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The Pauli algebra approach to relativity.

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THE PAULI ALGEBRA APPROACH TO RELATIVITY

By

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Abstract

The Pauli algebra for any Minkowski vector space is constructed and applied to the study of isometries of a Minkowski vector space, where it is shown that a subset of the Pauli algebra is the universal covering group of the restricted Lorentz group. The Pauli algebra theory of spacetime connections and curvature is developed and used to calculate the connection and curvature for spherically symmetric spacetimes. Spinors for Minkowski spacetime are shown to reside naturally in the Pauli algebra and other geometrical objects are built up from these spinors.

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1 Introduction

The Pauli algebra approach to relativity uses an algebra constructed from and including 4-vectors to elegantly formulate concepts and applications in relativity. It originated (Baylis 1980 and references therein) with the recognition that the set of all 2X2 hermitian complex matrices can be associated with Minkowski spacetime: the square of the Minkowski norm of any 4-vector is just the determinant of the corresponding matrix and Lorentz transformations are effected by using unimodular 2X2 matrices. When this method is studied in detail it becomes apparent that the matrices themselves are not really needed: the algebraic multiplication properties of the matrices are all that is required. Thus, the Pauli algebra approach takes as its basic structure an abstract algebra of 4-vectors having these properties.

The goals of this dissertation are: to construct the Pauli algebra from the vector space of 4-vectors in a fairly rigorous mathematical manner; to show how fundamental concepts in special relativity are implemented in this algebraic approach; and to extend this approach to the domain of general relativity. In other words this dissertation hopes to show that the Pauli algebra is applicable to a wide variety of problems in relativity, from kinematical concepts in special relativity to Einstein's equations in general relativity. It is also hoped that the efficiency and ease of use of the Pauli algebra is amply demonstrated here.

The Pauli algebra deals in a fundamental way with observers in spacetime, in fact it can't even be constructed without taking into account an observer. This might lead one to believe that the Pauli algebra formulation of relativity is not covariant, but this is erroneous because the observer used is initially left arbitrary. The notion of an observer is not adequately dealt with in most elementary treatments of general relativity. It is only advanced expositions on general relativity such as De Felice and Clarke (1990), that present detailed consideration of observers. Therefore, the role of observers in the Pauli algebra is actually a benefit, not a detriment.

The methods developed here have some similarities with other work. For a treatment of the connection and curvature using the Clifford algebra for spacetime, see Hestenes (1966), Hestenes (1986), and Hestenes and Sobczyk (1985). For a presentation using differential forms and the Clifford algebra of differential forms, see Benn and Tucker (1988). For an elementary but systematic treatment of the connection and curvature using differential forms, see Hughston and Tod (1990). Finally, the work with which this dissertation has the most in common is Rastall (1964), which uses the complex quaternions to study relativity. The Pauli algebra is isomorphic to the algebra of complex quaternions, but the construction of the Pauli algebra developed in this thesis is far different than Rastall's approach using quadruples of complex numbers. Also Rastall stops at the connection and does not consider curvature and Einstein's equations.

2 The Pauli Algebra of a Minkowski Vector Space

2.1 Minkowski Vector Spaces

A Minkowski vector space (Geroch 1985) V is a four dimensional real vector space along with a symmetric non-degenerate bilinear form g (i.e. an element of $V^* \otimes V^*$) that satisfies the conditions 1) and 2) below. A vector $v \in V$ is called timelike, lightlike, or spacelike if $g(v, v)$ is positive, zero, or negative respectively. For each Minkowski vector space:

1) timelike vectors exist;

2) v timelike, $g(v, w) = 0 \;\Rightarrow\; w$ spacelike.

The elements of a Minkowski vector space are called 4-vectors. These conditions are just a basis independent way of saying that there exists a basis $\{e_0, e_1, e_2, e_3\}$ of V with

$$1 = g(e_0, e_0) = -g(e_1, e_1) = -g(e_2, e_2) = -g(e_3, e_3). \tag{2.1}$$

Such a basis is termed orthonormal.

Let u, v be timelike 4-vectors and write $u \sim v$ if $g(u, v) > 0$. Assume, for now, that $\sim$ is an equivalence relation on the set of all timelike 4-vectors. Select an arbitrary timelike 4-vector w and note that w and $-w$ are in different equivalence classes. Any other timelike 4-vector v is either equivalent to w or $-w$, and so there are exactly two equivalence classes of timelike 4-vectors. Because there are only two equivalence classes, clearly

$$sgn[g(u, w)] = sgn[g(u, v)]sgn[g(v, w)] \tag{2.2}$$

for all timelike u, v, and w. Choosing the 4-vectors in one of these classes to be future pointing and the 4-vectors in the other class to be past pointing is called a time orientation for V.

It remains to verify that $\sim$ is an equivalence relation. Let u, v, and w be timelike 4-vectors. Trivially, $\sim$ is reflexive and symmetric, that is $u \sim u$, and $u \sim v \Rightarrow v \sim u$. To show that it is also transitive assume that $u \sim w$ and $w \sim v$, and let $r = g(w, v)u - g(u, w)v$. From this,

$$g(w, r) = 0$$
$$\Rightarrow g(r, r) < 0,$$

because w is timelike. But

$$g(r, r) = g(w, v)^2 g(u, u) - 2g(w, v)g(u, w)g(u, v) + g(u, w)^2 g(v, v) \tag{2.3}$$

and hence $g(u, v) \leq 0$ would imply $g(r, r) > 0$, a contradiction. Therefore $g(u, v) > 0$ and $u \sim v$.

Two other types of orientation can be defined on a Minkowski vector space. An equivalence relation can be defined on the set of all bases for V by saying that two bases are equivalent if the determinant of the change of basis transformation is positive. A spacetime orientation is a choice of one of the two resulting equivalence classes as the proper bases for V. Lastly, a spatial orientation results by calling a tetrad right handed if: the tetrad is proper, and its timelike element is future pointing; or the tetrad is improper, and its timelike element is past pointing.

As will be seen below, the tangent space at each point of any spacetime manifold is a Minkowski vector space, and every observer is represented by a curve whose tangent vector is a future directed, timelike unit 4-vector field. Thus, in a Minkowski vector space, future directed, unit timelike 4-vectors have particular physical significance. Fix a future directed, unit timelike 4-vector, e_0 and use this 4-vector to decompose V into

$$V = T \oplus S \tag{2.4}$$

where V is the subspace of 4-vectors proportional to $\mathbf{e}_0$ and S is the subspace of 4-vectors $\mathbf{g}$ orthogonal to $\mathbf{e}_0$. Denote the vectors in S by an $\bar{\ }$ and call them spatial vectors. Which 4-vectors are spatial clearly depends on the $\mathbf{e}_0$ used to effect the spacetime split; all spatial vectors are necessarily spacelike, but not all spacelike vectors are spatial. Hence for any 4-vector $\mathbf{v}$

$$\mathbf{v} = v^0 \mathbf{e}_0 + \bar{v}. \qquad (2.5)$$

The "dot product" on S is defined as the negative of $\mathbf{g}$ restricted to S.

If $\{\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is a basis for V then the set of linear functions on V $\{\omega^0, \omega^1, \omega^2, \omega^3\}$ defined by

$$\omega^\mu(\mathbf{e}_\nu) := \delta^\mu_\nu \qquad (2.6)$$

is a basis for V^*, the algebraic dual of V. An isomorphism can be established between V and V^* by identifying each ω^μ and $\mathbf{e}_\mu$, but this isomorphism is not natural since it is basis dependent. The metric gives rise to a natural (basis independent) isomorphism $\tilde{\ }$ between V and its algebraic dual defined by

$$\tilde{\mathbf{v}}(\mathbf{w}) = \mathbf{g}(\mathbf{v}, \mathbf{w}) \qquad (2.7)$$

for every $\mathbf{v}, \mathbf{w} \in V$. The inverse of this isomorphism is also denoted by $\tilde{\ }$, so context is used to indicate which is being used. Applying this to the dual basis gives a new basis, $\{\mathbf{e}^0 = \tilde{\omega}^0, \mathbf{e}^1 = \tilde{\omega}^1, \mathbf{e}^2 = \tilde{\omega}^2, \mathbf{e}^3 = \tilde{\omega}^3\}$, for V. Each element of this new basis can be expressed as a linear combination $\mathbf{e}^\mu = g^{\mu\nu}\mathbf{e}_\nu$ of the original basis elements. The coefficients $g^{\mu\nu}$ are related to the coffecients $g_{\mu\nu} := \mathbf{g}(\mathbf{e}_\mu, \mathbf{e}_\nu)$ by

$$\begin{aligned} \delta^\mu_\nu &= \omega^\mu(\mathbf{e}_\nu) \\ &= \tilde{\mathbf{e}}^\mu(\mathbf{e}_\nu) \\ &= g^{\mu\alpha} g_{\alpha\nu}. \end{aligned} \qquad (2.8)$$

If $\mathbf{v} = v^\mu \mathbf{e}_\mu$ and $\tilde{\mathbf{v}} = v_\mu \omega^\mu$ then

$$\begin{aligned} \tilde{\mathbf{v}} &= v^\mu \tilde{\mathbf{e}}_\mu \\ &= v^\mu g_{\mu\nu} \tilde{\mathbf{e}}^\nu \end{aligned} \qquad (2.9)$$

Thus, $\tilde{\iota}_\nu = \iota^\mu g_{\mu\nu}$, and in index notation $\tilde{}$ corresponds to index lowering (of components) and its inverse corresponds to index raising (of components).

2.2 The Pauli Algebra

It is well known (Baylis 1980 and references therein) that 2X2 complex matrices can be used to represent 4-vectors, Lorentz transformations, and the electromagnetic field tensor. If an abstract algebra $\mathcal{P}$ is constructed that contains these objects, and that has a faithful irreducible 2X2 matrix representation, then the need for the matrices is eliminated and attention can be focused on the properties of the algebra. This dissertation does this, and shows also that this algebra is quite useful in the study of general relativity.

Let V be a Minkowski vector space. The properties (2.5) and (2.7) of a Minkowski vector space hint at the desired properties for the abstract algebra referred to above. Specifically the algebra $\mathcal{P}$ is required to:

1) contain V (i.e., there is an injection from V into $\mathcal{P}$);

2) contain V^*, the dual space of V;

3) contain elements of T as scalars;

4) have $\tilde{v}\,v = \tilde{v}(v)$ for every $v \in V$.

$\mathcal{P}$ is now constructed as follows: find the "freest" algebra that contains both V and V^*, use 3) and 4) to generate an ideal of that algebra, and then form the quotient algebra. The freest algebra containing V and V^* is the full tensor algebra for spacetime

$$\mathcal{T} = \mathbf{R} \oplus V \oplus V^* \oplus (V \oplus V^*) \otimes (V \oplus V^*) \oplus \cdots \tag{2.10}$$

and the ideal generated by 3) and 4) is

$$I = \{ \textit{linear combinations of } \mathbf{t}_1 \otimes \mathbf{d} \otimes \mathbf{t}_2 \mid \mathbf{t}_1, \mathbf{t}_2 \in \mathcal{T} \textit{ and } \mathbf{d} \in D \} \tag{2.11}$$

where

$$D = \{ \tilde{v} \otimes v - g(v, v) \mid v \in V \} \cup \{ \mathbf{e}_0 - 1 \}. \tag{2.12}$$

Thus

$$\mathcal{P} = \mathcal{T} / I . \tag{2.13}$$

The canonical projection π from $\mathcal{T}$ onto $\mathcal{P}$

$$\pi(t) = t + I \tag{2.14}$$

has I as its kernel. Since $I \cap V = \{0\}$ and $I \cap V^* = \{0\}$, π is injective both when restricted to V and when restricted to V^*. Therefore no distinction is made between a 4-vector in V and its image in $\mathcal{P}$. If $\bar{v} \in V^*$ then its image in $\mathcal{P}$ is denoted $\overline{v}$.

2.3 Properties of the Pauli Algebra

Suppose $v, w \in V$, $\{e_0, e_1, e_2, e_3\}$ is a proper orthonormal basis for V, and e_0 is used to split V into space and time. Thus $\{e_1, e_2, e_3\}$ is a basis for S. As is shown below, $\mathcal{P}$ has the following properties:

1) $e_0 = 1$;
2) $\overline{v} v = g(v, v)$;
3) $v = v^0 e_0 + \vec{v} \Rightarrow \overline{v} = v^0 e_0 - \vec{v}$;
4) $v \overline{v} = \overline{v} v$;
5) $i = e_0 \overline{e_1} e_2 \overline{e_3} = e_1 e_2 e_3$ is independent of proper orthonormal basis;
6) $i^2 = -1$ and i is in the centre of $\mathcal{P}$;
7) $\vec{v}\vec{w} = \vec{v} \cdot \vec{w} + i \vec{v} \times \vec{w}$.

Properties 1) and 2) follow immediately from the definition of $\mathcal{P}$. From 2)

$$\begin{aligned} g(v, w) &= \frac{1}{2}(g((v+w),(v+w)) - g(v,v) - g(w,w)) \\ &= \frac{1}{2}((\overline{v+w})(v+w) - \overline{v}v - \overline{w}w) \\ &= \frac{1}{2}(\overline{v}w + \overline{w}v). \end{aligned} \tag{2.15}$$

Property 3) follows from the fact that

$$
\begin{aligned}
1 &= g(\mathbf{e}_0, \mathbf{e}_0) \\
&= \overline{\mathbf{e}_0}\,\mathbf{e}_0 \\
&= \overline{\mathbf{e}_0}
\end{aligned}
\tag{2.16}
$$

and

$$
\begin{aligned}
0 &= g(\mathbf{e}_0 + \mathbf{e}_i, \mathbf{e}_0 + \mathbf{e}_i) \\
&= (\overline{\mathbf{e}_0 + \mathbf{e}_i})(\mathbf{e}_0 + \mathbf{e}_i) \\
&= \overline{\mathbf{e}_0}\,\mathbf{e}_0 + \overline{\mathbf{e}_i}\,\mathbf{e}_0 + \overline{\mathbf{e}_0}\,\mathbf{e}_i + \overline{\mathbf{e}_i}\,\mathbf{e}_i \\
&= \overline{\mathbf{e}_i} + \mathbf{e}_i .
\end{aligned}
\tag{2.17}
$$

Since

$$
\begin{aligned}
\mathbf{v}\overline{\mathbf{v}} &= (v^0 + \vec{v})(v^0 - \vec{v}) \\
&= (\overline{v^0 - \vec{v}})(v^0 - \vec{v}) \\
&= g(v^0 \mathbf{e}_0 - \vec{v}, v^0 \mathbf{e}_0 - \vec{v}) \\
&= g(v^0 \mathbf{e}_0 + \vec{v}, v^0 \mathbf{e}_0 + \vec{v}) \\
&= \overline{\mathbf{v}}\mathbf{v} .
\end{aligned}
\tag{2.18}
$$

property 4) is true in the Pauli algebra.

Suppose $\{\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $\{\mathbf{e}'_0, \mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ are two proper orthonormal bases for V. Then there is a unique restricted Lorentz transformation, L, such that $\mathbf{e}'_\mu = L(\mathbf{e}_\mu)$. If $\mathbf{i} = \mathbf{e}_0 \overline{\mathbf{e}_1} \mathbf{e}_2 \overline{\mathbf{e}_3}$ and $\mathbf{i}' = \mathbf{e}'_0 \overline{\mathbf{e}'_1} \mathbf{e}'_2 \overline{\mathbf{e}'_3}$, then because $\mathbf{e}_0 \overline{\mathbf{e}_1} \mathbf{e}_2 \overline{\mathbf{e}_3}$ is antisymmetric under the interchange of any two indices, $\mathbf{i}' = det(L)\mathbf{i}$. But any linear transformation relating two proper has a determinant of plus one. Thus $\mathbf{i}' = \mathbf{i}$ and property 5) is demonstated.

Let $\mathbf{i} = \mathbf{e}_0 \overline{\mathbf{e}_1} \mathbf{e}_2 \overline{\mathbf{e}_3}$. Then

$$
\begin{aligned}
i^2 &= e_0\overline{e_1}e_2\overline{e_3}e_0\overline{e_1}e_2\overline{e_3} \\
&= (-1)^3 e_0\overline{e_0}e_1\overline{e_2}e_3\overline{e_1}e_2\overline{e_3} \\
&= (-1)^5 e_0\overline{e_0}e_1\overline{e_1}e_2\overline{e_3}e_2\overline{e_3} \\
&= (-1)^6 e_0\overline{e_0}e_1\overline{e_1}e_2\overline{e_2}e_3\overline{e_3} \\
&= -1 .
\end{aligned}
\tag{2.19}
$$

Because $i e_\mu = e_\mu i$, i is in the centre of $\mathcal{P}$. Thus property 6) holds in $\mathcal{P}$ and will be used to define a complex structure on $\mathcal{P}$.

To establish 7) it suffices to show that $\vec{v}$ proportional to $\vec{w}$ implies

$$\vec{v}\vec{w} = \vec{v}\cdot\vec{w}$$

and $\vec{v}$ orthogonal to $\vec{w}$ gives

$$\vec{v}\vec{w} = i\,\vec{v}\times\vec{w}.$$

Using properties 2) and 3),

$$
\begin{aligned}
\vec{v}\vec{v} &= -\overline{\vec{v}}\vec{v} \\
&= -g(\vec{v},\vec{v}) \\
&= \vec{v}\cdot\vec{v}.
\end{aligned}
\tag{2.20}
$$

Let $\vec{v}$ and $\vec{w}$ be orthogonal unit vectors. Hence $\{e_0, \vec{v}, \vec{w}, \vec{v}\times\vec{w}\}$ is a proper orthonormal basis for V. Therefore

$$\vec{v}\vec{w}(\vec{v}\times\vec{w}) = i \Rightarrow \vec{v}\vec{w} = i\,\vec{v}\times\vec{w}$$

and $\vec{v}\vec{w} = -\vec{w}\vec{v}$.

2.4 Isomorphism of $\mathcal{P}$ with the Algebra of the Pauli Spin Matrices

The algebra, $\mathcal{P}'$, generated by the Pauli spin matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

is a four dimensional complex vector space having $\{1, \sigma_1, \sigma_2, \sigma_3\}$ as a basis. It is easily verified that the Pauli spin matrices obey

$$\sigma_i \sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k . \qquad (1.1)$$

Any complex vector space V can be considered to be a real vector space (Geroch 1985) by "forgetting" that for $v \in V$, v and iv are linearly dependent. Thus $\mathcal{P}'$ is also a real vector space with

$$B_{\mathcal{P}'} = \{1, \sigma_1, \sigma_2, \sigma_3, \sigma_1\sigma_2 = i\sigma_3, \sigma_1\sigma_3 = -i\sigma_2, \sigma_2\sigma_3 = i\sigma_1, \sigma_1\sigma_2\sigma_3 = i\} \qquad (2.21)$$

as a basis.

The proof that the Pauli algebra of a Minkowski vector space and the algebra of the Pauli spin matrices are isomorphic is presented in two stages. First, using a homomorphism β, from $\mathcal{T}$ onto $\mathcal{P}'$, a homomorphism α, from $\mathcal{P}$ onto $\mathcal{P}'$ is established. This homomorphism is then utilized to find a linearly independent subset of $\mathcal{P}$ which has the same cardinality as a basis for $\mathcal{P}'$. Next, this set is shown to span $\mathcal{P}$. This demonstrates that $\mathcal{P}$ and $\mathcal{P}'$ have the same dimension, which together with the fact that α is onto gives that α is an isomorphism.

Define the homomorphism $\beta : \mathcal{T} \to \mathcal{P}'$ by:

1) β is linear;
2) $\beta(1) = \beta(\mathbf{e}_0) = \beta(\bar{\mathbf{e}}_0) = 1$;
3) $\beta(\mathbf{e}_i) = -\beta(\bar{\mathbf{e}}_i) = \sigma_i$;
4) $\beta(\mathbf{t}_1 \otimes \mathbf{t}_2) = \beta(\mathbf{t}_1)\beta(\mathbf{t}_2)$.

Now

$$\beta(\mathbf{e}_0 - 1) = 0 \qquad (2.22)$$

and

$$\begin{aligned} \beta(\bar{\mathbf{v}} \otimes \mathbf{v} - g(\mathbf{v}, \mathbf{v})) &= \beta(\bar{\mathbf{v}})\beta(\mathbf{v}) - g(\mathbf{v}, \mathbf{v}) \qquad (2.23) \\ &= (v^0)^2 - \beta(\vec{v})\beta(\vec{v}) - g(\mathbf{v}, \mathbf{v}) \\ &= 0. \end{aligned}$$

Since these elements generate I, $I \subset ker(\beta)$.

Define $\alpha : \mathcal{P} = \mathcal{T}/I \to \mathcal{P}'$ by

$$\alpha(t + I) = \beta(t). \tag{2.24}$$

It must be checked that the action of α is independent of the element of $\mathcal{T}$ used to pick out a particular coset and that α is a homomorphism. Therefore, assume $t_1 + I = t_2 + I$;

$$\Rightarrow t_1 - t_2 \in I \subset ker(\beta)$$
$$\Rightarrow \beta(t_1) = \beta(t_2)$$
$$\Rightarrow \alpha(t_1 + I) = \alpha(t_2 + I).$$

Also

$$\begin{aligned} \alpha((t_1 + I)(t_2 + I)) &= \alpha(t_1 \otimes t_2 + I) \\ &= \beta(t_1 \otimes t_2) \\ &= \beta(t_1)\beta(t_2) \\ &= \alpha(t_1 + I)\alpha(t_2 + I). \end{aligned} \tag{2.25}$$

Thus α is a well defined homomorphism.

Let

$$B_{\mathcal{P}} = \{1, e_1, e_2, e_3, e_1e_2, e_1e_3, e_2e_3, e_1e_2e_3\} \tag{2.26}$$

and assume

$$0 = \sum_A c_A e_A; \quad c_A \in \mathbb{R}, \quad e_A \in B_{\mathcal{P}} \quad distinct. \tag{2.27}$$

$$\begin{aligned} \Rightarrow 0 &= \alpha\Big(\sum_A c_A e_A\Big) \\ &= \sum_A c_A \sigma_A \end{aligned} \tag{2.28}$$

Therefore, since the σ_A are linearly independent, the e_A are as well.

Because $\{1, e_0, e_1, e_2, e_3, \bar{e}_0, \bar{e}_1, \bar{e}_2, \bar{e}_3\}$ generates $\mathcal{T}$, $\{1, e_0, e_1, e_2, e_3, \bar{e}_0, \bar{e}_1, \bar{e}_2, \bar{e}_3\}$ generates $\mathcal{P}$. But $1 = e_0 = \bar{e}_0 = e_i^2$ and $\bar{e}_i = -e_i$.

Thus $\{e_1, e_2, e_3\}$ generates $\mathcal{P}$. Since e_i commutes with itself and anticommutes with e_j for $i \neq j$, the e_i's in any product of the e_i's can be arbitrarily permuted at the expense of at most a factor of -1. Hence any product of the e_i's can be written

$$e_{i_1} e_{i_2} \dots e_{i_p} = (-1)^k e_1^{l_1} e_2^{l_2} e_3^{l_3} \tag{2.29}$$

$$= (-1)^k e_1^{l_1 (mod 2)} e_2^{l_2 (mod 2)} e_3^{l_3 (mod 2)}$$

where l_i is the number of times each e_i occurs in the product and $k \in \{0, 1\}$. The last equality follows from the property that $e_i^2 = 1$. Therefore any product of the e_i's can be expressed as a product of three or fewer e_i's with the subscripts increasing from left to right, $B_{\mathcal{P}}$ spans $\mathcal{P}$, and thus $\mathcal{P}$ and $\mathcal{P}'$ are isomorphic. Because such an isomorphism exists, $\mathcal{P}$ is called the Pauli algebra.

Let $\mathcal{P}_0 = span\{1\}$, $\mathcal{P}_1 = span\{e_1, e_2, e_3\}$, $\mathcal{P}_2 = span\{e_1 e_2, e_1 e_3, e_2 e_3\}$, and $\mathcal{P}_3 = span\{e_1 e_2 e_3\}$. Then, from the above,

$$\mathcal{P} = \mathcal{P}_0 \oplus \mathcal{P}_1 \oplus \mathcal{P}_2 \oplus \mathcal{P}_3. \tag{2.30}$$

The elements of $\mathcal{P}_0$, $\mathcal{P}_1$, $\mathcal{P}_2$, and $\mathcal{P}_3$ are called scalars, vectors, pseudovectors, and pseudoscalars respectively.

2.5 The Complex Structure and the Main Involutions of $\mathcal{P}$

The real Pauli algebra possesses a complex structure (Geroch 1985) i given by

$$\mathrm{i}(p) = \mathbf{i} p \tag{2.31}$$

for every p in $\mathcal{P}$. This allows the Pauli algebra to be viewed as a complex vector space by defining the product of a complex scalar $a + ib$ with the Pauli algebra element p as

$$(a + ib) p = a p + b \mathrm{i}(p), \tag{2.32}$$

i.e., replace $\mathbf{i}$ with the imaginary scalar i. Thus a complex basis for $\mathcal{P}$ is $\{1, e_1, e_2, e_3\}$ and an arbitrary element of the Pauli algebra is

$$p = p_0 + \overline{p} \tag{2.33}$$

with p_0 and $\vec{p}$ both complex.

There are two important involutions on the Pauli algebra: hermitian conjugation, "dagger"; and spatial reversal , "bar". Under hermitian conjugation elements of $\mathcal{P}_0 \oplus \mathcal{P}_1$ remain unchanged while elements of $\mathcal{P}_2 \oplus \mathcal{P}_3$ change sign. Elements of the "dagger " invariant subspace $\mathcal{P}_0 \oplus \mathcal{P}_1$ are called real and is identified with the space of 4-vectors, or the space of covectors depending on the context. Two other useful operations on $\mathcal{P}$ are defined from "dagger";the real part of $p \in \mathcal{P}$ is

$$Re\{p\} = \frac{1}{2}(p + p^{\dagger}) \tag{2.34}$$

and

$$Im\{p\} = \frac{1}{2i}(p - p^{\dagger}) \tag{2.35}$$

is the imaginary part of p. Under spatial reversal p goes to

$$\overline{p} = p_0 - \vec{p}. \tag{2.36}$$

These involutions are both anti-homomorphisms (when $\mathcal{P}$ is considered a real vector space), i.e., they reverse the order of algebraic products, and they commute with each other. The action of "bar" on a 4-vector is to take it to the corresponding dual vector, hence dual vectors are (usually) denoted by "bars" instead of "tildes". The compostition of "bar" with "dagger" is the grade involution for $\mathcal{P}$. This involution is the identity on $\mathcal{P}_0 \oplus \mathcal{P}_2$ (even elements of $\mathcal{P}$) and minus the identity on $\mathcal{P}_1 \oplus \mathcal{P}_3$ (odd elements of $\mathcal{P}$).

Two other products, the dot product and the wedge product, are defined between Pauli algebra elements using "bar" and the algebraic product: the dot product between p and q is

$$p \cdot q := \frac{1}{2}(pq + \overline{pq}); \tag{2.37}$$

and wedge product between p and q is

$$p \cdot q := \frac{1}{2}(pq - \overline{pq}). \tag{2.38}$$

This dot product, when restricted to spatial vectors, corresponds to the dot product defined above. The Lorentz "inner product" of two 4-vectors is given by

$$g(\mathbf{u}, \mathbf{v}) = \mathbf{u} \cdot \bar{\mathbf{v}} \tag{2.39}$$

and the scalar part of any Pauli element is given by $p \cdot 1$.

2.6 Valence Two Antisymmetric Tensors and the Pauli Algebra

A valence two (element of $V \otimes V$) tensor F is said to be antisymmetric if

$$F(\bar{u}, \bar{v}) = -F(\bar{v}, \bar{u}) \tag{2.40}$$

for every $\mathbf{u}$, $\mathbf{v}$ in V. Clearly the set of all valence two antisymmetric tensors is a subspace of $V \otimes V$. This subspace of $V \otimes V$ can be identified with a subspace Λ of $V \otimes V^*$ by using the metric induced map

$$\mathbf{u} \otimes \mathbf{v} \to \mathbf{u} \otimes \bar{\mathbf{v}}. \tag{2.41}$$

The space Λ is in turn mapped into the Pauli algebra by π (2.14). Assume $\{\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is an orthonormal basis for V and

$$\mathbf{F} = F^{\mu\nu}\mathbf{e}_\mu \otimes \bar{\mathbf{e}}_\nu \tag{2.42}$$

is in Λ, that is $F^{\mu\nu} = -F^{\nu\mu}$. If

$$\begin{aligned} 0 &= \pi(\mathbf{F}) \\ &= F^{\mu\nu}\mathbf{e}_\mu\overline{\mathbf{e}_\nu} \\ &= 2F^{\mu\nu}\mathbf{e}_\mu\overline{\mathbf{e}_\nu}, \quad \mu < \nu \end{aligned} \tag{2.43}$$

then, since $\{\mathbf{e}_\mu\overline{\mathbf{e}_\nu} \mid \mu < \nu\}$ is a basis for $\mathcal{P}_1 \oplus \mathcal{P}_2$, every

$$F^{\mu\nu} = 0 \tag{2.44}$$

and π restricted to Λ is a bijective map onto $\mathcal{P}_1 \oplus \mathcal{P}_2$. Elements of $\mathcal{P}_1 \oplus \mathcal{P}_2$ are called 6-vectors, under "bar" every 6-vector changes sign, and every 6-vector can be written in the form $\vec{a} + i\vec{b}$ where $\vec{a}$ and $\vec{b}$ are both real.

If $F = \pi(F)$ then for any 4-vector $\mathbf{u}$, $\mathbf{v}$ given by

$$\mathbf{v} = \frac{1}{2}(F\mathbf{u} + \mathbf{u}F^{*}) = \mathcal{R}e\{F\mathbf{u}\} \tag{2.45}$$

is real and hence a candidate for a 4-vector. To find out what $\mathbf{v}$ represents calculate

$$\begin{aligned} v^{\mu}\mathbf{e}_{\mu} &= \frac{1}{2}F^{\mu\nu}u^{\alpha}\mathcal{R}e\{\mathbf{e}_{\mu}\overline{\mathbf{e}_{\nu}}\mathbf{e}_{\alpha} - \mathbf{e}_{\nu}\overline{\mathbf{e}_{\mu}}\mathbf{e}_{\alpha}\} \\ &= \frac{1}{2}F^{\mu\nu}u^{\alpha}\mathcal{R}e\{2\eta_{\nu\alpha}\mathbf{e}_{\mu} - \mathbf{e}_{\mu}\overline{\mathbf{e}_{\alpha}}\mathbf{e}_{\nu} - \mathbf{e}_{\nu}\overline{\mathbf{e}_{\mu}}\mathbf{e}_{\alpha}\} \\ &= \frac{1}{2}F^{\mu\nu}u^{\alpha}\mathcal{R}e\{2\eta_{\nu\alpha}\mathbf{e}_{\mu} - 2\eta_{\mu\alpha}\mathbf{e}_{\mu} + \mathbf{e}_{\alpha}\overline{\mathbf{e}_{\mu}}\mathbf{e}_{\nu} - \mathbf{e}_{\nu}\overline{\mathbf{e}_{\mu}}\mathbf{e}_{\alpha}\} \\ &= 2F^{\mu\nu}u_{\nu}\mathbf{e}_{\mu}. \end{aligned} \tag{2.46}$$

Thus, $\mathbf{v}$ is twice the contraction of F with $\mathbf{u}$.

An identity that will be used on a number of occasions in what follows in this dissertation is

$$\overline{\mathbf{e}_{\alpha}}F\mathbf{e}^{\alpha} = 0 \tag{2.47}$$

for any 6-vector F. This is shown by the fact that for $\mu \neq \nu$,

$$\begin{aligned} \overline{\mathbf{e}_{\alpha}}\mathbf{e}_{\mu}\overline{\mathbf{e}_{\nu}}\mathbf{e}^{\alpha} &= (2\eta_{\alpha\mu} - \mathbf{e}_{\mu}\overline{\mathbf{e}_{\alpha}})(2\delta^{\alpha}_{\nu} - \mathbf{e}^{\alpha}\overline{\mathbf{e}_{\nu}}) \\ &= 4\eta_{\nu\mu} - 2\mathbf{e}_{\mu}\overline{\mathbf{e}_{\nu}} - 2\mathbf{e}_{\mu}\overline{\mathbf{e}_{\nu}} + 4\mathbf{e}_{\mu}\overline{\mathbf{e}_{\nu}} \\ &= 0. \end{aligned} \tag{2.48}$$

This implies that

$$\overline{\mathbf{e}_{\alpha}}p\mathbf{e}^{\alpha} = 4p \cdot 1 \tag{1.2}$$

for every $p \in \mathcal{P}$.

3 Lorentz Transformations on a Minkowski Vector Space

3.1 Restricted Lorentz Transformations

A Lorentz transformation on a Minkowski vector space, (V, g), is a (linear) isomorphism, L, that preserves g; i.e., for every v and w in V

$$g(v, w) = g(L(v), L(w)) \tag{3.1}$$

or in the Pauli algebra

$$\frac{1}{2}(\bar{v}w + \bar{w}v) = \frac{1}{2}(\overline{L(v)}L(w) + \overline{L(w)}L(v)). \tag{3.2}$$

Clearly, a Lorentz transformation acting on an (orthonormal) basis for V will give another (orthonormal) basis for V. Under composition the set of all Lorentz transformations forms a group, the Lorentz group $\mathcal{L}$. The Lorentz group contains more transformations than are needed in this work; the most physically important Lorentz transformations are the ones that preserve the orientations of V defined in the previous chapter.

Because the product of the determinant of two linear operators is the determinant of the product of the operators, a Lorentz transformation L either takes all proper bases to proper bases, and all improper bases to improper bases (preserves the orientation of V); or it takes all proper bases to improper bases, and all improper bases to proper ones (changes the orientation of V). Therefore, it is sufficient to know the action of L with respect to the orientation of any basis, to find its action on all bases. Define

$$\mathcal{L}_{+} \equiv \{L \in \mathcal{L} \mid L \ \textit{doesn't change spacetime orientation}\};$$

$\mathcal{L}_{+}$ is a subgroup of $\mathcal{L}$ called the proper Lorentz group.

Let v, w be arbitrary timelike vectors, and L be a Lorentz transformation. Combining (3.1) and (2.2) gives

$$sgn[g(v, L(v))]sgn[g(w, L(w))] = 1 \tag{3.3}$$

Thus either L changes the class of every timelike vector, or L preserves the class of every timelike vector: so the action of L on one timelike vector gives its action on all timelike vectors with respect to time orientation. Define

$$\mathcal{L}^{\uparrow} = \{L \in \mathcal{L} \mid L \text{ doesn't change time orientation}\}.$$

$\mathcal{L}^{\uparrow}$ is called the orthochronous Lorentz group.

Define the restricted Lorentz group, $\mathcal{L}_{+}^{\uparrow}$, as the group of Lorentz transformations that are both proper and orthochronous, i.e.

$$\mathcal{L}_{+}^{\uparrow} = \mathcal{L}_{+} \cap \mathcal{L}^{\uparrow}. \tag{3.4}$$

The physical significance of elements of the restricted Lorentz group is discussed in the next chapter.

3.2 Reflections and Restricted Lorentz transformations

Any non-lightlike vector u can be used to determine a reflection $\mathcal{R}_u$ through the hyperplane orthogonal to u

$$\mathcal{R}_u(v) = v - 2\frac{g(u,v)}{g(u,u)}u \quad \forall v \in V \tag{3.5}$$

$$= -\frac{u\bar{v}u}{\bar{u}u}.$$

If $\mathcal{R}_u$ is an arbitrary reflection then

$$g(\mathcal{R}_u(v), \mathcal{R}_u(v)) = \frac{\bar{u}v\bar{u}u\bar{v}u}{u\bar{u}\bar{u}u} \tag{3.6}$$

$$= v\bar{v}$$

$$= g(v,v).$$

Thus every reflection is a Lorentz transformation. In fact, any Lorentz transformation can be written as the product of four or fewer reflections (Harvey 1990).

Assume $\mathcal{R}_u$ is any reflection and choose an orthogonal basis for V that includes u. Then under $\mathcal{R}_u$, $u \to -u$, while the other members of the orthogonal basis are left unchanged. Therefore the determinant of every reflection is -1 and hence the spacetime orientation of every basis changes under a reflection. Thus the elements of $\mathcal{L}_-$ have to be the product of an even number of reflections.

Now consider the effect of a reflection $\mathcal{R}_u$ on the time orientation of V. If e_0 is used for the spacetime decomposition (2.4) then

$$g(e_0, \mathcal{R}_u(e_0)) = -\frac{1}{2\bar{u}u}(uu + \bar{u}\bar{u}) \tag{3.7}$$

$$= -\frac{1}{\bar{u}u}(u_0^2 + \vec{u}\cdot\vec{u})$$

Thus if u is spacelike, $\mathcal{R}_u$ doesn't alter the time orientation, but if u is timelike, $\mathcal{R}_u$ changes the time orientation. Hence every $L \in \mathcal{L}^{\uparrow}$ when expressed as the product of reflections has to contain an even number of (or zero) timelike reflections.

From the above, it follows that every restricted Lorentz transformation L is the compositon of an even number of reflections, an even number of these being timelike. Thus for any Lorentz transformation L there exist unit[1] 4-vectors u_1, u_2, u_3, and u_4 such that

$$L(v) = \mathcal{R}_{u_4}(\mathcal{R}_{u_3}(\mathcal{R}_{u_2}(\mathcal{R}_{u_1}(v)))) \tag{3.8}$$

$$= u_4\bar{u}_3u_2\bar{u}_1 v \bar{u}_1 u_2 \bar{u}_3 u_4$$

$$= LvL^{*};\quad L = u_4\bar{u}_3u_2\bar{u}_1,\quad \bar{L}L = 1.$$

Conversely, suppose v is a 4-vector and $L \in \mathcal{P}$ with $\bar{L}L = 1$. Then

$$(LvL^{*})^{*} = LvL^{*} \tag{3.9}$$

shows that LvL^{*} is a 4-vector, and

$$g(LvL^{*}, LvL^{*}) = (\overline{LvL^{*}})LvL^{*} \tag{3.10}$$

$$= \bar{v}v$$

$$= g(v, v)$$

1 u is said to be a unit 4-vector if $\bar{u}u = \pm 1$.

establishes that L defines a Lorentz transformation $L(v) = L v L^{*}$. Thus the set of all $L \in \mathcal{P}$ with $\bar{L}L = 1$ is closely related to the group of restricted Lorentz transformations. This relationship is explored more fully in §3.4.

Since, under a restricted Lorentz transformation, $v \to L v L^{*}$ for any 4-vector v, under the same Lorentz transformation a barred 4-vector transforms as

$$\bar{v} \to \overline{L v L^{*}} = \bar{L}^{*} \bar{v} \bar{L}. \tag{3.11}$$

Hence, alternating products of 4-vectors and barred 4-vectors have particularly simple transformation properties, e.g.,

$$u \bar{v} w \to L u L^{*} \bar{L}^{*} \bar{v} \bar{L} L w L^{*} = L u \bar{v} w L^{*} \tag{3.12}$$

transforms in the same way as a 4-vector. Because $\{e_{\mu} \overline{e_{\nu}} \mid \mu < \nu\}$ is a basis for the space of 6-vectors, a 6-vector F transforms like

$$F \to L F \bar{L}. \tag{3.13}$$

Now, in the Pauli algebra, it is easily seen that since under any Lorentz transformation L, $i = e_0 \overline{e_1} e_2 \overline{e_3}$ goes to

$$i \to L i \bar{L} = i, \tag{3.14}$$

and since for any two orthonormal bases there is a unique Lorentz transformation connecting the bases, i is independent of the orthonormal basis used to define it.

For any finite dimensional abstract algebra $\mathcal{A}$ there exists an exponential operator

$$\exp : \mathcal{A} \to \mathcal{A} \tag{3.15}$$

defined by

$$\exp(\alpha) = \sum_{j=0}^{\infty} \frac{1}{j!} \alpha^{j}. \tag{3.16}$$

where the implied limit in the above infinite sum is with respect to the usual topology[2] of a finite dimensional vector space. Because, for any 6-vector $\overline{\Lambda}$ such that $\Lambda^2 = \overline{\Lambda} \cdot \overline{\Lambda} \neq 0$,

$$\overline{\exp(\overline{\Lambda})}\exp(\overline{\Lambda}) = \exp(\overline{\overline{\Lambda}})\exp(\overline{\Lambda}) \tag{3.17}$$
$$= 1$$

and

$$\begin{aligned} \exp(\overline{\Lambda}) &= \lim_{k \to \infty} \sum_{j=0}^{k} \frac{1}{j!} \overline{\Lambda}^j \\ &= \lim_{k \to \infty} \left[\sum_{j=0}^{k} \frac{\Lambda^{2j}}{(2j)!} + \sum_{j=0}^{k} \frac{\Lambda^{(2j+1)}}{(2j+1)!\,\Lambda} \overline{\Lambda} \right] \\ &= \sum_{j=0}^{\infty} \frac{\Lambda^{2j}}{(2j)!} + \sum_{j=0}^{\infty} \frac{\Lambda^{(2j+1)}}{(2j+1)!\,\Lambda} \overline{\Lambda} \\ &= \cosh \Lambda + \frac{\sinh(\Lambda)}{\Lambda} \overline{\Lambda}, \end{aligned} \tag{3.18}$$

the exponential operator for the Pauli algebra is useful for working with restricted Lorentz transformations.

A pertinent question, then, is whether or not every L with $\bar{L}L = 1$ can be expressed as the exponential of some 6-vector. To answer this, suppose such an $L \neq 1$ is given. Choose a 6-vector $\overline{\Lambda}$ such that $\frac{\sinh \Lambda}{\Lambda}\overline{\Lambda} = \vec{L}$. This implies that $(\sinh \Lambda)^2 = \vec{L} \cdot \vec{L}$, which together with $\bar{L}L = 1$ gives that $\cosh \Lambda = \pm L_0$. If $\cosh \Lambda = L_0$ then $L = \exp(\overline{\Lambda})$; if $\cosh \Lambda = -L_0$ then $L = -\exp(-\overline{\Lambda})$. However, L and $-L$ define the same Lorentz transformation and therefore only elements of the form $\exp(\overline{\Lambda})$ need be considered when dealing with restricted Lorentz transformations.

2 For any finite dimensional vector space, there is a **unique** topology such that the vector space together with the topology form a topological vector space (Geroch 1985, chap. 41).

3.3 Rotations and Boosts

The two main involutions of the Pauli algebra, "bar" and "dagger", are used to single out particularly meaningful Lorentz transformations. First, consider Lorentz transformations, defined by B, of the form (3.18) and with $B = B^{*}$. Such a transformation, called a boost, can thus be written in the form

$$B = \exp\left(\frac{\vec{w}}{2}\right) \tag{3.19}$$

$$= \cosh(w/2) + \hat{w}\sinh(w/2),$$

where $\vec{w}$ is real. Let v be an arbitrary 4-vector and define $v_{\parallel} = v^{0} + \vec{v}\cdot\hat{w}\hat{w}$, $v_{\perp} = v - v_{\parallel}$. Then, under B, an arbitrary 4-vector v goes to

$$BvB^{*} = B(v_{\parallel} + v_{\perp})B \tag{3.20}$$

$$= v_{\parallel}\exp(\vec{w}) + v_{\perp}.$$

Thus, only 4-vectors in

$$U = span\{e_0, \vec{w}\} \tag{3.21}$$

are altered by B. Because this space is two dimensional, this means that B can be expressed as the product of two reflections. Consider the product of two timelike reflections,

$$L = u_2\overline{u_1}, \quad \overline{u_1}u_1 = \overline{u_2}u_2 = 1, \tag{3.22}$$

with u_1 and u_2 both in U. Since U has only one spatial dimension orthogonal to e_0, the spatial parts of u_1 and u_2 are parallel and therefore $L = L^{*}$, that is L is a boost. Conversely, for any boost B, choose $u_1 = e_0$ and $u_2 = B$. Therefore, any boost can be expressed as the product of two timelike reflections.

It might seem logical to next consider Lorentz transformations defined by Pauli elements that are invariant under "bar", but it is easily seen that the only such transformation is the identity. There are nontrivial Pauli elements, R, with $\bar{R}R = 1$

(unimodular) and that are invariant under the compostition of "bar"and "dagger". Such elements, called rotations, can be written as the exponentials of imaginary 6-vectors, i.e.,

$$R = \exp\left(\frac{-i\vec{\theta}}{2}\right) \tag{3.23}$$

$$= \cos(\theta/2) - i\hat{\theta}\sin(\theta/2),$$

with $\vec{\theta}$ real. A 4-vector v, under R transforms to

$$RvR^{*} = R(v_{\parallel} + v_{\perp})R^{*} \tag{3.24}$$

$$= v_{\parallel} + v_{\perp}\exp(i\vec{\theta}).$$

Now, only vectors in the two dimensional purely spatial subspace

$$W = span\{\hat{a}, \hat{a}\times\hat{\theta}\}, \tag{3.25}$$

where $\hat{a}$ is such that $\hat{a}\cdot\hat{\theta} = 0$, are changed by the transformation. Therefore, like the boosts, rotations can be expressed as the product of two reflections. The product of two spatial reflections,

$$L = \hat{w}_2\overline{\hat{w}_1}, \tag{3.26}$$

is invariant under the compositon of "bar" and "dagger" and hence is a rotation. Also, any L of the form (3.26) has

$$-L = \hat{w}_1\cdot\hat{w}_2 - i\hat{w}_1\times\hat{w}_2. \tag{3.27}$$

Thus, any rotation can be written as the product of two spatial reflections. The physical significance of rotations and boosts is studied in the next chapter.

For a given decomposition of spacetime into space and time, the set of all rotations is a subgroup of the group of restricted Lorentz transformations. The set of all boosts, however, is not closed under group multiplication and therefore is not a subgroup; this accounts for the physical phenomenon of Thomas precession. As is shown next, any unimodular Pauli element (in particular the product of two boosts) can be expressed as the product of a boost and a rotation.

Let L be a unimodular Pauli element and define $B=(LL^{*})^{\frac{1}{2}}$, $R=\bar{B}L$. With these definitions, $B=B^{*}$, $\bar{B}B=(\overline{LL^{*}}LL^{*})^{\frac{1}{2}}=1$, which shows that B is a boost. Also, $B^{2}=LL^{*}$ implies that

$$\begin{aligned} 1 &= \bar{B}LL^{*}\bar{B} \\ &= \bar{B}BRR^{*}B\bar{B} \\ &= RR^{*} \end{aligned} \tag{3.28}$$

and $\bar{R}R=\bar{L}B\bar{B}L=1$. This completes the proof that any Lorentz transformation can be expressed as the compositon of a boost and a rotation. More specific results pertaining to the product of two boosts is found in Baylis and Jones [1988].

3.4 The Universal Covering Group of the Restricted Lorentz Group

The universal covering manifold (Naber 1992) of $\mathcal{L}_{+}^{\uparrow}$ is a connected, simply connected topological group, $Spin+(V)$, together with a covering map from $Spin+(V)$ onto $\mathcal{L}_{+}^{\uparrow}$. As shown below, the universal covering group of the restricted Lorentz group is just the set of unimodular Pauli elements that have been used thus far to effect Lorentz transformations. The mathematical and physical importance of this group, and its relevance to other Pauli algebra elements is analyzed in some detail in the last chapter of this work.

Define

$$Spin+(V)=\{L\in\mathcal{P}\mid\bar{L}L=1\}. \tag{3.29}$$

Clearly, $Spin+(V)$ together with the algebraic multiplication of $\mathcal{P}$ is a group. $Spin+(V)$ is isomorphic to $SL(2,C)$, the group of unimodular 2X2 complex matrices, and inherits a natural topology via this isomorphism. ($SL(2,C)$ is a topological subspace of C^{4}.) Define the surjective map

$$\mu:Spin+(V)\to\mathcal{L}_{+}^{\uparrow} \tag{3.30}$$

by

$$\mu(L)v = L v L^* \qquad (3.31)$$

for every 4-vector v. Suppose L is in the kernel of μ, i.e., $\mu(L) = 1$. Then, $v = 1$ gives $L^* = \bar{L}$. Hence $\vec{L} = i\vec{\Omega}$ with $\vec{\Omega}^* = \vec{\Omega}$. Now choose $\vec{v}$ such that $\vec{\Omega}\cdot\vec{v} = 0$. This gives that $L^2 = 1$, and thus $L = \pm 1$. Therefore the kernel of μ is $\{1, -1\}$ and μ is a 2 to 1 homomorphism of groups that for every $L \in Spin+(V)$ maps L and $-L$ onto the same element of $\mathcal{L}^{\uparrow}_{+}$, i.e.,

$$Spin+(V)/\{1,-1\} \cong \mathcal{L}^{\uparrow}_{+}. \qquad (3.32)$$

The natural topology for $\mathcal{L}^{\uparrow}_{+}$ is therefore the quotient topology (Geroch 1985). What follows in the rest of this section shows that $Spin+(V)$ is the universal covering manifold of the restricted Lorentz group.

First, it must be demonstrated that μ is a covering map. Let L be an arbitrary element of $\mathcal{L}^{\uparrow}_{+}$ and $\mathcal{N}_1$ be an open neighbourhood of L. If $L \in Spin+(V)$ is such that $\mu(L) = L$, then $\pm L \in \mathcal{N}_2 = \mu^{-1}(\mathcal{N}_1)$. Because $Spin+(V)$ is Hausdorff, $\mathcal{N}_2$ contains disjoint $\mathcal{N}_3$ and $\mathcal{N}_4$, which are open neighbourhoods of $-L$ and L respectively. Therefore, $\mathcal{N}_5 = \mu(\mathcal{N}_3)\cap\mu(\mathcal{N}_4)$ is an open (from the properties of the quotient topology) neighbourhood of L. Then, $\mu^{-1}(\mathcal{N}_5)$ is the union of disjoint open subsets of $Spin+(V)$, each of which is mapped homeomorphically onto $\mathcal{N}_5$, an open neighbourhood of L. It is shown below that $Spin+(V)$ is connected and simply connected. Thus, μ is a covering map.

Next, it must be shown that $Spin+(V)$ is connected. Let L_1 and L_2 be arbitrary elements of $Spin+(V)$. Any L in $Spin+(V)$ can be written uniquely in the form

$$L = \exp\left(\frac{\vec{w}}{2}\right)\exp\left(i\frac{\vec{\Omega}}{2}\right), \qquad (3.33)$$

where $\vec{w}$ and $\vec{\Omega}$ are real 3-dimensional vectors. Thus, write $L_1 = \exp\left(\frac{\vec{w}_1}{2}\right)\exp\left(i\frac{\vec{\Omega}_1}{2}\right)$ and $L_2 = \exp\left(\frac{\vec{w}_2}{2}\right)\exp\left(i\frac{\vec{\Omega}_2}{2}\right)$. Let

$$f:[0,1]\to Spin-(V) \tag{3.34}$$

be given by

$$f(t)=\exp\left(\frac{(1-t)\vec{w}_1+t\vec{w}_2}{2}\right)\exp\left(i\left\{\frac{(1-t)\overline{\Omega}_1+t\overline{\Omega}_2}{2}\right\}\right). \tag{3.35}$$

This defines a path running from L_1 to L_2; therefore, $Spin-(V)$ is (path) connected.

Let L be an arbitrary loop in $Spin+(V)$, i.e.

$$L:[0,1]\to Spin+(V). \tag{3.36}$$

L is continuous, and $L(0)=L(1)$. Define a new loop, L', that starts and ends at the identity, by

$$L'(t)=L(0)^{-1}L(t) \tag{3.37}$$

$$=\exp\left\{\frac{\vec{w}'(t)}{2}\right\}\exp\left\{-i\frac{\overline{\Omega}'(t)}{2}\right\}.$$

Now,

$$L'(0)=1\Rightarrow\vec{w}'(0)=0,\quad \Omega'(0)=4\pi n,\quad n\in Z \tag{3.38}$$

and

$$L'(1)=1\Rightarrow\vec{w}'(1)=0,\quad \Omega'(1)=4\pi m,\quad m\in Z. \tag{3.39}$$

Next, define a family of loops

$$L'_\alpha:[0,1]\to Spin+(V),\quad 0\le\alpha\le 1 \tag{3.40}$$

by

$$L'_\alpha(t)=\exp\left\{(1-\alpha)\frac{\vec{w}'(t)}{2}\right\}\exp\left\{-i\frac{\overline{\Omega}'(t)}{4}\right\} \tag{3.41}$$

$$\exp\left\{-\frac{i}{4}\left(e^{-i\frac{\pi}{2}\alpha\hat{v}(t)}\overline{\Omega}'(t)e^{i\frac{\pi}{2}\alpha\hat{v}(t)}\right)\right\},$$

where $\hat{v}(t)$ varies continuously with t and is orthogonal to $\overline{\Omega}'(t)$. Since n odd gives

$$\exp\left\{-i\frac{\overline{\Omega}'(0)}{4}\right\}=\exp\left\{-\frac{i}{4}\left(e^{-i\frac{\pi}{2}\alpha\hat{v}(0)}\overline{\Omega}'(0)e^{i\frac{\pi}{2}\alpha\hat{v}(0)}\right)\right\}=-1 \tag{3.42}$$

and n even implies

$$\exp\left\{-i\frac{\overline{\Omega}'(0)}{4}\right\} = \exp\left\{-\frac{i}{4}\left(e^{-i\frac{n}{2}\alpha t(0)}\overline{\Omega}'(0)e^{i\frac{n}{2}\alpha t(0)}\right)\right\} = 1\,. \tag{3.43}$$

$L'_\alpha(0) = 1$ for every α. Similarly, $L'_\alpha(1) = 1$ for every α. Thus, every L'_α is a loop in $Spin+(V)$ that begins and ends at the identity. Also note that $L'_0 = L'$ and $L'_1(t) = 1$.

Finally, a family of loops at $L(0)$ is defined by

$$L_\alpha(t) = L(0)L'_\alpha(t)\,. \tag{3.44}$$

The first member ($\alpha = 0$) of this family is the original loop L, and the last member ($\alpha = 1$) is the trivial loop at $L(0)$. The existence of this family of loops shows that L is homotopic ("continuously deformable") to the trivial loop at $L(0)$ and hence $Spin+(V)$ is simply connected. This concludes the demonstration that the group of unimodular Pauli elements is the universal covering group of the restricted Lorentz group.

4 Spacetime

4.1 Spacetime Manifolds

Classical physics requires that spacetime, the set of all possible events occurring in space and time, possesses various types of structure. Firstly, spacetime appears to be a four dimensional continuum, that is, any event is contained in a region (possibly quite "small") of spacetime that can be continuously labelled by four coordinates. (These coordinates may or may not have physical significance.) Thus spacetime has the structure of a topological manifold. Many physical laws are formulated in terms of differential equations, hence spacetime must possess a differential structure, i.e., spacetime is a differentiable manifold. It seems fundamental that there should exist a continuous curve between any two points in spacetime, otherwise the two points would be in totally distinct universes. Therefore spacetime is connected. Spacetime also appears to have a natural light cone structure. This is given by a metric tensor field which has the property that the metric tensor for any event in spacetime together with the tangent space at that event forms a Minkowski vector space.

In addition, a spacetime manifold must satisfy a causality condition. If closed timelike curves exist, then a person would be able to traverse one, and after a non-zero amount of time run into herself. Such situations must surely be excluded. Since the metric, a physical quantity established by measurements, determines the causal structure of spacetime, curves that can be made timelike and closed by "small perturbations" of the metric must also be excluded. A spacetime satisfying these requirements is said to be stably causal (Naber 1988). It is interesting to note that any compact manifold allows closed timelike curves (Naber 1988), so it seems that as a four dimensional manifold our universe cannot be compact. However, compact three dimensional submanifolds are not ruled out, i.e.,our universe might be "closed".

Since, as seen in the previous chapter, any Minkoswki vector space admits a time orientation and a spatial orientation, the tangent space at each event in spacetime admits these orientations. The physical observations that there is a preferred direction of time and a preferred handedness for space requires that the orientations for each tangent space be chosen in a continuous manner, i.e., there is no discontinuous switching of the direction of time or the handedness of space as one moves from event to event.

Apparently spinor phenomena are observable at macroscopic levels (Aharonov and Suskind 1967); therefore the final requirement (in this thesis) for a spacetime manifold is that it admit global spinor structure. A necessary and sufficient condition for a non-compact spacetime manifold to admit global spinor structure is that there exists a global tetrad (Geroch 1968, Penrose and Rindler 1984), i.e., four smooth orthonormal vector fields. The causality condition above excludes compact spacetime manifolds, thus spacetime is required to possess a global tetrad.

4.2 The Pauli Algebra and Spacetime Manifolds

Since each tangent space is a Minkowski vector space, a Pauli algebra can be constructed from each tangent space, and the existence of a global tetrad allows for the continuous tying together of these algebras. Therefore, the existence of a global spinor structure for the spacetime manifold means that smooth fields of Pauli algebra elements exist. Using the concept of observers in spacetime (defined below), this contruction can now be given a physical interpretation.

Observers and particles in spacetime are represented by future directed timelike curves that are parametrized by proper time, i.e.,an observer is a map

$$c:[a,b]\rightarrow \mathcal{M} \tag{4.1}$$

that has a future directed unit timelike tangent vector. Physically, c maps the time on the observer's (ideal) wristwatch into the spacetime event which correspnds to that reading of his watch. It should be emphasized that because two different curves can have the same image, the mathematical model of the observer is the map c, not its image. If two observers are coincidental at an event, then their tangent vectors are clearly related by a (nonunique) Lorentz transformation. Using the tangent vector of the observer, the tangent space at each event on an observer curve is decomposed as in §1.1, splitting spacetime into time and space for the observer. This splitting of spacetime gives a clear physical basis for the construction of the Pauli algebra.

4.3 Minkowski Spacetime

The spacetime of special relativity is Minkowski spacetime, M. In Minkowski spacetime any pair of spacetime events, p and q say, has (independent of coordinates) a unique 4-vector connecting them, the 4-position of q with respect to p; thus M has the structure of an affine space (Dodson and Poston 1991, Kopczynski and Trautman 1992) that has as its associated vector space a Minkowski vector space $(V, \mathbf{g})$. The 4-vector between any pair of spacetime events is given by

$$d : M \times M \rightarrow V. \tag{4.2}$$

The square of the interval between any $p, q \in M$ is defined to be

$$\tau^2(p,q) = \mathbf{g}(d(p,q), d(p,q)). \tag{4.3}$$

Any affine space is a differentiable manifold (Dodson and Poston 1991); thus any event in spacetime is contained in a local coordinate system. Minkowski spacetime, however, also has a preferred set of global coordinate systems, the inertial coordinate systems. An inertial coordinate system is induced by the choice of an event p of spacetime and a proper orthonormal basis $\{\mathbf{e}_\mu\}$ of $(V, \mathbf{g})$ as follows. For any event q there exists a $\mathbf{v} \in V$ such that

$$\begin{aligned} d(p,q) &= \mathbf{v} \\ &= v_q^\mu \mathbf{e}_\mu . \end{aligned} \tag{4.4}$$

Now define inertial coordinates

$$x^\mu : M \rightarrow \mathrm{R} : \quad \mu = 0, 1, 2, 3 \tag{4.5}$$

by

$$x^\mu(q) = v_q^\mu. \tag{4.6}$$

This coordinate system has the event p at its origin.

Suppose $(p, \{\mathbf{e}_\mu\})$ and $(p', \{\mathbf{e}_{\mu'}\})$ are used to define two inertial coordinate systems, and suppose q is any arbitary event. Let: $\mathbf{v}$ be the 4-vector of q with respect to p; $\mathbf{v}'$ be the 4-vector of q with respect to p'; $\mathbf{t}$ be the 4-vector of p with respect to p'; and L be the unique restricted Lorentz transformation that takes the primed basis into the unprimed basis. Then

$$\mathbf{v}' = \mathbf{t} + \mathbf{v} \tag{4.7}$$

or in terms of the bases

$$\begin{aligned} v^{\mu'}\mathbf{e}_{\mu'} &= t^{\mu'}\mathbf{e}_{\mu'} + v^{\mu}\mathbf{e}_{\mu} \\ &= t^{\mu'}\mathbf{e}_{\mu'} + v^{\mu}L^{\nu'}{}_{\mu}\mathbf{e}_{\nu'}. \end{aligned} \tag{4.8}$$

Thus the relationship between the two inertial coordinate systems is

$$x^{\mu'}(q) = t^{\mu'} + x^{\nu}(q)L^{\mu'}{}_{\nu}. \tag{4.9}$$

Any coordinate system for a patch of a manifold induces a coordinate basis for the tangent space at each point in the patch. Since any inertial coordinate system $\{x^\mu\}$ is global, it induces a coordinate basis $\left\{\frac{\partial}{\partial x^\mu}\right\}$ for the tangent space at each event in spacetime. If $(p, \{\mathbf{e}_\mu\})$ is used to construct the inertial coordinate system, then the identification

$$\frac{\partial}{\partial x^\mu} \leftrightarrow \mathbf{e}_\mu. \tag{4.10}$$

gives an isomorphism between the tangent space at each event and the Minkowski vector space, V. From the above, these isomorphisms are independent of the inertial coordinate system used for their formulation and therefore the inertial structure of spacetime allows for the comparison of vectors and tensors at different points of spacetime. Hence, vector fields can be thought of as mapping from spacetime into V (instead of the tangent bundle) and it is possible to talk about constant vector and tensor fields. Also Pauli element fields can be defined as mappings from M into $\mathcal{P}$, the Pauli algebra constructed from V.

An inertial observer is a map that has as its tangent vector a constant future directed unit timelike vector, i.e., an inertial observer moves in a straight line in any inertial coordinate system. This constant tangent vector is called the inertial reference frame associated with the observer. Physically, inertial observers in special relativity correspond to observers that are under the influence of no external forces. An inertial observer α with tangent vector $\mathbf{u}$ can at any event p on its worldline choose an orthonormal basis $\{\mathbf{e}_\mu\}$ for V with time axis $\mathbf{e}_0 = \mathbf{u}$ and from this can construct an inertial coordinate system for all of spacetime. This is how any observer labels events in spacetime with space and time coordinates.

Let α and b be inertial observers having tangent vectors $\mathbf{e}_0$ and $\mathbf{u}$ respectively and suppose α constructs an inertial coordinate system as above. Then, from the identification (4.10),

$$\mathbf{u} = \frac{dx^\mu}{d\tau}\mathbf{e}_\mu, \tag{4.11}$$

where τ[3] is the curve parameter (proper time) for b. Because $\mathbf{e}_0$ and $\mathbf{u}$ are both future-directed timelike vectors,

$$0 < \mathbf{g}(\mathbf{u}, \mathbf{e}_0) = \frac{dx^0}{d\tau}. \tag{4.12}$$

Since $\mathbf{u}$ is a unit timelike vector,

$$\begin{aligned} 1 &= \left(\frac{dx^0}{d\tau}\right)^2 - \left[\left(\frac{dx^1}{d\tau}\right)^2 + \left(\frac{dx^2}{d\tau}\right)^2 + \left(\frac{dx^3}{d\tau}\right)^2\right] \\ &= \left(\frac{dx^0}{d\tau}\right)^2 \left\{1 - \left[\left(\frac{dx^1}{dx^0}\right)^2 + \left(\frac{dx^2}{dx^0}\right)^2 + \left(\frac{dx^3}{dx^0}\right)^2\right]\right\}, \end{aligned} \tag{4.13}$$

which together with (4.12) gives

3 Here a common abuse of notation has been used; $\frac{dx^\mu}{d\tau}$ is really $\frac{d}{d\tau}(x^\mu \circ b)$.

$$\gamma := \frac{dx^0}{d\tau} = \left\{ 1 - \left[\left(\frac{dx^1}{dx^0}\right)^2 + \left(\frac{dx^2}{dx^0}\right)^2 + \left(\frac{dx^3}{dx^0}\right)^2 \right] \right\}^{-\frac{1}{2}}. \tag{4.14}$$

Thus, if $\mathbf{e}_0$ is used to construct the Pauli algebra for Γ,

$$\mathbf{u} = \gamma(1 + \vec{v}), \tag{4.15}$$

where $\vec{v} = \frac{dx^i}{dx^0}\mathbf{e}_i$ is the spatial velocity with which b moves through a's coordinate system. Hence, the tangent vector of any observer curve (not just an inertial observer and not just in Minkowski spacetime) is called the observer's 4-velocity. If the two inertial observers have the same 4-velocity, i.e.,they share the same inertial reference frame, then (3.11) shows that time passes at the same rate for both of them, and that their relative spatial velocity is zero.

Suppose the two inertial observers are as above and that they have a common event p on their worldlines which they both choose as the origin of their global inertial coordinate systems. Suppose further that the proper orthonormal tetrads that they use to set up their coordinate axes are related by the Pauli algebra rotation $R = \exp\left(\frac{-i\vec{\theta}}{2}\right)$. This means that

$$\begin{aligned} \mathbf{u} &= R\mathbf{e}_0 R^* \\ &= \mathbf{e}_0 R R^{-1} \\ &= \mathbf{e}_0 \end{aligned} \tag{4.16}$$

and so they have no relative spatial velocity with respect to each other. Now let $\vec{v}$ be any one of a's spatial axes and $\vec{v}'$ be the corresponding spatial axis for b. Equation (3.24) gives

$$\begin{aligned} \vec{v}' &= \vec{v}_{\parallel} + \vec{v}_{\perp}(\cos\theta + i\hat{\theta}\sin\theta) \\ &= \vec{v}_{\parallel} + \vec{v}_{\perp}\cos\theta + \hat{\theta}\times\vec{v}_{\perp}. \end{aligned} \tag{4.17}$$

This shows that the spatial axes of b are obtained by rotating each of a's spatial axes about the axis $\hat{\theta}$ by an angle of θ.

Now suppose that the proper orthonormal tetrads that they use to set up their coordinate axes are related by the Pauli algebra boost $B = \exp\left(\frac{\vec{w}}{2}\right)$. Then

$$\begin{aligned} \mathbf{u} &= B\mathbf{e}_0 B^{*} \\ &= \mathbf{e}_0 B^2 \\ &= \cosh w - \hat{w}\sinh w. \end{aligned} \tag{4.18}$$

With (4.14) and (4.15) this demonstrates that the rates at which time passes for the observers differ by a factor of $\cosh w$ and that the spatial velocity of b with respect to a is

$$\vec{v} = \hat{w}\tanh w. \tag{4.19}$$

Thus b 's reference frame is obtained by "boosting" a 's.

For c an arbitrary observer in Minkowski spacetime, the tangent vector $\mathbf{u}$ of c is not necessarily constant. Therefore define the 4-acceleration $\mathbf{a}$ of c as

$$\mathbf{a} = \frac{d\mathbf{u}}{d\tau}. \tag{4.20}$$

Differentiating $1 = \mathbf{u}\bar{\mathbf{u}}$ shows that

$$0 = \mathbf{u}\cdot\bar{\mathbf{a}}, \tag{4.21}$$

so an observer's 4-acceleration is always a spacelike vector orthogonal to its 4-velocity.

The theory of electromagnetism in Minkowski spacetime is particularly elegant when expressed in the Pauli algebra. Only the fundamental formalism is presented here; for more details and applications see Baylis and Jones (1989 a,b) and (Baylis et. al. 1993).

First, define the Pauli algebra differential operator ∂ by

$$\partial := \mathbf{e}^{\mu}\frac{\partial}{\partial x^{\mu}}, \tag{4.22}$$

where $\{x_\mu\}$ is global inertial coordinate system. This definition is independent of the choice of inertial coordinate system. This operator maps Pauli element fields into Pauli element fields (in general of different types) and partial derivatives are understood only to act on the scalar coefficients of a given basis element when expressed in terms of a tetrad. Using this differential operator, the electromagnetic field 6-vector F is given in terms of the vector potential A by

$$F = \frac{1}{2}(\partial\overline{A} - \overline{\partial\overline{A}}) \qquad (4.23)$$
$$= \vec{E} + i\vec{B}.$$

Maxwell's equation(s) is (are) then just

$$\bar{\partial} F = 4\pi \bar{j} \qquad (4.24)$$

Finally the Lorentz force equation is

$$\frac{d\mathrm{P}}{d\tau} = q\mathcal{R}e\{\mathrm{u}\} \qquad (4.25)$$

for a charge q with 4-momentum $\mathrm{p} = m\mathrm{u}$. This Pauli algebra equation has the advantage over the conventional approach in that it offers coordinate free solutions to the motion of charged particles.

4.4 The Spacetime Connection

Because in general (unlike Minkowski spacetime) there is no natural way to compare vectors in tangent spaces of distinct events, the structures defined thus far on the spacetime manifold do not allow for an adequate generalization of the concept of a directional derivative. Thus a new structure on spacetime, called a connection, is required. The properties of the connection are abstracted from the properties of the directional derivative in flat spacetime. Therefore, a connection is a map that maps any ordered pair of vector fields into a vector field,

$$(\mathrm{u}, \mathrm{v}) \to \nabla_{\mathrm{u}}\mathrm{v}, \qquad (4.26)$$

which has the following properties:

$$\nabla_{f\mathrm{u}+\mathrm{v}}\mathrm{w} = f\nabla_{\mathrm{u}}\mathrm{w} + \nabla_{\mathrm{v}}\mathrm{w} \qquad (4.27)$$

$$\nabla_{\mathrm{u}}(f\mathrm{v} + \mathrm{w}) = \mathrm{u}(f)\mathrm{v} + f\nabla_{\mathrm{u}}\mathrm{v} + \nabla_{\mathrm{u}}\mathrm{w} \qquad (4.28)$$

where f is a scalar field and $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vector fields. $\nabla_{\mathbf{u}}\mathbf{v}$ is called the covariant derivative of $\mathbf{v}$ in the direction of $\mathbf{u}$. If $\{\mathbf{e}_\mu\}$ is a set of linearly independent vector fields, the connection coefficients, $\Gamma^{\alpha}{}_{\mu\nu}$, are given by[4]

$$\nabla_\mu \mathbf{e}_\nu = \Gamma^{\alpha}{}_{\nu\mu}\mathbf{e}_\alpha . \tag{4.29}$$

Covariant differentiation is extended to arbitrary tensor fields by requiring that

$$\nabla_{\mathbf{u}} f = \mathbf{u}(f) \tag{4.30}$$

$$\nabla_{\mathbf{u}}(A \otimes B) = (\nabla_{\mathbf{u}} A) \otimes B + A \otimes (\nabla_{\mathbf{u}} B) \tag{4.31}$$

$$\nabla_{\mathbf{u}}(C(A)) = C(\nabla_{\mathbf{u}} A) \tag{4.32}$$

where f is a scalar field, A and B are arbitrary tensor fields, and C is a contraction map. The spacetime connection is required to satisfy two more conditions. Firstly, the connection is metric compatible, i.e.,

$$\nabla_{\mathbf{u}}\mathbf{g} = 0 . \tag{4.33}$$

and secondly the connection is torsion free, i.e.,

$$\nabla_{\mathbf{u}}\mathbf{v} - \nabla_{\mathbf{v}}\mathbf{u} = [\mathbf{u}, \mathbf{v}] \tag{4.34}$$

for all vector fields $\mathbf{u}$ and $\mathbf{v}$. From these properties it is easily shown that

$$\nabla_{\mathbf{u}}\bar{\mathbf{v}} = \overline{\nabla_{\mathbf{u}}\mathbf{v}} . \tag{4.35}$$

When expressed with respect to a tetrad $\{\mathbf{e}_{\hat{\mu}}\}$, the connection coefficients have the antisymmetry property

$$\Gamma_{\hat{\alpha}\hat{\beta}\hat{\gamma}} = -\Gamma_{\hat{\beta}\hat{\alpha}\hat{\gamma}} , \tag{4.36}$$

which is shown by differentiating

$$\begin{aligned} \eta_{\hat{\beta}\hat{\alpha}} &= \mathbf{g}(\mathbf{e}_{\hat{\beta}}, \mathbf{e}_{\hat{\alpha}}) \\ &= C \circ C[\mathbf{g} \otimes \mathbf{e}_{\hat{\beta}} \otimes \mathbf{e}_{\hat{\alpha}}] , \end{aligned} \tag{4.37}$$

where C is a contraction map.

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4 In this dissertation $\nabla_\mu := \nabla_{\mathbf{e}_\mu}$. In particular, if $\left\{\mathbf{e}_\mu = \frac{\partial}{\partial x^\mu}\right\}$ and $\mathbf{v} = v^\mu \mathbf{e}_\mu$ then $\nabla_\mu v^\nu = \frac{\partial v^\nu}{\partial x^\mu}$.

Using the connection it is possible to define the parallel transport of a vector along a curve γ. A vector field v is said to be parallel transported along γ if

$$\nabla_{\gamma} v = 0 \tag{4.38}$$

along γ. Given events p and q in the image of γ and $v \in T_p(\mathcal{M})$, then there is exactly one $v' \in T_q(\mathcal{M})$ with v' equal to the parallel transport of v. Hence, parallel transport gives a natural curve dependent isomorphism between tangent spaces that can be used to define the covariant derivative as a limit analogously to the definition of the normal derivative. A geodesic is a curve γ that, with suitable parameterization, parallel transports its own tangent vector, that is

$$\nabla_{\dot{\gamma}} \dot{\gamma} = 0, \tag{4.39}$$

with dot denoting differentiation with respect to the curve parameter. As stated in the previous chapter, an observer is a curve γ that is parameterized by proper time with $g(\dot{\gamma}, \dot{\gamma}) = 1$; a freely falling (or locally inertial) observer is any such curve that is also a geodesic.

4.5 The Connection 6-vectors

Covariant differentiation is defined on Pauli algebra elements by noting that any Pauli element can be expressed in a covariant way as a linear combination of products of 4-vectors and barred 4-vectors and then requiring that, e.g.

$$\nabla_{\hat{\alpha}}(\mathbf{e}_{\hat{\beta}}\overline{\mathbf{e}_{\hat{\gamma}}}\mathbf{e}_{\hat{\delta}}) = (\nabla_{\hat{\alpha}}\mathbf{e}_{\hat{\beta}})\overline{\mathbf{e}_{\hat{\gamma}}}\mathbf{e}_{\hat{\delta}} + \mathbf{e}_{\hat{\beta}}(\nabla_{\hat{\alpha}}\overline{\mathbf{e}_{\hat{\gamma}}})\mathbf{e}_{\hat{\delta}} + \mathbf{e}_{\hat{\beta}}\overline{\mathbf{e}_{\hat{\gamma}}}(\nabla_{\hat{\alpha}}\mathbf{e}_{\hat{\delta}}) \tag{4.40}$$

for products of the members of the tetrad[5] $\{\mathbf{e}_{\hat{\mu}}\}$. From this it is easily shown that for any two Pauli elements a and b,

$$\nabla_{\hat{\alpha}}(a \cdot \bar{b}) = (\nabla_{\hat{\alpha}} a) \cdot \bar{b} + a \cdot (\overline{\nabla_{\hat{\alpha}} b}) \tag{4.41}$$

$$\nabla_{\hat{\alpha}}(a \wedge \bar{b}) = (\nabla_{\hat{\alpha}} a) \wedge \bar{b} + a \wedge (\overline{\nabla_{\hat{\alpha}} b}). \tag{4.42}$$

For any tetrad $\{\mathbf{e}_{\hat{\mu}}\}$ and any 4-vector v define the connection 6-vector

5 The hat notation is used from now on to refer to tetrads as opposed to the coordinate bases frequently used in general relativity.

$$\Gamma_v = \frac{1}{2}(\nabla_v \mathbf{e}_{\hat\mu})\overline{\mathbf{e}^{\mu}} \tag{4.43}$$

$$= \frac{1}{2}\upsilon^{\dot v}\Gamma^{\dot\alpha}{}_{\dot\mu\dot v}\mathbf{e}_{\dot\alpha}\overline{\mathbf{e}^{\mu}}.$$

Then from (4.43) and (2.46),

$$\nabla_v \mathbf{e}_{\hat\mu} = \frac{1}{2}(\Gamma_v \mathbf{e}_{\hat\mu} + \mathbf{e}_{\hat\mu}\Gamma_v^{*}) \tag{4.44}$$

$$= \mathcal{R}e\{\Gamma_v \mathbf{e}_{\hat\mu}\}.$$

It remains to show that each Γ_v is actually a 6-vector. Differentiating $4 = \mathbf{e}^{\mu}\overline{\mathbf{e}_{\hat\mu}}$ gives

$$0 = (\nabla_v \mathbf{e}_{\hat\mu})\overline{\mathbf{e}^{\mu}} + \mathbf{e}^{\hat\mu}(\overline{\nabla_v \mathbf{e}_{\hat\mu}}) \tag{4.45}$$

$$= 2(\Gamma_v + \overline{\Gamma_v})$$

and therefore Γ_v is a 6-vector.

Suppose $\{\mathbf{e}_{\hat\mu}\}$ is a tetrad, a second tetrad is

$$\{\mathbf{e}_{\hat\mu'} = L\mathbf{e}_{\hat\mu}L^{*}\}, \tag{4.46}$$

and v is a 4-vector. Then each tetrad can be used to define a connection 6-vector, i.e.,

$$\Gamma_v = \frac{1}{2}(\nabla_v \mathbf{e}_{\hat\mu})\overline{\mathbf{e}^{\mu}} \tag{4.47}$$

and

$$\Gamma'_v = \frac{1}{2}(\nabla_v \mathbf{e}_{\hat\mu'})\overline{\mathbf{e}^{\mu'}}. \tag{4.48}$$

To find the transformation properties of the connection 6-vectors, use (4.46) in (4.48) to give

$$\Gamma'_v = \frac{1}{2}\left[(\nabla_v L)\mathbf{e}_{\hat\mu}\overline{\mathbf{e}^{\mu}}\overline{L} + L(\nabla_v \mathbf{e}_{\hat\mu})\overline{\mathbf{e}^{\mu}}\overline{L} + L\mathbf{e}_{\hat\mu}(\nabla_v L^{*})\overline{L^{*}\mathbf{e}^{\mu}}\overline{L}\right] \tag{4.49}$$

$$= L\Gamma_v\overline{L} + 2(\nabla_v L)\overline{L}.$$

Therefore, if $\Gamma_v = 0$ then one has the spinor like equation

$$\nabla_{\mathbf{v}} \mathrm{L} = \frac{1}{2}\Gamma'_{\mathbf{v}} \mathrm{L}. \tag{4.50}$$

which has particular physical relevance when L relates the frame of an arbitrary observer to the frames of a set of freely falling observers coincidental to him. Note that $1 = \mathrm{L}\overline{\mathrm{L}}$ implies that

$$0 = (\nabla_{\mathbf{v}} \mathrm{L})\overline{\mathrm{L}} + \mathrm{L}(\overline{\nabla_{\mathbf{v}} \mathrm{L}}) \tag{4.51}$$

and thus $2(\nabla_{\mathbf{v}} \mathrm{L})\overline{\mathrm{L}}$ is a 6-vector.

In practice, the metric is given in terms of a coordinate chart, i.e.,

$$\mathbf{g} = g_{\mu\nu}\, dx^{\mu} \otimes dx^{\nu}, \tag{4.52}$$

and functions $h_{\hat{\mu}}{}^{\nu}$ are found such that

$$\mathbf{e}_{\hat{\mu}} = h_{\hat{\mu}}{}^{\nu} \frac{\partial}{\partial x^{\nu}} \tag{4.53}$$

is a tetrad. The structure coefficients of the tetrad are then found via

$$[\mathbf{e}_{\hat{\mu}}, \mathbf{e}_{\hat{\nu}}] = C^{\hat{\alpha}}{}_{\hat{\mu}\hat{\nu}} \mathbf{e}_{\hat{\alpha}}, \tag{4.54}$$

which are in turn used to calculate the connection 6-vectors

$$\Gamma_{\hat{\mu}} = \frac{1}{2}\Gamma_{\hat{\alpha}\hat{\beta}\hat{\mu}} \overline{\mathbf{e}^{\hat{\alpha}}\mathbf{e}^{\hat{\beta}}} \tag{4.55}$$

$$= \frac{1}{4}(-C_{\hat{\alpha}\hat{\beta}\hat{\mu}} - C_{\hat{\beta}\hat{\mu}\hat{\alpha}} + C_{\hat{\mu}\hat{\alpha}\hat{\beta}}) \overline{\mathbf{e}^{\hat{\alpha}}\mathbf{e}^{\hat{\beta}}}.$$

All the information about the connection is contained in the twenty-four components of these four 6-vectors, as contrasted to the forty components calculated by the traditional coordinate based approach.

4.6 The Spacetime Curvature and Einstein's Equation

Associated with any two vector fields u and v there is a curvature operator (Misner, Thorne, and Wheeler 1973) $\mathcal{R}(\mathbf{u}, \mathbf{v})$ that maps the set of all vector fields into itself and is defined by

$$\mathcal{R}(\mathbf{u}, \mathbf{v})\mathbf{w} := (\nabla_{\mathbf{u}}\nabla_{\mathbf{v}} - \nabla_{\mathbf{v}}\nabla_{\mathbf{u}} - \nabla_{[\mathbf{u}, \mathbf{v}]})\mathbf{w} \tag{4.56}$$

for w an arbitrary vector field. Applying this to a member of a tetrad one obtains

$$\mathcal{R}(u,v)e_{\dot\alpha} = (\nabla_u\nabla_v - \nabla_v\nabla_u - \nabla_{[u,v]})e_{\dot\alpha} \tag{4.57}$$

$$= \mathcal{R}e\{\nabla_u(\Gamma_v e_{\dot\alpha}) - \nabla_v(\Gamma_u e_{\dot\alpha}) - \Gamma_{[u,v]}e_{\dot\alpha}\}$$

$$= \mathcal{R}e\{(\nabla_u\Gamma_v)e_{\dot\alpha} + \frac{1}{2}(\Gamma_v\Gamma_u e_{\dot\alpha} + \Gamma_v e_{\dot\alpha}\bar\Gamma_u) - (\nabla_v\Gamma_u)e_{\dot\alpha}$$

$$-\frac{1}{2}(\Gamma_u\Gamma_v e_{\dot\alpha} + \Gamma_u e_{\dot\alpha}\bar\Gamma_v) - \Gamma_{[u,v]}e_{\dot\alpha}\}$$

$$= \mathcal{R}e\left\{\left(\nabla_u\Gamma_v - \nabla_v\Gamma_u - \frac{1}{2}(\Gamma_u\Gamma_v - \Gamma_v\Gamma_u) - \Gamma_{[u,v]}\right)e_{\dot\alpha}\right\}$$

Therefore, all the information about the curvature of spacetime is contained in the curvature 6-vectors

$$\mathcal{R}_{u,v} := \nabla_u\Gamma_v - \nabla_v\Gamma_u - \frac{1}{2}(\Gamma_u\Gamma_v - \Gamma_v\Gamma_u) - \Gamma_{[u,v]}. \tag{4.58}$$

The components of the curvature tensor are defined by

$$R^{\beta}{}_{\dot\alpha\dot\mu\dot\nu}e_\beta := \mathcal{R}(e_{\dot\mu}, e_{\dot\nu})e_{\dot\alpha} \tag{4.59}$$

$$= \mathcal{R}e\{\mathcal{R}_{\dot\mu\dot\nu}e_{\dot\alpha}\}.$$

Multiplying this equation by $e^{\bar\alpha}$ gives the curvature 6-vectors in terms of the components of the curvature tensor:

$$\mathcal{R}_{\dot\mu\dot\nu} = \frac{1}{2}R_{\beta\dot\alpha\dot\mu\dot\nu}e^{\beta}\overline{e^{\alpha}}. \tag{4.60}$$

In order to express Einstein's equation in the Pauli algebra, Pauli algebra equivalents of the Ricci tensor and the curvature scalar must be found. Define the Ricci 4-vectors

$$\mathcal{R}_{\dot\mu} := \mathcal{R}e\{\mathcal{R}_{\dot\mu\dot\nu}e^{\dot\nu}\} \tag{4.61}$$

$$= R^{\dot\nu}{}_{\dot\alpha\dot\nu\dot\mu}e^{\dot\alpha}$$

$$= R_{\dot\alpha\dot\mu}e^{\dot\alpha}$$

(use 1.45 for the penultimate equality) and the curvature scalar

$$R := \mathcal{R}_{\hat{\mu}} \cdot \overline{\mathbf{e}^{\mu}} \tag{4.62}$$

$$= R^{\hat{\mu}}{}_{\hat{\mu}}.$$

From these form the Einstein 4-vectors

$$\mathcal{G}_{\hat{\mu}} := \mathcal{R}_{\hat{\mu}} - \frac{1}{2} R \mathbf{e}_{\hat{\mu}} : \tag{4.63}$$

thus Einstein's equations are

$$\mathcal{G}_{\hat{\mu}} = 8\pi \mathcal{T}_{\hat{\mu}} \tag{4.64}$$

where $\mathcal{T}_{\hat{\mu}} := T_{\hat{\alpha}\hat{\mu}} \mathbf{e}^{\hat{\alpha}}$ are the energy-momentum 4-vectors.

4.7 Example: Spherically Symmetric Static Spacetime

To illustrate the techniques involved without getting bogged down in the technical details, choose as an example one of the simplest physically relevant metrics, the metric for spherically symmetric static spacetime as given in Schutz (1990). Spherically symmetric static metrics have the general form

$$\mathbf{g} = e^{2\Phi} dt \otimes dt - e^{2\Lambda} dr \otimes dr + r^2 (d\theta \otimes d\theta + \sin^2\theta d\phi \otimes d\phi), \tag{4.65}$$

where Φ and Λ are functions of r only. Choose the orthonormal frame

$$\mathbf{e}_0 = e^{-\Phi} \frac{\partial}{\partial t}, \quad \mathbf{e}_1 = e^{-\Lambda} \frac{\partial}{\partial r}, \tag{4.66}$$

$$\mathbf{e}_2 = \frac{1}{r} \frac{\partial}{\partial \theta}, \quad \mathbf{e}_3 = \frac{1}{r \sin\theta} \frac{\partial}{\partial \phi}.$$

Now, let prime denote differentiation with respect to r and calculate the six commutators $[\mathbf{e}_{\hat{\mu}}, \mathbf{e}_{\hat{\nu}}]$ for $\hat{\mu} < \hat{\nu}$.

One finds that the only independent non-zero structure constants are

$$C_{001} = \Phi' e^{-\Lambda}, \quad C_{212} = C_{313} = \frac{e^{-\Lambda}}{r}, \quad C_{323} = \frac{\cot\theta}{r}. \tag{4.67}$$

Then, from (4.55) the connection 6-vectors are

$$\Gamma_0 = \Phi' e^{-\Lambda} \mathbf{e}^0 \overline{\mathbf{e}^1}, \quad \Gamma_1 = 0, \tag{4.68}$$

$$\Gamma_2 = \frac{e^{-\Lambda}}{r} \mathbf{e}^1 \overline{\mathbf{e}^2}, \quad \Gamma_3 = \frac{e^{-\Lambda}}{r} \mathbf{e}^1 \overline{\mathbf{e}^3} + \frac{\cot\theta}{r} \mathbf{e}^2 \overline{\mathbf{e}^3}.$$

It simplifies the calculation of the curvature 6-vectors to notice that for $\hat{\mu} \neq \hat{\nu}$

$$\nabla_{\hat{\alpha}}(\mathbf{e}^{\mu}\overline{\mathbf{e}^{\nu}}) = \frac{1}{2}\left(\Gamma_{\hat{\alpha}}\mathbf{e}^{\mu}\overline{\mathbf{e}^{\nu}} + \mathbf{e}^{\mu}\overline{\Gamma_{\hat{\alpha}}\mathbf{e}^{\nu}} + \mathbf{e}^{\mu}\overline{\mathbf{e}^{\nu}\Gamma_{\hat{\alpha}}} + \mathbf{e}^{\mu}\overline{\Gamma_{\hat{\alpha}}\mathbf{e}^{\nu}}\right) \tag{4.69}$$

$$= \frac{1}{2}\left(\Gamma_{\hat{\alpha}}\mathbf{e}^{\mu}\overline{\mathbf{e}^{\nu}} - \mathbf{e}^{\mu}\overline{\mathbf{e}^{\nu}}\Gamma_{\hat{\alpha}}\right)$$

$$= \left(\Gamma_{\hat{\alpha}}\mathbf{e}^{\mu}\overline{\mathbf{e}^{\nu}}\right)\cdot 1 .$$

Also, for any two 6-vectors A and B,

$$\frac{1}{2}(AB - BA) = \frac{1}{2}(AB - \overline{B}\,\overline{A}) \tag{4.70}$$

$$= \frac{1}{2}(AB - \overline{AB}),$$

so just calculate the product AB and forget about the scalar part. With these shortcuts it is fairly straight forward to calculate the curvature 6-vectors

$$\mathcal{R}_{01} = (-\Phi'' + \Phi'\Lambda' - \Phi'^2)e^{-2\Lambda}\mathbf{e}^0\overline{\mathbf{e}^1}, \quad \mathcal{R}_{02} = -\frac{\Phi' e^{-2\Lambda}}{r}\mathbf{e}^0\overline{\mathbf{e}^2} \tag{4.71}$$

$$\mathcal{R}_{03} = \frac{-\Phi' e^{-2\Lambda}}{r}\mathbf{e}^0\overline{\mathbf{e}^3}, \quad \mathcal{R}_{12} = -\frac{\Lambda' e^{-2\Lambda}}{r}\mathbf{e}^1\overline{\mathbf{e}^2}$$

$$\mathcal{R}_{13} = -\frac{\Lambda' e^{-2\Lambda}}{r}\mathbf{e}^1\overline{\mathbf{e}^3}, \quad \mathcal{R}_{23} = \frac{1}{r^2}(e^{-2\Lambda} - 1)\mathbf{e}^2\overline{\mathbf{e}^3}.$$

The components of the curvature tensor can be read off the curvature 6-vectors and since these are components with respect to a tetrad these give the tidal forces that an (not necessarily freely falling) observer feels. If one wants the tidal forces for some other observer, then all that needs to be done is to Lorentz tranform the curvature 6-vectors to the correct frame.

Finally, calculating the Einstein 4-vectors results in

$$\mathcal{G}_0 = \left[\frac{2\Lambda' e^{-2\Lambda}}{r} + \frac{1}{r^2}(1 - e^{-2\Lambda}) \right] \mathbf{e}_0, \tag{4.72}$$

$$\mathcal{G}_1 = \left[\frac{-2\Phi' e^{-2\Lambda}}{r} + \frac{1}{r^2}(1 - e^{-2\Lambda}) \right] \mathbf{e}_1,$$

$$\mathcal{G}_2 = \left(-\Phi'' + \Phi'\Lambda' - \Phi'^2 + \frac{\Phi'}{r} + \frac{\Lambda'}{r} \right) e^{-2\Lambda} \mathbf{e}_2$$

$$\mathcal{G}_3 = \left[-\Phi'' + \Phi'\Lambda' - \Phi'^2 - \frac{\Phi'}{r} - \frac{1}{r^2}(1 - e^{2\Lambda}) \right] e^{-2\Lambda} \mathbf{e}_3.$$

5 Spinors for Minkowski Spacetime in the Pauli Algebra

5.1 Motivation

Tensors have a physically natural role as geometrical objects as well as a mathematical role as elements of carrier spaces of representations of the restricted Lorentz group, $\mathcal{L}_+^\uparrow$. Essentially, the transformation rule for tensors of a given valence is equivalent (Naber 1992) to a representation of $\mathcal{L}_+^\uparrow$ on the space of all tensors of that valence. A possible question is, then, whether or not all geometrical objects are realizable as representation spaces for the restricted Lorentz group. Much to the surprise and possible chagrin of the physics community, Dirac, in 1928 proposed a fundamental physical law that employed mathematical objects found not in representation spaces of $\mathcal{L}_+^\uparrow$, but in representation spaces of the universal covering group of $Spin+(V)$, of the restricted Lorentz group. Since then, these objects, called spinors, have been directly observed at the microscopic level in neutron inteferometry experiments, and at the macroscopic level, a gedanken experiment has been devised by Aharonov and Susskind (1967). Thus, spinors have fundamental physical significance.

Mathematically, spinors are also more fundamental than tensors. This is because the compositon of any representation of $\mathcal{L}_+^\uparrow$ with the covering map from $Spin+(V)$ to $\mathcal{L}_+^\uparrow$ naturally induces a representation of $Spin+(V)$; however, not all representations of $Spin+(V)$ arise in this way. Representations of $Spin+(V)$ not induced by representations of $\mathcal{L}_+^\uparrow$ are called "spinor" or double valued representations of the restricted Lorentz group. For an excellent, more detailed and slightly different motivational account (both physical and mathematical) of this, see Wald (1984).

In the literature there are seemingly different ways of realizing spinor spaces (Figueiredo et. al. 1990), and recently there has been some controversy over what types of spaces can be used as carrier spaces of representations of $Spin+(V)$ [Piazesse 1993]. The concern is that if the representation space is a subspace of a geometrical algebra generated by 4-vectors, then the transformation law for spinors is incompatible with the transformation law for 4-vectors. The rest of this chapter shows that an algebraic definition of spinors (Chevalley 1954, Riesz 1958) allows for the construction from spinors of geometrical objects having various types of transformation laws (Jones and Baylis 1993).

5.2 Spinors as Elements of the Pauli Algebra

The left regular representation [Budinich and Trautman 1988] of the Pauli algebra,

$$\rho : \mathcal{P} \rightarrow End_C(\mathcal{P}), \tag{5.1}$$

defined by

$$\rho(a)b = ab \tag{5.2}$$

is itself reducible because the Pauli algebra contains left ideals that are invariant under this representation. These reduced representations are themselves irreducible if the left ideal is minimal, i.e., if it does not contain any other left ideal as a proper subset. The representations of $Spin+(V)$ considered here are constructed by restricting irreducible representations of $\mathcal{P}$ to $Spin+(V) \subset \mathcal{P}$.

Let $\hat{v}$ be a unit spatial vector and define the left ideal of $\mathcal{P}$

$$S = \mathcal{P}\frac{1}{2}(1 + \hat{v}). \tag{5.3}$$

The reduced left regular represention of $\mathcal{P}$ on S, when restricted to $Spin+(V)$ has the natural action

$$\eta \rightarrow L\eta \tag{5.4}$$

for $\eta \in S$ and $L \in Spin+(V)$. Let

$$\alpha_0 = \frac{1}{2}(1 + \hat{v}) \tag{5.5}$$

and

$$\alpha_1 = \hat{u}\alpha_0. \tag{5.6}$$

where $\hat{u}$ is any spatial vector orthogonal to $\hat{v}$. It is easily shown that α_0 is idempotent, i.e., $(\alpha_0)^2 = \alpha_0$, α_1 is nilpotent, i.e., $(\alpha_1)^2 = 0$, and that $\alpha_0\alpha_1 = 0$. Any element of $\mathcal{S}$ can be written in the form $(a^0 + \vec{a})\frac{1}{2}(1 + \hat{v})$ for some $a^0 + \vec{a}$ in $\mathcal{P}$. But

$$(a^0 + \vec{a})\frac{1}{2}(1 + \hat{v}) = \left(a^0 + \vec{a}_\perp + \vec{a}\cdot\hat{v}\right)\frac{1}{2}(1 + \hat{v}) \tag{5.7}$$

$$= \left(\hat{u}\cdot\vec{a}_\perp + i\left(\hat{u}\times\vec{a}_\perp\right)\cdot\hat{v}\right)\alpha_1$$

$$+ (a^0 + \vec{a}\cdot\hat{v})\alpha_0,$$

where $\vec{a}_\perp = \vec{a} - \vec{a}\cdot\hat{v}\hat{v}$. Therefore, $\{\alpha_0, \alpha_1\}$ spans $\mathcal{S}$. Suppose $c^0, c^1 \in \mathbf{C}$ and

$$0 = c^0\alpha_0 + c^1\alpha_1. \tag{5.8}$$

Multilplying this equation by α_0 on the right gives that c_0 is zero, which in turn gives that c_1 is also zero. Thus, $\{\alpha_0, \alpha_1\}$ is a basis for $\mathcal{S}$.

Any left ideal of $\mathcal{P}$ constructed as above is a two dimensional complex vector space; therefore, if $\mathcal{S}$ contains a left ideal $\mathcal{L}$ as a proper subspace, $\mathcal{L}$ must have dimension one. Suppose $\mathcal{L} = \mathbf{C}$. There are Pauli elements that when multilplied by a complex scalar give a nonscalar; hence, $\mathbf{C}$ is not a left ideal of the Pauli algebra. Now suppose $\mathcal{L} = span_{\mathbf{C}}\{a\}$, where a is a nonscalar Pauli element. If a is invertible then $a^{-1}a = 1$ is not in $\mathcal{L}$, and if a is not invertible then $\bar{a}a = 0$ is also not in $\mathcal{L}$. These considerations demonstrate that it is not possible to have a one dimensional left ideal of $\mathcal{P}$; thus $\mathcal{S}$ is a minimal left ideal of $\mathcal{P}$ and the representation of $\mathcal{P}$ defined on $\mathcal{S}$ is irreducible. As seems plausible, the representation of $Spin+(V)$ on $\mathcal{S}$ is also irreducible, but this will not be shown here.

Elements of the space $\mathcal{S}$ are traditionaly referred to in the physics literature as two component contravariant undotted spinors. The product of a barred spinor with a spinor is invariant under the action of $Spin+(V)$, $\overline{\alpha_0}\alpha_0 = \overline{\alpha_1}\alpha_1 = 0$, and $\overline{\alpha_1}\alpha_0 = -\overline{\alpha_0}\alpha_1$. This allows the definition of a $Spin+(V)$ invariant skew-symmetric inner product.

$$< \, . \, > : S \times S \to \mathbf{C}. \tag{5.9}$$

by

$$\overline{\eta}\xi = < \eta , \xi > \overline{\alpha_0}\alpha_1 . \tag{5.10}$$

The basis $\{\alpha_0 , \alpha_1\}$ has the property that $< \alpha_0 , \alpha_1 > = 1$; any basis for S that has this characteristic is called a spin frame.

When restricted to S, "bar" is an isomorphism onto the minimal right ideal of

$$S^* := \overline{\alpha_0}\mathcal{P} \tag{5.11}$$

$$= \frac{1}{2}(1 - \hat{v})\mathcal{P}.$$

the space of undotted covariant spinors. If $\eta \in S$, then under $L \in Spin^+(V)$, $\overline{\eta} \in S^*$ transforms like

$$\overline{\eta} \to \overline{\eta}\overline{L}. \tag{5.12}$$

This defines a representation of $Spin^+(V)$ that is equivalent (Isham 1989) to the representation defined on S with bar playing the role of the intertwining operator. The dual space of S, that is the space of linear functionals on S is identified with S by defining,

$$\overline{\eta}(\xi) = < \eta , \xi > \tag{5.13}$$

for every $\overline{\eta} \in S^*$ and every $\xi \in S$. The basis for S^* dual to $\{\alpha_0 , \alpha_1\}$ is $\{\alpha^0 = -\overline{\alpha_1} , \alpha^1 = \overline{\alpha_0}\}$, i.e.,

$$\alpha^0(\alpha_0) = \alpha^1(\alpha_1) = 1 \tag{5.14}$$

and

$$\alpha^0(\alpha_1) = \alpha^1(\alpha_0) = 0. \tag{5.15}$$

If $\eta = \eta^A \alpha_A$ then $\overline{\eta} = \eta_A \alpha^A$, where

$$\eta_A = \eta^B \epsilon_{BA} . \tag{5.16}$$

The space of dotted contravariant spinors is the minimal right ideal

$$S^{\cdot} := \alpha_0^{\cdot}\mathcal{P} \tag{5.17}$$

$$= \frac{1}{2}(1 - \hat{v})\mathcal{P}.$$

S and $S^{\cdot}$ are mapped bijectively but anti-linearly by "dagger". The action of $\bar{L} \in Spin+(V)$ on $S^{\cdot}$ is

$$\eta^{\cdot} \to \eta^{\cdot} L^{\cdot}. \tag{5.18}$$

Because "dagger" is anti-linear, the representation of $Spin+(V)$ on $S^{\cdot}$ is inequivalent to the representations defined on S and S^{*}. The basis for $S^{\cdot}$ that corresponds with the $\{\alpha_0, \alpha_1\}$ basis of S is $\{\alpha_0^{\cdot}, \alpha_1^{\cdot}\}$ and $\eta^{\cdot} = \eta^{\dot{A}}\alpha_{\dot{A}}^{\cdot}$ with

$$\eta^{\dot{A}} = \eta^{A*}. \tag{5.19}$$

The space of dotted covariant spinors is the minimal left ideal

$$S^{\cdot *} := \mathcal{P}\overline{\alpha_0}^{\cdot} \tag{5.20}$$

$$= \mathcal{P}\frac{1}{2}(1 - \hat{v}).$$

The left action of $L \in Spin+(V)$ on $S^{\cdot *}$ is

$$\bar{\eta}^{\cdot} \to \bar{L}^{\cdot}\bar{\eta}^{\cdot}. \tag{5.21}$$

Since the compostion of "bar" with "dagger" is an anti-linear bijection from S to $S^{\cdot *}$, the representation thus defined is equivalent to the one defined on $S^{\cdot}$ and inequivalent to the ones defined on S and S^{*}. The basis for $S^{\cdot}$ that corresponds with the $\{\alpha_0, \alpha_1\}$ basis of S is $\{\alpha^{0\cdot}, \alpha^{1\cdot}\}$ and $\bar{\eta}^{\cdot} = \eta_{\dot{A}}\alpha^{A\cdot}$ with

$$\eta_{\dot{A}} = \eta_A{}^{*}. \tag{5.22}$$

The space of dotted covariant spinors is associated with the dual space of $S^{\cdot}$ via

$$\bar{\eta}^{\cdot}(\xi^{\cdot}) = \langle \eta, \xi \rangle^{*} \tag{5.23}$$

and $\{\alpha^{0\cdot}, \alpha^{1\cdot}\}$ is the basis dual to $\{\alpha_0^{\cdot}, \alpha_1^{\cdot}\}$.

5.3 Minkowski Vector Space from Spinor Spaces

As outlined in the first section of this chapter, spinors seem to be more fundamental, both physically and mathematically than tensors, so now 4-vectors will be built up from spinors. From the first chapter, the space of 4-vectors, Minkowski vector space, is just the subspace of the Pauli algebra that is invariant under "dagger", and from the second chapter, an element v of this space transforms like

$$v \to L v L^{*} \tag{5.24}$$

under $L \in Spin-(1^{*})$. Under L, the product of any $\eta \in S$ with any $\xi^{*} \in S^{*}$ clearly transforms like

$$\eta\xi^{*} \to L\eta\xi^{*}L^{*}, \tag{5.25}$$

which is just the transformation law required of 4-vectors. Not all such products are candidates for 4-vectors, however, because they are not necessarily invariant under "dagger". In fact, as shown next, every Pauli element can be expressed as sums of spinor products.

Form the products

$$l = \alpha_0\alpha_0^{*}, \quad n = \alpha_1\alpha_1^{*}, \quad m = \alpha_0\alpha_1^{*}, \quad m^{*} = \alpha_1\alpha_0^{*} \tag{5.26}$$

of spinor basis elements that transform like this. Set

$$c_1 l + c_2 m + c_3 n + c_4 m^{*} = 0, \tag{5.27}$$

with every c_α a complex scalar. Multiplying this by $\overline{\alpha_0}$ on both sides and using (5.6) gives that

$$\begin{aligned} 0 &= c_3\overline{\alpha_0}\bar{u}\alpha_0\bar{u}\overline{\alpha_0} \\ &= c_3\overline{\alpha_0}, \end{aligned} \tag{5.28}$$

i.e., $c_3 = 0$. By other similar such manipulations, every c_α is seen to be zero and hence $\{l, m, n, m^{*}\}$ is a linearly independent set and therefore a basis for the Pauli algebra. This set is called a null tetrad (De Felice and Clarke 1990), because

$$l \cdot \bar{l} = m \cdot \overline{m} = n \cdot \bar{n} = 0, \tag{5.29}$$

$$l \cdot \bar{n} = -m \cdot \overline{m}^* = 1,$$

$$l \cdot \overline{m} = l \cdot \overline{m}^* = n \cdot \overline{m} = n \cdot \overline{m}^* = 0.$$

A real tetrad is formed from the null tetrad by choosing

$$\mathbf{e}_0 = l + n, \quad \mathbf{e}_1 = m + m^*, \tag{5.30}$$

$$\mathbf{e}_2 = i(m^* - m), \quad \mathbf{e}_3 = l - n,$$

and $span_{\mathbb{R}}\{\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is a Minkowski vector space. Thus, the Pauli algebra can be thought of as a complexified Minkowski vector space. Since $\{\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $\{l, m, n, m^*\}$ are both bases (over $\mathbb{C}$) for $\mathcal{P}$, they are related by a change of basis matrix, i.e.,

$$\mathbf{e}_\mu = \sigma_\mu{}^{AA}\alpha_A\alpha_A. \tag{5.31}$$

The elements of this matrix are called the Infeld-van der Waerden symbols (Penrose and Rindler 1984).

Other geometrical objects such as 6-vectors can be constructed as before from the products of 4-vectors and barred 4-vectors, but now the 4-vectors themselves are built up from spinors.

6 Conclusions

This dissertation has demonstrated the utility and effectiveness of the Pauli algebra approach to relativity, including the calculation of spacetime curvature. Observers were found to have a natural role in the Pauli algebra formalism and additional work needs to be done on more general spacetimes, e.g. axially symmetric spacetimes, using this aspect of the Pauli algebra. The method derived here for calculating spacetime connections and curvatures is very algorithmic and should be implementable on symbolic computer systems such as MAPLE. Geometrical objects in Minkowski spacetime having various transformation properties, including 4-vectors, were found to be constructible in the Pauli algebra from spinors. This algebraic treatment of spinors should be extended to the realm of general relativity, where a spinorial treatment of the connection 6-vectors should show an association with the Newman-Penrose formalism.

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