ANNALES DE L'INSTITUT FOURIER

Raúl FELIPE *&* GLORIA MARÍ BEFFA

The pentagram map on Grassmannians

CENTRE
MERSENNE

THE PENTAGRAM MAP ON GRASSMANNIANS

by Raúl FELIPE & Gloria MARÍ BEFFA

————

ABSTRACT. — In this paper we define a generalization of the pentagram map to a map on twisted polygons in the Grassmannian space $\mathrm{Gr}(n, mn)$. We define invariants of Grassmannian twisted polygons under the natural action of $\mathrm{SL}(nm)$, invariants that define coordinates in the moduli space of twisted polygons. We then prove that when written in terms of the moduli space coordinates, the pentagram map is preserved by a certain scaling. The scaling is then used to construct a Lax representation for the map that can be used for integration.

RÉSUMÉ. — Dans cet article nous définissons une généralisation de la carte pentagramme à une carte sur des polygones tordus dans l'espace Grassmannien $\mathrm{Gr}(n, nm)$. Nous définissons les invariants des polygones tordus Grassmanniens sous l'action naturelle de $\mathrm{SL}(nm)$, invariants qui définissent les coordonnées dans l'espace des modules des polygones torsadés. Nous prouvons ensuite que lorsqu'il est écrit en termes de coordonnées d'espace de modules, la carte de pentagramme est préservée par une certaine mise à l'échelle. La mise à l'échelle est ensuite utilisée pour construire une représentation Lax pour la carte qui peut être utilisée pour l'intégration.

1. Introduction

In the last five years there has been a lot of activity around the study of the pentagram map, its generalizations and some related maps. The map was originally defined by Richard Schwartz over two decades ago ([9]) and after a dormant period it came back with the publication of [7], where the authors proved that the map, when defined on *twisted* polygons, was completely integrable. The literature on the subject is quite sizable by now, as different authors proved that the original map on closed polygons was also completely integrable ([8, 11]); worked on generalizations to polygons in higher dimensions and their integrability ([1, 2, 4, 5]); and studied the

integrability of other related maps ([6, 10]). The subject has also branched into geometry and combinatorics, this bibliography refers only to some geometric generalizations of the map and is by no means exhaustive.

The success of the map is perhaps due to its simplicity. The original map is defined on closed convex polygons in $\mathbb{RP}^2$. The map takes a convex polygon in the projective plane to the one formed by the intersection of the lines that join every other vertex, as in the figure. The mathematical consequences of such a simple construction are astonishing (in particular, the pentagram map is a double discretization of the Boussinesq equation, a well-known completely integrable system modeling waves, see [7]). Integrability is studied not for the map itself, but for the map induced by it on the moduli space of planar projective polygons, that is on the space of equivalence classes of polygons up to a projective transformation. In [7] the authors defined it on *twisted* polygons, or polygons with a monodromy after a period N, and proved that the map induced on the moduli space is completely integrable. (The map is equivariant under projective transformations, thus the existence of the moduli induced map is guaranteed.)

In this paper we look at the generalization of the map from the Grassmannian point of view. If we think of $\mathbb{RP}^2$ as the Grassmaniann $\mathrm{Gr}(1, 3)$, that is, the space of homogeneous lines in $\mathbb{R}^3$, then the polygon would be a polygon in the Grassmannian under the usual action of $\mathrm{SL}(2 + 1)$, with each side representing a homogeneous plane as in the picture. The case of $\mathbb{RP}^{m-1}$ was studied in [1] where the authors proved that the generalized pentagram map was integrable for low dimensions, and conjectured that a scaling existed for the map that ensured the existence of a Lax pair and its integrability. The conjecture was proved in [5]. We can also consider this case as $\mathrm{Gr}(1, m)$, $m \geqslant 3$. From this point of view, it is natural to investigate the generalized map defined on polygons in $\mathrm{Gr}(n, mn)$, where m and n are positive integers, $m \geqslant 3$. In this paper we define and study the generalization of the pentagram map to twisted Grassmannian polygons in $\mathrm{Gr}(n, mn)$, $m \geqslant 3$.

The first step is to define the map on the moduli space of Grassmannian twisted polygons, that is, on the space of equivalence classes of Grassmannian twisted polygons, under the classical action of $\mathrm{SL}((m-1)n + n)$ that generalizes the projective action of $PSL(m)$ on $\mathbb{RP}^{m-1}$. We do that by carefully studying the moduli space and finding generic coordinates that can be used to write the map in a convenient way (as in the case of the original pentagram map, the map can only be defined generically). The coordinates are found with the use of a discrete moving frame constructed
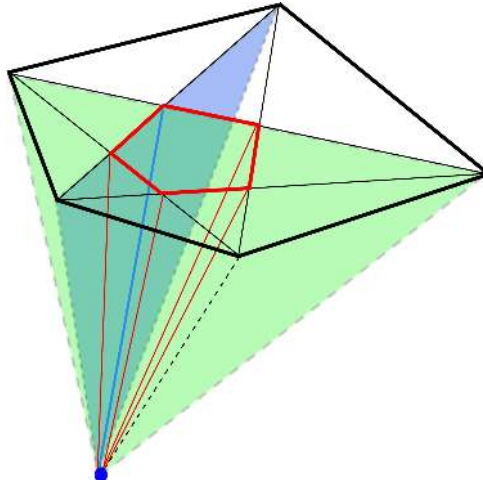
Figure 1.1. the pentagram map on pentagons in $\mathrm{Gr}(1,3)$

through a normalization process similar to the one described in [3]. The classification of invariants under this action is, as far as we know, unknown, and it is completed in Section 3.

In Section 4 we study the case $m = 2s$. In a parallel fashion to the study in [5], we proceed to write the pentagram map on the moduli space in the chosen coordinates, and we show that it can be written as the solution of a linear system of equations. We use that description and Cramer's rule to prove that the map is invariant under a certain scaling. As it was the case in [5], a critical part of the study is a fundamental lemma that decomposes the coefficient matrix of the system into terms that are homogeneous with respect to the scaling. This is Lemma 4 for the even dimensional case, and Lemma 9 for the odd dimensional one. The proofs of the rest of the results are supported by those two lemmas. Once the invariance under scaling is proved, the construction of a Lax representation is immediate when we introduce the scaling into a natural parameter-free Lax representation that exists for any map induced on the moduli space by a map on polygons. In Section 5 we prove the case $m = 2s + 1$.

The existence of a Lax representation with twisted boundary conditions does not guarantee the integrability in the geometric sense, although it is often taken as definition of integrability in some sectors of the community. Because of the existence of the spectral parameter λ in the Lax representation, and because the conjugation class of the monodromy is preserved

by the map, the map will preserve the characteristic equation of the monodromy, and hence it lies on the Riemann surfaces $\det(M(\lambda) - \mu I) = 0$, for the different values of the complex parameters λ and μ. The coefficients of this equation will be invariants of the map. This paper generalizes results in [1] and [5]. In [1] the authors described the existence of a Lax representation for the projective case (Grassmannian of dimension 1) assuming that the map is invariant under certain scaling - a fact they proved for $\mathbb{RP}^3$ and $\mathbb{RP}^5$. The invariance under scaling for the general case was proved in [5]. In [1] the authors also described the complete geometric integration for the lowest dimension $n = 3$ (the well-known pentagram map is the case $n = 2$). The lowest interesting dimension in our study is already very high dimensional as the dimension of the Grassmannian increases fast, and we do not include further work in that direction.

2. Definitions and notations

Let $\mathrm{Gr}(p, q)$ be the set of all p-dimensional subspaces of $V = \mathbb{R}^q$ or $V = \mathbb{C}^q$. Each $l \in \mathrm{Gr}(p, q)$ can be represented by a matrix X_l of size $q \times p$ such that the columns form a basis for l. We denote this relation by $l = \langle X_l \rangle$. Clearly $l = \langle X_l \rangle = \langle X_l d \rangle$ for any $d \in \mathrm{GL}(p)$, and the representation is not unique. Hence, $\mathrm{Gr}(p, q)$ can be viewed as the space of equivalence classes of $q \times p$ matrices, where two matrices are equivalent if their columns generate the same subspace. An element of this class, X_l is called *a lift of l*. The name reflects $\mathrm{Gr}(p, q)$ admitting the structure of a homogeneous space of dimension $p(q - p)$. Indeed, consider the Lie group $\mathrm{SL}(q + p)$, represented by block matrices of the form

$$\begin{pmatrix} A_{q-p \times q-p} & B_{q-p \times p} \\ C_{p \times q-p} & E_{p \times p} \end{pmatrix}.$$

Let H be the subspace defined by $B_{q-p \times p} = 0$. One can show that $\mathrm{SL}((q - p) + p)/H$ is isomorphic to $\mathrm{Gr}(p, q)$ and the natural action of $\mathrm{SL}((q - p) + p)$ on $\mathrm{Gr}(p, q)$ is given by

$$g \cdot \langle X \rangle = \langle g X \rangle.$$

Consider $\mathrm{Gr}(n, mn)$ for any positive integers n, m, and let

$$\mathrm{SL}((m-1)n + n) \times \mathrm{Gr}(n, mn) \longrightarrow \mathrm{Gr}(n, mn)$$

be the natural action of the group $\mathrm{SL}(mn)$ on $\mathrm{Gr}(n, mn)$.

A *twisted N-gon* in $\mathrm{Gr}(n, mn)$ is a map $\phi : \mathbb{Z} \longrightarrow \mathrm{Gr}(n, mn)$ such that $\phi(k + N) = M \cdot \phi(k)$ for all $k \in \mathbb{Z}$ and for some $M \in \mathrm{SL}(mn)$. The matrix M is called the *monodromy* of the polygon and N is the *period*. We will also denote an N-gon by $\wp = (l_k)$, where $l_k = \phi(k)$.

Let $X = (X_k)$ be an arbitrary lift for an N-gon $\wp = (l_k)$ with monodromy M, and choose X so it is also twisted, that is, $X_{N+k} = MX_k$ for all k. For any discrete closed N-polygon $d = (d_k)$ in $\mathrm{GL}(n)$ (i.e. satisfying $d_{k+N} = d_k$), we have that $Xd = (X_k d_k)$ is also a lift for the same polygon, with the same monodromy M.

Let us denote by $\mathcal{P}_N$ the moduli space of twisted N-gons in $\mathrm{Gr}(n, mn)$, that is, the space of equivalence classes of twisted polygons under the natural action of $SL(mn)$. We will also denote by $\mathcal{P}l_N$ the moduli space of N-gons in $\mathbb{R}^{mn \times n}$ (or $\mathbb{C}^{mn \times n}$, wherever the lifts live), under the linear action of $\mathrm{SL}(mn)$.

A N-gon $\wp = (l_k)$ is called *regular* if the matrix

$$\rho_k = (X_k \ X_{k+1} \ldots X_{k+m-2} \ X_{k+m-1})$$

satisfies the following condition

$$(2.1) \qquad \det \rho_k = |X_k \ X_{k+1} \ldots X_{k+m-2} \ X_{k+m-1}| \neq 0,$$

for any $k \in \mathbb{Z}$ and any lift X (clearly, it suffices to check the condition for one particular lift). In other words, the columns of the matrix constitute a basis of $\mathbb{R}^{mn}$ (or $\mathbb{C}^{mn}$) for all $k \in \mathbb{Z}$.

3. The moduli space of twisted polygons in $\mathrm{Gr}(n, mn)$

In this section we will prove that the moduli space of regular twisted polygons, $\mathcal{P}_N$, is a $N(m-1)n$-dimensional manifold and will define local coordinates.

Assume $\wp = (l_k)$, $l_k \in \mathrm{Gr}(n, mn)$ is a regular twisted N-gon and let $\{X_k\}$ be any twisted lift. By dimension counting, and given that $\wp$ is regular, for any $k = 0, \ldots, N-1$ we can find $n \times n$ matrices a_k^i, $i = 0, \ldots, m-1$ such that

$$(3.1) \qquad X_{k+m} = X_{m+k-1}a_k^{m-1} + \cdots + X_{k+1}a_k^1 + X_k a_k^0.$$

Notice that if ρ_k is as in (2.1), then

$$\text{(3.2)} \qquad \rho_{k+1} = \rho_k \begin{pmatrix} O_n & O_n & \dots & O_n & a_k^0 \\ I_n & O_n & \dots & O_n & a_k^1 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ O_n & \dots & I_n & O_n & a_k^{m-2} \\ O_n & \dots & O_n & I_n & a_k^{m-1} \end{pmatrix} = \rho_k Q_k,$$

where Q_k is the matrix above. Using (2.1), this implies that $\det a_k^0 \neq 0$, for all k. Notice that $\rho_N = \rho_0 Q_0 Q_1, \dots, Q_{N-1}$. Thus, if $\rho_0 = I$, the monodromy is given by $M = Q_0 \dots Q_{N-1}$, and for other choices $M = \rho_0 Q_0 \dots Q_{N-1} \rho_0^{-1}$. Thus, only the conjugation class of monodromy of the system defined by the matrices Q_k, $k = 0, \dots, N-1$, is well-defined, not the monodromy itself.

THEOREM 3.1. — *Assume m and N are coprime. Then, for any regular twisted N-gon, $\wp$, there exists a lift $V = (V_k)$ such that*

$$\text{(3.3)} \qquad \det(V_k, V_{k+1}, \dots, V_{k+m-1}) = 1,$$

for any $k = 0, \dots, N-1$, and such that if a_k^i are given as in (3.1), then

(1) *$a_k^0 = \text{diagonal}(r_k^1, \dots, r_k^s)$, where each r_k^s is an upper triangular Toeplitz matrix with $\det a_k^0 = 1$.*
(2) *We can choose V such that all a_k^{m-1}'s entries, for any k, are generated by $N(n^2 - n + 1)$ independent functions.*

The remaining $N(n-1)m$ entries of a_k^i, $i \neq 0, m-1$, together with those above, define a coordinate system on $\mathcal{P}_N$.

Proof. — Let $X = (X_k)$ be any twisted lift of the twisted polygon $\wp$. We will call $V_k = X_k d_k$ and show that we can find a closed polygon in $\text{GL}(n)$, $\{d_k\}$, such that the conditions of the theorem are satisfied. If $V_k = X_k d_k$ we have

$$\text{(3.4)} \quad (V_k, \dots, V_{k+m-1}) = (X_k, \dots, X_{k+m-1}) \, \text{diag}(d_k, d_{k+1}, \dots, d_{k+m-1}).$$

First of all we will show that condition (3.3) determines the values of $\delta_k = \det d_k$, for any k. Indeed, from (3.4) we have that

$$\prod_{i=k}^{k+m-1} \delta_i = Z_k,$$

where $Z_k = \det(X_k, \dots, X_{k+m-1})^{-1}$ is determined by the choice of lift. These equations determine δ_k uniquely whenever N and m are coprime, as

shown in [5]. Let us call b_k^i the invariants in (3.1) associated to X_k and a_k^i those associated to V_k. Then, substituting in (3.1), we have that

$$(3.5) \qquad a_k^i = d_{k+i}^{-1} b_k^i d_{k+m}.$$

Let $\mathbf{p} = \{p_k\}$ be a closed polygon in $\mathrm{GL}(n)$ and define the rth m-product to be the product of every m matrices starting at p_r until we get to the end of the period, that is

$$[p_r, \ldots, p_{r+jm}]_m = p_r p_{r+m} p_{r+2m} \cdots p_{r+jm},$$

with $r + (j+1)m \geqslant N$. If N and m are coprime, by repeatedly adding m to the subindex we can reach all N elements in $\{p_k\}$; that is, if N and m are coprime and $N = mq + s$, with $0 < s < m$, then all p_k, $k = 0, 1, \ldots, N-1$ appear in the product

$$(3.6) \quad \Pi_m(p_0)$$
$$= [p_0, \ldots, p_{N-s}]_m [p_{m-s}, \ldots,]_m \cdots [, \ldots, p_{N+s-m}]_m [p_s, \ldots, p_{N-m}]_m.$$

(To see this one can picture a circle with N marked points where we locate p_j. If we join with a segment every m points, we are sure to join all points with segments before closing the polygon. If the polygon closes leaving some vertices untouched, it means that a multiple of N can be divided into the union of disjoint orbits formed by joining every m points. This would imply that N and m are not co-prime.)

Let us call

$$(3.7) \quad A_r = \Pi_m(a_r^0) = [a_r^0, a_{r+m}^0 \ldots, a_{N+r-s}^0]_m \cdots [a_{s+r}^0, \ldots, a_{N-m+r}^0]_m.$$

We can see directly that $A_{r+m} = (a_r^0)^{-1} A_r a_r^0$, for all r. Once more, if N and m are coprimes, this property guarantees that *all A_r have the same Jordan form*, which we will call J.

Finally, notice that if $B_r = \Pi_m(b_r^0)$, then

$$(3.8) \qquad A_r = d_r^{-1} B_r d_r.$$

Let us choose d_r to be the matrix that conjugates B_r to its Jordan normal form J, so that A_r will all be in Jordan form. We can choose an order in the eigenvalues (for example, from smallest to largest) to ensure that the matrix is unique up to a factor that commutes with J. It is known that if a matrix commutes with a Jordan form matrix it must be block diagonal

$$\mathrm{diagonal}(r_1, \ldots, r_s),$$

where each r_s is a Toepliz matrix, upper triangular, whenever the corresponding Jordan block is of the form

$$\begin{pmatrix} \lambda & 1 & 0 & \ldots & 0 \\ 0 & \lambda & 1 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ldots & \vdots \\ 0 & \ldots & 0 & \lambda & 1 \\ 0 & \ldots & 0 & 0 & \lambda \end{pmatrix},$$

or it is diagonal if the Jordan block is diagonal. Thus, d_k are unique up to a block-diagonal matrix of this form.

Since

$$B_{k+m} = (b_k^0)^{-1} B_k b_k^0,$$

we have that

$$\begin{aligned} A_{k+m} = J &= d_{k+m}^{-1} B_{k+m} d_{k+m} = d_{k+m}^{-1} (b_k^0)^{-1} B_k b_k^0 d_{k+m} \\ &= d_{k+m}^{-1} (b_k^0)^{-1} d_k d_k^{-1} B_k d_k d_k^{-1} b_k^0 d_{k+m} = (a_k^0)^{-1} A_k a_k^0 = (a_k^0)^{-1} J a_k^0, \end{aligned}$$

for all k. Therefore, since a_k^0 commutes with the Jordan normal form, it must be a Toeplitz matrix of the form stated in the theorem, for all k.

Finally, d_k is unique up to a matrix commuting with J, lets call it q_k. We now turn our attention to the transformation of b_k^{m-1} under the change of lifting, namely

(3.9) $$a_k^{m-1} = q_{k+m-1}^{-1} \widehat{b}_k^{m-1} q_{k+m},$$

where $\widehat{b}_k^{m-1} = d_{k+m-1}^{-1} b_k^{m-1} d_{k+m}$ (d_k found above), and q_r Toeplitz and commuting with J.

How to determine which entries generate the others depend very much on the particular point in the Grassmannian. In the generic case q_r will all be diagonal; we will next describe the process generically. Using (3.9) we see that

$$a_{k-m+1}^{m-1} a_{k-m+2}^{m-1} \ldots a_{N+k-m}^{m-1} = q_k^{-1} \left(b_{k-m+1}^{m-1} b_{k-m+2}^{m-1} \ldots b_{N+k-m}^{m-1} \right) q_k,$$

for $k = 0, \ldots, N-1$.

Before we describe the normalizations that will generate the syzygies, we recall that the determinants of q_k are determined by (3.3) for any $k = 0, \ldots, N-1$, whenever N and m are coprime. Let us call $\det d_k = \delta_k$.

The last round of normalizations will be chosen by equating those entries in place $(i, i+1)$, $i = 1, 2 \ldots, m-1$ with the entry $(2, 1)$.

If we denote by $q_k = \mathrm{diag}(q_k^1, \ldots, q_k^n)$, and we denote the entries of

$$b^k = b_{k-m+1}^{m-1} b_{k-m+2}^{m-1} \ldots b_{N+k-m}^{m-1}$$

by $b_{i,j}^k$, then these normalizations result in equations of the form

$$(3.10) \qquad \frac{q_k^{i+1}}{q_k^i} b_{i,i+1}^k = \frac{q_k^2}{q_k^1} b_{1,2}^k,$$

for $i = 1, \ldots, N-1$, and

$$(3.11) \qquad \frac{q_k^1}{q_k^2} b_{2,1}^k = \frac{q_k^2}{q_k^1} b_{1,2}^k.$$

(3.11) solves for q_k^2 in terms of q_k^1 (if $b_{1,2}/b_{2,1}$ is not positive, we would need to choose different normalizations), and substituting it in (3.10) we get an expression for any q_k^i in terms of q_k^1. Since $\det q_k = \det d_k^{-1} \delta_k$, where d_k was determined in the normalization of b_k^0, q_k^1 is also determined.

These last normalizations will produce as many syzygies in the entries in b_k^{m-1} (linear or quadratic) as indicated in the statement of the theorem. The fact that the entries of Q_k, $k = 0, \ldots, N-1$, generate all other invariants of polygons in $\mathrm{Gr}(n, mn)$ is a consequence of the work in [3]. $\qquad \square$

Remark 3.2. — It is very clear that these last normalizations could be chosen in many different ways (we could make entries constant, for example; or we could choose a different block, or relate entries from different blocks). Not all choices will work for us, and in order to be able to prove scaling invariance of the map, it is important that we choose the equations to be homogeneous in the entries of b_k^r. It is also simpler (although not necessary) if we choose entries from one block only to define the equations. The choice of b_k^{m-1} versus b_k^r, $r \neq 0, m-1$ is just more convenient, but we could choose any other $r \neq 0$ instead.

4. The Pentagram map on $\mathrm{Gr}(n, 2sn)$

4.1. Definition of the map

Next, we define the Pentagram map for the Grassmannian $\mathrm{Gr}(n, 2sn)$ for $s \geqslant 2$. The dimension of $\mathrm{Gr}(n, 2sn)$ is clearly $(2s-1)n^2$.

Let $X = (X_k)$ be a lift of a regular N-gon in $\mathrm{Gr}(n, 2sn)$, and define the following subspaces

$$\Pi_k = \langle X_k, X_{k+2} \ldots, X_{k+2(s-1)}, X_{k+2s} \rangle,$$

and

$$\Omega_k = \langle X_{k+1}, X_{k+3}, \ldots, X_{k+2s-3}, X_{k+2s-1} \rangle.$$

Note that $\dim \Pi_k = (s+1)n$ and $\dim \Omega_k = sn$. Therefore, generically, $\dim \Pi_k \cap \Omega_k = n$.

DEFINITION 4.1. — *Let $\wp = (l_k)$ be a twisted N-gon in $G(n, 2sn)$. Let $T(\wp)$ be the map taking the N-gon $\wp$ to the unique twisted N-gon whose vertices have a lift of the form $T(X_k) = \Pi_k \cap \Omega_k$. The map T is independent from the choice of the lift X. We call T the Grassmannian Pentagram map.*

Notice that we are abusing notation by calling T both the map on polygons and their lifts. We will go further and use the letter T to denote the image of other data associated to $\wp$ in $T(\wp)$ (invariants, frames, etc). It is immediate to check that $T(\wp)$ is also twisted, with the same monodromy as $\wp$, using the fact that $\Pi_{N+k} = M\Pi_k$ and $\Omega_{N+k} = M\Omega_k$.

Next we will define this map in the moduli space of polygons, with coordinates given by the invariants in our previous section. We will keep on using the letter T, defining $T : \mathcal{P}_N \longrightarrow \mathcal{P}_N$. We will specify the domain if needed. Let us assume that $V = (V_k)$ is the lift defined in Theorem 3.1 for a polygon $\wp$. Assume

$$(4.1) \quad V_{k+2s} = V_k a_k^0 + V_{k+1} a_k^1 + \cdots + V_{k+2(s-1)} a_k^{2s-2} + V_{k+2s-1} a_k^{2s-1},$$

as in (3.1) for $2s$ matrices $a_k^0, a_k^1, \ldots, a_k^{2s-1}$ of size $n \times n$ and with the properties described in Theorem 3.1.

Using the fact that $T(V_k) \in \Pi_k$, we know that generically there exist matrices c_i^j such

$$(4.2) \qquad T(V_k) = V_k c_k^0 + V_{k+2} c_k^2 + \cdots\cdots + V_{k+2s} c_k^{2s}.$$

If we now use the relation (4.1) we can replace V_{k+2s} in (4.2), and arrive to the following expression for $T(V_k)$:

$$T(V_k) = V_k c_k^0 + V_{k+2} c_k^2 + \cdots + V_{k+2s-2} c_k^{2s-2}$$
$$+ \left(V_k a_k^0 + V_{k+1} a_k^1 + \cdots + V_{k+2(s-1)} a_k^{2s-2} + V_{k+2s-1} a_k^{2s-1} \right) c_k^{2s}$$
$$= V_k(c_k^0 + a_k^0 c_k^{2s}) + V_{k+1} a_k^1 c_k^{2s} + V_{k+2}(c_k^2 + a_k^2 c_k^{2s})$$
$$+ \cdots + V_{k+2s-1} a_k^{2s-1} c_k^{2s}.$$

Since we also assumed that $T(V_k) \in \Omega_k$ then $c_k^{2r} + a_k^{2r} c_k^{2s} = 0$, for $r = 0, \ldots, s-1$, and

$$(4.3) \quad T(V_k)$$
$$= \left(V_{k+1} a_k^1 + V_{k+3} a_k^3 + \cdots + V_{k+2s-3} a_k^{2s-3} + V_{k+2s-1} a_k^{2s-1} \right) c_k^{2s}.$$

Although the matrix c_k^{2s} seems to be arbitrary, it is uniquely determined by the fact that the right hand side of (4.3), not only for k, but also for $k+1, \ldots, k+2s-1$, must be a lift for the image polygon $T(\wp)$, with the same properties as those found in Theorem 3.1. Once c_k^{2s} are chosen that

way, we will be able to find the image of the matrices a_i^j under the map T, as follows. Let us call $c_k^{2s} = \lambda_k$, so that we can write

$$T(V_k) = \rho_k \mathbf{r}_k \lambda_k,$$

with $\rho_k = (V_k, V_{k+1}, V_{k+2}, \ldots, V_{k+2s-1})$,

$$(4.4) \qquad \mathbf{r}_k = \begin{pmatrix} O_n \\ a_k^1 \\ O_n \\ a_k^3 \\ \vdots \\ O_n \\ a_k^{2s-1} \end{pmatrix},$$

and where λ_k are uniquely chosen so that $\{T(V_k)\}$ is the lift of the image polygon described in Theorem 3.1.

Example 4.2. — In particular, (4.3) implies that when $s = 2$ the Pentagram map takes of following form

$$T(V_k) = [V_{k+1}a_k^1 + X_{k+3}a_k^3]\lambda_k = (V_k\ V_{k+1}\ V_{k+2}\ V_{k+3}) \begin{pmatrix} O_n \\ a_k^1 \\ O_n \\ a_k^3 \end{pmatrix} \lambda_k = \rho_k \mathbf{r}_k \lambda_k,$$

for all $k \in \mathbb{Z}$.

Extending the map T using, as usual, the pullback, we have that

$$T(\rho_k) = (T(V_k), \ldots, T(V_{k+2s-1}))$$
$$= \rho_k(\mathbf{r}_k\lambda_k, R_k\mathbf{r}_{k+1}\lambda_{k+1}, \ldots, R_{k+2s-2}\mathbf{r}_{k+2s-1}\lambda_{k+2s-1}),$$

where $R_{k+r} = Q_kQ_{k+1}\ldots Q_{k+r}$ and Q_k is given as in (3.2). This expression can be written as

$$(4.5) \qquad T(\rho_k) = \rho_k N_k \Lambda_k,$$

where

$$(4.6) \qquad N_k = (\mathbf{r}_k, R_k\mathbf{r}_{k+1}, R_{k+1}\mathbf{r}_{k+2}, \ldots, R_{k+2s-2}\mathbf{r}_{k+2s-1}),$$

and

$$(4.7) \qquad \Lambda_k = \begin{pmatrix} \lambda_k & O_n & \cdots & O_n \\ O_n & \lambda_{k+1} & \cdots & O_n \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ O_n & \cdots & O_n & \lambda_{k+2s-1} \end{pmatrix}.$$

One can recognize equations (3.2) and (4.5), that is

$$(4.8) \qquad \rho_{k+1} = \rho_k Q_k, \qquad T(\rho_k) = \rho_k N_k \Lambda_k,$$

as a parameter free Lax representation for the map T. The compatibility conditions are given by

$$(4.9) \qquad T(Q_k) = \Lambda_k^{-1} N_k^{-1} Q_k N_{k+1} \Lambda_{k+1}.$$

The last block-column of this equation defines the map T on the moduli space of Grassmannian polygons, written in coordinates given by the invariants a_i^j. The question we will resolve in the next subsection is how to introduce an spectral parameter in (4.8).

4.2. A Lax representation for the pentagram map on $\mathrm{Gr}(n, 2sn)$

In this section we will prove that one can introduce a parameter μ in (4.8) in such a way that (4.9) will be independent from μ. This will define a true Lax representation that can be used for integration of the map. As it was done in [5], we will prove that the map T is invariant under the scaling

$$(4.10) \qquad a_k^{2r+1} \to \mu a_k^{2r+1}, \qquad a_k^{2r} \to a_k^{2r},$$

for any $r = 0, 1, \ldots, s - 1$ and any k (this implies that all entries of these $n \times n$ matrix scale equally). This will involve several steps.

Let us denote the block columns of (4.6) by F_r, so that $F_k = \mathbf{r}_k$, and

$$(4.11) \qquad F_{k+\ell} = R_{k+\ell-1} \mathbf{r}_{k+\ell} = Q_k Q_{k+1} \ldots Q_{k+\ell-1} \mathbf{r}_{k+\ell},$$

$\ell = 1, 2, \ldots$, with $\mathbf{r}$ as in (4.4). Our first lemma will allow us to decompose the block columns of N_r into homogeneous terms according to (4.10). The lemma is almost identical to Lemma 3.1 in [5]. Let us denote by Γ the matrix

$$(4.12) \qquad \Gamma = \begin{pmatrix} O_n & O_n & O_n & \ldots & O_n \\ I_n & O_n & O_n & \ldots & O_n \\ O_n & I_n & O_n & \ldots & O_n \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ O_n & \ldots & O_n & I_n & O_n \end{pmatrix},$$

where I_n is the $n \times n$ identity matrix. Let us also denote by $\mathcal{T}$ the shift operator, namely $\mathcal{T}(V_k) = V_{k+1}$. This shift operator can trivially be extended to invariants using $\mathcal{T}(a_k^i) = a_{k+1}^i$ and to functions depending on the invariants using the pullback. We can also extend it to matrices whose entries are invariants by applying it to each entry, as it is customary.

LEMMA 4.3. — *Let $F_{k+\ell}$ be given as in* (4.11). *Then, there exist $n \times n$ matrices α_i^j such that*

(4.13)
$$F_{k+2\ell} = \sum_{r=1}^{\ell} F_{k+2r-1}\alpha_{2r-1}^{2\ell} + G_{k+2\ell},$$

$$F_{k+2\ell+1} = \sum_{r=0}^{\ell} F_{k+2r}\alpha_{2r}^{2\ell+1} + \widehat{G}_{k+2\ell+1},$$

for $\ell \geqslant 1$, where

(4.14)
$$\widehat{G}_{k+2\ell+1} = \mathbf{p}_k \left(\mathcal{T}G_{k+2\ell}\right)_m + \Gamma\mathcal{T}G_{k+2\ell},$$
$$\alpha_{2r}^{2\ell+1} = \mathcal{T}\alpha_{2r-1}^{2\ell}, \ \alpha_0^{2\ell+1} = \left(\mathcal{T}G_{k+2\ell}\right)_m,$$

with

(4.15)
$$\mathbf{p}_k = \begin{pmatrix} a_k^0 \\ O_n \\ a_k^2 \\ O_n \\ \cdots \\ a_k^{m-2} \\ O_n \end{pmatrix},$$

and

(4.16) $\quad G_{k+2\ell+2} = \Gamma\mathcal{T}\widehat{G}_{k+2\ell+1}, \quad \alpha_{2r+1}^{2\ell+2} = \mathcal{T}\alpha_{2r}^{2\ell+1}, \quad G_k = F_k = \mathbf{r}_k.$

By A_m we mean the last $n \times n$ block entry of a matrix A.

From now on we will simplify our notation by denoting $F_{k+\ell}$ simply by F_ℓ. We will introduce the subindex k only if its removal creates confusion.

Proof. — First of all, notice that the last column of Q in (3.2) is given by $\mathbf{p} + \mathbf{r}$, as in (4.15) and (4.4). Notice also that, from the definition in (4.11) we have

$$F_\ell = Q\mathcal{T}F_{\ell-1}.$$

We proceed by induction. First of all, since $F = \mathbf{r}$,

$$F_1 = Q\mathcal{T}F = Q\mathbf{r}_1 = (\mathbf{p} + \mathbf{r})a_1^{m-1} + \Gamma\mathbf{r}_1 = Fa_1^{m-1} + \mathbf{p}a_1^{m-1} + \Gamma\mathbf{r}_1.$$

We simply need to call $\widehat{G}_1 = \mathbf{p}a_1^{m-1} + \Gamma r_1$, and $\alpha_0^1 = a_1^{m-1} = \mathcal{T}(F)_m$. Let's do the first even case also:

$$F_2 = Q\mathcal{T}F_1 = Q(\mathcal{T}\widehat{G}_1 + \mathcal{T}Fa_2^{m-1}) = F_1a_2^{m-1} + \Gamma\mathcal{T}\widehat{G}_1,$$

and we call $G_2 = \Gamma\mathcal{T}\widehat{G}_1$. Notice that $Q\mathcal{T}\widehat{G}_1 = \Gamma\mathcal{T}\widehat{G}_1$ since the last block of $\widehat{G}_1$ vanishes, as indicated by the hat.

Now, assume

$$F_{2\ell} = \sum_{r=1}^{\ell} F_{2r-1}\alpha_{2r-1}^{2\ell} + G_{2\ell}.$$

Then

$$F_{2\ell+1} = Q\mathcal{T}F_{2\ell} = \sum_{r=1}^{\ell} Q\mathcal{T}F_{2r-1}\mathcal{T}\alpha_{2r-1}^{2\ell} + Q\mathcal{T}G_{2\ell}.$$

Since $Q\mathcal{T}G_{2\ell} = \mathbf{p}\,(\mathcal{T}G_{2\ell})_m + \mathbf{r}\,(\mathcal{T}G_{2\ell})_m + \Gamma\mathcal{T}G_{2\ell}$ and $\mathbf{r} = F$, if

$$\widehat{G}_{2\ell+1} = \mathbf{p}\,(\mathcal{T}G_{2\ell})_m + \Gamma\mathcal{T}G_{2\ell},$$

$$\alpha_{2r}^{2\ell+1} = \mathcal{T}\alpha_{2r-1}^{2\ell}, \ r = 1,\dots,\ell, \quad \alpha_0^{2\ell+1} = (\mathcal{T}G_{2\ell})_m,$$

then we have

$$F_{2\ell+1} = \sum_{r=0}^{\ell} F_{2r}\alpha_{2r}^{2\ell+1} + \widehat{G}_{2\ell+1}.$$

Looking into the even case, we have that

$$F_{2\ell+2} = Q\mathcal{T}F_{2\ell+1} = \sum_{r=0}^{\ell} Q\mathcal{T}F_{2r}\mathcal{T}\alpha_{2r}^{2\ell+1} + Q\mathcal{T}\widehat{G}_{2\ell+1},$$

and since $F_{2r+1} = Q\mathcal{T}F_{2r}$ and $Q\mathcal{T}\widehat{G}_{2\ell+1} = \Gamma\mathcal{T}\widehat{G}_{2\ell+1}$, if we call

$$G_{2\ell+2} = \Gamma\mathcal{T}\widehat{G}_{2\ell+1}, \quad \alpha_{2r+1}^{2\ell+2} = \mathcal{T}\alpha_{2r}^{2\ell+1},$$

we prove the lemma.   □

Once we have this lemma we can identify homogeneous terms in the expansion of block columns. Indeed, notice that $\mathbf{r}$ and $\mathbf{p}$ are homogeneous of degree 1 and 0, respectively, with respect to the scaling. Since the shift clearly preserves the degree, from the statement of the lemma we have that both $\widehat{G}_{k+2r+1}$ and G_{2r} are homogeneous of degree 1, for any r. Likewise, α_0^r are also homogeneous of degree 1, for any r, from its definition, and since all others are obtained by shifting these, they also are.

Therefore, if we denote $G = F$, iteratively applying the lemma we have that F_r are in all cases a combination of G_{2r} and $\widehat{G}_{2r+1}$ for the different values of r, with different types of factors of the form α_i^j, each of degree 1. One can also clearly see that if the columns of F_r generate R^{nm} for $r = 0, 1, \dots, m-1$, them the columns of G_{2r} and $\widehat{G}_{2r+1}$, $r = 0, \dots, s-1$ will also generate the same space since the change of basis matrix will be upper triangular with ones down the diagonal. This new basis will be crucial in the calculations that follow.

Finally, a comment as to the reason for our notation. Notice that both $G_{2\ell}$ and $\widehat{G}_{2\ell+1}$ have alternative zero and non-zero clocks, with $G_{2\ell}$ starting

with a zero block and $\widehat{G}_{2\ell+1}$ starting with a nonzero block. We are keeping that marked not only by the subindex but also by a hat, since as calculations become more involved it helps to have them be visibly different. It shows that the entire space can be written as a direct sum of two orthogonal subspaces, one generated by the block columns with hats and one generated by those without hats.

Next, assume that we drop the Λ_k factor and define

$$\overline{T}(V_k) = \rho_k \mathbf{r}_k.$$

Define further $c_k^i = \overline{T}(a_k^i)$, as given by the following compatibility formula, which is (4.9) after removing Λ_k

$$(4.17) \qquad \qquad \overline{T}(Q_k) = N_k^{-1} Q_k N_{k+1}.$$

Notice that c_k^i will need to be normalized by Λ_k before we can declare it to be $T(a_k^i)$. Let us call $\mathbf{a}_k$ the last block column in Q_k (the ith block will be a_k^{i-1}). Then, choosing the last block-column in both sides of the equation

$$\overline{T}(\mathbf{a}_k) = N_k^{-1} Q_k \mathcal{T} F_{k+2s-1} = N_k^{-1} F_{k+2s},$$

which can be written as

$$(4.18) \qquad \qquad N_k \overline{T}(\mathbf{a}_k) = F_{k+2s}.$$

Thus $\overline{T}(\mathbf{a}_k)$ can be interpreted as the solution of the linear equation (4.18). This will be crucial in what follows.

THEOREM 4.4. — *The matrices c_k^i are homogeneous with respect to the scaling (4.10), and*

$$d(c_k^{2\ell}) = 0, \qquad d(c_k^{2\ell+1}) = 1,$$

for any $\ell = 0, 1, \ldots, s-1$.

Proof. — As in the previous proof, we will drop the subindex k and introduce it only if needed.

First of all, let us analyze the homogeneity with respect to (4.10) of the determinant

$$D = \det N = \det(F, F_1, \ldots, F_{2s-1}).$$

From (4.13) we can rewrite it as

$$D = \det(F, \widehat{G}_1, G_2, \ldots, G_{2s-2}, \widehat{G}_{2s-1}),$$

and since $d(G_{2\ell}) = d(\widehat{G}_{2\ell-1}) = 1$ for all ℓ, we have that D is homogeneous and $d(D) = 2sn$.

Next, denote by f_r^j the jth column of F_r, and let $F_{r,i}^j$ be the block column whose individual columns are equal to those of F_r, except for the ith column which is equal to the jth column of F_{2s}, for any $r = 0, 1, \ldots, 2s - 1$.

Define next

$$D_{2\ell,i}^j = \det(F, F_1, \ldots, F_{2\ell-1}, F_{2\ell,i}^j, F_{2\ell+1}, \ldots, F_{2s-1}).$$

We first notice that using (4.13), we can substitute the ith column of $F_{2\ell,i}^j$ by the jth column of G_{2s}, since F_{2s} and G_{2s} differ in a linear combination of columns of F_{2r+1}, $r < s$. Let us call the new matrix $F_{2\ell,i}^{j,g}$. We then simplify the part of the determinant to the right of $F_{2\ell,i}^j$ using (4.13), to become

$$(4.19) \quad D_{2\ell,i}^j = \det\big(F, F_1, \ldots, F_{2\ell-1}, F_{2\ell,i}^{j,g}, \widehat{G}_{2\ell+1} + f_{2\ell}^i \left(\alpha_{2\ell}^{2\ell+1}\right)_i,$$
$$G_{2\ell}, \ldots, G_{2s-2}, \widehat{G}_{2s-1} + f_{2\ell}^i \left(\alpha_{2\ell}^{2s-1}\right)_i\big),$$

where $\left(\alpha_{2\ell}^r\right)_i$ denotes the ith row of $\alpha_{2\ell}^r$.

We now proceed to simplify the columns $f_{2\ell}^i$ which can be substituted by $g_{2\ell}^i$ (the ith column of $G_{2\ell}$) since their difference is generated by odd vectors with subindiced less that 2ℓ. We can then simplify the rest of the determinant, using (4.13) once more. We get

$$(4.20) \quad D_{2\ell,i}^j = \det\Big(F, \widehat{G}_1, \ldots, \widehat{G}_{2\ell-1}, G_{2\ell,i}^j, \widehat{G}_{2\ell+1} + g_{2\ell}^i \left(\alpha_{2\ell}^{2\ell+1}\right)_i,$$
$$G_{2\ell}, \ldots, G_{2s-2}, \widehat{G}_{2s-1} + g_{2\ell}^i \left(\alpha_{2\ell}^{2s-1}\right)_i\Big),$$

where $G_{2\ell,i}^j$ indicates the matrix equal to $G_{2\ell}$, except for the ith column which is equal to g_{2s}^j. Our last step is to notice that we have enough G_{2r} block-columns ($r = 0, \ldots, s-1$, $r \neq s$) that together with $G_{2\ell,i}^j$ generically generate the entire subspace generated by G_{2r}, $r = 0, \ldots, s-2$. But the vector $g_{2\ell}^i$ belongs to this subspace, and hence it will be a combination of the columns of those blocks. Thus

$$D_{2\ell,i}^j = \det(F, \widehat{G}_1, \ldots, \widehat{G}_{2\ell-1}, G_{2\ell,i}^j, \widehat{G}_{2\ell+1}, G_{2\ell}, \ldots, G_{2s-2}, \widehat{G}_{2s-1}),$$

which clearly shows that $D_{2\ell,i}^j$ is homogeneous, and since $d(G_{2\ell,i}^j) = 1$, $d(D_{2\ell,i}^j) = 2sn$ also.

Finally, $\overline{T}(\mathbf{a})$ is the solution of system of linear equations (4.18), and so, by Cramer's rule, the (j,i) entry of $c^{2\ell} = \overline{T}(a^{2\ell})$ is of the form

$$\frac{D_{2\ell,i}^j}{D}.$$

Therefore, $c^{2\ell}$ is homogeneous and $d(c^{2\ell}) = 0$, $\ell = 0, \ldots, s-1$.

We now study $c^{2\ell+1}$. Consider the determinant

$$D^j_{2\ell+1,i} = \det(F, F_1, \ldots, F_{2\ell}, F^j_{2\ell+1,i}, F_{2\ell+2}, \ldots, F_{2s-1}),$$

where $F^j_{2\ell+1,i}$ is defined as $F_{2\ell+1}$ substituting the ith column with the jth column of F_{2s}. As before, using (4.13), we start by noticing that we can substitute the ith column of $F^j_{2\ell+1,i}$ by the jth column of G_{2s}, call it g^j_{2s}, plus $f^i_{2\ell+1}\left(\alpha^{2s}_{2\ell+1}\right)_{i,j}$ that comes from the expansion of F_{2s} in terms of odd terms, and the fact that the ith column of $F_{2\ell+1}$ is missing. The expression $\left(\alpha^{2s}_{2\ell+1}\right)_{i,j}$ is the (i,j) entry of $\alpha^{2s}_{2\ell+1}$. We call the resulting matrix $F^{j,g}_{2\ell+1,i}$. If we simplify the right hand side of the determinant it becomes

$$D^j_{2\ell+1,i} = \det\Big(F, F_1, \ldots, F_{2\ell}, F^{j,g}_{2\ell+1,i}, G_{2\ell+2} + f^i_{2\ell+1}\left(\alpha^{2\ell+2}_{2\ell+1}\right)_i,$$
$$\widehat{G}_{2\ell+3}, \ldots, G_{2s-2} + f^i_{2\ell+1}\left(\alpha^{2s-2}_{2\ell+1}\right)_i, \widehat{G}_{2s-1}\Big)$$
$$= \left(\alpha^{2s}_{2\ell+1}\right)_{i,j} \det(F, F_1, \ldots, F_{2\ell}, F_{2\ell+1}, G_{2\ell+2}, \ldots, \widehat{G}_{2s-1})$$
$$+ \det\Big(F, F_1, \ldots, F_{2\ell}, \widehat{G}^j_{2\ell+1,i}, G_{2\ell+2} + f^i_{2\ell+1}\left(\alpha^{2\ell+2}_{2\ell+1}\right)_i,$$
$$\widehat{G}_{2\ell+3}, \ldots, G_{2s-2} + f^i_{2\ell+1}\left(\alpha^{2s-2}_{2\ell+1}\right)_i, \widehat{G}_{2s-1}\Big),$$

where $\widehat{G}^j_{2\ell+1,i}$ is equal to $\widehat{G}_{2\ell+1}$ except for the ith column which is equal to g^j_{2s}.

As before, $f^i_{2\ell+1}$ and $\widehat{g}^i_{2\ell+1}$ differ in a sum of columns of F_{2r}, $r \leqslant \ell$. Thus, we can substitute $f^i_{2\ell+1}$ by $\widehat{g}^i_{2\ell+1}$ in the determinant. After that, we proceed to simplify the rest of the determinant obtaining

$$D^j_{2\ell+1,i} = \left(\alpha^{2s}_{2\ell+1}\right)_{i,j} D$$
$$+ \det\Big(F, \widehat{G}_1, \ldots, G_{2\ell}, \widehat{G}^j_{2\ell+1,i}, G_{2\ell+2} + \widehat{g}^i_{2\ell+1}\left(\alpha^{2\ell+2}_{2\ell+1}\right)_i,$$
$$\widehat{G}_{2\ell+3}, \ldots, G_{2s-2} + \widehat{g}^i_{2\ell+1}\left(\alpha^{2s-2}_{2\ell+1}\right)_i, \widehat{G}_{2s-1}\Big),$$

Unlike the previous case, this time one of the columns of $\widehat{G}^j_{2\ell+1,i}$ is even, and hence, the odd columns (other than $\widehat{g}^i_{2\ell+1}$) do not generate the odd orthogonal subspace since they are one dimension short. Thus, including $\widehat{g}^i_{2\ell+1}$, we have an equal number of odd and even columns and we need to expand.

The term that includes no $\widehat{g}^i_{2\ell+1}$ in the expansion is given by

$$\det(F, \widehat{G}_1, G_2, \ldots, G_{2\ell}, \widehat{G}^j_{2\ell+1,i}, G_{2\ell+2}, \ldots G_{2s-2}, \widehat{G}_{2s-1}) = 0,$$

since, as we said before, there are more even columns that odd columns. The remaining terms in the expansion are

$$\sum_{r=\ell+1}^{s-1} \sum_{p=1}^{n} (\alpha_{2\ell+1}^{2r})_{i,p} \det\left(F, \widehat{G}_1, \ldots, \widehat{G}_{2\ell+1,i}^j, \ldots, \widehat{G}_{2r-1}, G_{2r}^i + \widehat{g}_{2\ell+1}^i e_p^T, \right.$$

$$\left. \widehat{G}_{2r-1}, \ldots, G_{2s-2}, \widehat{G}_{2s-1} \right),$$

where $(\alpha_{2\ell+1}^{2r})_{i,p}$ is the (i,p) entry of $\alpha_{2\ell+1}^{2r}$, G_{2r}^i has zero ith column and where e_p is the standard canonical basis of $\mathbb{R}^n$ with a 1 in the pth entry and zero elsewhere. Each one of these determinants has an equal number of odd and even columns. Each column is homogeneous of degree 1, and so each determinant is homogeneous of degree $2sn$. But, like D (also of degree $2sn$), they are multiplied by $(\alpha_{2\ell+1}^{2r})_{i,p}$, homogeneous of degree 1. Hence, $D_{2\ell+1,i}^j$ is homogeneous and $d(D_{2\ell+1,i}^j) = 2sn + 1$.

Finally, since, according to (4.18), the (j,i) entry of $\overline{T}(a^{2\ell+1}) = c^{2\ell+1}$ is equal to

$$\frac{D_{2\ell+1,i}^j}{D},$$

we conclude that $c^{2\ell+1}$ is homogeneous and $d(c^{2\ell+1}) = 1$. This concludes the proof of the theorem. $\square$

Our final step is to introduce the normalization matrices λ_k and to study how they might affect the scaling degree of $T(a_k^i)$. Recall that λ_k has two factors: d_k, used to normalize c_k^0 and to transform them into their Jordan form (as in (3.5)); and q_k, in the generic case, a diagonal matrix used to define syzygies among the entries of c_k^{m-1}, as in (3.9) (c_k^{m-1} plays the role of b_k^{m-1} in (3.9)).

LEMMA 4.5. — *The matrices λ_k are homogeneous with respect to (4.10) and*

$$d(\lambda_k) = -1,$$

for all k.

Proof. — Since $\lambda_k = d_k q_k$, we will look at each factor separately.

The first factor d_k is determined by the normalization of B_k as in (3.8), where, in our case, $B_k = \Pi_m(c_k^0)$ as in (3.6). But, given that c_k^0 are invariant under the scaling, B_k will also be, and hence so will d_k.

The second factor, q_k, is found by using a number of equations of the form (3.10)–(3.11), which finds each entry of q_k as fuctions of the first entry q_k^1. Since c_k^{m-1} (which plays the role of b_r^{m-1}) is homogeneous with respect to the scaling, equations (3.10)–(3.11) imply that q_k^i are homogeneous also,

with equal degree. Also, since $\det q_k = \det d_k^{-1} \delta_k$, where $\delta_k = \det \lambda_k$, each entry of q_k will have degree equal to $d(\delta_k)/n$. Hence, to prove the lemma we need to show that $d(\delta_k) = -n$.

But this follows from the fact that $N_k \Lambda_k = \rho_k^{-1} T(\rho_k)$, where Λ_k is as in (4.7), must have determinant equals 1 since ρ_k does. Therefore,

$$\det(N_k)\delta_k \delta_{k+1}, \dots, \delta_{k+m-1} = 1.$$

If we now apply the scaling, and having in mind that $d(N_k) = nm$, we get

$$\mu^{nm} \det(N_k)\hat{\delta}_k \hat{\delta}_{k+1}, \dots, \hat{\delta}_{k+m-1} = 1,$$

where $\hat{\delta}_{k+1}$ is the scaled determinant. As show in [5], this system has a unique solution whenever N and m are coprime. But

$$\hat{\delta}_{k+r} = \delta_{k+r}\mu^{-n},$$

for all r, is clearly a solution. Hence δ_k are homogeneous and $d(\delta_k) = -n$. This concludes the proof. $\qquad\square$

We are now in position to prove our main theorem.

THEOREM 4.6. — *The Grassmannian pentagram map on the moduli space $\mathcal{P}_N$ defined by (4.9) is invariant under the scaling (4.10).*

Proof. — We need to show that $T(a_k^i)$ are homogeneous, and $d(T(a_k^{2\ell})) = 0$, $d(T(a_k^{2\ell+1})) = 1$ for $\ell = 0, \dots, s-1$. As in previous proofs, we will drop the subindex unless there could be some confusion.

Using (4.9), and denoting by $\mathbf{a}$ the last column of Q, we can write $T(\mathbf{a})$ as

$$T(\mathbf{a}) = \Lambda^{-1} N^{-1} Q \mathcal{T} N \begin{pmatrix} O_n \\ \vdots \\ O_n \\ \lambda_{m+1} \end{pmatrix},$$

or as the solution of the linear system of equations

$$N\Lambda T(\mathbf{a}) = Q\mathcal{T}N \begin{pmatrix} O_n \\ \vdots \\ O_n \\ \lambda_{m+1} \end{pmatrix}.$$

Since we plan to use Cramer's rule once more, we will study the associated determinants.

To start with, we know that $\det(N\Lambda)$ is a homogeneous function of degree $nm - nm = 0$. Define

$$D_{r,i}^j = \det(F\lambda, F_1\lambda_1, \ldots, F_{r-1}\lambda_{r-1}, F_{r,i}^{j,\lambda}, F_{r+1}\lambda_{r+1}, \ldots, F_{2s-1}\lambda_{2s-1})$$

$$= \det(F, F_1, \ldots, F_{r-1}, F_{r,i}^{j,\lambda}, F_{r+1}, \ldots, F_{2s-1})\delta_1 \ldots \delta_{r-1}\delta_{r+1} \ldots \delta_{2s-1},$$

where $F_{r,i}^{j,\lambda}$ has all the columns equal to $F_r\lambda_r$, except for the ith column which is given by the jth column of $F_{2s}\lambda_{2s}$, that is, by $F_{2s}\lambda_{2s}^j = F_{2s}d_{2s}q_{2s}^j e_j$, with d_{2s} and q_{2s} as in (3.8) and (3.9).

Assume $r = 2\ell$. — As in the first lemma, we can use (4.13) to write down the determinants in terms of homogeneous components. For example, the jth column coming from F_{2s} can be replaced by that of G_{2s} in $F_{r,i}^{j,\lambda}$ and we can simplify the terms to the left of it, including the remaining columns of $F_{2\ell}$. We obtain

$$\det(F, F_1, \ldots, F_{2\ell-1}, F_{2\ell,i}^{j,\lambda}, F_{2\ell+1}, \ldots, F_{2s-1})$$

$$= \det\Big(F, F_1, \ldots, F_{2\ell-1}, G_{2\ell,i}^{j,\lambda}, \widehat{G}_{2\ell+1} + f_{2\ell}^i(\alpha_{2\ell}^{2\ell+1})_i,$$

$$G_{2\ell+2}, \ldots, \widehat{G}_{2s-1} + f_{2\ell}^i(\alpha_{2\ell}^{2s-1})_i\Big).$$

The block column $G_{2\ell,i}^{j,\lambda}$ is equal to $G_{2\ell}\lambda_{2\ell}$ except for the ith column, which is equal to $G_{2s}\lambda_{2s}^j$. We can further use (4.13) to substitute $f_{2\ell}^i$ with $g_{2\ell}^i$. Once we do that, we see that the even orthogonal subspace is generated by the columns of G_{2r}, $r = 0, \ldots, s-1$ except for the extra column in $G_{2\ell}$, which in this case is generically covered by combination of columns in $G_{2s}\lambda_{2s}^j$. Thus, as before, we can remove the $g_{2\ell}^i$ terms from the determinant and simplify to the left of the 2ℓ position.

We obtain

$$D_{r,i}^j = \det\Big(F, \widehat{G}_1, G_2 \ldots, \widehat{G}_{2\ell-1}, G_{2\ell,i}^{j,\lambda}, \widehat{G}_{2\ell+1}, \ldots, \widehat{G}_{2s-1}\Big)$$
$$\delta\delta_1 \ldots \delta_{2\ell-1}\delta_{2\ell+1} \ldots \delta_{2s-1},$$

with hats and non-hats alternating. All of the columns of G_r are homogeneous of degree 1 for any r. On the other hand, the columns of $G_{2\ell,i}^{j,\lambda}$ have degree zero since $G_{2\ell}$ has degree 1 and $\lambda_{2\ell}$ has degree -1, so does $G_{2s}\lambda_{2s}^j = G_{2s}d_{2s}q_{2s}^j e_j$. Therefore,

$$d(D_{r,i}^j) = n(m-1) - n(m-1) = 0.$$

The (j, i) entry of $T(a^{2\ell})$ is given by $\frac{D_{2\ell,i}^j}{D}$, and hence $d(T(a^{2\ell})) = 0$.

Assume $r = 2\ell + 1$. — In this case, and always using (4.13), the determinant

$$D_{r,i}^j = \det(F\lambda, F_1\lambda_1, \ldots, F_{2\ell}\lambda_{2\ell}, F_{2\ell+1,i}^{j,\lambda}, F_{2\ell+2}\lambda_{2\ell+2}, \ldots, F_{2s-1}\lambda_{2s-1}),$$

can be further simplified replacing the ith column of $F_{2\ell+1,i}^{j,\lambda}$, given by $F_{2s}\lambda_{2s}^j$, by

$$h_{2s}^{j,\lambda} = G_{2s}\lambda_{2s}^j + F_{2\ell+1}\alpha_{2\ell+1}^{2s}\lambda_{2s}^j.$$

We can then simplify the side of the determinant to the right of $F_{2\ell+1,i}^{j,\lambda}$ so that $F_{2r+1} \to \widehat{G}_{2r+1}$ and $F_{2r} \to G_{2r} + f_{2\ell+1}^i \left(\alpha_{2\ell+1}^{2r}\right)_i$. After this simplification we can also substitute $f_{2\ell+1}^i$ with $\widehat{g}_{2\ell+1}^i$ and $F_{2\ell+1,i}^{j,\lambda}$ by $\widehat{G}_{2\ell+1,i}^{j,\lambda}$, where all columns of $\widehat{G}_{2\ell+1,i}^{j,\lambda}$ are equal to those of $\widehat{G}_{2\ell+1}\lambda_{2\ell+1}$, except for the ith column, given by

$$g_{2s}^{j,\lambda} = G_{2s}\lambda_{2s}^j + \widehat{G}_{2\ell+1}\alpha_{2\ell+1}^{2s}\lambda_{2s}^j.$$

We then continue to simplify the part of the determinant to the left of $\widehat{G}_{2\ell+1,i}^{j,\lambda}$. The result of all these simplifications is the determinant

$$\det\Big(F, \widehat{G}_1, G_2, \ldots, G_{2\ell}, \widehat{G}_{2\ell+1,i}^{j,\lambda}, G_{2\ell+2} + \widehat{g}_{2\ell+1}^i \left(\alpha_{2\ell+1}^{2\ell+2}\right)_i,$$

$$\widehat{G}_{2\ell+3}, \ldots, G_{2s-2} + \widehat{g}_{2\ell+1}^i \left(\alpha_{2\ell+1}^{2s-2}\right)_i, \widehat{G}_{2s-1}\Big).$$

First of all, notice that if we expand this determinant, the term in the expansion without any $g_{2\ell+1}^i$ is given by

$$\det(F, \widehat{G}_1, \ldots, G_{2\ell}, X^\lambda, G_{2\ell+2}, \ldots, \widehat{G}_{2s-1}),$$

where X^λ equals $\widehat{G}_{2\ell+1}\lambda_{2\ell+1}$ except for the ith column given by $\widehat{G}_{2\ell+1}\alpha_{2\ell+1}^{2s}\lambda_{2s}^j$. All columns of $\widehat{G}_{2\ell+1}\lambda_{2\ell+1}$ have degree zero, except for the ith column which has degree 1 $(d(\widehat{G}_{2\ell+1}) = d(\alpha_{2\ell+1}^{2s}) = 1$ while $d(\lambda_{2s}^j) = -1)$. Therefore, the degree of this determinant is $n(m-1) + 1$.

If we now look at any of the terms in the expansion containing $g_{2\ell+1}^i$, we have

$$\sum_{r=\ell+1}^{s-1} \sum_{p=1}^{n} \left(\alpha_{2\ell+1}^{2r}\right)_{i,p} \det\Big(F, \widehat{G}_1, \ldots, Y^\lambda, \ldots, \widehat{G}_{2r-1}, G_{2r}^i + \widehat{g}_{2\ell+1}^i e_p^T,$$

$$\widehat{G}_{2r-1}, \ldots, G_{2s-2}, \widehat{G}_{2s-1}\Big),$$

where $\left(\alpha_{2\ell+1}^{2r}\right)_{i,p}$ is the (i,p) entry of $\alpha_{2\ell+1}^{2r}$, G_{2r}^i has zero ith column and where e_p is the standard canonical basis of $\mathbb{R}^n$. The matrix Y^λ is equal to $\widehat{G}_{2\ell+1}\lambda_{2\ell+1}$ except for the ith column, which is equal to $g_{2s}^{j,\lambda}$. We can

further simplify $g_{2s}^{j,\lambda}$ to become $G_{2s}\lambda_{2s}^j$ since its odd term is generated by the other columns.

Each column has degree 1, except for those in Y^λ, which have degree 0. Since each term is multiplied by $\left(\alpha_{2\ell+1}^{2r}\right)_{i,p}$, of degree 1, each term has degree $n(m-1)+1$, and so does the determinant. From here

$$d(D_{2\ell+1,i}^j) = n(m-1) + 1 - n(m-1) = 1.$$

Since the (j,i) entry of $T(a^{2\ell+1})$ is given by $\frac{D_{2\ell+1,i}^j}{D}$, we have

$$d(T(a^{2\ell+1})) = 1,$$

which concludes the proof of the theorem. □

This theorem allows us to define the Lax representation for the map T on $\mathcal{P}_N$. Indeed, if we define

$$Q_k(\mu) = \begin{pmatrix} O_n & O_n & \dots & O_n & a_k^0 \\ I_n & O_n & \dots & O_n & \mu a_k^1 \\ O_n & I_n & \dots & O_n & a_k^2 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ O_n & \dots & O_n & I_n & \mu a_k^{2s-1} \end{pmatrix},$$

then there is a unique matrix $N_k(\mu)$ such that

(4.21) $$T(Q_k(\mu)) = N_k(\mu)^{-1}\Lambda_k^{-1}Q_k(\mu)\Lambda_n N_k(\mu).$$

The matrix $N_k(\mu)$, which is invariant and hence depends on a_k^r, is simply the matrix N_k in (4.6) rescaled by (4.10), that is

$$N_k(\mu) = \mu\Big(\mathbf{r}_k, Q_k(\mu)\mathbf{r}_{k+1}, Q_k(\mu)Q_{k+1}(\mu)\mathbf{r}_{k+2}, \dots,$$
$$\big[Q_k(\mu)\dots Q_{k+2s-2}(\mu)\big]\mathbf{r}_{k+2s-1}\Big).$$

(We can ignore the factor μ in front.) The system of equations

$$T(\eta_k) = \eta_k N_k(\mu); \qquad \eta_{k+1} = \eta_k Q_k(\mu),$$

has (4.21) as compatibility condition. (4.21) must be independent from μ since T is defined by its last column, and it is preserved by the scaling, while the rest of the entries are zero or I_n, and hence independent from μ. Hence, this system is a standard *Lax representation for the Pentagram map on Grassmannian*, which can be used as usual to generate invariants of the map. Indeed, the conjugation class of the monodromy is preserved by the map $T : \mathcal{P}_N \to \mathcal{P}_N$, as we saw before. Since a representative of the class is given by $M = Q_0 Q_1 \dots Q_{N-1}$, we obtain the following theorem.

THEOREM 4.7. — *The map $T : \mathcal{P}_N \to \mathcal{P}_N$ lies on the Riemann surface*

$$\det(Q_0(\mu)Q_1(\mu)\dots Q_{N-1}(\mu) - \eta I_{nm}) = 0$$

with $\mu, \eta \in \mathbb{C}$.

5. Pentagram map on $\mathrm{Gr}(n, (2s+1)n)$

5.1. Definition of the map

In this section we will define the Pentagram map for the Grassmannian $\mathrm{Gr}(n, (2s+1)n)$ for $s \geqslant 1$. Recall that the dimension of $\mathrm{Gr}(n, (2s+1)n)$ is $2sn^2$.

Assume that $\wp = (l_k)$ is a twisted polygon on $\mathrm{Gr}(n, (2s+1)n)$ and let X_k be any twisted lift. Let Π_k be the unique $n(s+1)$ linear subspace containing the following subspaces

$$\Pi_k = \langle X_k, X_{k+2}, \dots, X_{k+2s} \rangle.$$

We define the pentagram map to be the map T taking the polygon $\wp$ to the unique twisted polygon (with the same monodromy) whose kth vertex has a lift given by the intersections $\Pi_k \cap \Pi_{k+1}$. This map can be defined either on the space of polygons, or on the vertices. The map T is well defined and independent from the lift X. In fact, from the Grassmann formula we get

$$
\begin{aligned}
(2s+1)n = \dim(\Pi_{k-1} + \Pi_k) &= \dim \Pi_{k-1} + \dim \Pi_k - \dim(\Pi_{k-1} \cap \Pi_k) \\
&= (s+1)n + (s+1)n - \dim(\Pi_{k-1} \cap \Pi_k) \\
&= 2(s+1)n - \dim(\Pi_{k-1} \cap \Pi_k),
\end{aligned}
$$

which shows that $\dim(\Pi_{k-1} \cap \Pi_k) = n$ for any k, and hence $\Pi_k \cap \Pi_{k+1}$ is a lift of a unique element in $\mathrm{Gr}(n, (2s+1)n)$, an element equals to the kth vertex of $T(\wp)$. As before, we will abuse notation and use T equally for the map on polygons, on their lifts, on frames or on the moduli space.

Clearly the pentagram map is invariant under the action of the projective group (linear on lifts), and therefore one is able to write it as a map on the moduli space of Grassmannian polygons, as represented by the invariants we found in Section 3. This is what we do next.

Let us consider a twisted normalized lift $V = (V_k)$ of a regular N-gon, $\wp = (l_k)$ as in Theorem 3.1. Then, using dimension counting, there exist $2s + 1$ squared $n \times n$ matrices $a_k^0, a_k^1, \dots, a_k^{2s-1}, a_k^{2s}$ such that

(5.1) $\qquad V_{k+2s+1} = V_k a_k^0 + V_{k+1} a_k^1 + \dots + V_{k+2s-1} a_k^{2s-1} + V_{k+2s} a_k^{2s}.$

The blocks will be normalized so that a_k^0 is diagonal or Toeplitz, and the entries of a_k^{2s} have a number of syzygies that relate them.

If the lift is twisted, then a_k^i will be N-periodic for $i = 0, 1, 2, \cdots, 2s$; that is

$$(5.2) \qquad a_{k+N}^i = a_k^i,$$

for any k.

Since $T(V_k) \in \Pi_{k+1}$, we can assume that there exists c_k^i, $n \times n$ matrices such that

$$(5.3) \quad T(V_k) = V_{k+1}c_k^1 + V_{k+3}c_k^3 + \cdots + V_{k+2s-1}c_k^{2s-1} + V_{k+2s+1}c_k^{2s+1},$$

for all $k \in \mathbb{Z}$.

On other hand, we can replace V_{k+2s+1} in the last term of (5.3) by

$$(5.4) \qquad V_{k+2s+1} = V_k a_k^0 + V_{k+1}a_k^1 + \cdots + V_{k+2s-1}a_k^{2s-1} + V_{k+2s}a_k^{2s}.$$

It follows that

$$\begin{aligned}
T(V_k) &= V_{k+1}c_k^1 + V_{k+3}c_k^3 + \cdots + V_{k+2s-1}c_k^{2s-1} \\
&\quad + (V_k a_k^0 + V_{k+1}a_k^1 + \cdots + V_{k+2s-1}a_k^{2s-1} + V_{k+2s}a_k^{2s})c_k^{2s+1} \\
(5.5) \quad &= V_k a_k^0 c_k^{2s+1} + V_{k+1}[c_k^1 + a_k^1 c_k^{2s+1}] + V_{k+2}a_k^2 c_k^{2s+1} \\
&\quad + V_{k+3}[c_k^3 + a_k^3 c_k^{2s+1}] + \cdots\cdots + V_{k+2s-2}a_k^{2s-2}c_k^{2s+1} \\
&\quad + V_{k+2s-1}[c_k^{2s-1} + a_k^{2s-1}c_k^{2s+1}] + V_{k+2s}a_k^{2s}c_k^{2s+1}.
\end{aligned}$$

Since we also have $T(V_k) \in \Pi_k$, it follows that $c_k^{2\ell+1} = -a_k^{2\ell+1}c_k^{2s+1}$ for any $\ell = 0, \ldots, s-1$, and

$$T(V_k) = \left[V_k a_k^0 + V_{k+2}a_k^2 + \cdots + V_{k+2s-2}a_k^{2s-2} + V_{k+2s}a_k^{2s}\right]c_k^{2s+1}.$$

Remark 5.1. — As in the previous case, the matrix invariant matrix c_k^{2s+1} has no apparent restrictions, but in fact, it is completely determined. In order to be able to define the pentagram map on the moduli space coordinates given by the matrices a_k^j, we need to guarantee that $\{T(V_k)\}_{k=1}^N$ is the lift of $T(l_k)$ as described by Theorem 3.1, as far as $T(\wp)$ is generic. As we showed in Theorem 3.1, there is such a unique lift, and $c_k^{2s+1} = \lambda_k$ will be the proportional matrix that appears in the theorem. In fact, we have not shown that if $\wp$ is regular, so is $T(\wp)$. As it was the case with the original pentagram map ([9]), the map is only generically defined.)

Example 5.2. — For $s = 1$, that is, on $\mathrm{Gr}(n, 3n)$ the Pentagram map is

$$T(V_k) = (V_k a_k^0 + V_{k+2}a_k^2)\lambda_k = (V_k, V_{k+1}, V_{k+2}) \begin{pmatrix} a_k^0 \\ O_n \\ a_k^2 \end{pmatrix} \lambda_k = \rho_k \mathbf{r}_k \lambda_k,$$

for any $k \in \mathbb{Z}$.

Now, define

(5.6)
$$\rho_k = (V_k \ V_{k+1} \ \ldots \ V_{k+2s-1} \ V_{k+2s}),$$

so that for any $k \in \mathbb{Z}$

$$T(V_k) = \rho_k \begin{pmatrix} a_k^0 \\ O_n \\ a_k^2 \\ O_n \\ \vdots \\ O_n \\ a_k^{2s} \end{pmatrix} \lambda_k.$$

As before, if $\rho_{k+1} = \rho_k Q_k$, and

$$\mathbf{r}_k = \begin{pmatrix} a_k^0 \\ O_n \\ a_k^2 \\ \vdots \\ a_k^{2s-2} \\ O_n \\ a_k^{2s} \end{pmatrix},$$

and if we extend T to ρ_k by applying it to each block-column, we can write

(5.7) $\quad T(\rho_k) = \rho_k \left(\mathbf{r}_k \lambda_k \quad R_{k+1} \mathbf{r}_{k+1} \lambda_{k+1} \quad \cdots \quad R_{k+2s} \mathbf{r}_{k+2s} \lambda_{k+2s} \right),$

where, if Q_k is given as in (3.2), then $R_{k+i} = Q_k Q_{k+1} \cdots Q_{k+i-1}$, for $i = 1, 2, \ldots$.

Now, as we did before, for any $k \in \mathbb{Z}$ define

$$N_k = \left(\mathbf{r}_k \quad R_{k+1} \mathbf{r}_{k+1} \quad R_{k+2} \mathbf{r}_{k+2} \quad \cdots \quad R_{k+2s} \mathbf{r}_{k+2s} \right).$$

It follows that

(5.8)
$$T(\rho_k) = \rho_k N_k \Lambda_k, \qquad \rho_{k+1} = \rho_k Q_k,$$

where

$$\Lambda_k = \begin{pmatrix} \lambda_k & O_n & \cdots & O_n \\ O_n & \lambda_{k+1} & \cdots & O_n \\ \ddots & \ddots & \ddots & \ddots \\ O_n & \cdots & O_n & \lambda_{k+2s} \end{pmatrix}.$$

As before, the compatibility condition of these two natural maps is given by

$$\mathcal{T}(T(\rho_k)) = \mathcal{T}(\rho_k N_k) = \rho_{k+1}N_{k+1} = T(\mathcal{T}(\rho_k))$$
$$= T(\rho_{k+1}) = T(\rho_k Q_k) = T(\rho_k)T(Q_k).$$

Hence

$$\rho_{k+1}N_{k+1}\Lambda_{k+1} = T(\rho_k)T(Q_k) = \rho_k N_k \Lambda_k T(Q_k),$$

for all $k \in \mathbb{Z}$. It shows that (5.8), together with

(5.9)
$$T(Q_k) = \Lambda_k^{-1} N_k^{-1} Q_k N_{k+1}\Lambda_{k+1},$$

holds true for any $k \in \mathbb{Z}$ and describes a discrete, parameter free, Lax representation for the map T defined on the moduli space as represented by the invariants that appear in the last column of Q_k. Notice that from (5.7) we now that λ_k are also periodic, that is $\lambda_{k+N} = \lambda_k$ for any k. And from the definition in (3.2) so are both Q_k and R_k.

5.2. A Lax representation for the pentagram map on $\mathrm{Gr}(n, (2s+1)n)$

As we did for the even dimensional case, in this section we will prove that one can introduce a parameter μ in (5.8) in such a way that (5.9) will be independent from μ. This will define a true Lax representation that can be used for integration of the map. As it was done in [5], we will prove that the map T is invariant under a scaling, this time given by

(5.10)
$$a_k^{2r+1} \to \mu^{-1+r/s} a_k^{2r+1}, \quad r = 0, \ldots s - 1$$
$$a_k^{2r} \to \mu^{r/s} a_k^{2r}, \qquad r = 0, 1, \ldots, s.$$

We will follow the same steps as in the even dimensional case. The first steps involve proving that the map defined without the proportional matrix λ_k is invariant under the scaling. We will then calculate the degree of λ_k using these results and incorporate the proportional matrices λ_k to the map to finally calculate the degree of $T(a_k^i)$.

First of all, notice that if, as before, we denote by F_{k+r} the $r+1$ block-column of N_k, the analogous to Lemma 4.3 still holds true. We cite it here without proof, since the proof is identical.

LEMMA 5.3. — Let $F_k = \mathbf{r}_k$ and $F_{k+\ell} = R_{k+\ell}\mathbf{r}_{k+\ell}$, $\ell = 1, \ldots$ as above. Then, there exist $n \times n$ matrices α_i^j such that

(5.11)
$$F_{k+2\ell} = \sum_{r=1}^{\ell} F_{k+2r-1}\alpha_{2r-1}^{2\ell} + G_{k+2\ell},$$

$$F_{k+2\ell+1} = \sum_{r=0}^{\ell} F_{k+2r}\alpha_{2r}^{2\ell+1} + \widehat{G}_{k+2\ell+1},$$

for $\ell \geqslant 1$, where

(5.12)
$$\widehat{G}_{k+2\ell+1} = \mathbf{p}_k \left(\mathcal{T}G_{k+2\ell} \right)_{2s} + \Gamma\mathcal{T}G_{k+2\ell},$$
$$\alpha_{2r}^{2\ell+1} = \mathcal{T}\alpha_{2r-1}^{2\ell}, \quad \alpha_0^{2\ell+1} = \left(\mathcal{T}G_{k+2\ell} \right)_{2s},$$

with

(5.13)
$$\mathbf{p}_k = \begin{pmatrix} O_n \\ a_k^1 \\ O_n \\ a_k^3 \\ O_n \\ \vdots \\ a_k^{2s-1} \\ O_n \end{pmatrix},$$

and

(5.14) $\quad G_{k+2\ell+2} = \Gamma\mathcal{T}\widehat{G}_{k+2\ell+1}, \quad \alpha_{2r+1}^{2\ell+2} = \mathcal{T}\alpha_{2r}^{2\ell+1}, \quad G_k = F_k = \mathbf{r}_k.$

By A_{2s} we mean the last $n \times n$ block entry of a matrix A.

The main difference with the even dimensional case is that the even block-columns $G_{2\ell}$ start now with a non-zero block (while before it started with a zero one), and generate a $(s+1)n$ dimensional subspace, orthogonal to those generated by the odd ones $\widehat{G}_{2\ell+1}$, which start with a zero block and generate a sn dimensional subspace. Also, in this case the first two blocks of $G_{2\ell}$ are zero since $G_{k+2\ell} = \Gamma\mathcal{T}\widehat{G}_{k+2\ell-1}$ and Γ shifts all the blocks once downwards.

Assume first that we drop the Λ_k factor and define

$$\overline{T}(V_k) = \rho_k\mathbf{r}_k.$$

Define also $c_k^i = \overline{T}(a_k^i)$, which is given by the compatibility formula below, the analogous to (4.17)

$$\overline{T}(Q_k) = N_k^{-1}Q_k N_{k+1}.$$

Then
$$\overline{T}(\mathbf{a}_k) = N_k^{-1} Q_k \mathcal{T} F_{k+2s} = N_k^{-1} F_{k+2s+1},$$
which can be written as

(5.15) $$N_k \overline{T}(\mathbf{a}_k) = F_{k+2s+1}.$$

Once more $\overline{T}(\mathbf{a}_k)$ can be interpreted as the solution of the linear equation (5.15).

THEOREM 5.4. — *The matrices c_k^i are homogenous with respect to the scaling* (5.10), *and*
$$d(c_k^{2\ell}) = \frac{\ell}{s}, \quad d(c_k^{2\ell+1}) = -1 + \frac{\ell}{s}.$$

Notice that the degree of c_k^r coincides with that of a_k^r. Later we will show that λ_k are all invariant under scaling and this theorem will essentially prove the invariance of the map under (5.10).

Let us once more drop the subindex k unless needed. As we did in the previous case, we will work with determinants of the form $D_{r,i}^j$, simplifying them down to their homogeneous component, and calculating their degree. Because they are solution of (5.15), each entry of c^r will be a quotient of these determinants and D, and this way we will be able to determine their degree. The study comes in a number of lemmas.

LEMMA 5.5. — *The determinant*
$$D = \det(F, F_1, \ldots, F_{2s}),$$
is invariant under (5.10).

Proof. — Using (5.11) we have that
$$D = \det(F, \widehat{G}_1, G_2, \ldots, \widehat{G}_{2s-1}, G_{2s}).$$
This time all $G_{2r} = \Gamma \mathcal{T} \widehat{G}_{2r-1}$ have the first two blocks equal O_n; therefore, the blocks in the first row of D are all zero, except for the first block which is the first block of $F = \mathbf{r}$, i.e. a^0, and which is invariant under scaling. If we simplify using the first n rows, we have the determinant of a matrix that looks like
$$\begin{pmatrix}
A_{1,1} & O_n & A_{1,2} & O_n & \ldots & A_{1,s} & O_n \\
O_n & B_{1,1} & O_n & B_{1,2} & \ldots & O_n & B_{1,s} \\
A_{2,1} & O_n & A_{2,2} & O_n & \ldots & A_{2,s} & O_n \\
O_n & B_{2,1} & O_n & B_{2,2} & \ldots & O_n & B_{2,s} \\
\vdots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
A_{s,1} & O_n & A_{s,2} & O_n & \ldots & A_{s,s} & O_n \\
O_n & B_{s,1} & O_n & B_{s,2} & \ldots & O_n & B_{s,s}
\end{pmatrix},$$

where $A_{i,j}$ are the nonzero blocks of $\widehat{G}_{2j-1}$ and $B_{i,j}$ are the nonzero blocks of G_{2j}. Using $n^2 s(s-1)$ exchanges of rows and columns, this determinant can be easily transformed into

$$\det \begin{pmatrix} A & O_{sn} \\ O_{sn} & B \end{pmatrix},$$

with $A = (A_{i,j})$ and $B = (B_{i,j})$. Also, since $G_{2\ell} = \Gamma \mathcal{T} \widehat{G}_{2\ell-1}$, we have that $B = \mathcal{T} A$ and $D = \det A \det \mathcal{T} A$.

We will next show that $d(A_{i,j}) = \frac{i-j}{s}$. This will imply, from the definition of determinant, that $d(\det A) = n \sum_{i=1}^s \sum_{j=1}^s \frac{i-j}{s} = 0$, concluding the proof.

Indeed, from (5.12) we have

$$\widehat{G}_{2\ell+1} = \mathbf{p} \left(\mathcal{T} G_{2\ell} \right)_{2s} + \Gamma \mathcal{T} G_{2\ell} = \mathbf{p} \left(\mathcal{T}^2 \widehat{G}_{2\ell-1} \right)_{2s-1} + \Gamma^2 \mathcal{T}^2 \widehat{G}_{2\ell-1},$$

and using this we conclude that

$$(5.16) \qquad A_{k,\ell+1} = a^{2k-1} \mathcal{T} A_{s,\ell} + \mathcal{T}^2 A_{k-1,\ell}.$$

The degrees of the nonzero blocks of $\mathbf{p}$ are given by

$$-1, \quad -1 + \frac{1}{s}, \quad -1 + \frac{2}{s}, \quad \dots, \quad -1 + \frac{s-1}{s},$$

while the degree of a^{2s} is 1. Thus,

$$d(A_{k,1}) = d(a^{2k-1} a^{2s}) = d(\mathcal{T} a^{2k-2}) = \frac{k-1}{s}.$$

We now use induction. Assume that $d(A_{i,j}) = \frac{i-j}{s}$ for all $i = 1, \dots s$ and all $j < \ell$. From (5.16) we have that, since $d(a^{2k-1}) = -1 + \frac{k-1}{s}$,

$$d(a^{2k-1} \mathcal{T}^2 A_{s,\ell}) = -1 + \frac{k-1}{s} + \frac{s-\ell}{s} = \frac{k-\ell-1}{s};$$

$$d(\mathcal{T}^2 A_{k-1,\ell}) = \frac{k-1-\ell}{s},$$

and so

$$d(A_{k,\ell+1}) = \frac{k-(\ell+1)}{s},$$

concluding the proof of the lemma. $\qquad\qquad\qquad\qquad\qquad\qquad \square$

Let us denote by $D^j_{r,i}$ the determinant given by

$$D^j_{r,i} = \det(F, F_1, \dots, F_{r-1}, F^j_{r,i}, F_{r+1}, \dots, F_{2s}),$$

where $F^j_{r,i}$ is the block-column obtained from F_r by substituting the ith column, f^i_r, with the jth column of F_{2s+1}, f^j_{2s+1}.

LEMMA 5.6. — *Determinant $D^j_{2\ell+1,i}$ is homogeneous for all $i, j = 1, \ldots n$, and $\ell = 0, \ldots, s-1$, and*

$$d(D^j_{2\ell+1,i}) = -1 + \frac{\ell}{s}.$$

Proof. — As in the previous cases we will make heavy use of (5.11) to reduce the determinants to their homogeneous components before calculating their degree. First of all we will substitute the ith column of $F^j_{2\ell+1,i}$ in

$$\det(F, F_1, \ldots, F_{2\ell}, F^j_{2\ell+1,i}, F_{2\ell+2}, \ldots, F_{2s}),$$

by that of $\widehat{G}_{2s+1}$, $\widehat{g}^j_{2s+1}$, since the difference is a combination of even columns, which are all present in the determinant. We can then simplify the columns to the right of this and substitute them by either $\widehat{G}_{2r+1}$ or by

$$G_{2r} + \widehat{g}^i_{2\ell+1} \left(\alpha^{2r}_{2\ell+1} \right)_i,$$

where a super index indicate the column, a subindex the row. We can then substitute all the remaining columns of $F^j_{2\ell+1,i}$ and F_r, $r \leqslant 2\ell$ by those of G_r or $\widehat{G}_r$, depending on parity, to obtain

$$D^j_{r,i} = \det\Big(F, \widehat{G}_1, \ldots, G_{2\ell}, \widehat{G}^j_{2\ell+1,i}, G_{2\ell+2} + \widehat{g}^i_{2\ell+1} \left(\alpha^{2\ell+2}_{2\ell+1} \right)_i,$$

$$\widehat{G}_{2\ell+3}, \ldots, G_{2s} + \widehat{g}^i_{2\ell+1} \left(\alpha^{2s}_{2\ell+1} \right)_i \Big),$$

where $\widehat{G}^j_{2\ell+1,i}$ is equal to $\widehat{G}_{2\ell+1}$ except for its ith column which has been substituted by the jth column of $\widehat{G}_{2s+1}$. Since the missing column in $\widehat{G}_{2\ell+1}$ has been substituted by another odd column, if we expand this determinant all the terms that include any extra column $\widehat{g}^i_{2\ell+1}$ will vanish since we already have odd columns equal to half the dimension in their standard position. Therefore

$$D^j_{r,i} = \det(F, \widehat{G}_1, \ldots, G_{2\ell}, \widehat{G}^j_{2\ell+1,i}, G_{2\ell+2}, \widehat{G}_{2\ell+3}, \ldots, G_{2s}).$$

Consider now the following product

$$(5.17) \qquad \widehat{g}^j_{2s+1} = \widehat{g}^i_{2\ell+1} \otimes (\widehat{g}^i_{2\ell+1})^{-1} \otimes \widehat{g}^j_{2s+1},$$

where by $(\widehat{g}^i_{2\ell+1})^{-1}$ we mean the vector whose entries are the inverses of the entries of $\widehat{g}^i_{2\ell+1}$, and where the product represents the individual entries product, as customary. We claim that $(\widehat{g}^i_{2\ell+1})^{-1} \otimes \widehat{g}^j_{2s+1}$ is homogeneous, each entry with the same degree. Indeed, using the fact that $d(A_{i,j}) = \frac{i-j}{s}$ (recall that $A_{i,j}$, $i = 1, \ldots s$, are the nonzero blocks of $\widehat{G}_{2j-1}$, $j = 1, \ldots, s$),

we can see that

$$d((\widehat{g}^i_{2\ell+1})^{-1} \otimes \widehat{g}^j_{2s+1}) = d(\widehat{g}^j_{2s+1}) - d((\widehat{g}^i_{2\ell+1})$$

$$= \left(*, -1, *, -1 + \frac{1}{s}, *, \ldots, *, -1 + \frac{s-1}{s}, *\right)$$

$$- \left(*, -\frac{\ell}{s}, *, \frac{1-\ell}{s}, *, \ldots, *, \frac{s-1-\ell}{s}, *\right)$$

$$= \left(*, -1 + \frac{\ell}{s}, *, -1 + \frac{\ell}{s}, *, \ldots, *, -1 + \frac{\ell}{s}, *\right),$$

where $*$ indicates the position of a zero block. Therefore, substituting $\widehat{g}^j_{2s+1}$ by (5.17) in the determinant, expanding the determinant using this column, and using the fact that D is invariant under scaling gives us

$$d(D^j_{2\ell+1,i}) = -1 + \frac{\ell}{s},$$

as claimed. $\qquad\square$

LEMMA 5.7. — Determinant $D^j_{2\ell,i}$ is also homogeneous for all $i, j = 1, \ldots, n$, $\ell = 0, \ldots, s-1$ and

$$d(D^j_{2\ell,i}) = \frac{\ell}{s}.$$

Proof. — The proof of this case is a bit more complicated. We need to look at the determinant

$$D^j_{2\ell,i} = \det(F, F_1, \ldots, F_{2\ell-1}, F^j_{2\ell,i}, F_{2\ell+1}, \ldots, F_{2s}).$$

As before, we simplify the ith column of $F^j_{2\ell,i}$, using (5.11). That is, f^j_{2s+1} will be substituted by

$$(5.18) \qquad\qquad \widehat{g}^j_{2s+1} + f^i_{2\ell} \left(\alpha^{2s+1}_{2\ell}\right)_{i,j}.$$

We can then substitute the columns to the right of this one: we substitute F_{2r+1} by $\widehat{G}_{2r+1} + f^i_{2\ell} \left(\alpha^{2r+1}_{2\ell}\right)_i$ and F_{2r} by G_{2r}. We can also substitute $f^i_{2\ell}$ in the expression of F_{2r+1} by $g^i_{2\ell}$. We then change the remaining columns in $F^j_{2\ell,i}$, including $f^i_{2\ell}$, by those of $G_{2\ell}$ (we call the resulting matrix $G^j_{2\ell,i}$), and substitute the F's blocks to the left of $G^j_{2\ell,i}$ by G-blocks. Then, column (5.18) becomes

$$(5.19) \qquad\qquad \widehat{g}^j_{2s+1} + g^i_{2\ell} \left(\alpha^{2s+1}_{2\ell}\right)_{i,j}.$$

and the resulting determinant is given by

$$\det\left(F, \widehat{G}_1, G_2 \dots, \widehat{G}_{2\ell-1}, G^j_{2\ell,i}, \widehat{G}_{2\ell+1} + g^i_{2\ell}\left(\alpha^{2\ell+1}_{2\ell}\right)_i,\right.$$
$$\left. G_{2\ell+2}, \dots, \widehat{G}_{2s-1} + g^i_{2\ell}\left(\alpha^{2s+1}_{2\ell}\right)_i G_{2s}\right).$$

If we now expand the determinant using the ith column of $G^j_{2\ell,i}$, i.e. (5.19), we have

$$D^j_{2\ell,i} = \left(\alpha^{2s+1}_{2\ell}\right)_{i,j} D$$
$$+ \det\left(F, \widehat{G}_1, G_2 \dots, \widehat{G}_{2\ell-1}, G^j_{2\ell,i}, \widehat{G}_{2\ell+1} + g^i_{2\ell}\left(\alpha^{2\ell+1}_{2\ell}\right)_i,\right.$$
$$\left. G_{2\ell+2}, \dots, \widehat{G}_{2s-1} + g^i_{2\ell}\left(\alpha^{2s+1}_{2\ell}\right)_i G_{2s}\right),$$

where $G^j_{2\ell,i}$ has as ith column $\widehat{g}^j_{2s+1}$ and $G_{2\ell}$ elsewhere.

Notice that, once more, if we expand this determinant, the terms with no $G^i_{2\ell}$ will be zero since we have more odd columns that are needed to generate the odd subspace (indicated with a hat) given that $G^j_{2\ell,i}$ contains one. Thus, the determinant expands as

$$\sum_{r=\ell+1}^{s-1} \sum_{p=1}^{n} \left(\alpha^{2r+1}_{2\ell}\right)_{i,p} \det\left(F, \widehat{G}_1, G_2, \dots, G^j_{2\ell,i}, \widehat{G}_{2\ell+1}, \dots,\right.$$
$$\left. \widehat{G}^p_{2r+1} + g^i_{2\ell} e^T_p, G_{2\ell+1}, \dots, G_{2s}\right),$$

where $\widehat{G}^p_{2r+1}$ indicates $\widehat{G}_{2r+1}$ with a zero p column. Each one of these terms can transformed by shifting the p column of $\widehat{G}^p_{2r+1} + g^i_{2\ell} e^T_p$ and the ith column of $G^j_{2\ell,i}$. The expansion becomes

$$(5.20) \quad \sum_{r=\ell+1}^{s-1} \sum_{p=1}^{n} \pm \left(\alpha^{2r+1}_{2\ell}\right)_{i,p} \det(F, \widehat{G}_1, G_2, \dots, G_{2\ell}, \widehat{G}_{2\ell+1}, \dots,$$
$$\widehat{G}^p_{2r+1} + \widehat{g}^j_{2s+1} e^T_p, G_{2\ell+1}, \dots, G_{2s}),$$

and now we are ready to calculate degrees. Substitute $\widehat{g}^j_{2s+1}$ by

$$\widehat{g}^j_{2s+1} = \widehat{g}^j_{2s+1} \otimes (\widehat{g}^p_{2r+1})^{-1} \otimes \widehat{g}^p_{2r+1},$$

as before, and notice that the degree of $\widehat{g}^j_{2s+1} \otimes (\widehat{g}^p_{2r+1})^{-1}$ is the same for all of its block-entries and equal to $-1 + \frac{r}{s}$

$$d(\widehat{g}^j_{2s+1} \otimes (\widehat{g}^p_{2r+1})^{-1}) = d(\widehat{g}^j_{2s+1}) - d(\widehat{g}^p_{2r+1}) = \begin{pmatrix} * \\ \frac{1-(s+1)}{s} \\ * \\ \frac{2-(s+1)}{s} \\ \vdots \\ \frac{s-(s+1)}{s} \\ * \end{pmatrix} - \begin{pmatrix} * \\ \frac{1-(r+1)}{s} \\ * \\ \frac{2-(r+1)}{s} \\ \vdots \\ \frac{s-(r+1)}{s} \\ * \end{pmatrix}.$$

We now need to calculate the degree of α^j_i. This is simple from (5.12) and (5.14). We see that

$$d(\alpha^{2r+1}_0) = \frac{s-r}{s}, \qquad d(\alpha^{2r+1}_{2\ell}) = d(\alpha^{2(r-\ell)+1}_0) = 1 - \frac{r-\ell}{s},$$

$$d(\alpha^{2r}_{2\ell+1}) = d(\alpha^{2(r-\ell-1)+1}_0) = 1 - \frac{r-\ell-1}{s}.$$

With this, every term in the expansion (5.20) has degree

$$-1 + \frac{r}{s} + d((\alpha^{2r+1}_{2\ell})_{i,p}) = -1 + \frac{r}{s} + 1 - \frac{r-\ell}{s} = \frac{\ell}{s},$$

and since

$$d\left((\alpha^{2s+1}_{2\ell})_{i,j} D\right) = d\left((\alpha^{2s+1}_{2\ell})_{i,j}\right) = 1 - \frac{s-\ell}{s} = \frac{\ell}{s},$$

we conclude the proof of the lemma. □

After these results, the proof of Theorem 5.4 is immediate since the entry (j,i) of c^r is the quotient

$$\frac{D^j_{r,i}}{D},$$

and so $d(c^r) = d(D^j_{r,i})$, which coincide with the statement of the theorem.

Finally, the following theorem is a consequence of 5.4.

THEOREM 5.8. — *If N and m are coprime, the map T is invariant under the scaling (5.10).*

Proof. — To prove this result we simply need to prove that λ_k are invariant under scaling. After this fact is proved, computations similar to those in the proof of Theorem 4.6 will show that even as we introduce λ_k in the different block columns of determinants D and $D^j_{\ell,i}$, they do alter neither the homogeneity nor the degree of the determinants because they

are invariant under the scaling and all columns of a block-column have the same degree (and so do their linear combinations). Therefore

$$d(c_k^\tau) = d(T(a_k^\tau)) = d(a_k^\tau),$$

for any k and $r = 0, \dots, m-1$. We will avoid further details of those computations since they are almost identical to those in 4.6 and the interested reader can reproduce them.

To show that λ_k are invariant under scaling is not hard: if N and m are coprime, their determinants, $\delta_k = \det \lambda_k$, are the unique solution of

$$\det N_k \delta_k \delta_{k+1} \dots \delta_{k+m-1} = 1,$$

and since $\det N_k = D_k$, $\det N_k$ is invariant under the scaling (since $d(D_k) = 0$) and so are δ_k for any k. We next look at each factor in the splitting of $\lambda_k = d_k q_k$. As in the previous case, the factors d_k are determined by the normalization of $B_k = \Pi_m(c_k^0)$ as in (3.6), and since c_k^0 are invariant under the scaling, so will d_k.

Finally, q_k are uniquely determined by equations of the form (3.10) and (3.11) for c_k^{m-1}, and by their determinants. Since $c_{i,j}^k = c_{k-m+1}^{m-1} c_{k-m+1}^{m-1} \dots c_{N+k-m}^{m-1}$ is homogeneous, equations (3.10)–(3.11) are scaling invariant. Furthermore, $\det q_k = \det d_k^{-1} \delta_k$ and both δ_k and $\det d_k$ are scaling invariant, so is $\det q_k$. Therefore, q_k are scaling invariant and so are λ_k. $\qquad\square$

As in the even dimensional case, this theorem allows us to define the Lax representation for the map $T : \mathcal{P}_N \to \mathcal{P}_N$. Indeed, if we define

$$Q_k(\mu) = \begin{pmatrix} O_n & O_n & O_n & \dots & O_n & O_n & a_k^0 \\ I_n & O_n & O_n & \dots & O_n & O_n & \mu^{-1} a_k^1 \\ O_n & I_n & O_n & \dots & O_n & O_n & \mu^{\frac{1}{s}} a_k^2 \\ O_n & O_n & I_n & \dots & O_n & O_n & \mu^{-1+\frac{1}{s}} a_k^3 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ O_n & O_n & \dots & O_n & I_n & O_n & \mu^{-1+\frac{s-1}{s}} a_k^{2s-1} \\ O_n & O_n & \dots & O_n & O_n & I_n & \mu a_k^{2s} \end{pmatrix},$$

then we can define

$$N_k(\mu) = (\mathbf{r}_k(\mu), R_k(\mu)\mathbf{r}_{k+1}, R_{k+1}(\mu)\mathbf{r}_{k+2}(\mu), \dots, R_{k+m-2}(\mu)\mathbf{r}_{k+m-1}(\mu)),$$

where

$$\mathbf{r}_k(\mu) = \begin{pmatrix} a_r^0 \\ O_n \\ \mu^{\frac{1}{s}} a_r^2 \\ O_n \\ \mu^{\frac{2}{s}} a_r^4 \\ \vdots \\ O_n \\ \mu a_k^s \end{pmatrix},$$

and $R_{k+r}(\mu) = Q_k(\mu)Q_{k+1}(\mu)\ldots Q_{k+r}(\mu)$. The compatibility condition of the system

$$\rho_{k+1} = \rho_k Q_k(\mu), \qquad T(\rho_k) = \rho_k N_k(\mu),$$

will be given by

$$(5.21) \qquad T(Q_k(\mu)) = N_k(\mu)^{-1}\Lambda_k^{-1}Q_k(\mu)\Lambda_n N_k(\mu).$$

(5.21) must be independent from μ since T is defined by its last column, and it is preserved by the scaling, while the rest of the entries are zero or I_n, and hence independent from μ. Hence, this system is a standard *Lax representation for the Pentagram map on Grassmannian* in the case $m = 2s + 1$, which can also be used as usual to generate invariants of the map. With this new scaling, Theorem 4.7 is also true in the case when m is odd.

BIBLIOGRAPHY

[1] B. Khesin & F. Soloviev, "Integrability of higher pentagram maps", *Math. Ann.* **357** (2013), no. 3, p. 1005-1047.

[2] ———, "The geometry of dented pentagram maps", *J. Eur. Math. Soc.* **18** (2016), no. 1, p. 147-179.

[3] E. Mansfield, G. Marí Beffa & J. P. Wang, "Discrete moving frames and integrable systems", *Found. Comput. Math.* **13** (2013), no. 4, p. 545-582.

[4] G. Marí Beffa, "On generalizations of the pentagram map: discretizations of AGD flows", *J. Nonlinear Sci.* **23** (2013), no. 2, p. 303-334.

[5] ———, "On Integrable Generalizations of the Pentagram Map", *Int. Math. Res. Not.* **2015** (2015), no. 12, p. 3669-3693.

[6] ———, "On the integrability of the shift map on twisted pentagram spirals", *J. Phys. A, Math. Theor.* **48** (2015), no. 28, Art. ID 285202, 25 p.

[7] V. Ovsienko, R. E. Schwartz & S. Tabachnikov, "The Pentagram map: A discrete integrable system", *Commun. Math. Phys.* **299** (2010), no. 2, p. 409-446.

[8] ———, "Liouville–Arnold integrability of the pentagram map on closed polygons", *Duke Math. J.* **162** (2012), no. 12, p. 2149-2196.

[9] R. E. Schwartz, "The Pentagram Map", *Exp. Math.* **1** (1992), no. 1, p. 85-90.

[10] ———, "The Projective Heat Map Acting on Pentagons", *Research monograph* **1** (2014), p. 85-90.

[11] F. SOLOVIEV, "Integrability of the Pentagram Map", *Duke Math. J.* **162** (2013), no. 15, p. 2815-2853.

Raúl FELIPE
CIMAT
Callejón Jalisco s/n, Valenciana
36023 Guanajuato (Mexico)
raulf@cimat.mx

Gloria MARÍ BEFFA
University of Wisconsin - Mathematics Department
480 Lincoln Dr
Madison, WI 53706 (USA)
maribeff@math.wisc.edu