The Pentagram Map

Richard Schwartz

We consider the pentagram map on the space of plane convex pentagons obtained by drawing a pentagon's diagonals, recovering unpublished results of Conway and proving new ones. We generalize this to a ''pentagram map'' on convex polygons of more than five sides, showing that iterated images of any initial polygon converge exponentially fast to a point. We conjecture that the asymptotic behavior of this convergence is the same as under a projective transformation. Finally, we show a connection between the pentagram map and a certain flow defined on parametrized curves.

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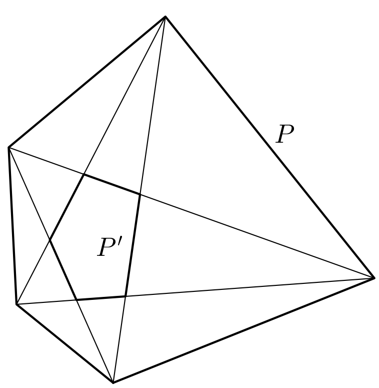

INTRODUCTION

A pentagon P, like the one shown on the left, cries out to have its pentagram drawn. Looking inside the pentagram, one sees a new pentagon P'. We call the transformation $P \to P'$ the *pentagram map*. The pentagram map can be defined more generally. Joining every other vertex of any strictly convex n-gon P produces a "pentagram" whose center is always another convex n-gon P'.

There is a question of labeling: Strictly speaking, $P \to P'$ is a map between *unlabeled* polygons. The composite map $P \to P''$ can unambiguously be considered as a map between labeled polygons, as in Figure 1. If n is odd, P' can be labeled in a unique way so that $P \to P'$ is the square root of $P \to P''$ as maps between labeled polygons.

The purpose of this paper is to describe some results and conjectures about the sequence P, P', P'', $P^{(3)}, \ldots$ Here is an overview. In Section 1, we review some basic facts about projective geometry. The pentagram map, which is defined entirely in terms of lines, behaves naturally with respect to any group action on the plane (or parts of the plane) that takes lines to lines. The projective group, which is the largest such group, figures prominently in our study.

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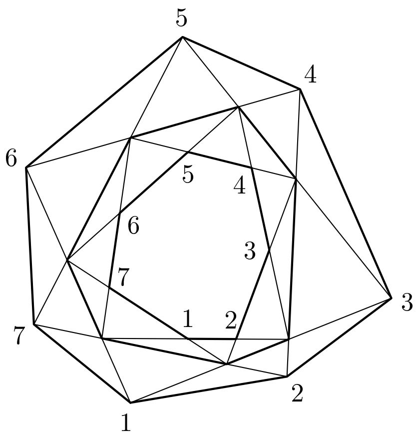

FIGURE 1

In Section 2, we describe what happens for pentagons and hexagons. The basic fact is that the pentagons P and P' are always projectively equivalent, and the hexagons H and $H^{(4)}$ are always projectively equivalent. This is to say that there are projective transformations T_P and T_H of the projective plane for which $T_P(P) = P'$ and $T_H(H) = H^{(4)}$. We study the correspondences $P \to T_P$ in detail and the correspondence $H \to T_H$ in much less detail.

In Section 3, we present what we know about the pentagram map in general. Our main result is that, for every strictly convex n-gon P, the sequence $\{P^{(k)}\}$ shrinks exponentially fast to a single point, which thus constitutes a projectively natural "center" for P. This result follows from the fact that the sequence lies in a compact subset of the space of projective classes of n-gons.

The facts run out after Section 3. In Section 4, we present a conjectural picture of the general behavior of the pentagram map on convex polygons. The picture leads to the statement that *asymptotically* the sequence $P, P', P'', \ldots$ behaves like the sequence $P, T_P(P), T_P^2(P), \ldots$ for some projective transformation T_P. Section 4 also contains a description of what happens when the pentagram map is defined in the complex projective plane. In Section 5, we show how the pentagram map is a discrete approximation to a certain flow defined on parametrized curves and give a heuristic derivation of the equation of this flow. The resulting flow seems to be of current interest to physicists. If the discrete process is indeed a good approximation to the flow, it probably would be useful for computer simulations.

1. PROJECTIVE GEOMETRY

The following is a list of facts about projective geometry that we will use throughout the paper. Most of them can be found in [Hilbert and Cohn-Vossen 1952] or are easy to derive from scratch.

$\mathbf{RP}^2$ is the space of lines through the origin in $\mathbf{R}^3$. Similarly, $\mathbf{CP}^2$ is the space of complex lines through the origin in $\mathbf{C}^3$. For simplicity, we will speak of the real case. Everything we say applies verbatim to the complex case. By intersecting the lines of $\mathbf{R}^3$ with the plane $\{z = 1\}$, we consider $\mathbf{R}^2$ as a subset of $\mathbf{RP}^2$. The set $\mathbf{RP}^2 - \mathbf{R}^2$ is called the *line at infinity*.

The general linear group $\mathrm{GL}_3(\mathbf{R})$ acts on $\mathbf{RP}^2$ because linear transformations take lines to lines. If we quotient $\mathrm{GL}_3(\mathbf{R})$ by scalar multiplication, the resulting group $\mathrm{PGL}_3(\mathbf{R})$ still acts on $\mathbf{RP}^2$. Elements of this group are called *projective transformations*. Two sets are said to be *projectively equivalent* if there is a projective transformation taking one to the other.

A line of $\mathbf{RP}^2$ is the set of lines of $\mathbf{R}^3$ contained in a two-dimensional linear subspace. In particular, the line at infinity corresponds to the subspace $\{z = 0\}$. Projective transformations permute these lines. The projective transformations that preserve the line at infinity restrict to give the affine transformations of $\mathbf{R}^2$.

A set of points is said to be in general position in the projective plane if no three points lie on the same line. In particular, a quadrilateral in $\mathbf{RP}^2$ is a set of four general position points. Here is a convenient characterization of the projective group: There is a unique projective transformation that takes one specified quadrilateral to another specified quadrilateral.

Suppose $l_1, \ldots, l_4$ are four linear subspaces of $\mathbf{R}^2$. Suppose further that s_j is the slope of l_j. Then the *cross ratio*

$$X(l_1, l_2, l_3, l_4) = \frac{(s_1 - s_2)(s_3 - s_4)}{(s_1 - s_3)(s_2 - s_4)}$$

is invariant under projective transformations that preserve the origin of $\mathbf{R}^2$. This implies that X extends to a well-defined invariant of four cyclically ordered lines of $\mathbf{RP}^2$, which contain a common point. Moreover, if m is any line that does not contain the origin, and m_1, m_2, m_3, m_4 are its intersections with l_1, l_2, l_3, l_4, we have

$$X(l_1, l_2, l_3, l_4) = \frac{\|m_1 - m_2\| \, \|m_3 - m_4\|}{\|m_1 - m_3\| \, \|m_2 - m_4\|}.$$

This formula defines the cross ratio of four cyclically ordered collinear points. The definition is independent of the choice of the lines l_j and is again projectively natural.

In particular, suppose m_1 and m_2 are two lines and p is a point belonging to neither line. Then p defines a projection map $m_p : m_1 \to m_2$ (Figure 2). It is not hard to verify that m_p preserves cross ratios of points.

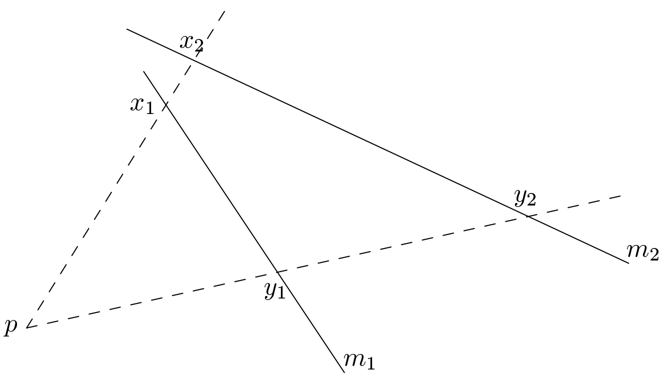

FIGURE 2. The projection m_p with center p takes x_1 to y_1 and x_2 to y_2.

In the next section, we will work with conjugacy classes of (diagonalizable) projective transformations. Let T be an element of $\mathrm{PGL}_3(\mathbf{R})$ and let $\hat{T}$ be a representative of T in $\mathrm{GL}_3(\mathbf{R})$. Define $\sigma_j(\hat{T})$ by the formula

$$\det(\hat{T} - \lambda I) = -\lambda^3 + \sigma_1(\hat{T})\lambda^2 - \sigma_2(\hat{T})\lambda + \sigma_3(\hat{T}).$$

Then $\sigma_1\sigma_2/\sigma_3$ and σ_1^3/σ_3 only depend on the conjugacy class of T in $\mathrm{PGL}_3(\mathbf{R})$, and completely determine the conjugacy class of a diagonalizable element of $\mathrm{PGL}_3(\mathbf{R})$.

2. PENTAGONS AND HEXAGONS

Pentagons

The pentagram map is defined on a pentagon P—even a nonconvex one—as long as the five points of P are in general position in the (complex) projective plane.

The basic fact for pentagons is the following.

Theorem 2.1. *The pentagons P and P' are projectively equivalent.*

This means that there is a projective transformation T_P so that $T_P(P) = P'$. I discovered this experimentally, and the fact is easy to prove algebraically. John Conway (personal communication) also discovered this fact years ago. He has a slick proof, which we sketch here.

Proof: Given a pentagon $P = (p_1, \ldots, p_5)$, let $l_{i,j}$ be the line containing p_i and p_j. Define

$$X_k = X(l_{k-1,k}, l_{k-2,k}, l_{k+2,k}, l_{k+1,k}),$$

where X is the cross ratio (Section 1). We call X_k a *corner invariant* of P. It is easy to verify that the projective class of P is completely determined by its corner invariants. By symmetry, we just have to see that $X_5(P) = X_5(P')$, for example. Figure 3 shows that projection from the point p_5

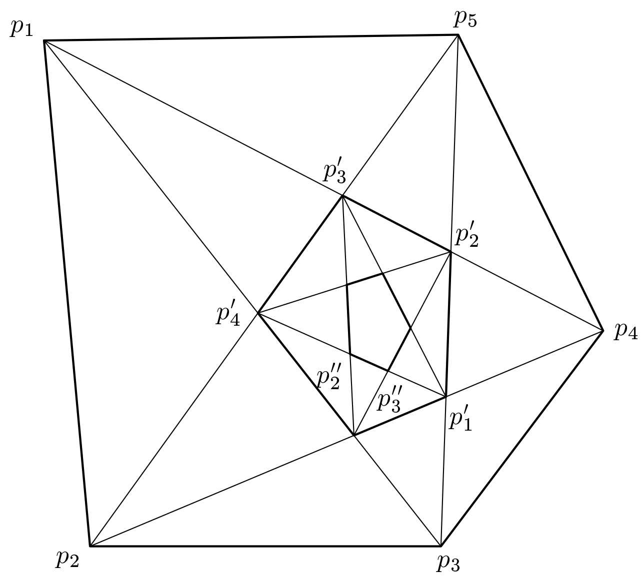

FIGURE 3

takes (p_1, p'_3, p'_2, p_4) to $(p'_4, p''_2, p''_3, p'_1)$. Therefore,

$$\begin{aligned} X_5(P) &= X(p_1, p'_3, p'_2, p_4) = X(p'_4, p''_2, p''_3, p'_1) \\ &= X(p'_1, p''_3, p''_2, p'_4) = X_5(P'). \end{aligned}$$

The third equality is a general symmetry of the cross ratio. This proof works just as well in the nonconvex case. □

Since the pentagram map commutes with projective transformations, we have

$$P^{(n)} = T_P^n(P).$$

If T_P is known, this formula tells practically all there is to know about the successive images of P under the pentagram map.

First we discuss T_P when P is a convex planar pentagon. In this case, we will usually take P to be the planar domain consisting of both the "pentagon proper" and its interior. In our situation, we have $P' \subset P \subset P^{(-1)}$. From this, it is easy to deduce that T_P fixes a unique point $c(P) \in P$ and a unique line $l(P)$ that misses P entirely. Furthermore, $P^{(n)}$ shrinks to the point $c(P)$ as $n \to \infty$. The "asymptotic shape" of $P^{(n)}$ is determined by the differential

$$d(P) = dT_P|_{c(P)}.$$

Theorem 2.2. *For a convex pentagon P, the derivative $d(P)$ is diagonalizable.*

In fact, the eigenvalues of $d(P)$ are equal if and only if P is projectively equivalent to the regular pentagon (this will be shown later in the formulas). This means that the pentagram map shrinks a random convex pentagon exponentially fast to a point at two different exponential rates. If the successive images of such a pentagon are rescaled to have constant area, they become exponentially long and thin in shape.

Proof: We first move P by a projective transformation so that $c(P) = 0$ and $l(P)$ is the line at infinity. Then $d(P) = T_P$. If $d(P)$ is not diagonalizable, we can, by conjugation, assume either that $d(P)$ is upper triangular or that $d(P)$ is a rotation by $\theta \neq 180°$. We will argue that the latter case cannot happen. A slight modification of the same argument rules out the former case as well.

If $d(P) = T_P$ is a rotation by, say, more than $180°$, the two lines $\overline{p_2p_3}$ and $\overline{p_2'p_3'}$ (in the notation of Figure 3) converge to the right of P. This means that

$$\text{area}\,\Delta p_2p_3p_4 > \text{area}\,\Delta p_2p_3p_1.$$

This argument can be repeated beginning with any of the five sides of P. Going counterclockwise, we obtain

$$\begin{aligned}\text{area}\,\Delta p_2p_3p_4 &> \text{area}\,\Delta p_1p_2p_3 > \text{area}\,\Delta p_5p_1p_2\\ &> \text{area}\,\Delta p_4p_5p_1 > \text{area}\,\Delta p_3p_4p_5 > \text{area}\,\Delta p_2p_3p_4,\end{aligned}$$

which is a contradiction. □

Now consider a general pentagon P. If we want to see what the successive images of P look like under the pentagram map, it suffices to consider the behavior of the pentagram map on any projective image of P. Choosing different projective images means looking from different perspectives. Concretely, we have the formula

$$T_{G(P)} = G \circ T_P \circ G^{-1}$$

for any projective transformation G. This gives us a correspondence between projective classes of pentagons and conjugacy classes of projective transformations.

This correspondence carries over to pentagons in the complex projective plane. One just puts the word "complex" before every statement. Let Ω be the set of (complex) projective classes of pentagons. We will abbreviate this correspondence by $[P] \to [T_P]$. If a pentagon P is normalized so that its first four points coincide with the points $(1,0)$, $(0,0)$, $(0,1)$ and $(1,1)$, the fifth point uniquely determines the conjugacy class. To satisfy the general position condition, this fifth point must miss the six (complex) lines joining these four points two by two. In other words, Ω can be identified with the complement of six particular (complex) lines in the (complex) projective plane.

Recall from Section 1 that the functions $\sigma_1\sigma_2/\sigma_3$ and σ_1^3/σ_3 uniquely determine the conjugacy class of a projective transformation. Therefore our correspondence $[P] \to [T_P]$ can be described by these two functions. I found the following equations experimentally:

$$\frac{\sigma_1\sigma_2}{\sigma_3} \equiv -1;$$

$$\psi(x,y) = \frac{\sigma_1^3(x,y)}{\sigma_3(x,y)} = \frac{x(x-y)(x+y-1)}{y(y-1)(x-1)}.$$

The equation for ψ extends by continuity to points in $\Omega - \mathbf{C}^2$, giving rise to a holomorphic function. We omit the derivation of these equations. They can easily be verified by computer or derived by using, say, Mathematica [Wolfram 1991].

Our formulas say that the correpondence $[P] \to [T_P]$ is determined by the *single* holomorphic function ψ. Furthermore, the formula for ψ allows one to compute the value of ψ explicitly for any given projective class of pentagons. In particular, one can verify the statements made earlier about the action of the pentagram map on a random convex pentagon.

Hexagons

Let H be a labeled hexagon, that is, a sextuple of points in general position in the (complex) projective plane. Let $H^{\#}$ denote the hexagon obtained by permuting the points of H according to the permutation $(123456) \to (456123)$. Note that $H^{\#\#} = H$.

The pentagram map is well-defined only on an open dense set of labeled hexagons, because for a sextuple of points in general position the intersections of diagonals need not be in general position.

The basic fact for hexagons is the following.

Theorem 2.3. *The hexagons $H^{\#}$ and H'' are projectively equivalent.*

Again I discovered this by computer. The algebra required to verify this fact is monstrous, but possible to do. One can construct an alternate "complex analysis" proof, based on the technique presented in the next section. We omit the details, because they are long and tedious. Another reason for the omission is that neither the algebraic proof nor the analytic proof gives any clue to why this theorem should be true. It would be nice to have a conceptual proof.

Let T_H be the projective transformation satisfying $T_H(H^{\#}) = H''$. Then we have the formula

$$H^{(4n)} = T_H^n(H).$$

I have not spent much time analyzing the correspondence $[H] \to [T_H]$, but I can make several remarks, analogous to those for pentagons.

In the case of convex hexagons, the fact that $H' \subset H \subset H^{-1}$ implies that T_H fixes a point $c(H) \in H$ and a line $l(H)$ missing H entirely. This means again that the long-term behavior of the pentagram map on H is determined by the differential $d(H) = T_H|_{c(H)}$. Computer experiments show that $d(H)$ is not diagonalizable in general—both other possible conjugacy classes of maps (shears and rotations) occur. Hexagons for which $d(H)$ is not diagonalizable can be found by looking at random perturbations of the regular hexagon. They're hard to find, but they're there.

For general hexagons, the correspondence $[H] \to [T_H]$ ought to have a description in terms of the two functions $\sigma_1\sigma_2/\sigma_3$ and σ_1^3/σ_3. Both functions are holomorphic and nonconstant, but I have not worked out the formulas.

3. CONVEX POLYGONS

In this section, we consider the pentagram map on strictly convex planar polygons. For short, we will usually just say "convex" instead of "strictly convex." Sometimes it is convenient to use the projective plane instead of the Euclidean plane. We will indicate when we switch back and forth.

Computer experiments show that, in general, the pentagram map on projective classes of polygons is not periodic. In other words, if Σ_n denotes the manifold of convex n-gons up to projective equivalence, the induced map $[P] \to [P']$ on Σ_n does not return to the identity under iteration.

By normalizing the first four points of an n-gon, it is easy to see that Σ_n is an open $(2n-8)$-dimensional ball. One can get a rough impression of the orbits in Σ_n by looking at the image of a single point. This means looking at certain two-dimensional projections of the orbit. The first thing I noticed was that the orbit of any point in Σ_n is always contained in a compact subset of Σ_n.

Assuming this for the moment, we can prove that the pentagram map shrinks convex polygons to points exponentially fast.

Theorem 3.1. *If P is a convex planar polygon, there is some $\eta_P < 1$ for which* $\operatorname{diam} P^{(k)} < \eta_P^k \operatorname{diam} P$.

Proof: The diameter of a convex polygon is equal to the largest distance between two vertices. If, contrary to the theorem's statement, the ratio between the diameters of consecutive polygons can get arbitrarily close to 1, it follows that the quantity

$$\max_s \frac{\|b_s - c_s\|}{\|a_s - d_s\|},$$

where a_s, b_s, c_s, d_s are the points shown in Figure 4 and s ranges over all pairs of distinct vertices, also gets arbitrarily close to 1. But it is easy to see that this maximum ratio is bounded away from 1 in an open neighborhood of any convex polygon. Since, by the compactness assumption, we can cover the orbit of p with a finite number of such open neighborhoods, we arrive at a contradiction. □

To complete the proof of Theorem 3.1, we have to show that the orbit of P in $\Sigma = \Sigma_n$ indeed has compact closure. The following proof relies essen-

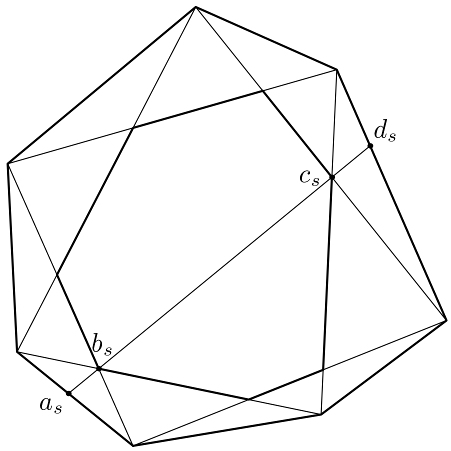

FIGURE 4

tially on a miracle. While searching for invariants of the pentagram map, I noticed that the equation

$$\prod_{p\in P} X_p(P) = \prod_{p'\in P'} X_{p'}(P')$$

holds true for every convex polygon, where $X_p(P)$ is the corner invariant introduced in the beginning of Section 2. The proof is instructive because it illustrates one method for establishing the truth of a computer-generated equation that seems impossibly large.

Lemma 3.2. *The orbit of any polygon P under the pentagram map has compact closure in Σ.*

Proof: Set $f(P) = \prod_{p\in P} X_p(P)$. The lemma is an immediate consequence of the following properties of the function f:

1. The level sets of f are compact in Σ.
2. $f(P) = f(P')$.

The function f is the product of n functions, each being the cross ratio of four lines determined by P. To study f, we need information on how the cross ratio of four given lines changes as the lines change.

Given two functions g_1 and g_2, we write $g_1 \sim g_2$ if there are positive constants C and ε such that

$$\frac{1}{C} \le \frac{g_1(t)}{g_2(t)} \le C$$

for $-\varepsilon \le t \le \varepsilon$. Let $S(t) = \{l_1(t), l_2(t), l_3(t), l_4(t)\}$ be a one-parameter family of ordered quadruples of lines such that each individual quadruple contains a common intersection point. We say that $S(t)$ has a simple (i,j) degeneration if $\angle(l_i(t), l_j(t)) \sim t$ and if $\angle(l_p(t), l_q(t)) \sim 1$ for all remaining pairs (p,q). It is easy to deduce the following information about the behavior of the function $Z(t) = X(l_1(t), l_2(t), l_3(t), l_4(t))$ as $t \to 0$.

Degeneration	Asymptotics
$(1,2)$	$Z \sim t$
$(1,3)$	$Z \sim 1/t$
$(1,4)$	$Z \sim 1$
$(2,3)$	$Z \sim 1$
$(2,4)$	$Z \sim 1/t$
$(3,4)$	$Z \sim t$

Proof of property 1: We show that, if $\{P_n\} \in \Sigma$ is a sequence with no convergent subsequence, $f(P_n)$ converges to 0, and therefore $\{P_k\}$ cannot be contained in a level set of f; thus the level sets of f are compact.

We normalize so that the first four vertices of P_n are $(1,0)$, $(0,0)$, $(0,1)$ and $(1,1)$. Let $p_5(n)$ be the fifth vertex of P_n; then $p_5(n)$ must lie in the interior of the triangle $T \in \mathbf{RP}^2$ bounded by the lines $y=0$, $y=1$ and $x=0$. By taking subsequences and possibly relabeling, it suffices to prove that $\{p_5(n)\} \to \tau \in \partial T$ implies $f(P_n) \to 0$.

Referring to Figure 5, we see that if $p_5 = p_5(n)$ approaches any point of ∂T on the line $y=0$ (including the point at infinity), the corner invariant X_2 approaches 0, and consequently so does f (since, for a strictly convex polygon, every corner

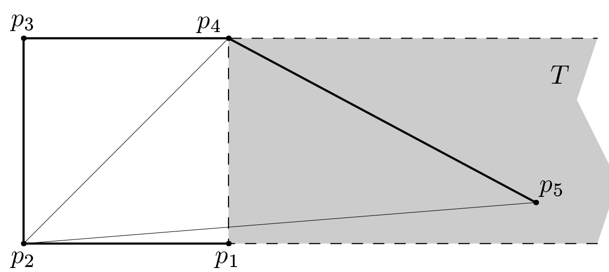

FIGURE 5

invariant is positive and less than 1). Likewise, if p_5 approaches the line $y=1$, we have $X_3 \to 0$. Finally, if p_5 approaches the line $x=1$, consideration of the sixth vertex in P_n (which lies in the triangle formed by the line $x=0$, the segment p_1p_5 and the extension of the segment p_4p_5) shows that $X_4 \to 0$.

Proof of property 2: Given a strictly convex planar n-gon P, let P_x denote the set of n ordered points in $\mathbf{RP}^2$ obtained by replacing the last point of P by the point x. Beginning with the regular n-gon,

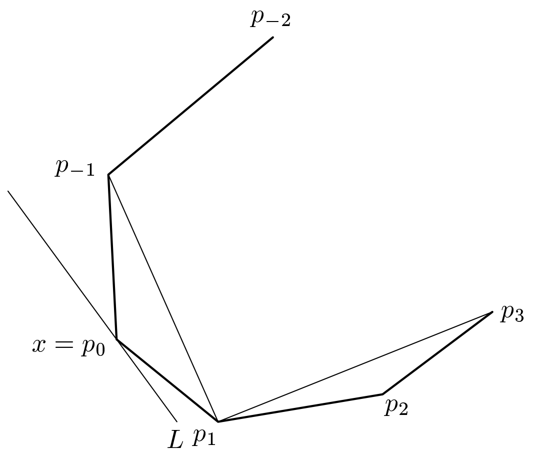

FIGURE 6. The cross ratio X_1 (say) has a zero if and only if the directions p_1p_0 and p_1p_{-1} coincide or the directions p_1p_2 and p_1p_3 coincide. As only $x = p_0$ varies, this happens if and only if p_0, p_1 and p_{-1} are aligned, that is, if and only if $p_0 = l_{1,-1} \cap L$. Similar arguments give the zeros and poles of all the X_i.

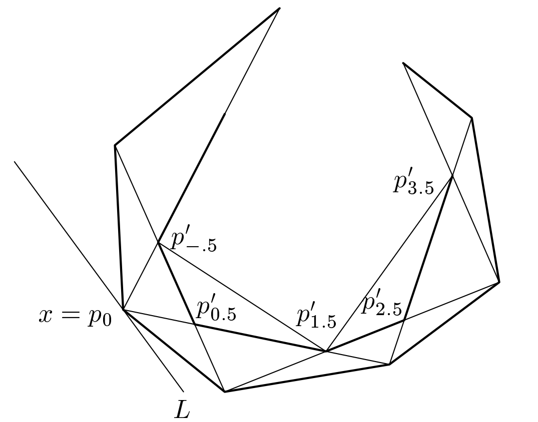

FIGURE 7. The cross ratio $X'_{1.5}$ has a zero if and only if the directions $p'_{1.5}p'_{.5}$ and $p'_{1.5}p_{-.5}$ coincide or the directions $p'_{1.5}p'_{2.5}$ and $p'_{1.5}p'_{3.5}$ coincide. As only $x = p_0$ varies, the second coincidence cannot occur, and the first entails $p'_{.5} = p'_{-.5}$, that is, $x = l_{1,-1} \cap L$. Similar arguments give the zeros and poles of all the X'_i.

we can move one point at a time until we obtain any other polygon P. Therefore, it suffices to prove that for every P, the function

$$\varphi_P(x) = \frac{f(P_x)}{f(P'_x)}$$

is constant wherever it is defined.

Let P be a fixed polygon and L a fixed line of $\mathbf{CP}^2$. Assume for the sake of exposition that P has enough points so that, in the argument below, all distinctly labeled points are in fact distinct. Also assume that L is in general position with respect to all lines mentioned below: There should be no triple intersections and no intersections at infinity. Since our constructions will always involve only finitely many lines, one can take L to be a small perturbation of any given line. Let $h(x) = f(P_x)|_L$ and $h'(x) = f(P'_x)|_L$. We prove that $\varphi_P|_L$ is constant by showing that the rational functions h and h' have the same singularities. Recall that $h = \prod X_j$ and $h' = \prod X'_j$, where $X'_j(x)$ is the j-th corner invariant of the polygon P'_x. To compute the singularities of h and h', we just have to compute the corresponding singularities of the X_j and the X'_j.

Let $q_{i,j} = l_{i,j} \cap L$, where $l_{i,j}$ is the line joining p_i and p_j. Away from the points $q_{i,j}$, h is finite and nonzero because every X_j is. As x varies on L, only the five functions X_{-2}, X_{-1}, X_0, X_1 and X_2 change at all, where the indices are taken modulo n. By reasoning as in Figure 6, or by using the table of asymptotics given earlier, we get for these functions the singularities listed in Table 1.

Function	Poles	Zeros
X_{-2}	$q_{-2,-3}$	$q_{-1,-2}$
X_{-1}	$q_{-1,-3}$	$q_{1,-1}$
X_0	$q_{2,-1}, q_{1,-2}$	$q_{1,2}, q_{-1,-2}$
X_1	$q_{1,3}$	$q_{1,-1}$
X_2	$q_{2,3}$	$q_{1,2}$

TABLE 1

Computing the singularities of the X'_i, though more tedious, is once more just a matter of going through all possibilities for coinciding directions; see Figure 7 for a typical case. It is convenient to label the vertices of P'_x with half-integers, so the functions comprising h' are $X'_{.5}, X'_{1.5}, \ldots$ As x varies on L, only the eight functions $X'_{-3.5}, \ldots, X'_{3.5}$ are affected. The resulting singularities are listed in Table 2.

From Tables 1 and 2, we see that $h = \prod X_j$ and $h' = \prod X'_j$ have the same singularities, and therefore that $\varphi_P|_L$ is constant. This implies that φ_P is constant everywhere, because any two points in the domain of φ_P can be connected to an appropriate third point by allowable lines L' and L''.

Our proof is complete except for the assumption that P has sufficiently many vertices. We complete the proof by showing that the invariance of f on convex $2k$-gons implies its invariance on convex

Function	Poles	Zeros
$X'_{-3.5}$	$q_{-2,-3}$	$q_{-2,-4}$
$X'_{-2.5}$	$q_{-2,-4}$	$q_{-2,-1}$
$X'_{-1.5}$	$q_{2,-1}$	$q_{1,-1}$
$X'_{-.5}$	$q_{1,3}$	$q_{1,2}$
$X'_{.5}$	$q_{-1,-3}$	$q_{-1,-2}$
$X'_{1.5}$	$q_{1,-2}$	$q_{1,-1}$
$X'_{2.5}$	$q_{2,4}$	$q_{1,2}$
$X'_{3.5}$	$q_{2,3}$	$q_{2,4}$

TABLE 2

k-gons. Choose a family $\{P_n\}$ of (nonconvex) $2k$-gons, each of which more and more nearly wraps twice around a fixed k-gon P. By analytic continuation, $f(P_n) = f(P'_n)$. By continuity,

$$f^2(P) = \lim f(P_n) = \lim f^2(P'_n) = f^2(P').$$

Since $f(P)$ and $f(P')$ are both positive, we get $f(P) = f(P')$.

This concludes the proof of Lemma 3.2, and also of Theorem 3.1. □

Theorem 3.1 gives us a way to assign to P a projectively natural center $c(P) = \bigcap P^{(n)}$, which is analogous to the center defined for pentagons in the previous section. It is easy to see that $c(P)$ varies continuously with the vertices of P. A projective center is like the center of mass, but it is more symmetric. The center of mass is natural with respect to the affine group, whereas a projective center is natural with respect to the larger projective group.

Open Questions

1. What are necessary and sufficient conditions on a convex polygon P in order for P and P'' to be projectively equivalent? I can prove that it suffices for P to be simultaneously inscribed in, and circumscribed about, a conic. Is this condition also necessary?
2. In Theorem 3.1, are there effective estimates for $\eta(P)$ based on the geometry of P?
3. Does the projective center $c(P)$ depend analytically on the vertices of P?
4. Does Theorem 3.1 have a conceptual proof, that is, one that doesn't require the aid of a miracle?

4. DYNAMICS

Saying that the orbits of points in Σ_n are precompact (Theorem 3.1) still leaves a lot to the imagination. Probably, more can be said.

Computer evidence, such as that shown in Figure 8, suggests that the pentagram map is recurrent on Σ_n. This is to say that every $[P] \in \Sigma_n$ is an accumulation point of the sequence $[P'], [P''], \ldots$ Another way of putting this is that infinitely many iterates of a convex polygon P are projectively equivalent to P, up to an arbitrarily small error.

Conjecture 4.1. *The pentagram map is recurrent on Σ_n.*

Here is a description of Figure 8. If P is a polygon, let $\hat{P}$ denote the projectively equivalent polygon whose first four points correspond with the unit square. Also, let P_k denote the k-th vertex of P. Then Figure 8 shows $\widehat{P^{(2n)}}_5$ for $1 \le n \le 200$. In other words, each point in Figure 8 is the pro-

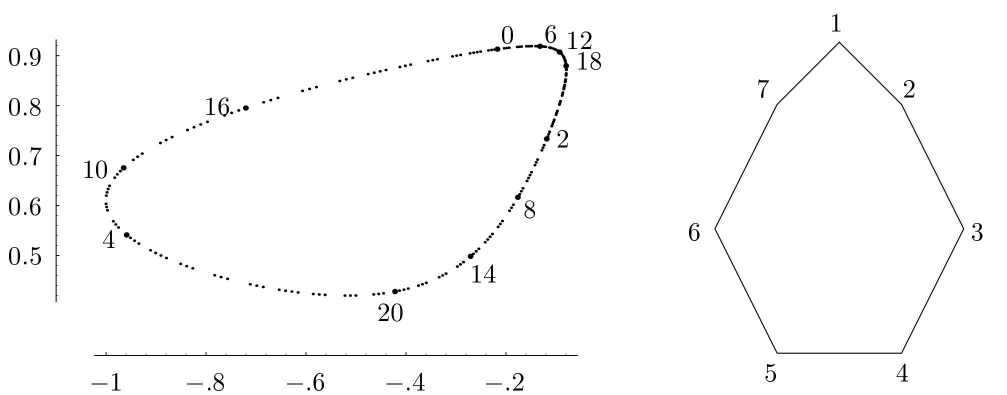

FIGURE 8. Projection of the orbit in Σ_7 of a bilaterally symmetric heptagon under the pentagram map, showing that the orbit is recurrent (Conjecture 4.1). The normalization is described in the paragraph following Conjecture 4.1.

jection of some point in Σ_n. To generate Figure 8, we took the simplest nontrivial example, which is a heptagon having bilateral symmetry. The projection of this orbit in Σ_n is a simple closed curve, indicating that the orbit of P in Σ_7 is a simple closed curve. Furthermore, the successive points of the orbit seem to be evenly spaced about this curve.

The next simplest case is that of a generic heptagon. A picture analogous to Figure 8 can be drawn for this case as well. One can best appreciate such a picture by watching it being drawn dynamically. I have done so, and my impression is that the orbit of a generic point in Σ_7 fills out a two-dimensional torus. This seems to be true in general, though the computer evidence is harder to interpret. Here is one form of this conjecture:

Conjecture 4.2. *Generically, the closure of an orbit in Σ_n is a torus (of some intermediate dimension). There is a natural flat metric on this torus for which the restriction of the pentagram map is an isometry.*

This says that the orbit of $[P]$ is recurrent in a very orderly way. If $[P^{(n)}]$ is very close to $[P]$, there is a projective transformation T_n for which $P^{(n)} = T_n(P)$ is very nearly true. Perhaps there is a well-defined limit

$$T_P = \lim_{n\to\infty} T_n^{1/n}.$$

If this is true, asymptotically the sequence P, P', P'', ... remains close to the sequence P, $T_P(P)$, $T_P^2(P)$, ... As for pentagons and hexagons, this would give a way to predict the general behavior of the pentagram map on P.

The pentagram map can be defined in a wider setting. Let S_n denote the set of n-tuples of general position points in the complex projective plane. Also let

$$S_n(k) = \{P :\ |j| \le k \Rightarrow P^{(j)} \in S_n\}.$$

It is easy to see that $S_n(k)$ is an open dense subset of $\mathbf{CP}^2 \times \cdots \times \mathbf{CP}^2$ that has full measure. (Its complement is a finite union of lower-dimensional subvarieties.) Furthermore, $S_n(k)$ is $\mathrm{PGL}_3(\mathbf{C})$-equivariant. Consequently, all forward and backward iterates of the pentagram map are well-defined on the full-measure equivariant subset $\bigcap S_n(k)$. Taking the quotient by the projective group, one has a well-defined iteration in

$$\Omega_n = \bigcap S_n(k)/\mathrm{PGL}_3(\mathbf{C}).$$

In some sense, Ω_n is the complexification of the set Σ_n discussed in Section 3.

Computations like the one shown in Figure 9 support the following conjecture:

Conjecture 4.3. *The pentagram map is recurrent on Ω_n.*

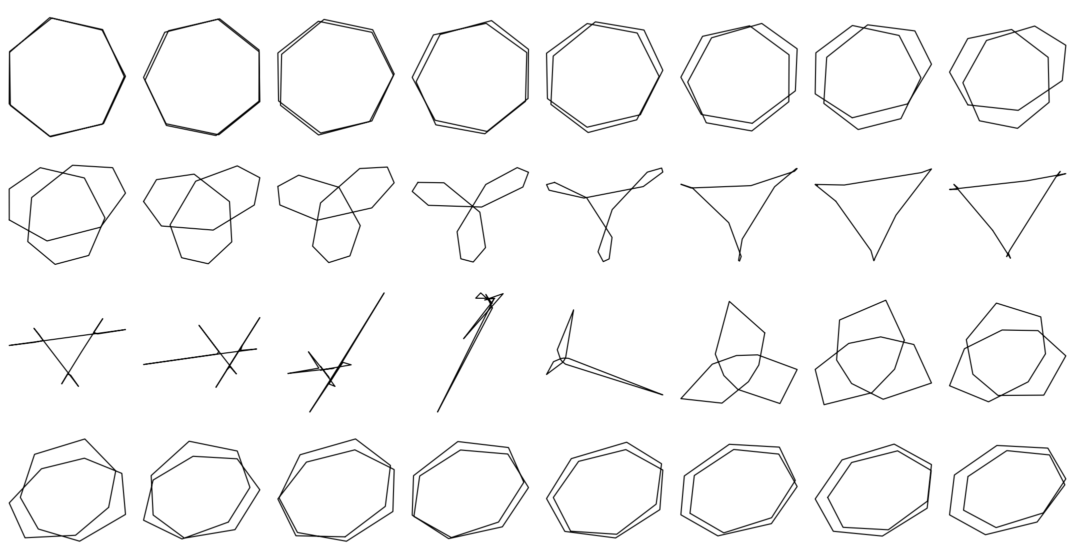

FIGURE 9. The action of the pentagram map on a 14-gon that wraps twice around the regular heptagon. (We take a small perturbation to produce some interesting behavior.) After about thirty iterations, the polygon nearly returns to its original shape (up to a projective transformation).

Imagine writing out the *Encyclopaedia Britannica* by hand and then taking a fine polygonal approximation. (You have to connect the last letter to the first one to get a closed polygon.) Applying the pentagram map several thousand times would turn the script into a seemingly random scribble. However, Conjecture 4.3 says that after an enormous number of additional iterations, the scribble would reassemble itself into another handwritten version of the *Encyclopaedia*!

5. THE LIMITING FLOW

The pentagram map seems to be a discrete analogue of a flow on parametrized curves: As we take finer and finer polygonal approximations to the curve, the action of the pentagram map appears to converge to a flow (Figure 10). We give here a heuristic derivation of the equation for this limiting flow.

Suppose P_n is an n-gon consisting of n points on the curve C that are evenly spaced according to the parameter. The corresponding vertices of P_n and P_n'' are about n^{-2} apart. To move P_n a distance t away from itself, one needs to make tn^2 iterations. If the pentagram map does converge to a flow C_t, we should have

$$\left.\frac{dC_t(s)}{dt}\right|_0 = \lim_{n\to\infty} n^2 N_n(s),$$

where $N_n(s)$ is the vector from $P_i^{(n)}$ (the i-th vertex of $P^{(n)}$, corresponding to the value s of the parameter) to $P_i^{(n+2)}$.

Computing this limit is quite messy algebraically. However, it is easy to see that $P_i^{(n+2)}$ is approximated by the intersection of the lines $P_{i-1}^{(n)}P_{i+2}^{(n)}$ and $P_{i+1}^{(n)}P_{i-2}^{(n)}$. Denoting by $M_n(s)$ the vector from $P_i^{(n)}$ to this intersection point, we have

$$\lim_{n\to\infty} n^2 N_n(s) = \lim_{n\to\infty} n^2 M_n(s).$$

This limit is much easier to compute:

$$\lim_{n\to\infty} n^2 M_n(s) = C''(s) - \tfrac{2}{3}W(s)C'(s),$$
$$W = \frac{\det(C', C''')}{\det(C', C'')},$$

the derivatives being with respect to the parameter s. Thus the evolution equation is

$$\left.\frac{dC_t(s)}{dt}\right|_0 = C''(s) - \tfrac{2}{3}W(s)C'(s).$$

Recall that our proof of Theorem 3.1 relies on the function

$$f(P) = \prod_{v\in P} X_v(P).$$

The equation $f(P) = f(P')$ has an analogue here. The quantity $f(P_n)$ tends to ∞ with n, so we have to be a little careful taking limits. Assuming that

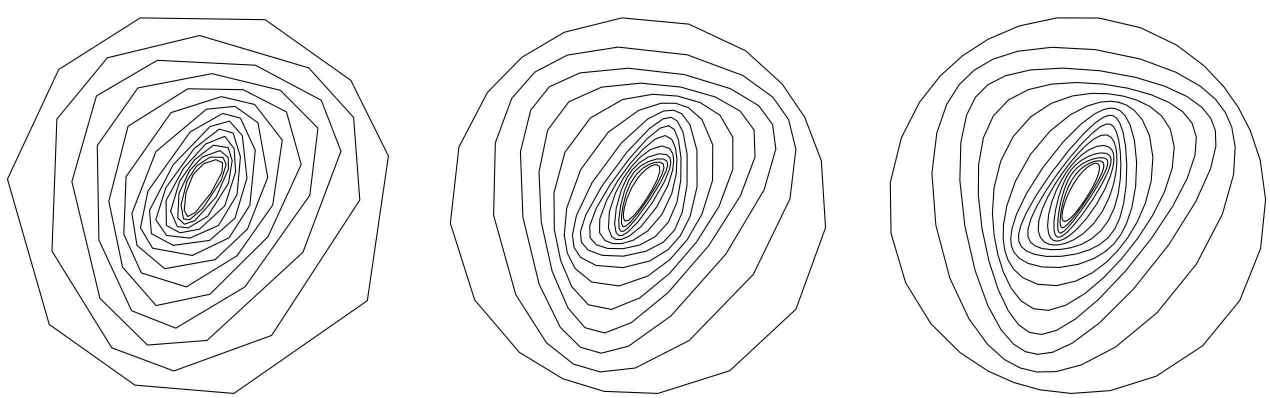

FIGURE 10. Action of the pentagram map on 10-, 19- and 30-sided polygonal approximations to the parametrized circle $(\cos\theta(t), \sin\theta(t))$, where $\theta(t) = t + .1\cos t + .07\sin(2t+\pi/3) + .1\cos(3t+\pi/5)$. The spacing between successive images appears to be the same in the three pictures because the time step has been renormalized: it corresponds to one iteration of the pentagram map in the first picture, four in the second and nine in the third.

$v = C(s)$ is always a vertex of P_n, the following limit is well-defined:

$$\psi(s) = \lim_{n\to\infty} n^2 \log(9\ X_v(P_n)).$$

It is not hard to work out the equations

$$\psi = \left(\frac{A_{13}}{A_{12}}\right)' - 3\left(\frac{A_{13}}{A_{12}}\right)^2 - 3\,\frac{A_{23}}{A_{12}},$$

where

$$A_{ij} = \det(C^{(i)}, C^{(j)}).$$

Then the integral $\int_{S_1} \psi(s)\, ds$ is conserved by the flow.

The invariant ψ is one of the two projective differential invariants of a (nondegenerate) curve immersed in the projective plane. Both the flow and the conserved quantity are known to physicists (see for example [Drinfeld and Sokolov 1985; DiFrancesco et al. 1990]). I don't know how well the pentagram map approximates the flow. It would be nice if the approximation was good, because then the discrete process would give an easy way to simulate the flow by computer.

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