2.3). Note that by (7.6), $S_{k,2}^2 - S_{k,1}^2 \to 1$ and hence $J_2(S) \to \sum_{k=1}^K Q_\Gamma(S'_{k,1})$ and $J_3(\zeta, S) \to F(\zeta')$. Then

$$\liminf_{a \to 0} \frac{1}{(\frac{a|D|}{K\pi})^{1/2}} (J(S,\zeta) - J(S^*, \zeta))$$

$$
\begin{aligned}
&= \liminf_{a\to 0} \log(\frac{K\pi}{a|D|})^{1/2}[J_1(S)-J_1(S^*)] + \lim_{a\to 0}[J_2(S)-J_2(S^*)] + \lim_{a\to 0}[J_3(\zeta,S)-J_3(\zeta,S^*)] \\
&\geq 0 + \sum_{k=1}^{K}[Q_\Gamma(S'_{k,1})-Q_\Gamma(T_1)] + F(\zeta')-F(\zeta') \ = \ \sum_{k=1}^{K}[Q_\Gamma(S'_{k,1})-Q_\Gamma(T_1)] \ > \ 0.
\end{aligned}
$$

The last inequality follows since Q_Γ is minimized at T_1 and at least one $S'_{k,1}$ is not T_1. We have a contradiction to the assumption that (ζ, S) minimizes J.

Finally we show that

$$F(\zeta') = \min_{\xi\in D} F(\xi). \tag{7.8}$$

Assume $F(\zeta') > F(\zeta^*)$ where $F(\zeta^*) = \min_{\xi\in D}F(\xi)$. Consider

$$\lim_{a\to 0}\frac{1}{(\frac{a|D|}{K\pi})^{1/2}}(J(\zeta,S)-J(\zeta^*,S)) = \lim_{a\to 0}[J_3(\zeta,S)-J_3(\zeta^*,S)] = F(\zeta')-F(\zeta^*) > 0,$$

again a contradiction. □

Although $J(E_{\varphi(\cdot,\xi,r)})$ is considered for $(\xi,R)\in\overline{U_1\times U_2}$, since $(\zeta^*,S^*)\in U_1\times U_2$, $(\zeta,S)\in U_1\times U_2$ when a is small. Naturally at (ζ,s), $\varphi(\cdot,\zeta,s)$ is a solution of (1.2). In other words, at $\xi=\zeta$ and $r=s$, $A_k=B_k=C_{k,\alpha}=0$, i.e., $\mathcal{S}(\varphi(\cdot,\zeta,s))=0$. For a proof of this intuitively clear fact, see [7, Lemma 7.4] and [21, Lemma 8.4].

Hence we have proved Theorem 2.1. The solution is close to E_0 which is the union of the P_k's, and each P_k is the ring $\{x\in R^2 : s_{k,1}\leq |x-\zeta_k|\leq s_{k,2}\}$. The difference between the exact solution and E_0 is measured by $\sqrt{r_{k,\alpha}^2+\varphi_{k,\alpha}}-r_{k,\alpha}$, and according to Lemma 6.1, $\|\varphi\|_{H^2}=O(a^{3/2})$. The inner and outer radii $s_{k,1}$ and $s_{k,2}$, by Lemma 7.2, have the properties

$$\frac{s_{k,1}}{(\frac{a|D|}{K\pi})^{1/2}}\to T_1 \text{ and } \frac{s_{k,2}}{(\frac{a|D|}{K\pi})^{1/2}}\to T_2, \quad \text{as } a\to 0.$$

Lemma 7.2 also states that ζ of the ring centers converges, possibly along a subsequence, to a global minimum of F.

The stability of the solution is determined in Lemma 5.3. If $\Gamma>\Gamma_0$, then (T_1,Γ) lies below all the W_n curves (Figure 3) and the operator $\Pi\mathcal{S}'(0)$ on $\mathcal{X}_*$ is positive. The fixed point $\varphi(\cdot,\xi,r)$ found in Lemma 6.1 locally minimizes J in $\mathcal{X}_*$. Since the solution $\varphi(\cdot,\zeta,s)$ is obtained by another minimization of $J(E_{\varphi(\cdot,\xi,r)})$ with respect to ξ and r, $\varphi(\cdot,\zeta,s)$ is a local minimum of J, and hence stable.

If $\Gamma\in(\Gamma_0,\Gamma_1)$, then (T_1,Γ) lies between W_2 and W_3 (Figure 3). The operator $\Pi\mathcal{S}'(0)$ is not positive. More precisely, in $\mathcal{X}_*$ the eigenvalues $\lambda_{k,2}$ $(k=1,2,...,K)$ of $\mathcal{L}$ are all negative. Therefore, the fixed point $\varphi(\cdot,\xi,r)$ does not locally minimize J in $\mathcal{X}_*$. Even though the solution $\varphi(\cdot,\zeta,s)$ is obtained by minimizing $J(E_{\varphi(\cdot,\xi,r)})$ with respect to ξ and r, $\varphi(\cdot,\zeta,s)$ is not a local minimum but a saddle point. More discussion regarding stability in this setting may be found in [22, 21, 23].

8 Coexistence

In Theorem 2.2 where both rings and discs appear, the index set $\{1,2,...,K\}$ for k that labeled the rings earlier must be replaced by two sets: I_r and I_d so that $I_r\cup I_d=\{1,2,...,K\}$ and $k\in I_r$ refers to a ring and $k \in I_d$ refers to a disc. When we deal with $k \in I_d$, much can be accomplished by setting $r_{k,1} = 0$ (and $R_{k,1} = 0$, $\phi_{k,1} = 0$) in the earlier argument.

The main difference appears in the analysis of $\mathcal{L}$, the dominating part of the linear operator. When $k \in I_d$, $\mathcal{L}_k$ only has one component instead of two:

$$\mathcal{L}_k(u)(\theta_k) = -\frac{1}{4r_k^3}(u_k''(\theta_k) + u_k(\theta_k)) - \frac{\gamma}{8\pi}\int_0^{2\pi} u_k(\omega)\log|r_k e^{i\theta_k} - r_k e^{i\omega}|\,d\omega - \frac{\gamma u_k(\theta_k)}{8} + l(u) \quad (8.1)$$

where r_k is the radius of the disc. In the Fourier space it acts like

$$\widehat{\mathcal{L}_k(u)}(n) = \Big[\frac{n^2-1}{4r_k^3} + \frac{\gamma}{8}(\frac{1}{|n|} - 1)\Big]\widehat{u_k}(n), \ n = \pm 2, \pm 3, ... \quad (8.2)$$

The range for r_k is

$$R_k \in [1-\delta, 1+\delta] \quad (8.3)$$

for some small $\delta > 0$, where $R_k = (\frac{a|D|}{K\pi})^{-1/2} r_k$ is the scaled version of r_k. Our construction requires that the eigenvalues $\frac{n^2-1}{4r_k^3} + \frac{\gamma}{8}(\frac{1}{|n|} - 1)$ be nonzero, i.e.

$$\Gamma \neq 2n(n+1), \ n = 2, 3, ... \quad (8.4)$$

Note that the curves W_n meet the Γ-axis at $2n(n+1)$ (Figure 3). The condition (8.4) is joined with the condition $\Gamma \in (\Gamma_0, \Gamma_1) \cup (\Gamma_1, \infty)$ which comes from the analysis of the rings.

Also, stability cannot be achieved. For a disc to be stable, we must have $\Gamma < 2n(n+1)$ for all $n = 2, 3, ...$, and for a ring to be stable, we need $\Gamma > \Gamma_1$. These two requirements are not compatible, since from Figure 3, $\Gamma_1 > 12$ where 12 is the smallest $2n(n+1)$.

Later when $J(E_\varphi)$ is minimized, the leading order $J_1(R)$ becomes a sum over rings plus a sum over discs:

$$J_1(R) = \sum_{k \in I_r} \frac{\Gamma\pi}{4}(R_{k,2}^2 - R_{k,1}^2)^2 + \sum_{k \in I_d} \frac{\Gamma\pi}{4} R_k^4. \quad (8.5)$$

Because of the convexity of the square function and Jessen's inequality, $S_k^2 \to 1$ as $a \to 0$. For the rings one still has $S_{k,2}^2 - S_{k,1}^2 \to 1$. This is the equal area observation. Asymptotically the rings and the discs have the same area in a coexistence solution.

9 Discussion

A solution of the geometric problem considered in this paper is a subset of a two-dimensional domain. It satisfies an equation that involves the curvature of its boundary and a quantity that depends globally on the subset. This problem serves as a reduced problem for both the Gierer-Meinhardt theory for morphogenesis and the Ohta-Kawasaki theory for block copolymers.

We constructed two types of solutions. A solution of the first type consists of small rings of nearly the same size, and the locations of the rings are determined by minimizing a function derived from the Green's function of $-\Delta$ with the Neumann boundary condition on the domain of the problem. Two threshold numbers, Γ_0 and Γ_1, were found. The solution exists if Γ, a scaled version of γ which is one of the two parameters of the problem, is in $(\Gamma_0, \Gamma_1) \cup (\Gamma_1, \infty)$, and a, the second parameter of the problem, is sufficiently small. Moreover, the solution is stable if $\Gamma \in (\Gamma_1, \infty)$ and unstable if $\Gamma \in (\Gamma_0, \Gamma_1)$.

A solution of the second type is a mix of small rings and small discs. The rings and the discs have approximately the same area; the rings have nearly the same inner and outer radii, and the discs have nearly the same radii. Here Γ must be in $(\Gamma_0, \infty)\backslash\{\Gamma_1, 2n(n+1) : \ n = 2, 3, 4, ...\}$, and a is again small. Any solution of this type is unstable.

We may adopt the viewpoint that a single ring is a building block of a multiple ring solution. The multiple ring solution is obtained by properly placing a number of small rings in the domain of the problem. If we focus on one ring and enlarge it to a size of order 1, then up to a small correction term, the ring is a solution of a geometric problem on the entire plane R^2. This problem looks for a set $E \subset R^2$ such that $|E| = \pi$ and on the boundary of E the equation

$$H(\partial E) + \Gamma N(E) = Const. \tag{9.1}$$

holds. On the right side of (9.1) *Const.* is a Lagrange multiplier corresponding to the constraint $|E| = \pi$, and on the left side $N(E)$ is the Newtonian potential of E given by

$$N(E)(x) = \frac{1}{2\pi} \int_E \log \frac{1}{|x-y|}\, dy. \tag{9.2}$$

The question is whether (9.1) admits a solution of the form

$$E = \{x \in R^2 : T_1 \le |x| \le T_2\}, \tag{9.3}$$

where T_1 and T_2 depend on Γ. Although not explicitly stated, it was essentially proved in this paper that there is indeed a solution to (9.1) of the form (9.3), provided $\Gamma > \Gamma_0$. Here Γ_0, and the inner and the outer radii T_1 and T_2 of the solution are given in Section 2. Moreover, the spectral property of this ring solution is also known. The linearized operator at the ring solution is invertible up to translation invariance, if and only if $\Gamma \neq \Gamma_1$ (where Γ_1 is also given in Section 2). Modulo translation, all eigenvalues of the linearized operator are positive if $\Gamma > \Gamma_1$. If $\Gamma \in (\Gamma_0, \Gamma_1)$, the linearized problem has exactly one negative eigenvalue, and the rest are all positive. The eigenvalues of the linear operator are the eigenvalues of the matrices

$$\begin{bmatrix} \frac{n^2-1}{4T_1^3} + \frac{\Gamma}{8n} & -\frac{\Gamma}{8n}(\frac{T_1}{T_2})^n \\ & \\ -\frac{\Gamma}{8n}(\frac{T_1}{T_2})^n & \frac{n^2-1}{4T_2^3} + \frac{\Gamma}{8}[\frac{1}{n} - 1 + (\frac{T_1}{T_2})^2] \end{bmatrix}, \quad n = 2, 3, 4, ... \tag{9.4}$$

which are analyzed in Section 5.

Once the existence and the spectral property of the ring solution of (9.1) are known, Theorem 2.1 uses it as an ansatz to build a solution of (1.2) with multiple rings. The ring solution of (9.1) is shrunk so that its area becomes something close to $\frac{a|D|}{K}$. The ring solution of (9.1) is a perfect ring, i.e., the inner and outer boundaries are exact circles; a ring in a solution of Theorem 2.1 is not perfect: its inner and outer boundaries are perturbed circles.

A natural question is whether the results in this paper also hold in three dimensions. For this purpose one must study the problem (9.1) in R^3 and see if it admits a shell-shaped solution of the form

$$E = \{x \in R^3 : T_1 \le |x| \le T_2\} \tag{9.5}$$

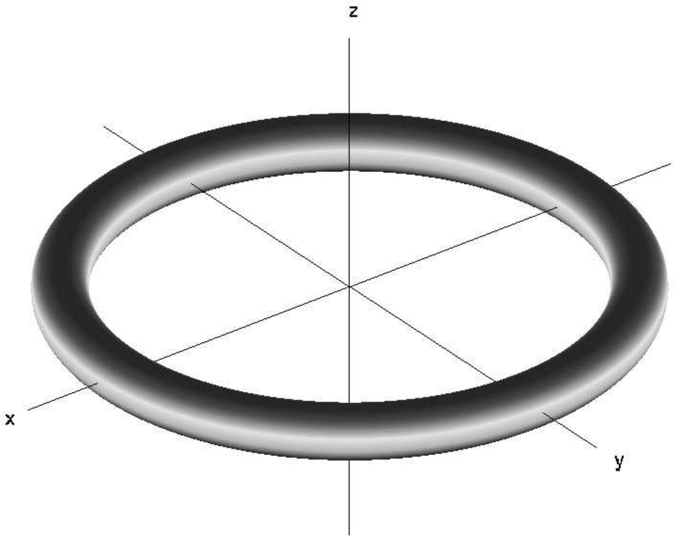

Figure 5: A toroidal-tube-like solution of (9.1) in R^3.

for some T_1, T_2, depending on Γ. In three dimensions $H(\partial E)$ is the mean curvature of the surface ∂E, and $N(E)$ is the Newtonian potential in R^3:

$$N(E)(x) = \frac{1}{4\pi} \int_E \frac{1}{|x-y|}\, dy. \tag{9.6}$$

This was done in [12]. There exists $\Gamma_0' > 0$ such that if $\Gamma > \Gamma_0'$, there exists a shell solution of the form (9.5), with T_1 and T_2 depending on Γ. However, the spectral property of this shell solution is very different. There exists a sequence $\{\Gamma_n' : n = 1, 2, 3, ...\}$ such that the linearized operator at the shell solution is invertible, modulo translation, if and only if $\Gamma \neq \Gamma_n'$, $n = 1, 2, 3, ...$ Moreover, for any $\Gamma > \Gamma_0'$, the linearized operator has at least one negative eigenvalue.

The three-dimensional analogy of Theorem 2.1 reads that a multiple shell solution exists on a bounded domain in R^3 if $\Gamma > \Gamma_0$, $\Gamma \neq \Gamma_n'$ where $n = 1, 2, 3, ...$, and a is small. In contrast to the two-dimensional case, this multiple shell solution is always unstable.

The lesson here is that before constructing a solution of (1.2) on a bounded domain with multiple small components, we should study (9.1) on the whole space. A solution on the whole space may be used as a building block to construct a solution of (1.2) with multiple components on a bounded domain.

This idea was first used in [21], where solutions of (1.2) with multiple small discs were found. The building block of such a solution is the unit disc $\{x \in R^2 : |x| < 1\}$ which is obviously a solution of (9.1). There also exist solutions of multiple small balls in a three-dimensional domain [23].

In addition to discs, rings, balls, and shells, Ren and Wei [25] recently found an interesting solution of (9.1) in R^3 which has the shape of a toroidal tube (see Figure 5).

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