THE PERIODICITY CONJECTURE FOR PAIRS OF DYNKIN DIAGRAMS

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Abstract. We prove the periodicity conjecture for pairs of Dynkin diagrams using Fomin-Zelevinsky's cluster algebras and their (additive) categorification via triangulated categories.

Contents

1. Introduction

1.1. **The periodicity conjecture.** The *Y-system* associated with a pair (Δ, Δ') of Dynkin diagrams is a certain infinite system of algebraic recurrence equations. The *periodicity conjecture* asserts that all solutions to this system are periodic of period dividing the double of the sum of the Coxeter numbers of Δ and Δ'. The conjecture was first formulated by Al. B. Zamolodchikov [69] (for (Δ, A_1), where Δ is simply laced) in his study of the thermodynamic Bethe ansatz. We refer to [24] for an introduction to the significance of the conjecture in physics and to [24] [11] [18] [58] for its applications to dilogarithm identities. Zamolodchikov's original conjecture was subsequently generalized

- by Ravanini-Valleriani-Tateo to (Δ, Δ'), where Δ and Δ' are simply laced or 'tadpoles' [60, (6.2)];
- by Kuniba-Nakanishi to (Δ, A_n), where Δ is not necessarily simply laced [54, (2a)], see also Kuniba-Nakanishi-Suzuki [55, B.6].

The conjecture was proved

- for (A_n, A_1) by Frenkel-Szenes [24] (who produced explicit solutions) and independently by Gliozzi-Tateo [31] (via volumes of threefolds computed using triangulations),

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- by Fomin-Zelevinsky [22] for (Δ, A_1), where Δ is not necessarily simply laced (via the philosophy of cluster algebras and a computer check for the exceptional types; a uniform proof can now be given using [68]),
- for (A_n, A_m) by Volkov [67], who exhibited explicit solutions using cross ratios, and by Szenes [64], who interpreted the system as a system of flat connections on a graph; an equivalent statement was proved by Henriques [34].

An original approach via representations of quantum affine algebras is due to Hernandez-Leclerc [35]. They treat the case (A_n, A_1) and obtain formulas for solutions in terms of q-characters.

In [43], we announced a proof of the conjecture in the general case and gave an outline of the proof. In this article, we provide the detailed proof. Notice that for the case (Δ, A_n), where Δ is not simply laced, there are two variants of the conjecture: the first one was stated by Kuniba-Nakanishi [54] and involves *dual Coxeter numbers* (*cf. e.g.* Chapter 6 of [40]). It is proved in the papers [36] and [37]. The second one is due to Fomin-Zelevinsky [22] and is proved at the end of this article. The conjecture has a refined version (half-periodicity) and an analogue for the so-called T-system. Both were proved in [38] in the simply laced case by adapting the method of [43].

1.2. **On the proof.** The proof we propose is based on a reformulation of the conjecture in terms of Fomin-Zelevinsky's cluster algebras [20] [21] [8] [23] [70] [19] and on the recent theory linking cluster algebras to representations of quivers (with relations).

The link between the conjecture and cluster algebras goes back to Fomin-Zelevinsky's fundamental work [22]. Here and in [23], they showed how the Y-system appearing in the conjecture is controlled by the evolution of Y-seeds in the bipartite belt of a cluster algebra. Let us point out that recently, other discrete dynamical systems have been linked to cluster algebras, in particular the T-system [38] and the Q-system [41] [15].

The theory linking cluster algebras to quiver representations was initiated by Marsh-Reineke-Zelevinsky [56] and subsequently developed in several variants and by many authors, *cf.* for example the surveys [6] [29] [42] [61] [63]. It is related to Kontsevich-Soibelman's interpretation of cluster transformations in their theory of Donaldson-Thomas invariants [52] [53] and, in fact, it provided one of the starting points for Kontsevich-Soibelman's work.

For our purposes, we use the *cluster categories* first introduced in [7] (for general quivers) and independently in [10] (for quivers of type A_n). These are certain triangulated categories which are 2-Calabi-Yau and contain a distinguished object with remarkable properties (a cluster tilting object). Originally, the cluster category $\mathcal{C}_A$ was defined for algebras A of global dimension at most one; however, in recent work, Amiot [1] has extended the construction to many algebras of global dimension at most 2. We show that Amiot's results apply in particular to tensor products $A = kQ \otimes kQ'$ of path algebras of quivers obtained by orienting the given Dynkin diagrams Δ and Δ'. Using Palu's [59] generalization of the Caldero-Chapoton map [9] we show that the resulting cluster category $\mathcal{C}_A$ is indeed an (additive) categorification of the cluster algebra which controls the Y-system associated with (Δ, Δ'). In the categorification $\mathcal{C}_A$, it is easy to write down an autoequivalence, the *Zamolodchikov transformation*, whose powers provide the solutions to the Y-system. We conclude by showing that the Zamolodchikov transformation is of order dividing the sum of the Coxeter numbers of Δ and Δ'.

This proof is effective in the sense that in prinicple, it yields explicit (periodic) formulas for the general solution of the Y-system. It is conceptual in the sense that the validity of the conjecture is obtained from a categorical periodicity theorem, which in turn follows from classical results of Gabriel and Happel. For $\Delta' = A_1$, the proof specializes to a new proof of the case due to Fomin-Zelevinsky [22].

1.3. **Contents.** In section 2, we state the conjecture for pairs of Dynkin diagrams (Δ, Δ') which may be multiply laced. We present the plan of the proof in section 2.5. We treat the case of simply laced Dynkin diagrams in sections 3 to 8. In section 3, we recall fundamental constructions from the theory of cluster algebras: quiver mutation and the mutation of Y-seeds. Then we introduce three types of products of quivers (section 3.3) and reformulate the conjecture as a statement about the periodicity of a sequence of mutations $\mu_\boxtimes$ applied to the initial Y-seed associated with the triangle product $Q \boxtimes Q'$ of two alternating quivers with underlying graphs Δ and Δ' (section 3.5). Section 4 is devoted to tropical Y-variables and F-polynomials: we recall how they are constructed and how they can be used to express the non tropical Y-variables (section 4.1); we also introduce g-vectors and cluster variables (section 4.4), which are useful in the application of our categorical model to the so-called T-systems.

In section 5, we recall the notions of 2-Calabi-Yau triangulated category $\mathcal{C}$ and of a cluster tilting object T in such a category. There is a canonical quiver associated with the datum of $\mathcal{C}$ and T, namely the *endoquiver* R of T (=quiver of the endomorphism algebra of T in $\mathcal{C}$). The pair $(\mathcal{C}, T)$ is called a *2-Calabi-Yau realization* of the quiver R. One expects the cluster combinatorics associated with the quiver R to be encoded in the category $\mathcal{C}$. We therefore construct a 2-Calabi-Yau realization $(\mathcal{C}, T)$ for the triangle product $R = Q \boxtimes Q'$ (section 5.8). In section 5.13, we describe the category $\mathcal{C}$ using a quiver with potential. This will later enable us to determine the endoquivers of new cluster tilting objects in $\mathcal{C}$.

In section 6, we show how to recover cluster combinatorial data associated with a quiver R from a 2-Calabi-Yau realization $(\mathcal{C}, T)$ of R. The most important step is to lift the mutation operation to the mutation of cluster tilting objects in such a way that the endoquivers transform as predicted by Fomin-Zelevinsky's mutation rule (section 6.1). In fact, this is not always possible because 2-cycles may appear in the quivers of the mutated cluster tilting objects. However, we show in sections 6.1, 6.7 and 6.13, that if no 2-cycles appear, then all cluster combinatorial data we need can be recovered from the category: quivers, tropical Y-variables and F-polynomials (as well as g-vectors and cluster variables). For a given 2-Calabi-Yau realization $(\mathcal{C}, T)$, the appearance of 2-cycles in the endoquivers of a sequence of mutations of T has to be excluded by an explicit computation. In section 7.1, we perform this computation for the 2-Calabi-Yau realizations associated with a class of quivers with potential (the (Q, Q')-constrained quivers with potential) which includes our 2-Calabi-Yau realization $\mathcal{C}$ of the triangle product $Q \boxtimes Q'$ and its canonical sequence of mutations $\mu_\boxtimes$. We then reduce the conjecture to the study of an auto-equivalence, the *Zamolodchikov transformation*, of the category $\mathcal{C}$ (sections 7.3 and 7.5). We conclude by showing that the order of the Zamolodchikov transformation is finite and divides the sum of the Coxeter numbers (section 8.3). In section 9, we reduce the non simply laced case of the conjecture (in the form due to Fomin-Zelevinsky) to the simply laced case using the classical folding technique.

In the final section 10, we show that our proof is effective in the sense that, in principle, it allows us to write down explicit (periodic) formulas for the general solution of the Y-system in terms of homological invariants and Euler characteristics of quiver Grassmannians.

Acknowledgments. The author is indebted to Bernard Leclerc, who rekindled his interest in the conjecture during a conversation at the MSRI program on combinatorial representation theory in May 2008. He thanks Carles Casacuberta, André Joyal, Joachim Kock and Amnon Neeman for an invitation to the Centre de Recerca Matemàtica, Barcelona, where the details of the proof were worked out and [43] was written down in June 2008. He is grateful Sarah Scherotzke, Lingyan Guo and Alfredo Nájera Chávez for pointing out misprints in an earlier version of this paper as well as to two anonymous referees for their careful reading of the manuscript and their helpful advice.

2. The conjecture

2.1. **Statement.** Let Δ and Δ' be two Dynkin diagrams with vertex sets I and I'. Let A and A' be the incidence matrices of Δ and Δ', *i.e.* if C is the Cartan matrix of Δ and J the identity matrix of the same format, then $A = 2J - C$. Let h and h' be the Coxeter numbers of Δ and Δ'.

The *Y-system of algebraic equations* associated with the pair of Dynkin diagrams (Δ, Δ') is a system of countably many recurrence relations in the variables $Y_{i,i',t}$, where (i, i') is a vertex of $\Delta \times \Delta'$ and t an integer. The system reads as follows:

$$Y_{i,i',t-1}Y_{i,i',t+1} = \frac{\prod_{j\in I}(1+Y_{j,i',t})^{a_{ij}}}{\prod_{j'\in I'}(1+Y^{-1}_{i,j',t})^{a'_{i'j'}}}. \tag{2.1.1}$$

Periodicity Conjecture 2.2. *All solutions to this system are periodic in t of period dividing $2(h + h')$.*

We refer to the introduction for a sketch of the history of the conjecture.

Theorem 2.3. *The periodicity conjecture 2.2 is true.*

Let us give an algebraic reformulation of the conjecture: Let K be the fraction field of the ring of integer polynomials in the variables $Y_{ii'}$, where i runs through the set of vertices I of Δ and i' through the set of vertices I' of Δ'. Since Δ is a tree, the set I is the disjoint union of two subsets I_+ and I_- such that there are no edges between any two vertices of I_+ and no edges between any two vertices of I_-. Analogously, I' is the disjoint union of two sets of vertices I'_+ and I'_-. For a vertex (i, i') of the product $I \times I'$, define $\varepsilon(i, i')$ to be 1 if (i, i') lies in $I_+ \times I'_+ \cup I_- \times I'_-$ and -1 otherwise. For $\varepsilon = \pm 1$, define an automorphism τ_ε of K by

$$\tau_\varepsilon(Y_{ii'}) = \begin{cases} Y_{ii'}\prod_j(1+Y_{ji'})^{a_{ij}}\prod_{j'}(1+Y^{-1}_{ij'})^{-a'_{i'j'}} & \text{if } \varepsilon(i,i') = \varepsilon; \\ Y^{-1}_{ii'} & \text{if } \varepsilon(i,i') = -\varepsilon. \end{cases} \tag{2.3.1}$$

Finally, define an automorphism φ of K by

$$\varphi = \tau_-\tau_+. \tag{2.3.2}$$

Then, as in [22], we have the following lemma:

Lemma 2.4. *The periodicity conjecture holds iff the order of the automorphism φ is finite and divides $h + h'$.*

Proof. We adapt the proof given in [22] for the case where $\Delta' = A_1$. First we notice that the equation 2.1.1 only involves variables $Y_{i,i',t}$ with a fixed 'parity' $\varepsilon(i, i')(-1)^t$. Therefore, the Y-system decomposes into two independent systems, an even one and an odd one, and it suffices to show periodicity for one of them, say the even one. Thus, without loss of generality, we may modify the odd system, and for the purposes of the proof, we choose to put

$$Y_{i,i',t} = Y^{-1}_{i,i',t-1} \text{ whenever } \varepsilon(i,i')(-1)^t = -1. \tag{2.4.1}$$

We combine 2.1.1 and 2.4.1 into
(2.4.2)

$$Y_{i,i',t+1} = \begin{cases} Y_{i,i',t} \prod_{j\in I}(1+Y_{j,i',t})^{a_{ij}} \prod_{j'\in I'}(1+Y_{i,j',t}^{-1})^{-a'_{i'j'}} & \text{if } \varepsilon(i,i')(-1)^{t+1}=1; \\ Y_{i,i',t}^{-1} & \text{if } \varepsilon(i,i')(-1)^{t+1}=-1. \end{cases}$$

Thus, if we put $Y_{i,i',t} = Y_{ii'}$, then we have

$$Y_{i,i',t+1} = \tau_{(-1)^{t+1}}(Y_{ii'}) \, ,$$

for all $i \in I$ and $i' \in I$, as we see by comparing 2.4.2 with 2.3.1. Now we set $Y_{i,i',0} = Y_{ii'}$ for $i \in I$, $i' \in I'$. By induction on t, we obtain, for all $t \geq 0$ and all $i \in I$ and $i' \in I'$,

$$Y_{i,i',t} = (\tau_-\tau_+ \dots \tau_\pm)(Y_{ii'}) \, ,$$

where the number of factors τ_+ and τ_- equals t. In particular, we obtain $Y_{i,i',2t} = \varphi^t(Y_{i,i',0})$ for all $t \geq 0$, $i \in I$ and $i' \in I'$, which clearly implies the assertion. $\surd$

2.5. **Plan of the proof.** We refer to the respective sections in the body of the paper for detailed explanations of the notions appearing in the following plan.

Let Δ and Δ' be two Dynkin diagrams. In section 9, we use the standard folding technique to reduce the conjecture to the case where Δ and Δ' are simply laced, which we assume from now on. We choose alternating quivers Q and Q' (*cf.* section 3.1) whose underlying graphs are Δ and Δ'. We define the square product $Q\square Q'$ and the triangle product $Q \boxtimes Q'$ as certain quivers whose vertex set is the product of the vertex sets of Q and Q' (*cf.* section 3.3). We associate canonical sequences of mutations $\mu_\square$ and $\mu_\boxtimes$ to these products. These yield restricted Y-patterns $\mathbf{y}_\square$ and $\mathbf{y}_\boxtimes$, *cf.* section 3.5. Let φ be as defined in equation 2.3.2.

Step 1. We have $\varphi^{h+h'} = \mathbf{1}$ iff the restricted Y-pattern $\mathbf{y}_\square$ is periodic of period dividing $h+h'$ iff this holds for the restricted Y-pattern $\mathbf{y}_\boxtimes$.

This step is proved in section 3.5 by adapting the methods of section 8 of [23]. Notice however that in the case $\Delta' = A_1$ considered there, the systems $\mathbf{y}_\square$ and $\mathbf{y}_\boxtimes$ are indistinguishable.

Step 2. The restricted Y-pattern $\mathbf{y}_\boxtimes$ is periodic of period dividing $h+h'$ iff such a periodicity holds for the sequences of the tropical Y-variables and of F-polynomials (*cf.* section 4.1) associated with the sequence of mutations $\mu_\boxtimes^p$, $p \in \mathbb{Z}$.

This follows from Proposition 3.12 of [23], *cf.* sections 4.1 and 4.3. We refer to sections 5.1 and 5.5 for the terminology used in the following step.

Step 3. There is a triangulated 2-Calabi-Yau category $\mathcal{C}$ with a cluster-tilting object T whose endoquiver (=quiver of its endomorphism algebra) is $Q \boxtimes Q'$.

We construct the category $\mathcal{C}$ as the (generalized) cluster category in the sense of Amiot [1] associated with the tensor product $kQ \otimes kQ'$ of the path algebras of the quivers Q and Q', *cf.* section 5.8. Thanks to Iyama-Yoshino's results [39], there is a well-defined mutation operation for cluster tilting objects in arbitrary 2-Calabi-Yau categories (*cf.* section 6.1).

Step 4. When we apply powers of the sequence of mutations $\mu_\boxtimes$ to the cluster tilting object T, no loops or 2-cycles appear in the endoquivers of the mutated cluster tilting objects.

We prove this by an explicit computation: First we show that $\mathcal{C}$ is equivalent to the cluster category associated [1] with a Jacobi-finite quiver with potential of the form $(Q \boxtimes Q', W)$ and that under this equivalence, the object T corresponds to the canonical cluster tilting object (section 5.13). Then we show that when we perform the sequence of mutations $\mu_\boxtimes$ on the quiver with potential $(Q \boxtimes Q', W)$ following [13] (*cf.* section 5.12) no loops or

2-cycles appear in the mutated quivers with potential. Another proof of this step, based on [1], was given in Proposition 4.35 of [38].

Step 5. If there is an isomorphism $\mu_{\boxtimes}^{h+h'}(T) \cong T$, then periodicity holds for the tropical Y-variables and the F-polynomials associated with the sequence $\mu_{\boxtimes}^{p}$, $p \in \mathbb{Z}$.

This is 'decategorification'. Thanks to steps 3 and 4, it follows essentially from Palu's multiplication formula [59] for the generalized Caldero-Chapoton map [9], *cf.* sections 6.7 and 6.13. Another proof of this step is given in Proposition 4.35 of [38].

Step 6. We have $\mu_{\boxtimes}(T) = \mathsf{Za}(T)$, where $\mathsf{Za} : \mathcal{C} \to \mathcal{C}$ is the Zamolodchikov transformation.

The Zamolodchikov transformation, defined in section 7.3, should be viewed as a categorification of the automorphism φ of equation 2.3.2. If Δ' equals A_1, it coincides with the (inverse) Auslander-Reiten translation (and also with the loop functor of the category $\mathcal{C}$).

Step 7. We have $\mathsf{Za}^{h+h'} = \mathbf{1}$ and therefore $\mu_{\boxtimes}^{h+h'}(T) \cong T$.

This follows easily from the 2-Calabi-Yau property of the category $\mathcal{C}$ and its construction from the tensor product of the path algebras of Q and Q', *cf.* section 8.3. Thanks to steps 5 and 2, this concludes the proof, *cf.* section 8.6.

3. Reformulation of the conjecture in terms of cluster combinatorics

3.1. **Quiver mutation.** A *quiver* Q is an oriented graph given by a set of vertices Q_0, a set of arrows Q_1 and two maps $s : Q_1 \to Q_0$ and $t : Q_1 \to Q_0$ taking an arrow to its source respectively its target. A quiver Q is *finite* if the sets Q_0 and Q_1 are finite. A vertex i of a quiver is a *source* (respectively, a *sink*) if there are no arrows α with target i (respectively, with source i). A quiver is *alternating* if each of its vertices is a source or a sink. A *Dynkin quiver* is a quiver whose underlying graph is a Dynkin diagram of type A_n, $n \geq 1$, D_n, $n \geq 4$, E_6, E_7 or E_8.

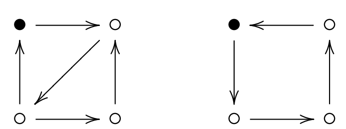
Let Q be a quiver. A *loop* of Q is an arrow α whose source and target coincide. A 2-*cycle* of Q is a pair of distinct arrows β and γ such that $s(\beta) = t(\gamma)$ and $t(\beta) = s(\gamma)$.

Let Q be a finite quiver without loops or 2-cycles. Let k be a vertex of Q. Following Fomin-Zelevinsky [20], we define the *mutated quiver* $\mu_k(Q)$: It has the same set of vertices as Q; its set of arrows is obtained from that of Q as follows:

1) for each subquiver $i \longrightarrow k \longrightarrow j$, add a new arrow $i \longrightarrow j$;
2) reverse all arrows with source or target k;
3) remove the arrows in a maximal set of pairwise disjoint 2-cycles.

It is not hard to check that we have $\mu_k^2(Q) = Q$ for each vertex k. For example, the following two quivers are obtained from each other by mutating at the black vertex

(3.1.1)



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From now and until the end of this section, we assume that the set of vertices of Q is the set of integers $1, \ldots, n$ for some $n \geq 1$.

There is a skew-symmetric integer matrix B associated with Q defined such that b_{ij} is the difference of the number of arrows from i to j minus the number of arrows from j to i in Q. Clearly, the map taking Q to B establishes a bijection between the quivers without loops or 2-cycles with vertex set $\{1, \ldots, n\}$ (modulo isomorphisms which are the identity on the vertices) and the skew-symmetric integer $n \times n$-matrices. Via this bijection, the above operation of quiver mutation corresponds to matrix mutation as originally defined by Fomin-Zelevinsky [20]. According to [23], the matrix B' corresponding to the mutated quiver is given by

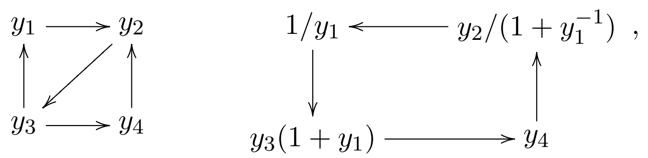
$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k \\ b_{ij} + \operatorname{sgn}(b_{ik}) \max(0, b_{ik}b_{kj}) & \text{otherwise.} \end{cases} \tag{3.1.2}$$

Let $\mathbb{T}_n$ be the *n-regular tree:* Its edges are labeled by the integers $1, \ldots, n$ such that the n edges emanating from each vertex carry different labels. Let t_0 be a vertex of $\mathbb{T}_n$. To each vertex t of $\mathbb{T}_n$ we associate a quiver $Q(t)$ such that at $t = t_0$, we have $Q(t) = Q$ and whenever t is linked to t' by an edge labeled i, we have $Q(t') = \mu_i Q(t)$. The family of quivers $Q(t)$, where t runs through the vertices of $\mathbb{T}_n$ is the *quiver pattern* associated with Q.

3.2. **Y-seeds.** We follow [23]. Let $n \geq 1$ be an integer. A *Y-seed* is a pair (Q, Y) formed by a finite quiver Q without loops or 2-cycles with vertex set $\{1, \ldots, n\}$ and by a free generating set $Y = \{Y_1, \ldots, Y_n\}$ of the field $\mathbb{Q}(y_1, \ldots, y_n)$ generated over $\mathbb{Q}$ by indeterminates $y_1, \ldots, y_n$. If (Q, Y) is a Y-seed and k a vertex of Q, the *mutated Y-seed* $\mu_k(Q, Y)$ is the Y-seed (Q', Y') where $Q' = \mu_k(Q)$ and, for $1 \leq j \leq n$, we have

$$Y'_j = \begin{cases} Y_k^{-1} & \text{if } j = k; \\ Y_j(1 + Y_k^{-1})^{-m} & \text{if there are } m \geq 0 \text{ arrows } k \to j \\ Y_j(1 + Y_k)^m & \text{if there are } m \geq 0 \text{ arrows } j \to k \end{cases}$$

One checks that $\mu_k^2(Q, Y) = (Q, Y)$. For example, the following Y-seeds are related by a mutation at the vertex 1



where we write the variable Y_i in place of the vertex i.

Let Q be a finite quiver without loops or 2-cycles with vertex set $\{1, \ldots, n\}$. The *initial Y-seed* associated with Q is $(Q, \{y_1, \ldots, y_n\})$. The *$Y$-pattern* associated with Q is the family of Y-seeds $(Q(t), Y(t))$ indexed by the vertices t of the n-regular tree $\mathbb{T}_n$ (*cf.* section 3.1) such that at the chosen initial vertex t_0, the Y-seed $(Q(t_0), Y(t_0))$ is the initial Y-seed associated with Q and whenever two vertices t and t' are linked by an edge labeled k, we have $(Q(t'), Y(t')) = \mu_k(Q(t), Y(t))$.

Let $\mathbf{v}$ be a sequence of vertices $v_1, \ldots, v_N$ of Q. We assume that the composed mutation

$$\mu_{\mathbf{v}} = \mu_{v_N} \ldots \mu_{v_2} \mu_{v_1}$$

transforms Q into itself. Then clearly the same holds for the inverse sequence

$$\mu_{\mathbf{v}}^{-1} = \mu_{v_1} \mu_{v_2} \ldots \mu_{v_N}.$$

Now the *restricted Y-pattern* associated with Q and $\mu_{\mathbf{v}}$ is the sequence of Y-seeds obtained from the initial Y-seed $\mathbf{y}_0$ associated with Q by applying all integer powers of $\mu_{\mathbf{v}}$. Thus this pattern is given by a sequence of seeds $\mathbf{y}_p$, $p \in \mathbb{Z}$, such that $\mathbf{y}_0$ is the initial Y-seed associated with Q and, for all $p \in \mathbb{Z}$, $\mathbf{y}_{p+1}$ is obtained from $\mathbf{y}_p$ by the sequence of mutations $\mu_{\mathbf{v}}$.

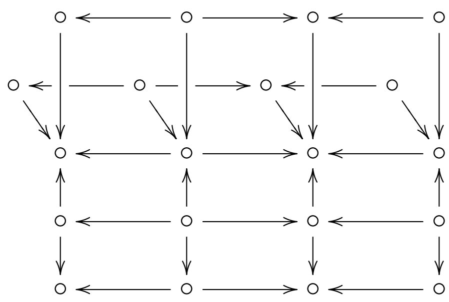


FIGURE 1. The quiver $\vec{A}_4 \otimes \vec{D}_5$

3.3. **Products of quivers.** Let Q and Q' be two finite quivers without oriented cycles. We define the *tensor product* $Q \otimes Q'$ to be the quiver whose set of vertices is the product $Q_0 \times Q'_0$ and where the number of arrows from a vertex (i, i') to a vertex (j, j')

a) is zero if $i \neq j$ and $i' \neq j'$;
b) equals the number of arrows from j to j' if $i = i'$;
c) equals the number of arrows from i to i' if $j = j'$.

Thus, for each vertex i', the full subquiver of $Q \otimes Q'$ formed by the vertices (i, i'), $i \in Q_0$, is isomorphic to Q by an isomorphism taking (i, i') to i and similarly, for each vertex i of Q, the full subquiver on the vertices (i, i'), $i' \in Q'_0$, is isomorphic to Q' by an isomorphism taking (i, i') to i'. In Lemma 5.9 below, we will see that the path algebra of a tensor product of two quivers is the tensor product of the path algebras. The tensor product of the quivers

$$\vec{A}_4 :\ 1 \longleftarrow 2 \longrightarrow 3 \longleftarrow 4\ ,$$

$$\vec{D}_5 :\ 1 \longleftarrow 2 \longrightarrow 3 \begin{array}{l} \swarrow 4 \\ \nwarrow 5. \end{array}$$

is depicted in figure 1.

We define the *triangle product* $Q \boxtimes Q'$ to be the quiver obtained from $Q \otimes Q'$ by adding rr' arrows from (j, j') to (i, i') whenever Q contains r arrows from i to j and Q' contains r' arrows from i' to j'. For example, the triangle product of the quivers $\vec{A}_4$ and $\vec{D}_5$ is depicted in figure 2. We will see a representation-theoretic interpretation of the triangle product in Corollary 5.11 below.

Now assume that Q and Q' are *alternating*, *i.e.* each vertex is a source or a sink. For example, the above quivers $\vec{A}_4$ and $\vec{D}_5$ are alternating. We define the *square product* $Q \square Q'$ to be the quiver obtained from $Q \otimes Q'$ by reversing all arrows in the full subquivers of the form $\{i\} \times Q'$ and $Q \times \{i'\}$, where i is a sink of Q and i' a source of Q'. The square product of the above quivers $\vec{A}_4$ and $\vec{D}_5$ is depicted in figure 2. The triangle product and the square product of two copies of the quiver $\vec{A}_2 : \circ \to \circ$ are given in 3.1.1.

Lemma 3.4. *Let Q and Q' be alternating and let M be the set of vertices (i, i') of $Q \square Q'$ such that i is a sink of Q and i' a source of Q'. Then there are no arrows between any two vertices of M and the composition of the mutations at the vertices of M transforms $Q \square Q'$ into $Q \boxtimes Q'$.*

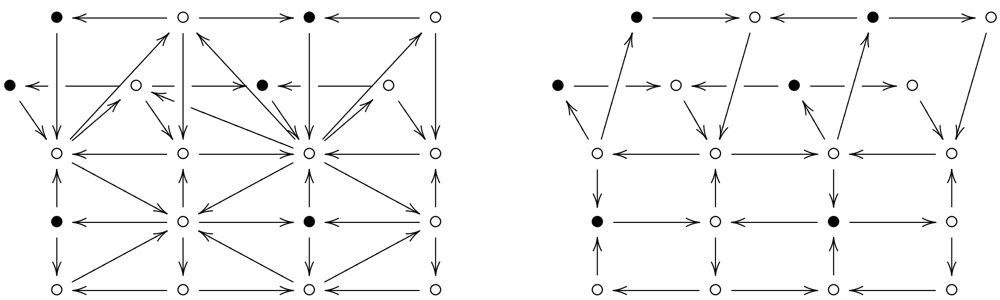


FIGURE 2. The quivers $\vec{A}_4 \boxtimes \vec{D}_5$ and $\vec{A}_4 \square \vec{D}_5$

We leave the easy proof to the reader. In figure 2, the vertices belonging to M are marked by $\bullet$.

3.5. **Reformulation of the conjecture.** Let Δ and Δ' be simply laced Dynkin diagrams. We choose alternating quivers Q and Q' whose underlying graphs are Δ and Δ'. If i is a vertex of Q or Q', we put $\varepsilon(i) = 1$ if i is a source and $\varepsilon(i) = -1$ if i is a sink. For example, we can consider the quivers $\vec{A}_4$ and $\vec{D}_5$ of section 3.3. For two elements σ, σ' of $\{+, -\}$, we define the following composed mutation of $Q \square Q'$:

$$\mu_{\sigma,\sigma'} = \prod_{\varepsilon(i)=\sigma, \varepsilon(i')=\sigma'} \mu_{(i,i')}.$$

Notice that there are no arrows between any two vertices of the index set so that the order in the product does not matter. Then it is easy to check that $\mu_{+,+}\mu_{-,-}$ and $\mu_{-,+}\mu_{+,-}$ both transform $Q \square Q'$ into $(Q \square Q')^{op}$ and vice versa. Thus the composed sequence of mutations

$$\mu_{\square} = \mu_{-,-}\mu_{+,+}\mu_{-,+}\mu_{+,-}$$

transforms $Q \square Q'$ into itself. We define the *Y-system* $\mathbf{y}_{\square}$ associated with $Q \square Q'$ to be the restricted Y-pattern (*cf.* section 3.2) associated with $Q \square Q'$ and $\mu_{\square}$. The following lemma is inspired from section 8 of [23].

Lemma 3.6. *The periodicity conjecture holds for Δ and Δ' if and only if the Y-system $\mathbf{y}_{\square}$ is periodic of period dividing $h + h'$.*

Proof. Let Σ_0 be the initial Y-seed associated with $Q \square Q'$ and, for $t \geq 1$, define the Y-seed Σ_t by

$$\Sigma_t = \begin{cases} \mu_{+,-}\mu_{-,+}(\Sigma_{t-1}) & \text{if } t \text{ is odd;} \\ \mu_{+,+}\mu_{-,-}(\Sigma_{t-1}) & \text{if } t \text{ is even.} \end{cases}$$

Let $(Y_{i,i',t})$ be the generating set of $\mathbb{Q}(y_1, \ldots, y_n)$ given by Σ_t. Then using equation 2.4.2 we see that the $Y_{i,i',t}$ are precisely those defined in the proof of Lemma 2.4. Therefore, the periodicity conjecture translates into the fact that $\Sigma_{2(h+h')} = \Sigma_0$ and this yields the assertion. $\surd$

By lemma 3.4, we have

$$\mu_{-+}(Q \square Q') = Q \boxtimes Q'.$$

Therefore, the periodicity of the restricted Y-system $\mathbf{y}_{\square}$ associated with $Q \square Q'$ and $\mu_{\square}$ is equivalent to that of the restricted Y-system $\mathbf{y}_{\boxtimes}$ associated with $Q \boxtimes Q'$ and

$$\mu_{\boxtimes} = \mu_{-,+}\mu_{+,+}\mu_{-,-}\mu_{+,-}. \tag{3.6.1}$$

So we finally obtain the following lemma.

Lemma 3.7. *The periodicity conjecture holds for Δ and Δ' if and only if the Y-system $\mathbf{y}_{\boxtimes}$ is periodic of period dividing $h+h'$.*

4. More cluster combinatorics

Let $n \geq 1$ be an integer and Q a finite quiver without loops or 2-cycles with vertex set $\{1, \ldots, n\}$. Let $\mathbb{T}_n$ be the n-regular tree and t_0 a distinguished vertex of $\mathbb{T}_n$. Let $(Q(t))$ be the quiver pattern associated with Q as in section 3.1.

4.1. **Tropical Y-variables and F-polynomials.** The *tropical semifield* $\mathsf{Trop}(y_1, \ldots, y_n)$ generated by the indeterminates $y_1, \ldots, y_n$ is the free multiplicative group generated by the y_i endowed with the auxiliary addition defined by

$$\prod_i y_i^{a_i} \oplus \prod_i y_i^{b_i} = \prod_i y_i^{\min(a_i, b_i)}.$$

The *tropical Y-variables* $\eta_i(t)$, $1 \leq i \leq n$, are elements of $\mathsf{Trop}(y_1, \ldots, y_n)$ associated to the vertices t of $\mathbb{T}_n$. They are defined recursively as follows: we put

$$\eta_i(t_0) = y_i \;,\; 1 \leq i \leq n \;,$$

and if t is linked to t' by an edge labeled k, we put

$$\eta_j(t') = \begin{cases} \eta_j(t)^{-1} & \text{if } j = k; \\ \eta_j(t)(1 \oplus \eta_k(t))^m & \text{if there are } m \geq 0 \text{ arrows } j \to k \text{ in } Q(t); \\ \eta_j(t)/(1 \oplus \eta_k(t)^{-1})^m & \text{if there are } m \geq 0 \text{ arrows } k \to j \text{ in } Q(t). \end{cases} \tag{4.1.1}$$

The *F-polynomials* $F_i(t)$, $1 \leq i \leq n$, are elements of the polynomial ring $\mathbb{Z}[y_1, \ldots, y_n]$ associated to the vertices t of $\mathbb{T}_n$. According to Proposition 5.1 of [23], they can be defined recursively as follows: we put

$$F_i(t_0) = 1 \;,\; 1 \leq i \leq n \;,$$

and if t is linked to t' by an edge labeled k, then $F_i(t') = F_i(t)$ for all $i \neq k$ and $F_k(t')$ is defined by the *exchange relation*

$$F_k(t)F_k(t') = \prod_{c_{ik}>0} y_i^{c_{ik}} \prod_{\substack{\text{arrows} \\ k \to j}} F_j(t) + \prod_{c_{ik}<0} y_i^{-c_{ik}} \prod_{\substack{\text{arrows} \\ i \to k}} F_i(t) \;, \tag{4.1.2}$$

where the products are taken over the arrows in the quiver $Q(t)$ and the c_{ik} are defined by

$$\eta_k(t) = \prod_{i=1}^{n} y_i^{c_{ik}}.$$

Notice that the group ring of the free multiplicative abelian group underlying the tropical semifield $\mathsf{Trop}(y_1, \ldots, y_n)$ is canonically isomorphic to the ring of Laurent polynomials

$$\mathbb{Z}[y_1^{\pm 1}, \ldots, y_n^{\pm 1}].$$

Thanks to this identification, it makes sense to multiply elements of the tropical semifield with elements of the field $\mathbb{Q}(y_1, \ldots, y_n)$. This is the multiplication that we use in the following proposition.

Proposition 4.2 (Proposition 3.12 of [23]). *At each vertex t of $\mathbb{T}_n$, and for each $1 \leq j \leq n$, we have*

$$Y_j(t) = \eta_j(t) \prod_{i=1}^{n} F_i(t)^{b_{ij}(t)} \tag{4.2.1}$$

where $(b_{ij}(t))$ is the skew-symmetric integer matrix associated with $Q(t)$.

4.3. **Consequence for the periodicity conjecture.** Now let Q and Q' be as in section 3.5. Let t_0 be the initial vertex of the N-regular tree $\mathbb{T}_N$ whose edges are labeled by the vertices of $Q \boxtimes Q'$. Let t_i, $i \in \mathbb{Z}$, be the sequence of vertices of $\mathbb{T}_N$ which are visited when performing the mutations in the integer powers of the composition

$$\mu_{\boxtimes} = \mu_{+,-}\mu_{-,-}\mu_{+,+}\mu_{-,+}$$

(for each factor $\mu_{\sigma,\sigma'}$, we choose some order of the mutations). Notice that $\mu_{\boxtimes}$ contains exactly one mutation at each of the N vertices of $Q\boxtimes Q'$. We already know from section 3.3 that $(Q \boxtimes Q')(t_{pN}) = Q \boxtimes Q'$ for all $p \in \mathbb{Z}$. Thus, by the above proposition, the periodicity conjecture holds for Q and Q' if, for each vertex j of $Q \boxtimes Q'$, the sequences $\eta_j(t_{pN})$ and $F_j(t_{pN})$ are periodic in p of period dividing $h + h'$.

4.4. **Cluster variables, g-vectors.** We will not need cluster variables for the proof of the conjecture. We nevertheless define them since they are useful in other applications of the categorical model that we will construct, notably in the study [38] of T-systems. We will use the categorical lift of the g-vectors to express the tropical Y-variables (cf. Corollary 6.10 and Corollary 6.11). In this way, the g-vectors do play a role in our proof.

The *cluster variables* $X_i(t)$, $1 \leq i \leq n$, associated to the vertices t of $\mathbb{T}_n$ lie in the field $\mathbb{Q}(x_1, \ldots, x_n)$ generated by n indeterminates $x_1, \ldots, x_n$; they are defined recursively by

$$X_i(t_0) = x_i \;,\; 1 \leq i \leq n \;, \tag{4.4.1}$$

and if t is linked to t' by an edge labeled k, then $X_i(t') = X_i(t)$ for all $i \neq k$ and $X_k(t')$ is determined by the *exchange relation*

$$X_k(t)X_k(t') = \prod_{\substack{\text{arrows} \\ i\to k}} X_i(t) + \prod_{\substack{\text{arrows} \\ k\to j}} X_j(t) \;, \tag{4.4.2}$$

where the products are taken over the arrows of $Q(t)$ with source, respectively, target k. By a fundamental theorem of Fomin-Zelevinsky [20], all cluster variables are Laurent polynomials with integer coefficients in the initial cluster variables $x_1, \ldots, x_n$.

The *g-vectors* $g_i^{t_0}(t)$, $1 \leq i \leq n$, are vectors in $\mathbb{Z}^n$ associated with the vertices t of $\mathbb{T}_n$. In the language of Fock-Goncharov [18], they can be interpreted as distinguished points of the tropical $\mathcal{X}$-variety associated with Q. To define them, we first define, for all vertices t_1 and t_2 of $\mathbb{T}_n$, a piecewise linear bijection

$$G(t_2, t_1) : \mathbb{Z}^n \to \mathbb{Z}^n$$

by induction on the distance between t_1 and t_2 in the tree $\mathbb{T}_n$ as follows: 1) We put $G(t_1, t_1)$ equal to the identity. 2) If t_1 and t_1' are linked by an edge labeled k, we put

$$G(t_1', t_1)(v) = \begin{cases} \varphi_+(v) & \text{if } v = \sum x_i e_i \text{ with } x_k \geq 0 \;, \\ \varphi_-(v) & \text{if } v = \sum x_i e_i \text{ with } x_k \leq 0 \;, \end{cases}$$

where φ_+ and φ_- are the linear automorphisms of $\mathbb{Z}^n$ with $\varphi_+(e_i) = e_i = \varphi_-(e_i)$ for $i \neq k$ and

$$\varphi_+(e_k) = -e_k + \sum_{i\to k} e_i \text{ and } \varphi_-(e_k) = -e_k + \sum_{k\to j} e_j \;,$$

where the sums are taken over the arrows $i \to k$ respectively $k \to j$ of the quiver $Q(t_1')$. One checks that $G(t_1, t_1')$ and $G(t_1', t_1)$ are inverse to each other. 3) If the shortest path linking t_1 to t_2 is of length greater than or equal to two, we define

$$G(t_2, t_1) = G(t_2, t_1') \circ G(t_1', t_1),$$

where t'_1 is the first vertex different from t_1 on the shortest path from t_1 to t_2. Now, for all vertices t and all $1 \leq i \leq n$, we define

$$g_i^{t_0}(t) = G(t, t_0)(e_i).$$

By Theorem 1.7 of [14], this definition agrees with the one given in section 6 of [23]. Notice that, like all the other data defined before, the g-vectors also depend on the initial vertex t_0. For later reference, notice that the g-vectors are characterized by

$$g_i^{t_0}(t_0) = e_i\ ,\ 1 \leq i \leq n\ , \tag{4.4.3}$$

and, whenever t_0 and t_1 are linked by an edge labeled k, we have

$$g_i^{t_1}(t) = \begin{cases} \varphi_+(v) & \text{if } x_k \geq 0\ , \\ \varphi_-(v) & \text{if } x_k \leq 0\ , \end{cases} \tag{4.4.4}$$

where v is short for $g_i^{t_0}(t)$, the integer x_k is the coefficient of e_k in v and the maps φ_+ and φ_- are the linear automorphisms of $\mathbb{Z}^n$ associated as above with the vertex k and the quiver $Q(t_0)$.

By Corollary 6.3 of [23], the cluster variables can be expressed in terms of the quiver pattern, the g-vectors and the F-polynomials as follows: For each vertex t of $\mathbb{T}_n$ and each integer $1 \leq i \leq n$, we have

$$X_i(t) = F_i(t)(\widehat{y}_1, \ldots, \widehat{y}_n) \prod_{j=1}^{n} x_j^{g_j}\ ,$$

where the g_j are the components of $g_i(t)$ and, if (b_{ij}) is the skew-symmetric integer matrix corresponding to $Q(t)$, the elements $\widehat{y}_j$ are given by

$$\widehat{y}_j = \prod_{i=1}^{n} x_i^{b_{ij}}.$$

5. Calabi-Yau triangulated categories

5.1. **Krull-Schmidt categories.** We briefly recall basic notions from the representation theory of finite-dimensional associative algebras. More details can be found in the books [2] [3] [27] [62]. An introduction with motivating examples from cluster theory is given in sections 5 and 6 of [48]. By a *module*, we will mean a right module.

Recall that an *additive category* is a category where 1) each finite family of objects admits a direct sum and 2) the morphism sets are endowed with structures of abelian groups such that the composition is bilinear. For example, the category of free modules over a ring is additive, and so are the category of projective modules and that of all modules. An additive category *has split idempotents* if each idempotent endomorphism e of an object X gives rise to a direct sum decomposition $Y \oplus Z \xrightarrow{\sim} X$ such that Y is a kernel for e. The category of free modules over a ring usually does not have split idempotents but the category of projective modules does. An object X in an additive category is *indecomposable* if in each direct sum decomposition $X \xrightarrow{\sim} Y \oplus Z$, the object Y or the object Z is a zero object. In the category of modules over a ring, the simple modules are indecomposable but usually there are many other indecomposable objects. For example, let k be a field, let $n \geq 1$ be an integer and let $T_n(k)$ be the associative algebra of *lower triangular* $n \times n$*–matrices over* k. Let e_{ij} denote the matrix whose (i, j)-coefficient equals 1 and whose other coefficients vanish. Let $A = T_2(k)$. Then the modules $P_1 = e_{11}A = [k\ 0]$ and $P_2 = e_{22}A = [k\ k]$ are indecomposable (and projective). The module P_1 is also simple but the module P_2 is not since it contains P_1 as a proper submodule. A *Krull-Schmidt category* is an additive category where the endomorphism rings of indecomposable objects are local and each object decomposes into a finite direct sum of indecomposable objects (which are then unique up to isomorphism and permutation). One can show that each Krull-Schmidt category has split idempotents. We *write* $\mathsf{indec}(\mathcal{C})$ for the set of isomorphism classes of indecomposable objects of a Krull-Schmidt category $\mathcal{C}$. For example, if k is a field, the category of finitely generated modules over the polynomial ring $k[x]$ is not a Krull-Schmidt category since the free module $k[x]$ is indecomposable but its endomorphism algebra, which is $k[x]$, is not local. On the other hand, the category of coherent sheaves over a projective variety over k is a Krull-Schmidt category. If A is a finite-dimensional associative algebra over k, then the category $\mathsf{mod}(A)$ of A-modules whose underlying k-vector spaces are finite-dimensional is a Krull-Schmidt category. So is its subcategory $\mathsf{proj}(A)$ of finitely generated projective modules. In rare cases, one can explicitly enumerate the isomorphism classes of indecomposable objects of $\mathsf{mod}(A)$ and $\mathsf{proj}(A)$. For example, if A is the algebra $T_2(k)$, then, up to isomorphism, the indecomposables of the category $\mathsf{mod}(A)$ are $P_1 = e_{11}A$, $P_2 = e_{22}A$ and $S_2 = P_2/P_1$. More generally, the indecomposable modules over the algebra $T_n(k)$ are, up to isomorphism, the quotients P_j/P_i, where $1 \leq i < j \leq n$ and $P_i = e_{ii}T_n(k)$. We refer to the books books [2] [3] [27] [62] and to section 5 of [48] for more examples.

Let $\mathcal{C}$ be a Krull-Schmidt category. An object X of $\mathcal{C}$ is *basic* if every indecomposable of $\mathcal{C}$ occurs with multiplicity ≤ 1 in X. For example, the category of finitely generated projective modules over $T_2(k)$ contains, up to isomorphism, exactly four basic objects: 0, P_1, P_2 and $P_1 \oplus P_2$. If an object X is basic, it is determined, up to isomorphism, by the full additive *subcategory* $\mathsf{add}(X)$ whose objects are the direct factors of finite direct sums of copies of X. The map $X \mapsto \mathsf{add}(X)$ yields a bijection between the isomorphism classes of basic objects and the full additive subcategories of $\mathcal{C}$ which are stable under taking direct factors and only contain finitely many indecomposables up to ismorphism.

From now on, let k be an algebraically closed field. A *k-category* is a category whose morphism sets are endowed with structures of k-vector spaces such that the composition maps are bilinear. For example, each full subcategory of the category of modules over a k-algebra is naturally a k-category. A k-category is Hom-*finite* if all of its morphism spaces are finite-dimensional. For example, the category of finitely generated projective modules over a k-algebra A is Hom-finite if and only if the algebra A is finite-dimensional over k. A *k-linear category* is a k-category which is additive. For example, a subcategory of the category of modules over a k-algebra is k-linear if and only if it is stable under forming finite direct sums (in particular it then contains the zero module, which is the sum over the empty family of modules). Let $\mathcal{C}$ be a k-linear Hom-finite category with split idempotents. One can show that $\mathcal{C}$ is then a Krull-Schmidt category. Let $\mathcal{T}$ be an additive subcategory of $\mathcal{C}$ stable under taking direct factors. The *quiver $Q = Q(\mathcal{T})$ of $\mathcal{T}$* is defined as follows: The vertices of Q are the isomorphism classes of indecomposable objects of $\mathcal{T}$ and the number of arrows from the isoclass of T_1 to that of T_2 equals the dimension of the *space of irreducible morphisms*

$$\mathsf{irr}(T_1, T_2) = \mathsf{rad}(T_1, T_2)/\mathsf{rad}^2(T_1, T_2)\ ,$$

where rad denotes the *radical of $\mathcal{T}$*, *i.e.* the ideal such that $\mathsf{rad}(T_1, T_2)$ is formed by all non isomorphisms from T_1 to T_2. We refer to the next section for examples. The *quiver of a finite-dimensional algebra A* is the quiver of the category of finitely generated projective A-modules. By lemma 5.4 below, the computation of the quiver of a category can often be reduced to the computation of the quiver of a finite-dimensional algebra A.

5.2. **The quiver of a finite-dimensional algebra, path algebras, representations.** In the case of the category of finitely generated projective modules over a finite-dimensional algebra A, one can describe the radical and its square more explicitly: A morphism

$f : P \to Q$ between finitely generated projective modules lies in the radical, respectively its square, iff it factors as $f = \pi g\, \iota$, where the morphism $\iota : P \to A^p$ is the inclusion of a direct summand of a free module, the morphism $\pi : A^q \to Q$ is the projection onto a direct summand of a free module and the morphism $g : A^p \to A^q$ has all its matrix coefficients in the Jacobson radical of the algebra A, respectively its square. For example, let A be the algebra $T_n(k)$ (section 5.1). Then the Jacobson radical of A is formed by the strictly lower triangular matrices. For $1 \leq i \leq n$, put $P_i = e_{ii}A$. Then, if $i + 1 \leq j \leq n$, all morphisms $P_i \to P_j$ lie in the radical. On the other hand, if $j = i + 1$, then the space of irreducible morphisms from P_i to P_j is one-dimensional. Thus, the quiver of the algebra $T_n(k)$ is the chain

$$[P_1] \longrightarrow [P_2] \longrightarrow \cdots \longrightarrow [P_n] \ ,$$

where $[P_i]$ denotes the isomorphism class of P_i (in the sequel, we will usually omit the brackets). One can generalize this example as follows: Let Q be any finite quiver. A *path of length l in Q* is a formal composition $p = (z|\alpha_l| \ldots |\alpha_2|\alpha_1|x)$ of $l \geq 0$ arrows forming a diagram

$$x \xrightarrow{\alpha_1} y \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_l} z \ .$$

The vertex x is the *source* and the vertex z the *target* of the path p. For each vertex x of Q we have the *lazy path* $e_x = (x|x)$ of length 0. Two paths p and q are *composable* if the target of q equals the source of p; in this case, their *composition* is obtained by concatenating p with q. The *path algebra kQ* is the vector space whose basis is formed by all paths and where the product of two paths is their composition if they are composable and zero otherwise. Notice that kQ is finite-dimensional if and only if Q does not have oriented cycles. For example, if Q is the quiver

$$1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow n \ ,$$

then the path algebra kQ is isomorphic to the algebra $T_n(k)$. For an arbitrary quiver Q without oriented cycles, the Jacobson radical of kQ is the two-sided ideal generated by all arrows. Using this it is not hard to show that the quiver of the category $\mathsf{proj}\, kQ$ is isomorphic to Q, where the isomorphism sends a vertex x of Q to the isomorphism class of the module $e_x kQ$.

For later use, let us record the classical equivalence (*cf.* [2] [3] [27] [62]) between the category of modules over the path algebra kQ and the category of representations of the opposite quiver Q^{op}: It sends a module M over the path algebra to the representation V of Q^{op} whose value at the vertex i is $V_i = Me_i$ and which maps an arrow $\alpha : i \to j$ to the linear map $V_\alpha : V_j \to V_i$ given by the right multiplication with α. For example, the modules P_i, $1 \leq i \leq 3$, over the path algebra of $Q : 1 \to 2 \to 3$ correspond to the representations

$$k \longleftarrow 0 \longleftarrow 0 \ , \ \ k \xleftarrow{\mathbf{1}} k \longleftarrow 0 \ , \ \ k \xleftarrow{\mathbf{1}} k \xleftarrow{\mathbf{1}} k \ .$$

5.3. **From objects to modules.** Let us fix $\mathcal{C}$, a k-linear Hom-finite category with split idempotents. Let T be a basic object of $\mathcal{C}$ and B its endomorphism algebra. Let $\mathsf{mod}(B)$ denote the category of k-finite-dimensional right B-modules. The following lemma is well-known and easy to prove. We denote by $D = \mathsf{Hom}_k(?, k)$ the duality over the ground field.

Lemma 5.4. *The functor*

$$\mathcal{C}(T, ?) : \mathcal{C} \to \mathsf{mod}(B)$$

induces an equivalence from $\mathsf{add}(T)$ *to the full subcategory* $\mathsf{proj}\, B$ *of finitely generated projective* B*-modules. Moreover, for each object* U *of* $\mathsf{add}(T)$ *and each object* X *of* $\mathcal{C}$*, the canonical map*

$$\mathcal{C}(U,X) \to \mathsf{Hom}_B(\mathcal{C}(T,U), \mathcal{C}(T,X))$$

is bijective. Dually, the functor

$$D\mathcal{C}(?,T) : \mathcal{C} \to \mathsf{mod}(B)$$

induces an equivalence from $\mathsf{add}(T)$ *to the full subcategory* $\mathsf{inj}\, B$ *of finitely generated injective* B*-modules. Moreover, for each object* U *of* $\mathsf{add}(T)$ *and each object* X *of* $\mathcal{C}$*, the canonical map*

$$\mathcal{C}(X,U) \to \mathsf{Hom}_B(D\mathcal{C}(X,T), D\mathcal{C}(U,T))$$

is bijective.

In particular, the lemma shows that the isomorphism classes of the indecomposable projective B-modules are represented by the $\mathcal{C}(T,T_i)$, $1 \leq i \leq n$, where the T_i are the indecomposable pairwise non isomorphic direct factors of T. Thus, the quiver of the endomorphism algebra $\mathsf{End}_{\mathcal{C}}(T)$ is canonically isomorphic to that of the category $\mathsf{add}(T)$: The isomorphism sends the indecomposable factor T_i to the indecomposable projective B-module $\mathcal{C}(T,T_i)$, $1 \leq i \leq n$. We sometimes refer to the quiver of $\mathsf{End}_{\mathcal{C}}(T)$ as the *endoquiver of* T.

5.5. **2-Calabi-Yau triangulated categories.** Let k be an algebraically closed field. Let $\mathcal{C}$ be a k-linear triangulated category with suspension functor Σ, *cf.* [66]. We refer to sections 5.6.1 and 5.6.2 below for examples. We assume that

(C1) $\mathcal{C}$ is Hom-*finite* and has split idempotents.

Thus, the category $\mathcal{C}$ is a Krull-Schmidt category. For objects X, Y of $\mathcal{C}$ and an integer i, we define

$$\mathsf{Ext}^i(X,Y) = \mathcal{C}(X, \Sigma^i Y).$$

An object X of $\mathcal{C}$ is *rigid* if $\mathsf{Ext}^1(X,X) = 0$.

Let d be an integer. The category $\mathcal{C}$ is *d-Calabi-Yau* if there exist bifunctorial isomorphisms

$$D\mathcal{C}(X,Y) \xrightarrow{\sim} \mathcal{C}(Y, \Sigma^d X)\ ,\ X, Y \in \mathcal{C}\ ,$$

where $D = \mathsf{Hom}_k(?,k)$ is the duality over the ground field (this definition suffices for our purposes; a refined definition is given in [65], *cf.* also [47]). For example, the derived category of the category of coherent sheaves on a smooth projective variety of dimension d over an algebraically closed field is d-Calabi-Yau if the canonical bundle is trivial. Let us assume that

(C2) $\mathcal{C}$ is 2-Calabi-Yau.

In the sequel, by a 2-*Calabi-Yau category*, we will mean a k-linear triangulated category satisfying (C1) and (C2).

A *cluster tilting object* is a basic object T of $\mathcal{C}$ such that T is rigid and each object X satisfying $\mathsf{Ext}^1(T,X) = 0$ belongs to $\mathsf{add}(T)$. If T is a cluster tilting object, we write Q_T for its endoquiver.

5.6. **2-Calabi-Yau realizations.** Let k be an algebraically closed field and Q a finite quiver. A 2-*Calabi-Yau realization of* Q is a 2-Calabi-Yau category $\mathcal{C}$ which admits a cluster tilting object T whose endoquiver Q_T is isomorphic to Q. We will see below that in this case, under suitable additional assumptions, the cluster combinatorics associated with Q have a categorical lift in $\mathcal{C}$. It is therefore important to construct 2-Calabi-Yau realizations.

5.6.1. *Cluster categories from quivers.* Assume that Q does not have oriented cycles. In this case, the *cluster category* $\mathcal{C}_Q$ provides a 2-Calabi-Yau realization of Q. Let us recall its construction (*cf.* section 5.6.2 below for an example): Let $A = kQ$ denote the path algebra of Q (cf. section 5.1) and $\mathsf{mod}(A)$ the category of k-finite-dimensional A-modules. Let $\mathcal{D}^b(A)$ denote the bounded derived category of $\mathsf{mod}(A)$, *cf.* [66] [33] [46]. Thus, the objects of $\mathcal{D}^b(A)$ are the bounded complexes

$$\cdots \longrightarrow M^p \longrightarrow M^{p+1} \longrightarrow \cdots$$

of k-finite-dimensional A-modules and its morphisms are obtained from the morphisms of complexes by formally inverting all quasi-isomorphisms; the triangles are 'induced' by short exact sequences of complexes. The category $\mathcal{D}^b(A)$ is a Hom-finite triangulated Krull-Schmidt category which admits a *Serre functor*, *i.e.* an auto-equivalence $S : \mathcal{D}^b(A) \to \mathcal{D}^b(A)$ such that there are bifunctorial isomorphisms

$$D\,\mathsf{Hom}(X,Y) \xrightarrow{\sim} \mathsf{Hom}(Y,SX)\ ,\ X,Y \in \mathcal{D}^b(A)\ ,$$

where $D = \mathsf{Hom}_k(?,k)$ is the duality over the ground field, *cf.* [33]. In fact, the Serre functor S is given by the derived tensor product with the bimodule DA which is k-dual to the bimodule A. The suspension functor Σ of $\mathcal{D}^b(A)$ is induced by the functor shifting a complex one degree to the left, *i.e.* $(\Sigma M)^p = M^{p+1}$, and changing the sign of its differential. The *Auslander-Reiten translation* is the auto-equivalence τ of $\mathcal{D}^b(A)$ which is defined by

$$\tau\Sigma = S.$$

Notice that $\Sigma\tau = \tau\Sigma$ since τ is a triangle functor. If we send a module M to the complex

$$\cdots \longrightarrow 0 \longrightarrow M \longrightarrow 0 \longrightarrow \cdots$$

concentrated in degree 0, we obtain a fully faithful embedding $\mathsf{mod}(A) \to \mathcal{D}^b(A)$. In this way, we identify modules with complexes. It follows from the fact that the algebra A is of global dimension at most one that each object of $D^b(A)$ is isomorphic to a direct sum of objects of the form $\Sigma^p M$, where $p \in \mathbb{Z}$ and M is a module. In particular, the indecomposable objects of $\mathcal{D}^b(A)$ are isomorphic to shifted indecomposable modules.

The cluster category was introduced in [7] and, independently in the case of quivers whose underlying graph is a Dynkin diagram of type A, in [10]. It is the orbit category

$$\mathcal{C}_A = \mathcal{D}^b(A)/(S^{-1}\Sigma^2)^{\mathbb{Z}} = \mathcal{D}^b(A)/(\tau^{-1}\Sigma)^{\mathbb{Z}}.$$

Thus, its objects are those of $\mathcal{D}^b(A)$ and its morphisms are defined by

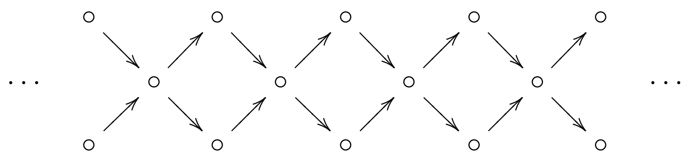
$$\mathsf{Hom}_{\mathcal{C}_A}(X,Y) = \bigoplus_{p\in\mathbb{Z}} \mathsf{Hom}_{\mathcal{D}^b(A)}(X, (\Sigma^2 S^{-1})^p Y)\ ,\ X,Y \in \mathcal{C}_A.$$

As shown in [45], the category $\mathcal{C}_A$ is canonically triangulated and 2-Calabi-Yau. Moreover, as shown in [7], it satisfies (C1) and (C2) and the image in $\mathcal{C}_A$ of the free A-module of rank one is a cluster tilting object T whose endoquiver is canonically isomorphic to Q.

5.6.2. *The example A_3.* As an example, let us consider the following quiver Q

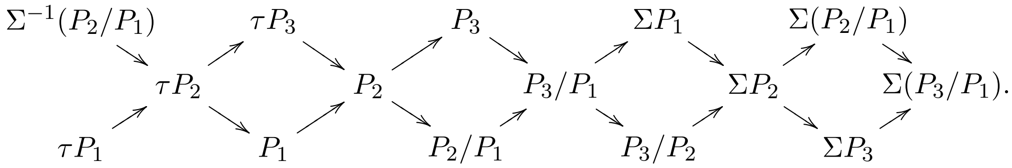
$$1 \longrightarrow 2 \longrightarrow 3\ ,$$

whose underlying graph is a Dynkin diagram of type A_3. As recalled above, the indecomposable objects of $\mathcal{D}^b(A)$ are isomorphic to shifted indecomposable modules. Hence, in our example, using the notations of section 5.1, the vertices of the quiver of the category $\mathcal{D}^b(A)$ correspond to the complexes $\Sigma^p(P_i/P_j)$, where $p \in \mathbb{Z}$ and $1 \le i < j \le 3$. Its arrows have been computed by Happel [32] [33]. It turns out that the quiver of $\mathcal{D}^b(A)$ is isomorphic to the *repetition* of Q, *cf.* [32] [46]. In our example, this is the infinite band:



The correspondence between its vertices and the indecomposable objects of $\mathcal{D}^b(A)$ is indicated by the following diagram:

(5.6.1)



The objects τP_i are the following complexes: $\tau P_3 = \Sigma^{-1}(P_3/P_2)$, $\tau P_2 = \Sigma^{-1}(P_3/P_1)$ and $\tau P_1 = \Sigma^{-1}P_3$. The fully faithful embedding $\mathsf{mod}(A) \to \mathcal{D}^b(A)$ identifies the quiver of $\mathsf{mod}(A)$ with the triangle formed by the objects P_i/P_j. Notice that the 'meshes' in this triangle correspond to short exact sequences and thus yield triangles in the derived category. More generally, each mesh of the whole diagram comes from a triangle in the derived category (in fact, it comes from a so-called Auslander-Reiten triangle, *cf.* [32]). The suspension functor Σ induces the glide reflection obtained by reflecting at the horizontal symmetry axis and translating by two units to the right. The Auslander-Reiten translation τ induces the shift by one unit to the left. The effect of the Serre functor $S = \tau\Sigma$ is the glide reflection whose translation is the shift by one unit to the right. The auto-equivalence

$$S^{-1}\Sigma^2 = \tau^{-1}\Sigma\,,$$

which appears in the definition of the cluster category, is the glide reflection taking the τP_i to the ΣP_i. As shown in [7], we obtain the quiver of the cluster category $\mathcal{C}_A$ as the quotient of the quiver of $\mathcal{D}^b(A)$ by the action of the automorphism induced by $S^{-1}\Sigma^2 = \tau^{-1}\Sigma$. Thus, in our example, this quiver is obtained by cutting out the fattened triangle bordered by the τP_i and the ΣP_i and then identifying each vertex τP_i with the corresponding ΣP_i. The quiver of $\mathcal{C}_A$ has nine vertices so that the cluster category $\mathcal{C}_A$ has nine indecomposables (up to isomorphism). The direct sum T of the images of the P_i in the cluster category is the canonical cluster-tilting object of $\mathcal{C}_A$.

5.6.3. *Cluster categories from algebras of global dimension* 2. Since k is algebraically closed, every finite-dimensional k-algebra of global dimension at most one is Morita-equivalent to kQ for some quiver Q without oriented cycles. Thus, the cluster category $\mathcal{C}_A$ is in fact defined for every finite-dimensional algebra A of global dimension at most one. If A is an algebra of finite but arbitrary global dimension, the derived category $\mathcal{D}^b(A)$ still has a Serre functor (given by the derived tensor product with DA) but the orbit category is no longer triangulated in general. In recent work [1], Amiot has extended the construction of the (triangulated) cluster category to certain algebras A of global dimension at most 2: Let A be such an algebra and let $\Pi_3(A)$ be its 3-Calabi-Yau completion in the sense of [49]. By definition, $\Pi_3(A)$ is the tensor algebra

$$T_A(X) = A \oplus X \oplus (X \otimes_A X) \oplus \ldots$$

of a cofibrant replacement of the complex of bimodules $X = \Sigma^2 \mathsf{RHom}_{A^e}(A, A^e)$, where $A^e = A^{op} \otimes_k A$. Its homology is given by

$$H^n(\Pi_3(A)) = \bigoplus_{p \geq 0} H^n((\Sigma^2 S^{-1})^p A).$$

In particular, it vanishes in degrees $n \leq 0$. Then Amiot defines the cluster category $\mathcal{C}_A$ as the triangle quotient

$$\mathsf{per}(\Pi_3(A))/\mathcal{D}_{fd}(\Pi_3(A)) ,$$

where, for a dg algebra B, the *perfect derived category* $\mathsf{per}(B)$ is the thick subcategory of the derived category $\mathcal{D}(B)$ generated by the free module B and the *finite-dimensional derived category* $\mathcal{D}_{fd}(B)$ is the full subcategory of $\mathcal{D}(B)$ whose objects are the dg modules M whose homology $H^*(M)$ is of finite total dimension. Amiot shows in [1] that if A is of global dimension at most one, this definition agrees with the definition given above. If A is of global dimension at most 2, then in general, the category $\mathcal{C}_A$ is not Hom-finite. However, we have the following theorem.

Theorem 5.7 (Amiot [1])**.** *Suppose that the functor*

$$\mathsf{Tor}_2^A(?, DA) : \mathsf{mod}\, A \to \mathsf{mod}\, A$$

*is nilpotent (*i.e. *it vanishes when raised to a sufficiently high power).*

a) *The category $\mathcal{C}_A$ is Hom-finite and 2-Calabi-Yau.*
b) *The image T of A in $\mathcal{C}_A$ is a cluster tilting object.*
c) *The endomorphism algebra of T in $\mathcal{C}_A$ is isomorphic to $H^0(\Pi_3(A))$ and the quiver of the endomorphism algebra is obtained from that of A by adding, for each pair of vertices (i, j), a number of arrows equal to*

$$\dim \mathsf{Tor}_2^A(S_j, S_i^{op})$$

from i to j, where S_j is the simple right module associated with j and S_i^{op} the simple left module associated with i.

As an example, let us consider the tensor product A of two copies of the path algebra kQ' of the quiver $1 \to 2$ (equivalently, the tensor product of two copies of the algebra $T_2(k)$ of lower triangular 2×2-matrices). This algebra is isomorphic to the quotient of the path algebra of the square

$$\begin{array}{ccc} (1,2) & \xrightarrow{\alpha} & (2,2) \\ {\scriptstyle\beta}\uparrow & & \uparrow{\scriptstyle\gamma} \\ (1,1) & \xrightarrow[\delta]{} & (2,1) \end{array}$$

by the two-sided ideal generated by the relator $\alpha\beta - \gamma\delta$. Using this description or lemma 5.9 below, one sees that the quiver of A is isomorphic to the tensor product $Q' \otimes Q'$. By a direct computation or using lemma 5.10 below, one sees that the algebra A satisfies the assumption of the theorem. So we obtain that the cluster category $\mathcal{C}_A$ is Hom-finite and 2-Calabi-Yau, that the image T of A in $\mathcal{C}_A$ is a cluster-tilting object and that the quiver of its endomorphism algebra is isomorphic to

$$\begin{array}{ccc} (1,2) & \xrightarrow{\alpha} & (2,2) \\ {\scriptstyle\beta}\uparrow & \swarrow{\scriptstyle\rho} & \uparrow{\scriptstyle\gamma} \\ (1,1) & \xrightarrow[\delta]{} & (2,1), \end{array}$$

that is, to the triangle product $Q' \boxtimes Q'$. By proposition 5.14 below, the endomorphism algebra itself is isomorphic to the quotient of the path algebra of this quiver by the two-sided ideal generated by the relators

$$\alpha\beta - \gamma\delta\ ,\ \beta\rho\ ,\ \rho\alpha\ ,\ \delta\rho\ ,\ \rho\gamma.$$

We have thus obtained a 2-Calabi-Yau realization of the triangle product $Q' \boxtimes Q'$ for the quiver $Q' : 1 \to 2$ and a precise description of the endomorphism algebra of a cluster-tilting object. In the following sections, our aim is to generalize this example to triangle products $Q' \boxtimes Q''$ of two arbitrary alternating Dynkin quivers.

5.8. **2-Calabi-Yau realizations of triangle products.** Let Q and Q' be two finite quivers without oriented cycles. Let k be a field. The path algebras kQ and kQ' are then finite-dimensional algebras of global dimension at most one and the tensor product $kQ \otimes_k kQ'$ is a finite-dimensional algebra of global dimension at most 2.

Lemma 5.9. *The quiver of the finite-dimensional k-algebra $kQ \otimes kQ'$ is isomorphic to $Q \otimes Q'$.*

Proof. Recall that if B is a finite-dimensional algebra whose simple modules are one-dimensional, the vertices of the quiver of B are in bijection with the isomorphism classes of the (simple) right B-modules and the number of arrows from the vertex of S to that of S' equals the dimension of the first extension group of S' by S. Equivalently, it equals the multiplicity of the projective cover P_S of S in the first term (not the zeroth term!) of a minimal projective presentation of S'. Now the simple modules of $kQ \otimes kQ'$ are the tensor products $S_i \otimes S_{i'}$, where S_i is the simple right kQ'-module associated with a vertex i of Q and similarly for $S_{i'}$. We obtain a minimal projective presentation of $S_i \otimes S_{i'}$ by tensoring a minimal projective presentation of S_i with a minimal projective presentation of $S_{i'}$. Now the minimal projective presentation of S_i is of the form

$$\bigoplus P_{s(\alpha)} \longrightarrow P_i$$

where the sum ranges over all arrows α with target i, the vertex $s(\alpha)$ is the source of the arrow α and $P_{s(\alpha)}$ the projective cover of the simple $S_{s(\alpha)}$. Thus, the minimal projective presentation of $S_i \otimes S_{i'}$ has its first term isomorphic to the direct sum of modules $P_{s(\alpha)} \otimes P_{i'}$ and $P_i \otimes P_{s(\alpha')}$, where α runs through the arrows of Q with target i and α' through the arrows of Q' with target i'. Clearly, this implies the assertion. $\surd$

Lemma 5.10. *Suppose Q and Q' are Dynkin quivers. Let $A = kQ \otimes_k kQ'$. Then the functor*

$$\mathsf{Tor}_2^A(?, DA) : \mathsf{mod}\, A \to \mathsf{mod}\, A$$

is nilpotent.

Proof. Let $S = ? \overset{L}{\otimes}_A DA$ be the Serre functor of $\mathcal{D}^b(A)$ and let $\mathcal{D}^b_{\geq 0}(A)$ be the right aisle of the canonical t-structure on $\mathcal{D}^b(A)$. For $p \in \mathbb{Z}$, put $\mathcal{D}^b_{\geq p}(A) = \Sigma^{-p}\mathcal{D}^b_{\geq 0}(A)$. We also use analogous notations for $\mathcal{D}^b(kQ)$ and $\mathcal{D}^b(kQ')$. By Proposition 4.9 of [1], to show that $\mathsf{Tor}_2^A(?, DA)$ is nilpotent, it suffices to show that

$$(\Sigma^{-2}S)^p(\mathcal{D}^b_{\geq 0}(A)) \subset \mathcal{D}^b_{\geq 1}(A)$$

for sufficiently large integers p. Now if L belongs to $\mathcal{D}^b(kQ)$ and M to $\mathcal{D}^b(kQ')$, then we have a canonical isomorphism

$$(\Sigma^{-2}S)(L \otimes M) = (\Sigma^{-1}SL) \otimes (\Sigma^{-1}SM) = (\tau L) \otimes (\tau M)\ ,$$

where $\tau = \Sigma^{-1}S$ is the Auslander-Reiten translation of the derived category of Q respectively Q'. Since Q and Q' are Dynkin quivers, it follows from Happel's description of the derived category of a Dynkin quiver [32], that there are integers N and N' such that

$$\tau^N(\mathcal{D}^b_{\geq 0}(kQ)) \subset \mathcal{D}^b_{\geq 1}(kQ) \quad \text{and} \quad \tau^{N'}(\mathcal{D}^b_{\geq 0}(kQ')) \subset \mathcal{D}^b_{\geq 1}(kQ').$$

Then we have

$$(\Sigma^{-2}S)^{N+N'}(L \otimes M) = (\tau^{N+N'}L) \otimes (\tau^{N+N'}M) \in \mathcal{D}^b_{\geq 2}(A)$$

for all $L \in \mathcal{D}^b_{\geq 0}(kQ)$ and $M \in \mathcal{D}^b_{\geq 0}(kQ')$. Since $\mathcal{D}^b_{\geq 0}(A)$ is the closure under Σ^{-1}, extensions and passage to direct factors of such objects $L \otimes M$, the claim follows. $\surd$

Corollary 5.11. *Suppose that Q and Q' are Dynkin quivers. Let $A = kQ \otimes_k kQ'$. Then the cluster category $\mathcal{C}_A$ is* Hom*-finite and 2-Calabi-Yau. Moreover, the image T of A in $\mathcal{C}_A$ is a cluster tilting object. The endoquiver of T is canonically isomorphic to the triangle product $Q \boxtimes Q'$.*

Proof. Except for the last claim, this follows directly from Theorem 5.7 and Lemma 5.10. To determine the endoquiver, by Theorem 5.7 and Lemma 5.9, it suffices to compute the second torsion groups between the simple A-modules. By the proof of Lemma 5.9, these are tensor products of simple modules. We compute

$$\mathsf{Tor}_2^A(S_j \otimes_k S_{j'}, S_i^{op} \otimes S_{i'}^{op}) = \mathsf{Tor}_1^{kQ}(S_j, S_i^{op}) \otimes \mathsf{Tor}_1^{kQ'}(S_{j'}, S_{i'}^{op}) = e_j(kQ_1)e_i \otimes e_{j'}(kQ'_1)e_{i'} \, ,$$

where we write kQ_1 for the vector space generated by the arrows of Q and consider it as a bimodule over the semi-simple subalgebra $\prod_{i\in Q_0} ke_i$ of kQ. This shows the claim. $\surd$

5.12. **Reminder on quivers with potential.** We recall results from Derksen-Weyman-Zelevinsky's fundamental article [13]. Let Q be a finite quiver and k a field. Let $\widehat{kQ}$ be the *completed path algebra*, *i.e.* the completion of the path algebra at the ideal generated by the arrows of Q. Thus, $\widehat{kQ}$ is a topological algebra and the paths of Q form a topologial basis so that the underlying vector space of $\widehat{kQ}$ is

$$\prod_{p \text{ path}} kp.$$

The *continuous zeroth Hochschild homology* of $\widehat{kQ}$ is the vector space HH_0 obtained as the quotient of $\widehat{kQ}$ by the closure of the subspace generated by all commutators. It admits a topological basis formed by the *cycles* of Q, *i.e.* the orbits of paths $p = (i|\alpha_m|\dots|\alpha_1|i)$ of any length $m \geq 0$ with identical source and target under the action of the cyclic group of order m. In particular, the space HH_0 is a product of copies of k indexed by the vertices if Q does not have oriented cycles. For each arrow a of Q, the *cyclic derivative with respect to a* is the unique linear map

$$\partial_a : \mathsf{HH}_0 \to \widehat{kQ}$$

which takes the class of a path p to the sum

$$\sum_{p=uav} vu$$

taken over all decompositions of p as a concatenation of paths u, a, v, where u and v are of length ≥ 0. A *potential* on Q is an element W of HH_0 whose expansion in the basis of cycles does not involve cycles of length ≤ 1. A potential is *reduced* if it does not involve cycles of length ≤ 2. The *Jacobian algebra* $\mathcal{P}(Q, W)$ associated to a quiver Q with potential W is the quotient of the completed path algebra by the closure of the 2-sided ideal generated by the cyclic derivatives of the elements of W. If the potential W is reduced and the Jacobian algebra $\mathcal{P}(Q,W)$ is finite-dimensional, its quiver is isomorphic to Q. Two quivers with potential (Q,W) and (Q',W') are *right equivalent* if $Q_0 = Q'_0$ and there exists a k-algebra isomorphism $\varphi : \widehat{kQ} \to \widehat{kQ'}$ such that φ induces the identity on the subalgebra $\prod_{Q_0} k$ and takes W to W'. One of the main theorems of [13] is the existence, for each quiver with potential (Q,W), of a reduced quiver with potential (Q_{red}, W_{red}), unique up to right equivalence, such that (Q,W) is right equivalent to the sum of (Q_{red}, W_{red}) with a trivial quiver with potential (*cf.* [13] for the definition). In particular, the Jacobian algebras of (Q,W) and (Q_{red}, W_{red}) are isomorphic. The quiver with potential (Q_{red}, W_{red}) is the *reduced part* of (Q,W).

Let (Q,W) be a quiver with potential such that Q does not have loops. Let i be a vertex of Q not lying on a 2-cycle. The *mutation* $\mu_i(Q,W)$ is defined as the reduced part of the quiver with potential $\widetilde{\mu}_i(Q,W) = (Q',W')$, which is defined as follows:

a) (i) To obtain Q' from Q, add a new arrow $[\alpha\beta]$ for each pair of arrows $\alpha : i \to j$ and $\beta : l \to i$ of Q and
   (ii) replace each arrow γ with source or target i by a new arrow γ^* with $s(\gamma^*) = t(\gamma)$ and $t(\gamma^*) = s(\gamma)$.

b) Put $W' = [W] + \Delta$, where
   (i) $[W]$ is obtained from W by replacing, in a representative of W without cycles passing through i, each occurrence of $\alpha\beta$ by $[\alpha\beta]$, for each pair of arrows $\alpha : i \to j$ and $\beta : l \to i$ of Q;
   (ii) Δ is the sum of the cycles $[\alpha\beta]\beta^*\alpha^*$ taken over all pairs of arrows $\alpha : i \to j$ and $\beta : l \to i$ of Q.

Then i is not contained in a 2-cycle of $\mu_i(Q,W)$ and $\mu_i(\mu_i(Q,W))$ is shown in [13] to be right equivalent to (Q,W). Note that if neither Q nor the quiver Q' in $(Q',W') = \mu_i(Q,W)$ have loops or 2-cycles, then Q and Q' are linked by the quiver mutation rule.

Let (Q,W) be a quiver with potential. The*Ginzburg dg algebra* $\Gamma = \Gamma(Q,W)$ associated with (Q,W) is due to V. Ginzburg [30]. We use the variant defined in [51]. The Ginzburg algebra is a topological differential graded algebra whose homology vanishes in (cohomological) degrees > 0 and whose zeroth homology is isomorphic to the Jacobian algebra $\mathcal{P}(Q,W)$. We refer to [51] for the definitions of the derived category $\mathcal{D}\Gamma$, the perfect derived category $\mathsf{per}(\Gamma)$ and the finite-dimensional derived category $\mathcal{D}_{fd}(\Gamma)$. All three are triangulated categories and $\mathcal{D}_{fd}(\Gamma)$ is a thick subcategory of $\mathsf{per}(\Gamma)$, which is a thick subcategory of $\mathcal{D}(\Gamma)$. The objects of $\mathcal{D}(\Gamma)$ are all differential graded right Γ-modules. The *cluster category* $\mathcal{C}_\Gamma$ is defined as the triangle quotient of $\mathsf{per}(\Gamma)$ by $\mathcal{D}_{fd}(\Gamma)$.

Let us assume that the Jacobian algebra of (Q,W) is finite-dimensional. Then the cluster category $\mathcal{C}_\Gamma$ is a (Hom-finite!) 2-Calabi-Yau category and the image T in $\mathcal{C}_\Gamma$ of the free right Γ-module of rank one is a cluster tilting object by [1]. Moreover, as shown in [1], the endomorphism algebra of T in $\mathcal{C}_\Gamma$ is isomorphic to the Jacobian algebra and thus, if (Q,W) is reduced, the quiver of the endomorphism algebra of T is isomorphic to Q.

Let (Q,W) be a quiver with potential and i a vertex of Q not lying on a 2-cycle. Let $\Gamma = \Gamma(Q,W)$ and $\Gamma' = \Gamma(\mu_i(Q,W))$. For each vertex j of Q, let e_j be the associated idempotent and $P_j = e_j\Gamma$ the right ideal generated by e_j. We use analogous notations for Γ'. It is shown in [51] that there is a triangle equivalence

$$F : \mathcal{D}(\Gamma') \xrightarrow{\sim} \mathcal{D}(\Gamma)$$

which induces triangle equivalences in the perfect and the finite-dimensional derived categories and sends the objects P'_j, $j \neq i$, to the P_j and the object P_i to the cone over the morphism of dg modules

$$P_i \to \bigoplus P_j ,$$

where the sum is taken over all arrows in Q with source i and the components of the morphism are the left multiplications with the corresponding arrows. If the Jacobian algebra of (Q, W) is finite-dimensional, the equivalence F induces a triangle equivalence

$$\mathcal{C}_{\Gamma'} \xrightarrow{\sim} \mathcal{C}_{\Gamma}$$

which sends $T' = \Gamma'$ to $\mu_i(T)$, by the exchange triangles of Lemma 6.3. In particular, the endomorphism algebra of $\mu_i(T)$ is isomorphic to the Jacobian algebra of $\mu_i(Q, W)$ (this also follows from Theorem 5.1 in [5]) and its quiver is the quiver Q' appearing in $(Q', W') = \mu_i(Q, W)$.

5.13. **Description of $\mathcal{C}_A$ by a quiver with potential.** Let k be a field and Q and Q' finite quivers without oriented cycles. Let $A = kQ \otimes_k kQ'$. The cluster category $\mathcal{C}_A$ defined in section 5.6 is associated with the 3-Calabi-Yau completion $\Pi_3(A)$. We can describe this dg algebra conveniently using quivers with potentials.

For arrows $\alpha : i \to j$ of Q and $\alpha' : i' \to j'$ of Q', we use notations as in the following full subquiver of $Q \boxtimes Q'$:

$$\begin{array}{ccc} (i,j') & \xrightarrow{(\alpha,j')} & (j,j') \\ {\scriptstyle (i,\alpha')}\uparrow & \swarrow{\scriptstyle \rho(\alpha,\alpha')} & \uparrow{\scriptstyle (j,\alpha')} \\ (i,i') & \xrightarrow[(\alpha,i')]{} & (j,i') \end{array}$$

Proposition 5.14. *The 3-Calabi-Yau completion $\Pi_3(A)$ is quasi-isomorphic to the Ginzburg algebra $\Gamma(Q \boxtimes Q', W)$ associated with the quiver $Q \boxtimes Q'$ and the potential*

$$W = \sum \rho(\alpha, \alpha') \circ ((j, \alpha') \circ (\alpha, i') - (\alpha, j') \circ (i, \alpha')) ,$$

where the sum ranges over all pairs of arrows α of Q and α' of Q'.

Proof. A closer examination of the proof of Corollary 5.11 shows that the minimal relations for the algebra A are precisely the commutativity relations $(\alpha, i') \circ (j, \alpha') - (\alpha, j') \circ (i, \alpha')$ associated with pairs of arrows α of Q and α' of Q'. Now Theorem 6.9 of [49] shows that $\Pi_3(A)$ is quasi-isomorphic to the non completed Ginzburg algebra $\Gamma'(Q \boxtimes Q', W)$. Since the homology in each degree of this dg algebra is finite-dimensional, the canonical morphism

$$\Gamma'(Q \boxtimes Q', W) \to \Gamma(Q \boxtimes Q', W)$$

is a quasi-isomorphism. $\surd$

6. Cluster combinatorics from Calabi-Yau triangulated categories

Here we describe how to associate cluster combinatorial data with objects in 2-Calabi-Yau categories with a cluster tilting object. We start with the categorical lift of the most basic operation: quiver mutation.

6.1. **Decategorification: quiver mutation.** Let k be an algebraically closed field and $\mathcal{C}$ a 2-Calabi-Yau category with cluster tilting object T. Let T_1 be an indecomposable direct factor of T.

Theorem 6.2 (Iyama-Yoshino [39]). *Up to isomorphism, there is a unique indecomposable object T_1^* not isomorphic to T_1 such that the object $\mu_1(T)$ obtained from T by replacing the indecomposable summand T_1 with T_1^* is cluster tilting.*

We call $\mu_1(T)$ the *mutation* of T at T_1. If $T_1, \ldots, T_n$ are the pairwise non isomorphic indecomposable direct summands of T and $\mathbb{T}_n$ is the n-regular tree with distinguished vertex t_0, we define cluster tilting objects $T(t)$ for each vertex t of $\mathbb{T}_n$ in such a way that $T(t_0) = T$ and, whenever t and t' are linked by an edge labeled k, we have $T(t') = \mu_k(T(t))$. Notice that the construction simultaneously yields a natural numbering of the indecomposable summands $T_i(t')$ of $T(t')$ in such a way that $T_i(t) \xrightarrow{\sim} T_i(t')$ for all $i \neq k$.

Lemma 6.3. *Suppose that endoquiver of T does not have a loop at the vertex corresponding to T_1. Then the space $\mathsf{Ext}^1(T_1, T_1^*)$ is one-dimensional and we have non split triangles*

$$T_1^* \xrightarrow{i} E \xrightarrow{p} T_1 \xrightarrow{\varepsilon} \Sigma T_1^* \quad \text{and} \quad T_1 \xrightarrow{i'} E' \xrightarrow{p'} T_1^* \xrightarrow{\varepsilon'} \Sigma T_1 \ , \tag{6.3.1}$$

where

$$E = \bigoplus_{\substack{arrows \\ i \to 1}} T_i \text{ and } E' = \bigoplus_{\substack{arrows \\ 1 \to j}} T_j$$

and the components of the morphisms i, i', p and p' represent the corresponding arrows of the endoquivers of T respectively $\mu_1(T)$.

The lemma is well-known to the experts. We include a proof for the convenience of the reader.

Proof. Let T' be the direct sum of the indecomposable direct factors of T not isomorphic to T_1. By the construction of T_1^* in [39], we have a triangle

$$T_1^* \xrightarrow{i} E \xrightarrow{p} T_1 \xrightarrow{\varepsilon} \Sigma T_1^* \ ,$$

where p is a minimal right $\mathsf{add}(T')$-approximation of T_1. Since T is rigid, we obtain an exact sequence

$$\mathcal{C}(T, E) \to \mathcal{C}(T, T_1) \to \mathcal{C}(T, \Sigma T_1^*) \to 0. \tag{6.3.2}$$

Since the endoquiver of T does not have a loop at T_1, each non isomorphism from T_1 to itself factors through E. Since k is algebraically closed, it follows that the space $\mathcal{C}(T, \Sigma T_1^*)$ is one-dimensional and isomorphic to the simple quotient of the indecomposable projective $\mathcal{C}(T,T)$-module $\mathcal{C}(T, T_1)$. Moreover, the sequence 6.3.2 is a minimal projective presentation of this simple quotient. This yields the description of E. By applying the same argument to $\mathcal{C}^{op}$, we obtain the description of E'. $\surd$

Theorem 6.4 (Buan-Iyama-Reiten-Scott [4])**.** *Suppose that the endoquivers Q and Q' of T and $T' = \mu_1(T)$ do not have loops nor 2-cycles. Then Q' is the mutation of Q at the vertex* 1*.*

We define a cluster tilting object T' to be *reachable from* T if there is a path

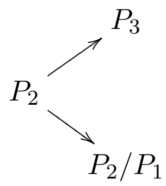
$$t_0 \text{ —— } t_1 \text{ —— } \cdots \text{ —— } t_N$$

in $\mathbb{T}_n$ such that $T(t_N) = T'$ and the quiver of $\mathsf{End}(T(t_i))$ does not have loops nor 2-cycles for all $0 \leq i \leq N$. It follows from the theorem above that in this case, for each $0 \leq i \leq N$, the endoquiver $Q_{T(t_i)}$ of $T(t_i)$ is obtained from the endoquiver of $T = T(t_0)$ by the corresponding sequence of mutations, *i.e.* we have $Q_{T(t_i)} = Q(t_i)$ for all $1 \leq i \leq N$. We define a rigid indecomposable object of $\mathcal{C}$ to be *reachable from* T if it is a direct summand of a reachable cluster tilting object.

Example 6.5. *Consider the cluster category $\mathcal{C}$ of the quiver $1 \to 2 \to 3$ from example 5.6.2 with its cluster tilting object T which is the sum of P_1, P_2 and P_3. If we mutate T at its summand $T_1 = P_1$, we find $T_1^* = P_2/P_1$ and the exchange triangles 6.3.1 are*

$$P_2/P_1 \longrightarrow 0 \longrightarrow P_1 \longrightarrow \Sigma(P_2/P_1) \quad \text{and} \quad P_1 \longrightarrow P_2 \longrightarrow P_2/P_1 \longrightarrow \Sigma P_1 \ .$$

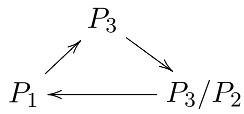
Notice that the third morphism of the first triangle is the composition of the isomorphism $P_1 \xrightarrow{\sim} \tau(P_2/P_1)$, already present in the derived category, with the isomorphism $\tau(P_2/P_1) \xrightarrow{\sim} \Sigma(P_2/P_1)$ obtained thanks to the passage to the cluster category. We obtain the new quiver



which is indeed isomorphic to the mutation of $1 \to 2 \to 3$ at the vertex 1. If we mutate T at its summand P_2, we obtain $P_2^* = P_3/P_1$ and the exchange triangles

$$P_3/P_2 \longrightarrow P_1 \longrightarrow P_2 \longrightarrow \Sigma(P_3/P_2) \quad \text{and} \quad P_2 \longrightarrow P_3 \longrightarrow P_3/P_2 \longrightarrow \Sigma P_2.$$

The new quiver is



and it is indeed isomorphic to the mutation of $1 \to 2 \to 3$ at the vertex 2.

Example 6.6. *Let us consider the following alternating Dynkin quiver of type A_3:*

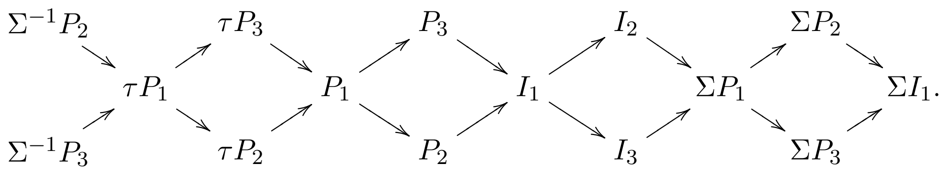
$$Q: \ 2 \longleftarrow 1 \longrightarrow 3 \ .$$

If we identify this quiver with the triangle product $Q \boxtimes Q'$, where Q' consists of a single vertex (considered as a source) without arrows, then the mutation sequence $\mu_\boxtimes$ defined in formula 3.6.1 simplifies to

$$\mu_\boxtimes = \mu_- \mu_+ \ ,$$

*where μ_+ is the mutation at the source 1 and μ_- the sequence of mutations $\mu_2\mu_3$ at the sinks 2 and 3. Let us lift the composition $\mu_\boxtimes$ to the categorical level and check that the action of $\mu_\boxtimes^{h+h'} = \mu_\boxtimes^{h+2}$ is indeed the identity, where $h = 4$ and $h' = 2$ are the Coxeter numbers of A_3 and A_1. Let A be the path algebra (*cf. *section 5.2) and for each vertex i, let $P_i = e_i A$ and $I_i = \mathsf{Hom}(Ae_i, k)$ (these are the indecomposable projective, respectively injective, A-modules, up to isomorphism). In analogy with example 5.6.2, we draw a piece of the quiver of the derived category $\mathcal{D}^b(A)$.*

(6.6.1)



Let us point out that the derived category in this example is equivalent to that in example 5.6.2 by the derived functor of the Bernstein-Gelfand-Ponomarev reflection functor associated with the vertex 1, cf. [32] [46]*. This explains the isomorphism between their* *quivers, which of course respects the actions of the automorphisms* Σ, S *and* τ. *If we mutate the initial cluster-tilting object* $T = P_1 \oplus P_2 \oplus P_3$ *at the summand* P_1, *we obtain the exchange triangles*

$$I_1 \longrightarrow 0 \longrightarrow P_1 \longrightarrow \Sigma I_1 \quad \textit{and} \quad P_1 \longrightarrow P_2 \oplus P_3 \longrightarrow I_1 \longrightarrow \Sigma P_1 \,.$$

and the new quiver

$$\begin{array}{ccc} P_3 & & \\ & \searrow & \\ & & I_1. \\ & \nearrow & \\ P_2 & & \end{array}$$

If we now successively mutate at the summands associated with the vertices 2 *and* 3 *of the original quiver, we successively obtain the quivers*

$$\begin{array}{ccc} P_3 & & \\ & \searrow & \\ & & I_1 \\ & & & \searrow & \\ & & & & I_3 \end{array} \qquad \textit{and} \qquad \begin{array}{ccc} & & I_2 \\ & \nearrow & \\ I_1 & & \\ & \searrow & \\ & & I_3. \end{array}$$

What we see in this example is that applying $\mu_{\boxtimes}$ *to the initial cluster-tilting object is tantamount to applying the autoequivalence* τ^{-1}. *Therefore, if we raise* $\mu_{\boxtimes}$ *to the power* $h + h' = 4 + 2$, *the resulting sequence of mutations acts on the initial cluster tilting object like the auto-equivalence* $\tau^{-h}\tau^{-2}$. *Now the functor* $\tau^{-h} : \mathcal{D}^b(A) \to \mathcal{D}^b(A)$ *is isomorphic to* Σ^2 *by the classical Theorem 8.2 below. On the other hand, the functor induced by* τ^{-1} *in the cluster category is isomorphic to that induced by* Σ^{-1}, *by the definition of the cluster category. So the effect of* $\tau^{-h}\tau^{-2}$ *in the cluster category is that of* $\Sigma^2\Sigma^{-2} = \mathbf{1}$. *This is a simple case of 'categorical periodicity', proved in general in section 5.6.1 below.*

6.7. **Decategorification: g-vectors and tropical Y-variables.** As in section 6.1, we assume that k is an algebrically closed field and $\mathcal{C}$ a 2-Calabi-Yau category with a cluster tilting object T.

Let $\mathcal{T} = \mathsf{add}(T)$ be the full subcategory whose objects are all direct factors of finite direct sums of copies of T. Let $K_0(\mathcal{T})$ be the Grothendieck group of the additive category $\mathcal{T}$. Thus, the group $K_0(\mathcal{T})$ is free abelian on the isomorphism classes of the indecomposable summands of T.

Lemma 6.8 ([50]). *For each object* L *of* $\mathcal{C}$, *there is a triangle*

$$T_1 \to T_0 \to L \to \Sigma T_1$$

such that T_0 *and* T_1 *belong to* $\mathcal{T}$. *The difference*

$$[T_0] - [T_1]$$

considered as an element of $K_0(\mathcal{T})$ *does not depend on the choice of this triangle.*

In the situation of the lemma, we define the *index* $\mathsf{ind}_T(L)$ *of* L as the element $[T_0] - [T_1]$ of $K_0(\mathcal{T})$.

Theorem 6.9 ([12]). a) *Two rigid objects are isomorphic iff their indices are equal.*

b) *The indices of the indecomposable summands of a cluster tilting object form a basis of* $K_0(\mathcal{T})$. *In particular, all cluster tilting objects have the same number of pairwise non isomorphic indecomposable summands.*

c) *In the situation of Lemma 6.3, if $T' = \mu_1(T)$ and L is an object of $\mathcal{C}$, we have*

$$\mathsf{ind}_{T'}(L) = \begin{cases} \varphi_+(\mathsf{ind}_T(L)) & \text{if } [\mathsf{ind}_T(L) : T_1] \geq 0 \\ \varphi_-(\mathsf{ind}_T(L)) & \text{if } [\mathsf{ind}_T(L) : T_1] \leq 0. \end{cases} \tag{6.9.1}$$

where $\varphi_\pm$ are the linear automorphisms of $K_0(\mathcal{T})$ which fix all classes of indecomposable factors of T not isomorphic to T_1 and send the class of T_1 to

$$\varphi_+([T_1]) = -[T_1] + [E] \text{ respectively } \varphi_-([T_1]) = -[T_1] + [E'].$$

Corollary 6.10 ([25])**.** *Let $T(t)$ be a cluster tilting object reachable from $T = T(t_0)$. For each $1 \leq j \leq n$, we have*

$$\mathsf{ind}_T(T_j(t)) = \sum_{i=1}^{n} g_{ij}^{t_0}(t)[T_i]. \tag{6.10.1}$$

Proof. This follows by induction from Theorem 6.9 and from the recursive characterization of the g-vectors in equations 4.4.3 and 4.4.4. √

Notice that a cluster tilting object T' of $\mathcal{C}$ is also a cluster tilting object of the opposite category $\mathcal{C}^{op}$ so that each object L in $\mathsf{obj}(\mathcal{C}) = \mathsf{obj}(\mathcal{C}^{op})$ also has a well-defined index in $\mathcal{C}^{op}$ with respect to T'; we denote it by $\mathsf{ind}^{op}_{T'}(L)$. If we identify the Grothendieck groups of $\mathsf{add}\, T'$ and $(\mathsf{add}(T'))^{op}$, this index identifies with $-\mathsf{ind}_{T'}(\Sigma L)$.

Corollary 6.11. *Let $T(t)$ be a cluster tilting object reachable from $T = T(t_0)$. We have*

$$\mathsf{ind}^{op}_{T(t)}(T_j) = \sum_{i=1}^{n} c_{ji}[T_i(t)] \,, \tag{6.11.1}$$

where the c_{ij} are given by the tropical Y-variable

$$\eta_i(t) = \prod_{j=1}^{n} y_j^{c_{ij}} \,,\ 1 \leq i \leq n. \tag{6.11.2}$$

Proof. This follows from the recursive definition 4.1.1 of the tropical Y-variables and Theorem 6.9. √

Example 6.12. *We continue example 6.6. Let $T = T(t_0)$ be the direct sum of P_1, P_2 and P_3 in the cluster category $\mathcal{C}_A$. Let $T' = \mu_\boxtimes(T)$. In example 6.6, we have obtained that $T_1' = I_1$, $T_2' = I_3$ (sic!) and $T_3' = I_2$. Let us compute the indices of the $T_j = P_j$ with respect to T' in the opposite of the cluster category. For this, we have to produce 'co-resolutions' of the T_j by objects belonging to $\mathsf{add}(T')$. Now we have the exact sequences*

$$0 \longrightarrow P_1 \longrightarrow I_1 \longrightarrow I_2 \oplus I_3 \longrightarrow 0 \ ,$$

$$0 \longrightarrow P_2 \longrightarrow I_1 \longrightarrow I_3 \longrightarrow 0 \ ,$$

$$0 \longrightarrow P_3 \longrightarrow I_1 \longrightarrow I_2 \longrightarrow 0 \ .$$

To obtain these, it is best to identify modules over the path algebra with representations of the quiver

$$Q^{op} : \ 2 \longrightarrow 1 \longleftarrow 3 \ ,$$

cf. *section 5.2. Specifically, these representations are:*

$$P_1 : \ 0 \longrightarrow k \longleftarrow 0 \ , \ P_2 : \ k \longrightarrow k \longleftarrow 0 \ , \ P_3 : \ 0 \longrightarrow k \longleftarrow k$$

$$I_1 : \ k \longrightarrow k \longleftarrow k \ , \ I_2 : \ k \longrightarrow 0 \longleftarrow 0 \ , \ I_3 : \ 0 \longrightarrow 0 \longleftarrow k$$

So we obtain that the matrix whose coefficients are the integers c_{ij} defined in the corollary is

$$\begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & 0 \\ -1 & 0 & -1 \end{bmatrix}.$$

Notice again that $T_2' = I_3$ and $T_3' = I_2$! An easy computation shows that the tropical Y-variables associated with the vertex $t = \mu_{\boxtimes}(t_0)$ of the regular tree $\mathbb{T}_3$ are indeed the Laurent monomials

$$y_1y_2y_3\ ,\ y_1^{-1}y_2^{-1}\ ,\ y_1^{-1}y_3^{-1}$$

whose exponents appear in the rows of this matrix.

6.13. **Decategorification: cluster variables and F-polynomials.** Let k be the field of complex numbers. Let $T_1, \ldots, T_n$ be the pairwise non isomorphic indecomposable direct summands of T and B its endomorphism algebra. Let $P_i = \mathsf{Hom}(T, T_i)$ be the indecomposable projective right B-module associated with T_i, $1 \leq i \leq n$. Let S_i be the simple quotient of P_i. For a right B-module M, the *dimension vector* is the n-tuple formed by the $\dim \mathsf{Hom}_B(P_i, M)$, $1 \leq i \leq n$.

For two finite-dimensional right B-modules L and M put

$$\langle L, M\rangle_a = \dim \mathsf{Hom}(L, M) - \dim \mathsf{Ext}^1(L, M) - \dim \mathsf{Hom}(M, L) + \dim \mathsf{Ext}^1(M, L).$$

This is the antisymmetrization of a truncated Euler form. A priori it is defined on the split Grothendieck group of the category $\mathsf{mod}\, B$ (*i.e.* the quotient of the free abelian group on the isomorphism classes divided by the subgroup generated by all relations obtained from direct sums in $\mathsf{mod}\, B$).

Proposition 6.14 (Palu [59]). *The form $\langle , \rangle_a$ descends to an antisymmetric form on $K_0(\mathsf{mod}\, B)$. Its matrix in the basis of the simples is the antisymmetric matrix associated with the quiver of B (loops and 2-cycles do not contribute to this matrix).*

In notational accordance with equation 6.10.1, for $L \in \mathcal{C}$, we define the integer $g_i(L)$ to be the multiplicity of $[T_i]$ in the index $\mathsf{ind}(L)$, $1 \leq i \leq n$. We define the element X_L of the field $\mathbb{Q}(x_1, \ldots, x_n)$ by

$$X_L = \prod_{i=1}^{n} x_i^{g_i(L)} \sum_e \chi(\mathsf{Gr}_e(\mathsf{Ext}^1(T, L))) \prod_{i=1}^{n} x_i^{\langle S_i, e\rangle_a}\ , \tag{6.14.1}$$

where the sum ranges over all n-tuples $e \in \mathbb{N}^n$, the *quiver Grassmannian* $\mathsf{Gr}_e(\mathsf{Ext}^1(T, L))$ is the variety of all B-submodules of the B-module $\mathsf{Ext}^1(T, L)$ whose dimension vector is e and χ denotes the Euler characteristic of singular cohomology with coefficients in $\mathbb{C}$. Notice that we have $X_{T_i} = x_i$, $1 \leq i \leq n$. The expression 6.14.1 is a vastly generalized form of Caldero-Chapoton's formula [9]. We define the *F-polynomial* associated with L as the integer polynomial in the indeterminates $y_1, \ldots, y_n$ given by

$$F_L = \sum_e \chi(\mathsf{Gr}_e(\mathsf{Ext}^1(T, L))) \prod_{i=1}^{n} y_i^{e_i}$$

Now let Q be the endoquiver of T in $\mathcal{C}$. We assume that Q does not have loops or 2-cycles. Let $\mathcal{A}_Q$ be the associated cluster algebra.

Theorem 6.15 (Palu [59]). *If L and M are objects of $\mathcal{C}$ such that $\mathsf{Ext}^1(L, M)$ is one-dimensional and*

$$L \xrightarrow{i} E \longrightarrow M \longrightarrow \Sigma L \quad \textit{and} \quad M \xrightarrow{i'} E' \longrightarrow L \longrightarrow \Sigma M$$

are 'the' two non split triangles, then we have

$$X_L X_M = X_E + X_{E'} \tag{6.15.1}$$

$$F_L F_M = F_E \prod_{i=1}^{n} y_i^{d_i} + F_{E'} \prod_{i=1}^{n} y_i^{d'_i} \,, \tag{6.15.2}$$

where

$$d_i = \dim \mathsf{ker}(\mathcal{C}(T_i, \Sigma L) \xrightarrow{i_*} \mathcal{C}(T_i, \Sigma E)) \text{ and } d'_i = \dim \mathsf{ker}(\mathcal{C}(T_i, \Sigma M) \xrightarrow{i'_*} \mathcal{C}(T_i, \Sigma E')).$$

Proof. For $L \in \mathcal{C}$, let X_L^{Palu} be the polynomial defined by Palu in [59]. We then have $X_L = X_{\Sigma L}^{\text{Palu}}$, as follows from the formula at the end of section 2 in [59]. By Theorem 4 of [59], the map $L \mapsto X_L^{\text{Palu}}$ satisfies the formula 6.15.1. Hence so does the map $L \mapsto X_L$. The formula 6.15.2 is implicit in section 5.1 of [59] and in particular in formula (2) of the proof of Lemma 16 of [59]. $\surd$

Corollary 6.16. *If the cluster tilting object $T(t)$ associated with a vertex t of $\mathbb{T}_n$ is reachable from $T(t_0)$, we have*

$$X_{T_i(t)} = X_i(t) \text{ and } F_{T_i(t)} = F_i(t)$$

for all $1 \leq i \leq n$.

Proof. Indeed, the first formula follows by induction from Theorem 6.15 and the description of the exchange triangles given in Lemma 6.3. The second formula similarly follows from Theorem 6.15 once we show that the integers d_i and d'_i coincide with the corresponding exponents in the tropical Y-variable. This results from the categorical interpretation of the tropical Y-variables in Corollary 6.11 and the following lemma. $\surd$

Lemma 6.17. *Let L be a rigid object of $\mathcal{C}$ and let $1 \leq k \leq n$. Let*

$$T_k^* \longrightarrow E \longrightarrow T_k \longrightarrow \Sigma T_k^* \text{ and } T_k \longrightarrow E' \longrightarrow T_k^* \longrightarrow \Sigma T_k$$

be 'the' exchange triangles. Let m be the multiplicity of $[T_k]$ in $\mathsf{ind}_T^{op}(L)$. Then we have

$$m = \begin{cases} \dim \mathsf{ker}(\mathcal{C}(L, \Sigma T_k^*) \to \mathcal{C}(L, \Sigma E)) & \text{if } m \geq 0; \\ \dim \mathsf{ker}(\mathcal{C}(L, \Sigma T_k) \to \mathcal{C}(L, \Sigma E')) & \text{if } m \leq 0. \end{cases} \tag{6.17.1}$$

Moreover, at least one of the two integers on the right hand side vanishes.

Proof. A triangle is *contractible* if it is a direct sum of triangles one of whose terms is zero. A triangle is *minimal* if it does not contain a contractible triangle as a direct factor. Every triangle is the sum of a contractible and a minimal triangle. In particular, we can choose the triangle

$$L \to T_L^0 \to T_L^1 \to \Sigma L \,,$$

of Lemma 6.8, where T_L^0 and T_L^1 lie in $\mathsf{add}(T)$, to be minimal. Let m_0 and m_1 be the multiplicities of T_k in these two objects. We will show that m_0 and m_1 agree with the two dimensions on the right hand side of equation 6.17.1. By definition, we have $m = m_0 - m_1$. Now by Proposition 2.1 of [12], the indecomposable T_k cannot occur in both T_L^0 and T_L^1 and so $m_0 = 0$ or $m_1 = 0$. Clearly, this will imply both assertions. It remains to prove that m_0 and m_1 equal the two dimensions. Let us first reinterpret m_0 and m_1. Let B be the endomorphism algebra of T and S_k the simple top of the indecomposable projective B-module $\mathcal{C}(T, T_k)$. Then we see from Lemma 5.4 combined with the 2-Calabi-Yau property and the rigidity of T that the multiplicity m_0 of T_k in T_L^0 is the multiplicity of S_k in the socle of the B-module $\mathcal{C}(T, \Sigma^2 L)$ and the multiplicity m_1 of T_k in T_L^1 is the multiplicity of S_k in the top of the B-module $\mathcal{C}(T, \Sigma L)$. We have to show that these multiplicities agree with the respective dimensions on the right hand side of equation 6.17.1. We first consider the kernel of

$$\mathcal{C}(\Sigma L, \Sigma^2 T_k) \to \mathcal{C}(\Sigma L, \Sigma^2 E').$$

By Lemma 5.4, it is isomorphic to the kernel of

$$\mathsf{Hom}_B(\mathcal{C}(T, \Sigma L), \mathcal{C}(T, \Sigma^2 T_k)) \to \mathsf{Hom}_B(\mathcal{C}(T, \Sigma L), \mathcal{C}(T, \Sigma^2 E'))$$

and thus to the value of $\mathsf{Hom}_B(\mathcal{C}(T, \Sigma L), ?)$ on the kernel of

$$\mathcal{C}(T, \Sigma^2 T_k) \to \mathcal{C}(T, \Sigma^2 E').$$

Because of the triangle

$$\Sigma E' \to \Sigma T_k^* \to \Sigma^2 T_k \to \Sigma^2 E'$$

and the fact that $\mathsf{Ext}^1(T_k, T_k^*)$ is one-dimensional, we have an exact sequence

$$0 \to S_k \to \mathcal{C}(T, \Sigma^2 T_k) \to \mathcal{C}(T, \Sigma^2 E')$$

and so the kernel of $\mathcal{C}(L, \Sigma T_k) \to \mathcal{C}(L, \Sigma E')$ is isomorphic to the space

$$\mathsf{Hom}_B(\mathcal{C}(T, \Sigma L), S_k) ,$$

whose dimension clearly equals the multiplicity of S_k in the head of $\mathcal{C}(T, \Sigma L)$. This is what we wanted to show. Now we consider the kernel of

$$\mathcal{C}(L, \Sigma T_k^*) \to \mathcal{C}(L, \Sigma E).$$

By the 2-Calabi-Yau property, it is isomorphic to the dual of the cokernel of

$$\mathcal{C}(E, \Sigma L) \to \mathcal{C}(T_k^*, \Sigma L).$$

By the triangle

$$\Sigma^{-1} E \to \Sigma^{-1} T_k \to T_k^* \to E ,$$

this cokernel is isomorphic to the kernel of

$$\mathcal{C}(T_k, \Sigma^2 L) \to \mathcal{C}(E, \Sigma^2 L).$$

By Lemma 5.4, this kernel is isomorphic to the kernel of

$$\mathsf{Hom}_B(\mathcal{C}(T, T_k), \mathcal{C}(T, \Sigma^2 L)) \to \mathsf{Hom}_B(\mathcal{C}(T, E), \mathcal{C}(T, \Sigma^2 L)).$$

Because of the triangle

$$E \to T_k \to \Sigma T_k^* \to \Sigma E ,$$

the rigidity of T and the fact that $\mathsf{Ext}^1(T_k, T_k^*)$ is one-dimensional, we have an exact sequence

$$\mathcal{C}(T, E) \to \mathcal{C}(T, T_k) \to S_k \to 0.$$

So the above kernel is isomorphic to the space

$$\mathsf{Hom}_B(S_k, \mathcal{C}(T, \Sigma^2 L)) ,$$

whose dimension clearly equals the multiplicity of S_k in the socle of $\mathcal{C}(T, \Sigma^2 L)$. This is what we had to show. $\surd$

Example 6.18. *We continue example 6.12. We keep the initial cluster tilting object $T = P_1 \oplus P_2 \oplus P_3$ and wish to compute the F-polynomials associated with the direct factors of $T' = \mu_{\boxtimes}(T) = I_1 \oplus I_3 \oplus I_2$. For this, we first compute*

$$\mathsf{Ext}^1_{\mathcal{C}_A}(T, T'_i) = \mathsf{Ext}^1_{\mathcal{C}_A}(T, \tau^{-1} T_i) = \mathsf{Hom}_{\mathcal{C}_A}(T, \Sigma \tau^{-1} T_i) = \mathsf{Hom}_{\mathcal{C}_A}(T, T_i).$$

By lemma 5.4, the space $\mathsf{Hom}_{\mathcal{C}_A}(T, T_i)$, considered as a module over $\mathsf{End}_{\mathcal{C}_A}(T) = A$, is isomorphic to P_i. Hence the F-polynomials we are looking for are the generating functions of the Euler characteristics of the quiver Grassmannians of the modules P_i, or equivalently *of the associated representations of* $Q^{op}: \; 2 \longrightarrow 1 \longleftarrow 3$ *. Now* P_1 *is simple and so has exactly two submodules, namely* 0 *and* P_1*, which leads to*

$$F_{T'_1} = 1 + y_1.$$

The module P_2 *has exactly three submodules, namely* 0*,* P_1 *and* P_2*, which leads to*

$$F_{T'_3} = 1 + y_1 + y_1 y_2.$$

Similarly, we find

$$F_{T'_3} = 1 + y_1 + y_1 y_3.$$

A simple computation shows that these are indeed the F*-polynomials associated with the vertex* $\mu_\boxtimes(t_0)$ *of* $\mathbb{T}_3$*. Thanks to Fomin-Zelevinsky's formula 4.2.1 expressing the non tropical* Y*-variables at a vertex* t *of* $\mathbb{T}_n$ *in terms of the tropical* Y*-variables, the* F*-polynomials and the quiver at* t*, we obtain a categorical expression for the (non tropical)* Y*-variables associated with* $\mu_\boxtimes(t_0)$ *by combining the results of this example with those of examples 6.6 and 6.12.*

6.19. **Consequence for the conjecture.** Let Δ and Δ' be simply laced Dynkin diagrams with Coxeter numbers h and h'. Let Q and Q' be quivers with underlying graphs Δ and Δ'. According to Lemma 3.7, in order to show the periodicity conjecture for (Δ, Δ'), it suffices to show that the restricted Y-pattern $\mathbf{y}_\boxtimes$ associated with $Q \boxtimes Q'$ and the sequence of mutations

(6.19.1) $$\mu_\boxtimes = \mu_{+,-}\mu_{-,-}\mu_{+,+}\mu_{-,+}.$$

is periodic of period dividing $h + h'$. By section 4.3, for this it suffices to show that, for each vertex v of $Q \boxtimes Q'$, the sequences $\eta_v(t_{pN})$ and $F_v(t_{pN})$ are periodic in p of period dividing $h+h'$, where the vertices t_i of $\mathbb{T}_N$ are those visited when going through an integer power of $\mu_\boxtimes$.

Now let A be the algebra $kQ \otimes kQ'$ and $\mathcal{C}_A$ its associated cluster category with canonical cluster tilting object T as recalled in section 5.8. By Corollary 5.11, the endoquiver of T is isomorphic to $Q \boxtimes Q'$ so that the category $\mathcal{C}_A$ yields a 2-Calabi-Yau realization of $Q \boxtimes Q'$. In particular, it makes sense to consider the sequence of cluster tilting objects

$$T = T(t_0) \;\text{——}\; T(t_1) \;\text{——}\; \cdots$$

associated with the vertices t_i. If we can show that the endoquivers of all the objects $T(t_i)$ do not have loops nor 2-cycles, it will follow from Corollaries 6.11 and 6.16 that, for all vertices v of $Q \boxtimes Q'$ and all i, we have

$$\eta_v(t_i) = \prod y_j^{\mathsf{ind}^{op}_{T(t_i)}(T_v)} \quad \text{and} \quad F_v(t_i) = F_{T_v(t_i)}.$$

To conclude that the sequences $\eta_v(t_{pN})$ and $F_v(t_{pN})$ are periodic of period dividing $h+h'$, it will then suffice to show that the sequence $T(t_i)$ is periodic of period dividing $(h+h')N$. In section 7 below, we will show that indeed, the endoquivers of the $T(t_i)$ do not have loops or 2-cycles and we will describe the objects $T(t_{pN}) = \mu^p_\boxtimes(T)$ using the Zamolodchikov transformation of $\mathcal{C}_A$. In section 8 below, we will show that the sequence of the $T(t_i)$ is periodic of period dividing $N(h + h')$ or, in other words, that $\mu_\boxtimes^{h+h'}(T) \cong T$.

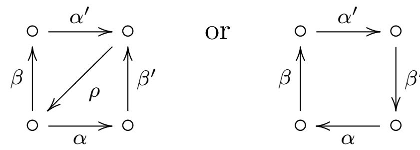
7. Mutations of products

7.1. **Constrained quivers with potential.** We want to study the effect of the mutations $\mu_\boxtimes$ of section 3.5 on the cluster tilting object of Corollary 5.11. We will use the description of this category by a quiver with potential constructed in Proposition 5.14. For this we introduce a class of quivers with potential containing the ones from Proposition 5.14.

Let Q and Q' be finite quivers without oriented cycles. To simplify the notations, let us suppose that between any two vertices of Q and Q', there is at most one arrow. Let Q_0 and Q'_0 denote their sets of vertices. Let R be a quiver whose vertex set is the product $Q_0 \times Q'_0$. An arrow $\alpha : (i, i') \to (j, j')$ of R is *horizontal* (respectively, *vertical*) if $i' = j'$ (respectively, if $i = j$). It is *diagonal* if it is neither horizontal nor vertical. The *non diagonal subquiver of* R is the subquiver formed by all vertices and by all the non diagonal arrows of R. The quiver R is (Q, Q')*-constrained* if

a) its non diagonal subquiver has the same underlying graph as $Q \otimes Q'$ (as defined in section 3.3) and
b) for any pair of arrows $i \to j$ of Q and $i' \to j'$ of Q', the full subquiver of R with vertex set $\{i, i'\} \times \{j, j'\}$ is isomorphic to

(7.1.1)



in such a way that α and α' correspond to horizontal arrows, β and β' to vertical arrows and ρ to a diagonal arrow.

We sometimes call the subquivers appearing in b) the *squares* of R. For example the quivers $Q\square Q'$ and $Q \boxtimes Q'$ are (Q, Q')-constrained and have the minimal, respectively maximal, number of diagonal arrows. Notice that a (Q, Q')-constrained quiver does not have loops nor 2-cycles.

A quiver with potential (R, W) is (Q, Q')*-constrained* if R is (Q, Q')-constrained and the potential W is the sum of non zero scalar multiples of all the cycles $\alpha'\beta\rho$ and $\alpha\rho\beta'$ appearing in a square as in the left diagram in (7.1.1) as well as all the cycles $\alpha\beta'\alpha'\beta$ appearing in a square as in the right diagram in (7.1.1). Recall that changing the starting point in a cycle does not change the superpotential. For example, the quivers with potential obtained from Proposition 5.14 are (Q, Q')-constrained.

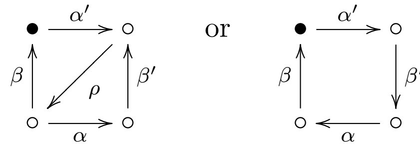
Let R be a (Q, Q')-constrained quiver. For a vertex (i, i') of R, the *horizontal slice through* (i, i') is the full subquiver $\mathsf{hrz}(R, i')$ formed by the vertices (j, i'), $j \in Q_0$, of R; the *vertical slice* $\mathsf{vrt}(R, i)$ *through* (i, i') is defined analogously. A vertex (i, i') of R is a *source-sink* if it is a source in its horizontal slice and a sink in its vertical slice and is not the source or the target of any diagonal arrow. Analogously, one defines *sink-sources*, … . Notice that two source-sinks are never linked by an arrow and that if (i, i') is a source-sink, each arrow with source (respectively target) (i, i') lies in the horizontal (respectively the vertical) slice passing through (i, i').

Lemma 7.2. *Let R be a (Q, Q')-constrained quiver and (i, i') a source-sink of R.*

a) *The mutated quiver $\mu_{(i,i')}(R)$ is still (Q, Q')-constrained. The horizontal slice of $\mu_{(i,i')}(R)$ passing through (i, i') is the mutation $\mu_{(i,i')}(\mathsf{hrz}(R, i'))$ and the vertical slice the mutation $\mu_{(i,i')}(\mathsf{vrt}(R, i))$.*
b) *If (R, W) is a (Q, Q')-constrained quiver with potential, then $\mu_{(i,i')}(R, W)$ is still (Q, Q')-constrained. In particular, it does not have loops or 2-cycles.*

Proof. a) The mutation at (i, i') reverses the arrows passing through (i, i') and does not create any diagonal arrows incident with (i, i'). Thus, (i, i') is a sink-source in the mutated quiver and the second assertion is clear. If (i, i') belongs to a square (where it has to be the vertex •)

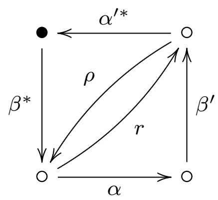
(7.2.1)



as in 7.1.1 then it transforms it into the other type of square; it leaves all squares not containing (i, i') unchanged. This shows the first assertion.

b) In the non reduced mutated quiver $\widetilde{\mu}_{(i,i')}(R, W)$ the squares of the first type containing (i, i') are modified as follows

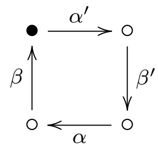
(7.2.2)



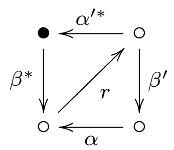
where $r = [\alpha'\beta]$. Their contribution to the potential is changed from $c_1\alpha'\beta\rho + c_2\beta'\alpha\rho$, where c_1 and c_2 are non zero scalars, to

$$c_1 r\rho + r\beta^*\alpha'^* + c_2\beta'\alpha\rho = (c_1 r + c_2\beta'\alpha)(\rho + c_1^{-1}\beta^*\alpha'^*) - c_2 c_1^{-1}\beta'\alpha\beta^*\alpha'^*.$$

Notice that r and ρ do not appear in any other terms of the potential. Therefore, after reduction, the square becomes



and its contribution to the potential becomes $-c_2 c_1^{-1}\beta'\alpha\beta^*\alpha'^*$. For the squares of the second type containing (i, i'), mutation changes them into



where $r = [\alpha'\beta]$, and their contribution to the potential changes from $c\alpha'\beta'\alpha\beta'$, where c is a non zero scalar, to

$$cr\alpha\beta' + r\beta^*\alpha'^*.$$

So we see that the effect of mutation is to exchange the two types of squares. The assertion follows. $\surd$

7.3. **The Zamolodchikov transformation.** Let Q be a finite quiver without oriented cycles. We order the vertices of Q such that $i \leq j$ iff there is a path (of length ≥ 0) leading from i to j. A *source sequence of* Q is an enumeration $i_1, i_2, \ldots, i_n$ of the vertices of Q which is non-decreasing with respect to $\leq$, i. e. if $1 \leq s \leq t \leq n$, then we have $i_s \leq i_t$ for the order on the vertices that we have just defined.

Let kQ be the path algebra of Q and $\mathsf{mod}\, kQ$ the category of finite-dimensional right kQ-modules. Let $\mathcal{D}^b(kQ)$ its bounded derived category and denote by $\tau = \Sigma^{-1}S$ its

Auslander-Reiten functor. As explained in [32], for each vertex i of Q, we have a canonical Auslander-Reiten triangle

$$P_i \longrightarrow (\bigoplus_{i\to j} P_j) \oplus (\bigoplus_{j\to i} \tau^{-1} P_j) \longrightarrow \tau^{-1} P_i \longrightarrow \Sigma P_i \tag{7.3.1}$$

Now let $\mathcal{C}_Q$ be the cluster category of Q, *cf.* section 5.6. It is a 2-Calabi-Yau category and the image T of the free module kQ is a cluster tilting object in $\mathcal{C}_Q$.

Lemma 7.4. *Let $i_1, \ldots, i_n$ be a source sequence of Q. For each $1 \le j \le n$, the mutated cluster tilting object*

$$\mu_{i_j}\mu_{i_{j-1}} \ldots \mu_{i_1}(T)$$

is the direct sum of the objects $\tau^{-1}P_{i_r}$, $1 \le r \le j$, and the objects P_{i_s}, $j < s \le n$. In particular, for $j = n$,the mutated object cluster tilting object

$$\mu_{i_n}\mu_{i_{n-1}} \ldots \mu_{i_1}(T)$$

is isomorphic to $\tau^{-1}T$.

Proof. This is an easy induction: For $j = 0$, there is nothing to show. For $j > 0$, we compare the above triangle 7.3.1 with the exchange triangles 6.3.1 to see that μ_j replaces the summand P_j by $\tau^{-1}P_j$. $\surd$

Now let Q' be another finite quiver without oriented cycles. For simplicity, we assume that in both, Q and Q', there is at most one arrow between any two given vertices. Let A be the finite-dimensional algebra $kQ \otimes_k kQ'$. We denote by $\mathsf{mod}(A)$ its category of k-finite-dimensional right A-modules, by $\mathcal{D}^b(A)$ its bounded derived category and by $\mathcal{C}_A$ the associated cluster category as recalled in section 5.8. We define $\tau \otimes \mathbf{1}$ to be the autoequivalence of $\mathcal{D}^b(A)$ given by left derived tensor product with the bimodule

$$(\Sigma^{-1}D(kQ)) \otimes_k kQ'$$

over $A = kQ \otimes_k kQ'$ and we define $\tau^{-1} \otimes \mathbf{1}$ to be its quasi-inverse. By Proposition 4.1 of [49], this functor induces an autoequivalence $\widetilde{\mathsf{Za}}$ of the perfect derived category $\mathsf{per}(\Pi_3(A))$ which preserves the finite-dimensional derived category. The *Zamolodchikov transformation* is the induced autoequivalence $\mathsf{Za} : \mathcal{C}_A \to \mathcal{C}_A$ of the cluster category. By abuse of notation, we still write $\mathsf{Za} = \tau^{-1} \otimes \mathbf{1}$. If the underlying graph of the quiver Q' is the Dynkin diagram A_1, the Zamolodchikov transformation coincides with the inverse Auslander-Reiten translation τ^{-1}.

7.5. **The Zamolodchikov transformation as a composition of mutations.** As in section 7.3, let Q and Q' be finite quivers without oriented cycles which, for simplicity of notation, are assumed to have at most one arrow between any two vertices. Let us order the vertices of $Q \boxtimes Q'$ such that $(i, i') \le (j, j')$ iff $i \le j$ and $i' \ge j'$ (sic!). Let $v_1, \ldots, v_N$ be a non decreasing enumeration of the vertices of $Q \boxtimes Q'$. For each $1 \le j \le N$, we put

$$(R(j), W(j)) = \mu_{v_j}\mu_{v_{j-1}} \ldots \mu_{v_1}(Q \boxtimes Q', W)$$

where W is the potential constructed in Proposition 5.14.

Lemma 7.6. *For each $0 \le j \le N$, we have*

a) *v_{j+1} is a source-sink of $R(j)$;*
b) *$(R(j), W(j))$ is (Q, Q')-constrained, and so $R(j)$ does not have loops nor 2-cycles;*
c) *for each vertex i' of Q', the quiver $\mathsf{hrz}(R(j), i')$ is isomorphic to $\mu_{i_s}\mu_{i_{s-1}} \ldots \mu_{i_1}(Q)$, where (i_1, i'), ..., (i_s, i') is the subsequence of the vertices of the form (x, i'), $x \in Q_0$, among the sequence v_1, ..., v_j;*

d) *for each $i \in Q_0$, the quiver* $\mathsf{vrt}(R(j), i)$ *is isomorphic to* $\mu_{i_s}\mu_{i_{s-1}} \dots \mu_{i_1}(Q')$, *where* $(i, i_1), \dots, (i, i_s)$ *is the subsequence of the vertices of the form* (i, y), $y \in Q'_0$, *among the sequence* $v_1, \dots, v_j$;

e) *The object*

$$\mu_{v_j}\mu_{v_{j-1}} \dots \mu_{v_1}(T)$$

is the direct sum of the objects $(\tau^{-1}P_i) \otimes P_{i'}$ *where* (i, i') *is among the* v_s, $1 \le s \le j$, *and of the* $P_i \otimes P_{i'}$, *where* (i, i') *is not among the* v_s, $1 \le s \le j$.

Proof. We prove a)-d) simultaneously by induction on j. For $j = 0$, all the assertions are clear. Assume the statement hold up to $j-1$. Then by a)$_{j-1}$, the vertex v_j is a source-sink of $R(j-1)$ and $(R(j-1), W(j-1))$ is (Q, Q')-constrained by b)$_{j-1}$. So $(R(j), W(j))$ is still (Q, Q')-constrained by Lemma 7.2. So we have proved b)$_j$. To prove c)$_j$, let i' be a vertex of Q'. If i' is not the second component of v_j, the sequence $i_1, \dots, i_s$ remains unchanged and so does the subquiver $\mathsf{hrz}(R(j), i')$ by Lemma 7.2. If i' is the second component of v_j, then the sequence $i_1, \dots, i_s$ is extended by adding $i_{s+1} \ge i_s$ and the claim of c)$_j$ still follows from Lemma 7.2. Similarly, one proves d). Finally, we have to show a)$_j$. Indeed, the vertex v_{j+1} is of the form (i, i'), and the first components $i_1, \dots, i_s, i_{s+1}$ of the $v_1, \dots, v_{j+1}$ which are of the form (x, i'), $x \in Q_0$, form a source sequence of $\mathsf{hrz}(R(0), i')$. So i is a source of $\mathsf{hrz}(R(j), i') = \mu_{i_s} \dots \mu_{i_1}(\mathsf{hrz}(R(0), i'))$. Similarly, one sees that (i, i') is a sink of $\mathsf{vrt}(R(j), i)$.

Now let us prove e) by induction on j. For $j = 0$, there is nothing to prove. Assume the assertion holds up to $j-1$. By a)$_{j-1}$, the vertex $v_j = (i, i')$ is a source-sink of $R(j-1)$. So in particular, the vertex v_j is a source of $\mathsf{hrz}(R(j-1), i')$. By the induction hypothesis, the direct summands of

$$\mu_{v_{j-1}} \dots \mu_{v_1}(T)$$

associated with the vertices of $\mathsf{hrz}(R(j-1), i')$ are the

$$\tau^{-1}P_{i_u} \otimes P_{i'}$$

for $1 \le u < s$ and the $P_j \otimes P_{i'}$ for j not among the i_u, $1 \le u < s$. Now in $\mathcal{D}^b(kQ)$, we have the sequence

$$P_i \longrightarrow (\textstyle\bigoplus_{i \to j} P_j) \oplus (\bigoplus_{j \to i} \tau^{-1}P_j) \longrightarrow \tau^{-1}P_i \longrightarrow \Sigma P_i \tag{7.6.1}$$

recalled in 7.3.1. By tensoring the sequence with $P_{i'}$ over k and taking the image in $\mathcal{C}_A$, we get the exchange triangle, which shows that the mutation at $v_j = (i, i')$ replaces the summand $P_i \otimes P_{i'}$ with $(\tau^{-1}P_i) \otimes P_{i'}$. $\surd$

Corollary 7.7. *Let $v_1, \dots, v_N$ be a non decreasing enumeration of the vertices of $Q \boxtimes Q'$ (for the order where $(i, i') \le (j, j')$ iff there is a path from i to j and a path from j' to i'!). Let μ_v be the composed mutation $\mu_{v_N}\mu_{v_{N-1}} \dots \mu_{v_1}$. For each vertex (i, i') of $Q \boxtimes Q'$, let $T_{i,i'}$ be the indecomposable summand $P_i \otimes P_{i'}$ of the canonical cluster tilting object T of the cluster category $\mathcal{C}_A$.*

a) *For any vertex t of $\mathbb{T}_n$ visited when performing an integer power of μ_v, the endoquiver of $T(t)$ does not have loops nor 2-cycles.*

b) *For each $p \ge 0$ and all (i, i'), we have an isomorphism in the cluster category*

$$\mathsf{Za}^p(T_{i,i'}) \cong (\mu_{\boxtimes}^p(T))_{i,i'}.$$

Proof. a) This follows from part b) of Lemma 7.6 and the fact that the endoquiver of a mutation of a cluster tilting object is the quiver appearing in the mutated quiver with potential as recalled at the end of section 5.12.

b) We proceed by induction on p. For $p = 0$, there is nothing to show. For $p = 1$, the assertion follows from part e) of Lemma 7.6. Now suppose that $p \geq 1$ and that we have an isomorphism

$$\mathsf{Za}^p(T_{i,i'}) \cong (\mu_v^p(T))_{i,i'}.$$

Since Za is an autoequivalence of $\mathcal{C}_A$, it follows that

$$\mathsf{Za}^{p+1}(T_{i,i'}) \cong \mathsf{Za}((\mu_v^p(T))_{i,i'}) \cong (\mu_v^p(\mathsf{Za}(T)))_{i,i'}.$$

Thanks to the case where $p = 1$, we conclude that

$$\mathsf{Za}^{p+1}(T_{i,i'}) \cong (\mu_v^p(\mu_v(T)))_{i,i'} \cong (\mu_v^{p+1}(T))_{i,i'}.$$

√

8. Categorical periodicity

8.1. **Categorification of the Coxeter element.** Let Δ be a simply laced Dynkin diagram with vertices 1, ..., n. Let α_i be the simple root associated with the vertex i, $1 \leq i \leq n$. Let h be the Coxeter number of Δ. Let Q be a quiver with underlying graph Δ. Let k be a field and $\mathcal{D}^b(kQ)$ the bounded derived category of finite-dimensional right modules over the path algebra kQ. Then $\mathcal{D}^b(kQ)$ is a Hom-finite triangulated category. We write Σ for its suspension functor, S for its Serre functor and $\tau = \Sigma^{-1}S$ for its Auslander-Reiten translation. For each $1 \leq i \leq n$, let S_i be the simple module associated with the vertex i.

Theorem 8.2 (Gabriel, Happel). a) *There is a canonical isomorphism from the Grothendieck group $K_0(\mathcal{D}^b(kQ))$ of the triangulated category $\mathcal{D}^b(kQ)$ to the root lattice of Δ which takes the class of the simple module $[S_i]$ to the simple root α_i.*

b) *Under this isomorphism, the (positive and negative) roots correspond precisely to the classes in $K_0(\mathcal{D}^b(kQ))$ of the indecomposable objects of $\mathcal{D}^b(kQ)$.*

c) *The automorphism of the Grothendieck group of the derived category induced by τ^{-1} corresponds to the action of a Coxeter element c on the root lattice. The identity $c^h = \mathbf{1}$ lifts to an isomorphism*

$$\tau^{-h} \xrightarrow{\sim} \Sigma^2$$

of k-linear functors $\mathcal{D}^b(kQ) \to \mathcal{D}^b(kQ)$.

Proof. Parts a) and b) follow immediately from Gabriel's theorem [26] and from Happel's description of the derived category $\mathcal{D}^b(kQ)$ in [32]. The isomorphism of functors in part c) follows from Happel's description of the category $\mathcal{D}^b(kQ)$ as the mesh category of the translation quiver $\mathbb{Z}Q$ and from Gabriel's description of the Serre functor (alias Nakayama functor) in Proposition 6.5 of [28]. A detailed proof of a more precise statement is given by Miyachi-Yekutieli in Theorem 4.1 of [57]. √

8.3. **On the order of the Zamolodchikov transformation.** As in section 7.3, let Q and Q' be finite quivers without oriented cycles such that between any two vertices, there is at most one arrow. Let $\mathsf{Za} = \tau^{-1} \otimes \mathbf{1}$ be the Zamolodchikov transformation of the cluster category $\mathcal{C}_A$ associated with $A = kQ \otimes_k kQ'$, *cf.* section 7.3.

Theorem 8.4. *Suppose that the graphs underlying Q and Q' are two simply laced Dynkin diagrams with Coxeter numbers h and h'. Then we have an isomorphism of functors from $\mathcal{C}_A$ to itself*

$$\mathsf{Za}^{h+h'} \xrightarrow{\sim} \mathbf{1}. \qquad (8.4.1)$$

Remark 8.5. *Let W be the Weyl group associated with Q and w_0 its longest element. The element w_0 takes all positive roots to negative roots. It is the product (in a suitable order) of the reflections at all positive roots. Let w_0' be the longest element of the Weyl group associated with Q'. One can refine the proof below to show that if both w_0 and w_0' act by multiplication with -1 on their respective root lattices, we have an isomorphism*

$$\mathsf{Za}^{(h+h')/2} \xrightarrow{\sim} \mathbf{1}. \tag{8.5.1}$$

Proof. Let $A_1 = kQ$. The Auslander-Reiten translation $\tau = \Sigma^{-1}S$ of $\mathcal{D}^b(A_1)$ is given by tensoring with the bimodule $\Sigma^{-1}DA_1$, where DA_1 is the k-dual of the bimodule A_1. By Theorem 8.2, we have an isomorphism $\tau^h(A_1) = \Sigma^{-2}(A_1)$ which is compatible with the left actions of A_1 on the two sides. By the main theorem of [44], this implies that we have an isomorphism in the derived category of bimodules $\mathcal{D}^b(A_1 \otimes A_1^{op})$

$$(\Sigma^{-1}DA_1)^{\otimes_{A_1} h} \xrightarrow{\sim} \Sigma^{-2}A_1,$$

where we write $\otimes_{A_1} h$ for the derived tensor power. This yields an isomorphism in the derived category of A-bimodules

$$(\Sigma^{-1}DA_1 \otimes A_2)^{\otimes_A h} \xrightarrow{\sim} \Sigma^{-2}A.$$

Whence an isomorphism of functors $(\tau \otimes \mathbf{1})^h = \Sigma^{-2}$ from $\mathcal{C}_A$ to itself. Similarly, we have an isomorphism in the derived category of A_2-bimodules

$$(\Sigma^{-1}DA_2)^{\otimes_{A_2} h'} \xrightarrow{\sim} \Sigma^{-2}A_2\,,$$

which yields an isomorphism of functors $(\mathbf{1} \otimes \tau)^{h'} = \Sigma^{-2}$ in $\mathcal{C}_A$. Now we have

$$(\Sigma^{-1}DA_1 \otimes A_2) \otimes_A (A_1 \otimes \Sigma^{-1}DA_2) = \Sigma^{-2}(DA_1 \otimes DA_2) = \Sigma^{-2}DA.$$

Since the category $\mathcal{C}_A$ is 2-Calabi-Yau, the Serre functor of $\mathcal{C}_A$ is isomorphic to Σ^2. Now the Serre functor is induced by the left derived functor of tensoring with DA. So tensoring with $\Sigma^{-2}DA$ induces the identity in $\mathcal{C}_A$ and so the derived tensor product with

$$(\Sigma^{-1}DA_1 \otimes A_2) \otimes_A (A_1 \otimes \Sigma^{-1}DA_2)$$

induces the identity in $\mathcal{C}_A$. In other words, we have

$$(\tau \otimes \mathbf{1})(\mathbf{1} \otimes \tau) = \mathbf{1}$$

as functors from $\mathcal{C}_A$ to itself. Finally, we find the following chain of isomorphisms of functors from $\mathcal{C}_A$ to itself:

$$(\tau \otimes \mathbf{1})^{h+h'} = (\tau \otimes \mathbf{1})^h (\tau \otimes \mathbf{1})^{h'} = (\tau \otimes \mathbf{1})^h (\mathbf{1} \otimes \tau)^{-h'} = \Sigma^{-2}\Sigma^2 = \mathbf{1}.$$

√

8.6. **Conclusion.** Let Δ and Δ' be simply laced Dynkin diagrams with Coxeter numbers h and h'. Let Q and Q' be quivers with underlying graphs Δ and Δ'. Let $\mathcal{C}_A$ be the cluster category associated with $A = kQ \otimes kQ'$ and T is canonical cluster tilting object. As announced in section 6.19, we have shown that in the sequence of cluster tilting objects associated with the powers of $\mu_{\boxtimes}$, the endoquivers do not have loops or 2-cycles and that we have

$$\mu_{\boxtimes}^{h+h'}(T) \cong T.$$

This implies the conjecture, as explained in section 6.19.

9. The non simply laced case

In this section, we reduce the general case of the conjecture to the one where the two Dynkin diagrams are simply laced. We use the classical folding technique in the spirit of section 2.4 of [22]. The material in sections 9.1 and 9.2 is adapted from [17].

9.1. **Valued quivers and skew-symmetrizable matrices.** A *valued quiver* is a quiver Q endowed with a function $v : Q_1 \to \mathbb{N}^2$ such that

a) there are no loops in Q,
b) there is at most one arrow between any two vertices of Q and
c) there is a function $d : Q_0 \to \mathbb{N}$ such that $d(i)$ is strictly positive for all vertices i and, for each arrow $\alpha : i \to j$, we have

$$v(\alpha)_1 d(i) = d(j) v(\alpha)_2 ,$$

where $v(\alpha) = (v(\alpha)_1, v(\alpha)_2)$.

For example, we have the valued quiver

$$\vec{B}_2 :\ 1 \xrightarrow{(2,1)} 2\ ,$$

where a possible function d is given by $d(1) = 1$, $d(2) = 2$. Let Q be a valued quiver with vertex set I. We associate an integer matrix $B = (b_{ij})_{i,j\in I}$ with it as follows

$$b_{ij} = \begin{cases} 0 & \text{if there is no arrow between } i \text{ and } j; \\ v(\alpha)_1 & \text{if there is an arrow } \alpha : i \to j; \\ -v(\alpha)_2 & \text{if there is an arrow } \alpha : j \to i. \end{cases}$$

If D is the diagonal $I \times I$-matrix with diagonal entries $d_{ii} = d(i)$, $i \in I$, then the matrix DB is antisymmetric. The existence of such a matrix D means that the matrix B is *antisymmetrizable*. It is easy to check that in this way, we obtain a bijection between the antisymmetrizable $I \times I$-matrices B and the valued quivers with vertex set I. Using this bijection, we define the *mutation of valued quivers* using Fomin-Zelevinsky's matrix mutation rule 3.1.2.

Let (Q, v) be a valued quiver with vertex set $I = Q_0$. Its *associated Cartan matrix* is the Cartan companion [21] of the antisymmetrizable matrix B associated with Q. Explicitly, it is the the $I \times I$-matrix C whose coefficient c_{ij} vanishes if there are no arrows between i and j, equals 2 if $i = j$, equals $-v(\alpha)_1$ if there is an arrow $\alpha : i \to j$ and equals $-v(\alpha)_2$ if there is an arrow $\alpha : j \to i$. Thus, the Cartan matrix associated with the above valued quiver $\vec{B}_2$ equals

$$\begin{bmatrix} 2 & -2 \\ -1 & 2 \end{bmatrix}.$$

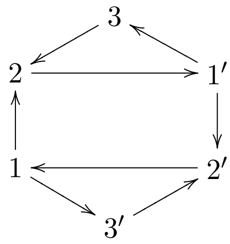
The *valued graph* [16] underlying the valued quiver (Q, v) is by definition the set I of vertices of Q together with the non negative integers e_{ij}, $i, j \in I$, defined by

$$e_{ij} = \begin{cases} -c_{ij} & i \neq j\ , \\ 0 & i = j. \end{cases}$$

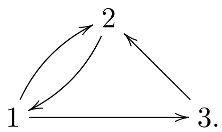
One checks easily that the pictorial representation (used in [16]) of the valued graph $(I, \mathbf{e})$ is obtained from that of (Q, v) by replacing all arrows with unoriented edges. In the sequel, we will identify Dynkin diagrams with the valued graphs corresponding to their Cartan matrices.

9.2. **Valued orbit quivers and their mutations.** Let $\widetilde{Q}$ be an (ordinary) quiver with vertex set $\widetilde{I}$ without loops or 2-cycles. Let G be a finite group of automorphisms of $\widetilde{Q}$. Let $\widetilde{B}$ be the antisymmetric matrix associated with $\widetilde{Q}$. The *orbit quiver* $\widetilde{Q}/G$ is the quiver with vertex set $I = \widetilde{I}/G$ and where there is an arrow from a vertex i to a vertex j if there is an arrow $\widetilde{i} \to \widetilde{j}$ for some vertices $\widetilde{i}$ in i and $\widetilde{j}$ in j. In the following example, due to A. Zelevinsky, the orbit quiver of

(9.2.1)



under the action of $\mathbb{Z}/2\mathbb{Z}$ which exchanges opposite vertices is



Notice the presence of a 2-cycle in the orbit quiver. The action of G on Q is *admissible* if the orbit quiver does not have loops or 2-cycles. In this case, we define the *valued orbit quiver*

$$Q = \widetilde{Q}/_v G$$

to be the valued quiver with vertex set $I = \widetilde{I}/G$ and whose associated antisymmetrizable $I \times I$-matrix is given by

$$b_{ij} = \sum_{\widetilde{i} \in i} \widetilde{b}_{\widetilde{i}\,\widetilde{j}} \, ,$$

where $\widetilde{j}$ is any representative of j. We obtain a function d for Q by sending an orbit i to the cardinality $d(i)$ of the stabilizer in G of any vertex $\widetilde{i}$ in i. For example, the above quiver $\vec{B}_2$ is isomorphic to the orbit quiver of

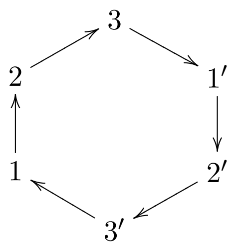
$$1 \longrightarrow 2 \longleftarrow 1'$$

under the action of $\mathbb{Z}/2\mathbb{Z}$ which fixes 2 and exchanges 1 and $1'$.

Now assume that the action of G on $\widetilde{Q}$ is admissible. Let k be a vertex of Q. Notice that between any two vertices $\widetilde{k}$ and $\widetilde{k}'$ of the orbit k, there are no arrows (because the orbit quiver has no loops). Thus, the mutated quiver

$$\prod_{\widetilde{k}} \mu_{\widetilde{k}}(\widetilde{Q})$$

is independent of the choice of the order of the vertices $\widetilde{k}$ in the orbit k. This quiver inherits a natural action of G. However, this action need not be admissible any more. For example, the following quiver



has an admissible action by $\mathbb{Z}/2\mathbb{Z}$ which exchanges opposite vertices but if we mutate at the vertices of the orbit of 3, we obtain the quiver of 9.2.1 with its non admissible action.

Lemma 9.3. *If the action of G on the mutated quiver $\prod_{\widetilde{k}} \mu_{\widetilde{k}}(\widetilde{Q})$ is admissible, there is a canonical isomorphism between the valued orbit quiver*

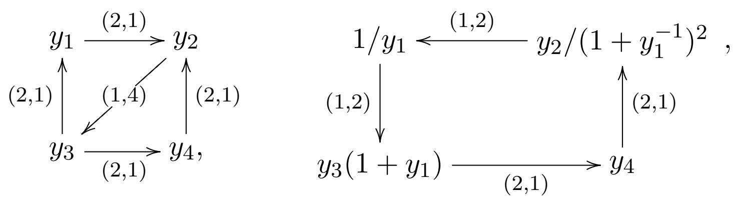
$$(\prod_{\widetilde{k}} \mu_{\widetilde{k}}(\widetilde{Q}))/_v G$$

and the mutated valued quiver $\mu_k(Q/_v G)$.

9.4. **Valued Y-seeds.** A *(valued) Y-seed* is a pair (Q, Y) formed by a finite valued quiver Q with vertex set I and a free generating set $Y = \{Y_i \mid i \in I\}$ of the semifield $\mathbb{Q}_{sf}(y_i \mid i \in I)$ generated over $\mathbb{Q}$ by indeterminates y_i, $i \in I$. If (Q, Y) is a Y-seed and k a vertex of Q, the *mutated Y-seed* $\mu_k(Q, Y)$ is (Q', Y'), where $Q' = \mu_k(Q)$ and, for $j \in I$, we have

$$Y'_j = \begin{cases} Y_k^{-1} & \text{if } j = k; \\ Y_i(1 + Y_k^{-1})^{-b_{kj}} & \text{if } b_{kj} \geq 0 \\ Y_i(1 + Y_k)^{-b_{kj}} & \text{if } b_{kj} \leq 0. \end{cases}$$

One checks that $\mu_k^2(Q, Y) = (Q, Y)$. For example, the following Y-seeds are related by a mutation at the vertex 1:



where we write the variable Y_i in place of the vertex i.

For a valued quiver Q, the *initial Y-seed*, the *Y-pattern* and the *restricted Y-pattern* associated with a sequence of vertices of Q are defined as in the simply laced case in section 3.2.

Now let $\widetilde{Q}$ be a quiver endowed with an admissible action of a finite group G. Let $Q = \widetilde{Q}/_v G$ be the valued orbit quiver. Let k be a vertex of Q and $\widetilde{Q}'$ the mutated quiver

$$\prod_{\widetilde{k} \in k} \mu_{\widetilde{k}} \widetilde{Q}.$$

Assume that the action of G on $\widetilde{Q}'$ is still admissible. Let

$$\pi : \mathbb{Q}_{sf}(y_{\widetilde{i}} | \widetilde{i} \in \widetilde{I}) \to \mathbb{Q}_{sf}(y_i | i \in I)$$

be the unique morphism of semifields such that $\pi(y_{\widetilde{i}}) = y_i$ for all vertices $\widetilde{i}$ of $\widetilde{Q}$. Let $\widetilde{Y}$ be the set of the $y_{\widetilde{i}}$, $\widetilde{i} \in \widetilde{I}$, and let Y be the set of the y_i, $i \in I$. Define Y' by $\mu_k(Q, Y) = (Q', Y')$ and $\widetilde{Y}'$ such that $(\widetilde{Q}', \widetilde{Y}')$ is the result of applying the mutations $\mu_{\widetilde{k}}$, $\widetilde{k} \in k$, to $(\widetilde{Q}, \widetilde{Y})$.

Lemma 9.5. *We have $\pi(\widetilde{Y}'_i) = Y'_i$ for all $i \in I$.*

The proof is a straightforward computation which we omit.

9.6. **Products of valued quivers.** Let Q and Q' be two valued quivers. The *tensor product* $Q \otimes Q'$ is defined as the tensor product of the underlying graphs endowed with the valuation v such that

$$v(\alpha, i') = v(\alpha) \text{ and } v(i, \alpha') = v(\alpha')$$

for all arrows α of Q and α' of Q' and for all vertices i' of Q' and i of Q. The *triangle product* $Q \boxtimes Q'$ is obtained from the tensor product by adding an arrow $(j, j') \to (i, i')$ of valuation

$$(v(\alpha)_2 v(\alpha')_2, v(\alpha)_1 v(\alpha')_1)$$

for each pair of arrows α of Q and α' of Q'. For example, the triangle product $\vec{B}_2 \boxtimes \vec{B}_2$ is given by

$$\begin{array}{ccc}
(1,2) & \xrightarrow{(2,1)} & (2,2) \\
{\scriptstyle (2,1)}\uparrow & \swarrow{\scriptstyle (1,4)} & \uparrow{\scriptstyle (2,1)} \\
(1,1) & \xrightarrow[(2,1)]{} & (1,2)
\end{array}$$

A valued quiver is *alternating* if its underlying ordinary quiver is alternating. The *opposite* of a valued quiver Q has the opposite underlying quiver and the valuation defined by

$$v^{op}(\alpha^{op}) = (v(\alpha)_2, v(\alpha)_1)$$

for each arrow α of Q. The *square product* $Q \square Q'$ of two alternating valued quivers is obtained from $Q \otimes Q'$ by replacing all full valued subquivers $\{i\} \otimes Q'$ and $Q \otimes \{i'\}$ by their opposites, where i runs through the sinks of Q and i' through the sources of Q'. Then the quivers $Q \boxtimes Q'$ and $Q \square Q'$ are related by the same sequence of mutations as in Lemma 3.4.

9.7. **Restricted Y-patterns for pairs of arbitrary Dynkin diagrams.** Let Δ be a Dynkin diagram and Q an alternating valued quiver with underlying valued graph Δ, cf. the end of section 9.1. If Δ is not simply laced, we write Q as the valued quotient quiver of a valued quiver $\widetilde{Q}$ with a group action by G as in the following list where each quiver Q is followed by the corresponding quiver $\widetilde{Q}$:

(9.7.1) $$\vec{B}_n : \; 1 \longrightarrow \cdots \longrightarrow (n-2) \longleftarrow (n-1) \xrightarrow{(2,1)} n$$

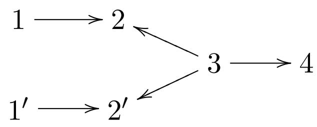
(9.7.2) $$\vec{A}_{2n-1} : \; \begin{array}{l} 1 \longrightarrow \cdots \longrightarrow (n-2) \longleftarrow (n-1) \searrow \\ \hspace{7cm} n \\ 1' \longrightarrow \cdots \longrightarrow (n-2)' \longleftarrow (n-1)' \nearrow \end{array}$$

(9.7.3) $$\vec{C}_n : \; 1 \longrightarrow \cdots \longrightarrow (n-2) \longleftarrow (n-1) \xrightarrow{(1,2)} n$$

(9.7.4) $$\vec{D}_{n+1} : \; \begin{array}{l} \hspace{6cm} \nearrow n \\ 1 \longrightarrow \cdots \longrightarrow (n-2) \longleftarrow (n-1) \\ \hspace{6cm} \searrow n' \end{array}$$

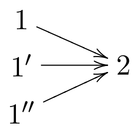
(9.7.5) $\vec{F}_4: \; 1 \longrightarrow 2 \overset{(2,1)}{\longleftarrow} 3 \longrightarrow 4$

(9.7.6) $\vec{E}_6:$



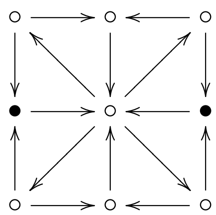
(9.7.7) $\vec{G}_2: \; 1 \overset{(3,1)}{\longrightarrow} 2$

(9.7.8) $\vec{D}_4:$



If Δ is simply laced, we consider $\widetilde{Q} = Q$ with the trivial group action. Notice that the Coxeter numbers of Δ and the underlying diagram of $\widetilde{Q}$ coincide. They are respectively equal to $2n$, $2n$, 12 and 6. In all cases except $\vec{G}_2$, the group acting is $\mathbb{Z}/2\mathbb{Z}$, for $\vec{G}_2$, it is $\mathbb{Z}/3\mathbb{Z}$.

Now let Δ' be another Dynkin diagram and $\widetilde{Q}'$ a quiver with a group action by G' defined similarly. The group $G \times G'$ acts on the each of the products $\widetilde{Q} \otimes \widetilde{Q}'$, $\widetilde{Q} \square \widetilde{Q}'$ and $\widetilde{Q} \boxtimes \widetilde{Q}'$ and the quotients are isomorphic to the respective products of the valued quivers Q and Q'. For example, the triangle product of two copies of $\vec{B}_2$ leads to the following quiver $\widetilde{Q} \boxtimes \widetilde{Q}'$:



The sequences of mutations $\mu_{\square}^{Q \square Q'}$ and $\mu_{\boxtimes}^{Q \boxtimes Q'}$ defined in section 3.5 make sense for the valued quivers $Q \square Q'$ and $Q \boxtimes Q'$. As in section 3.5, one checks that the Y-system associated with Δ and Δ' is periodic iff the restricted Y-pattern associated with $\mu_{\boxtimes}^{Q \boxtimes Q'}$ is periodic. Now the sequence of mutations $\mu_{\boxtimes}^{Q \boxtimes Q'}$ lifts to the sequence of mutations $\mu_{\boxtimes}^{\widetilde{Q} \boxtimes \widetilde{Q}'}$ associated with the simply laced quivers $\widetilde{Q}$ and $\widetilde{Q}'$. None of the mutations in this latter sequence introduces 2-cycles in the quotient quiver. Thus, by lemma 9.5, the fact that the restricted Y-pattern associated with $\mu_{\boxtimes}^{\widetilde{Q} \boxtimes \widetilde{Q}'}$ is periodic with period dividing $h_{\widetilde{Q}} + h_{\widetilde{Q}'}$ implies that the restricted Y-pattern associated with $\mu_{\boxtimes}^{Q \boxtimes Q'}$ is periodic with period dividing $h_{\widetilde{Q}} + h_{\widetilde{Q}'} = h_{\Delta} + h_{\Delta'}$.

10. Effectiveness

Let Δ and Δ' be simply laced Dynkin diagrams and Q and Q' alternating quivers with underlying graphs Δ and Δ'. In section 3.5, we have seen that in order to write down the explicit general solution of the Y-system associated with (Δ, Δ'), it suffices to write down the general seeds in the restricted Y-patterns $\mathbf{y}_{\square}$ or indeed in $\mathbf{y}_{\boxtimes}$. In Proposition 4.2, we have seen that the Y-variables at a vertex of the regular tree are determined by the tropical Y-variables and the F-polynomials. Thus, in order to write down the explicit general solution of the Y-system, it suffices to write down explicit expressions for the F-polynomials and the tropical Y-variables at the vertices $\mu_{\boxtimes}^{p}(t_0)$ obtained from the initial vertex t_0 of the regular tree by applying the sequence of mutations $\mu_{\boxtimes}^{p}$ for all $p \in Z$. Such explicit expressions can easily be extracted from the proof we gave, as we show now.

Let $k = \mathbb{C}$ and let $A = kQ$ and $A' = kQ'$ be the path algebras of Q and Q'. For a vertex i of Q, we write $P_i = e_i A$ for the corresponding indecomposable projective A-module and similarly $P_{i'}$ for a vertex i' of Q'. Let $\mathcal{D}$ be the derived category of the category $\mathsf{mod}(A \otimes A')$ of finite-dimensional right modules over $A \otimes A'$. Consider the finite-dimensional algebra

$$B = \bigoplus_{r \in \mathbb{Z}} \mathsf{Hom}_{\mathcal{D}}(A \otimes A', (\tau^{-r} A) \otimes \tau^{-r} A') ,$$

where τ is the Auslander-Reiten translation (in the derived category of right A-modules respectively A'-modules, *cf.* the proof of Theorem 8.4). Notice that since A and A' are hereditary, it is easy to determine these translations. By Theorem 5.7, the algebra B is isomorphic to $H^0(\Pi_3(A \otimes A'))$ which, by Proposition 5.14, is isomorphic to the Jacobian algebra of the quiver with potential $(Q \boxtimes Q', W)$ for the potential W constructed in the Proposition. Notice that in the definition of B, the summands indexed by the $r < 0$ vanish (because then the homology in degrees ≤ 0 of $\tau^{-r} A$ and $\tau^{-r} A'$ vanishes).

Given a vertex (i, i') of $Q \boxtimes Q'$ and an integer $p \in \mathbb{Z}$ we put

$$M(i, i', p) = \bigoplus_{r \in \mathbb{Z}} \mathsf{Hom}_{\mathcal{D}}(A \otimes A', (\tau^{-r-p} \Sigma P_i) \otimes (\tau^{-r} P_{i'})).$$

This is a right B-module. It follows from Corollary 6.16 and section 6.19 that the F-polynomial associated with (i, i') at the vertex $\mu_{\boxtimes}^{p}(t_0)$ can be expressed as

$$F_{(i,i')}(\mu_{\boxtimes}^{p}(t_0)) = \sum_{e} \chi(Gr_e(M(i, i', p))) y^e ,$$

where e runs through the dimension vectors of submodules of $M(i, i', p)$. Now we define another right B-module by

$$N(i, i', p) = \bigoplus_{r \in \mathbb{Z}} \mathsf{Hom}_{\mathcal{D}}(A \otimes A', (\tau^{-r-p} P_i) \otimes (\tau^{-r} P_{i'})).$$

For vertices (i, i') and (j, j') of $Q \boxtimes Q'$, put

$$c_{(i,i'),(j,j')} = \dim \mathsf{Hom}(N(i, i', -p), S_{(j,j')}) - \dim \mathsf{Ext}^1(N(i, i', -p), S_{(j,j')}).$$

Then it follows from Corollary 6.10 and section 6.19 that we have

$$\eta_{(i,i')}(\mu_{\boxtimes}^{p}(t_0)) = \prod_{(j,j')} y_{(j,j')}^{c_{(i,i'),(j,j')}}.$$

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