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The Pfaffian Calabi–Yau, its Mirror, and their Link to the Grassmannian G(2,7)

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Abstract. The rank 4 locus of a general skew-symmetric 7×7 matrix gives the Pfaffian variety in $\mathbf{P}^{20}$ which is not defined as a complete intersection. Intersecting this with a general $\mathbf{P}^6$ gives a Calabi–Yau manifold. An orbifold construction seems to give the 1-parameter mirror-family of this. However, corresponding to two points in the 1-parameter family of complex structures, both with maximally unipotent monodromy, are two different mirror-maps: one corresponding to the general Pfaffian section, the other to a general intersection of $G(2, 7) \subset \mathbf{P}^{20}$ with a $\mathbf{P}^{13}$. Apparently, the Pfaffian and $G(2, 7)$ sections constitute different parts of the A-model (Kähler structure related) moduli space, and, thus, represent different parts of the same conformal field theory moduli space.

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1. The Pfaffian Variety

Let E be a rank 7 vector space. For $N \in E \wedge E$ non-zero, we look at the locus of $\bigwedge^3 N = 0 \in \bigwedge^6 E$: the rank 4 locus of N if viewed as a skew-symmetric matrix. This defines a degree 14 variety of codimension 3 in $\mathbf{P}(E \wedge E) \cong \mathbf{P}^{20}$. As N is skew-symmetric, this variety is defined by the Pfaffians, i.e. square roots of the determinants, of the 6×6 diagonal minors of the matrix. Intersecting this with a general 6-plane in $\mathbf{P}(E \wedge E) \cong \mathbf{P}^{20}$ will give a three-dimensional Calabi–Yau ([7]). In coordinates x_i on $\mathbf{P}^6$, the matrix N can be written $N_A = \sum_{i=0}^{6} x_i A_i$ where the $A_i \in E \wedge E$ are skew-symmetric matrices spanning the $\mathbf{P}^6$. Denote this variety $X_A \subset \mathbf{P}^6$. The Pfaffian variety in $\mathbf{P}^{20}$ is smooth away from the rank 2 locus which has dimension 10. Hence, by Bertini's theorem, the variety X_A is smooth for general A.

DEFINITION 1. Let $N_A = \sum_{i=0}^{6} x_i A_i$ where A_i are 7×7 skew-symmetric matrices. Let $X_A \subset \mathbf{P}^6$ denote the zero-locus of the Pfaffians of the 6×6 diagonal minors of N_A: ie., the rank 4 locus of the matrix.

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For $P = N^3 \in \bigwedge^6 E$, $\mathcal{O} = \mathcal{O}_{\mathbf{P}(E)}$, there are exact sequences

$$\begin{aligned} 0 \longrightarrow (\bigwedge^7 E^\vee)^2 \otimes \mathcal{O}(-7) &\xrightarrow{P} \bigwedge^7 E^\vee \otimes E^\vee \otimes \mathcal{O}(-4) \\ &\xrightarrow{N} \bigwedge^7 E^\vee \otimes E \otimes \mathcal{O}(-3) \xrightarrow{P} \mathcal{O} \longrightarrow \mathcal{O}_X \longrightarrow 0 \end{aligned} \tag{1}$$

and

$$\begin{aligned} 0 \longrightarrow (\bigwedge^7 E^\vee)^2 \otimes \bigwedge^2 E^\vee \otimes \mathcal{O}(-8) &\xrightarrow{N\cdot} (\bigwedge^7 E^\vee)^2 \otimes (\mathrm{Hom}(E, E)/\mathrm{Id}_E) \otimes \mathcal{O}(-7) \\ &\xrightarrow{\cdot N} (\bigwedge^7 E^\vee)^2 \otimes S^2 E \otimes \mathcal{O}(-6) \xrightarrow{P^{\otimes 2}} \mathcal{J}_X^2 \longrightarrow 0 \end{aligned} \tag{2}$$

or more simply, for $P = [p_i]$ the Pfaffians with proper choice of sign and ordering,

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^6}(-7) \xrightarrow{P^{\mathrm{T}}} 7\mathcal{O}_{\mathbf{P}^6}(-4) \xrightarrow{N} 7\mathcal{O}_{\mathbf{P}^6}(-3) \xrightarrow{P} \mathcal{O}_{\mathbf{P}^6} \longrightarrow \mathcal{O}_X \longrightarrow 0 \tag{3}$$

and

$$0 \longrightarrow 21\mathcal{O}_{\mathbf{P}^6}(-8) \longrightarrow 48\mathcal{O}_{\mathbf{P}^6}(-7) \longrightarrow 28\mathcal{O}_{\mathbf{P}^6}(-6) \xrightarrow{P^{\otimes 2}} \mathcal{J}_X^2 \longrightarrow 0. \tag{4}$$

These sequences together with $0 \to \mathcal{J}_X^2 \to \mathcal{J}_X \to \mathcal{N}_X^\vee \to 0$, $0 \to \mathcal{N}_X^\vee \to \mathbf{\Omega}_{\mathbf{P}^6}|_X \to \mathbf{\Omega}_X \to 0$, and $0 \to \mathbf{\Omega}_{\mathbf{P}^6}|_X \to 7\mathcal{O}_X(-1) \to \mathcal{O}_X \to 0$ give the cohomology of the general, smooth manifold:

PROPOSITION 2. *The general variety $X_A \subset \mathbf{P}^6$ is smooth with $h^{1,0} = h^{2,0} = 0$, $h^{3,0} = 1$, $h^{1,1} = h^{2,2} = 1$, $h^{1,2} = h^{2,1} = 50$, $\chi = -98$, and $\omega_X \cong \mathcal{E}xt^3(\mathcal{O}_{X_A}, \omega_{\mathbf{P}^6}) \cong \mathcal{O}_{X_A}$; hence, it is a Calabi–Yau manifold. When X_A is singular, we have trivial dualizing sheaf, $\omega_{X_A}^\circ \cong \mathcal{O}_{X_A}$.*

2. The Canonical Bundle

In order to find the Picard–Fuchs operator, a global section of the canonical bundle is needed. In the case of a complete intersection, one could simply have used the dual of $\bigwedge_j \mathrm{d}p_j$ or its residue form $\bigwedge_i \mathrm{d}x_i / \prod_j p_j$. The Pfaffian variety, however, is not a complete intersection. For p_i the Pfaffian of N with row and column i removed, the polynomials $p_{\mu_0}, p_{\mu_1}, p_{\mu_2}$, μ_i a permutation of $\mathbf{Z}_7$, give a complete intersection wherever the submatrix $N_{\mu_3\mu_4\mu_5\mu_6}$ of N containing rows and columns μ_3, μ_4, μ_5, and μ_6 has rank 4: ie., its Pfaffian $(N_{[\mu_3\mu_4\mu_5\mu_6]})$ is different from zero. This follows from $N \cdot P = 0$. Hence,

$$(-1)^\mu \frac{\mathrm{d}p_{\mu_0} \wedge \mathrm{d}p_{\mu_1} \wedge \mathrm{d}p_{\mu_2}}{\mathrm{Pf}(N_{[\mu_3\mu_4\mu_5\mu_6]})} \tag{5}$$

gives a global section of $\mathcal{O}_{\mathbf{P}^6}(7) \otimes_{\mathcal{O}_{\mathbf{P}^6}} \bigwedge^3 \mathcal{N}_X^\vee \cong \omega_X^{\circ\,\vee}$. As $\omega_X^\circ \cong \mathcal{O}_X$, this section must

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be non-vanishing and independent of μ, that is, independent of μ up to a constant which proves to be $(-1)^{\mu}$: checked with Maple. Hence, the dual section in ω_X° is non-vanishing. For smooth varieties, the canonical and dualizing sheaves are identical, $\omega = \omega^\circ$, so we get:

PROPOSITION 3. *On the varieties X_A, we have a global section of the dualizing sheaf given by*⋆

$$\Omega = \frac{(-1)^{\mu}(2\pi i)^3 (N_{[\mu_3\mu_4\mu_5\mu_6]})\Omega_0}{\mathrm{d}p_{\mu_0} \wedge \mathrm{d}p_{\mu_1} \wedge \mathrm{d}p_{\mu_2}} = \mathrm{Res}\frac{(-1)^{\mu}(N_{[\mu_3\mu_4\mu_5\mu_6]})\Omega_0}{p_{\mu_0}p_{\mu_1}p_{\mu_2}}, \tag{6}$$

where Ω_0 is the global section of $\omega_{\mathbf{P}^6}(7) \cong \mathcal{O}_{\mathbf{P}^6}$ given by

$$\Omega_0 = \frac{x_0^7}{(2\pi i)^6} \bigwedge_{i=1}^{6} d\left(\frac{x_i}{x_0}\right). \tag{7}$$

The general X_A is smooth, making Ω a global section of the canonical bundle.

Actually, $\mathrm{Pf}(N_{[\mu_3\mu_4\mu_5\mu_6]}) \neq 0$ specifies the appropriate component of $p_{\mu_0} = p_{\mu_1} = p_{\mu_2} = 0$.

3. The Orbifold Construction

There are maps $\sigma\colon e_i \mapsto e_{i+1}$ and $\tau\colon e_i \mapsto e_i w^i$, where $w = e^{2\pi i/7}$ and (e_i) is a fixed basis for E, forming a group action on E. The commutator is multiplication with a constant, so in the projective setting, these two maps commute giving an abelian 7×7-group G: e.g., it gives an action on $\mathbf{P}(E \wedge E)$. We take the family of 6-planes in $\mathbf{P}(E \wedge E) \cong \mathbf{P}^{20}$ such that these maps restrict to them: i.e., $\mathrm{Span}\{\sum_{i+j\equiv k} y_{i-j} e_i \wedge e_j\}_{k\in\mathbf{Z}_7}$ or in matrix representation, $N = [x_{i+j}y_{i-j}]_{i,j\in\mathbf{Z}_7}$, where we take x_i to be coordinates on $\mathbf{P}^6$ and $y_i + y_{-i} = 0$. This gives a $\mathbf{P}^2$-family of 6-planes as parametrized by $[y_1 : y_2 : y_3]$, thus defining a $\mathbf{P}^2$-subfamily of X_A. These have double-points at the 49 points $[x_i]_{i\in\mathbf{Z}_7} \in \{g([y_i]_{i\in\mathbf{Z}_7}) | g \in G\}$.

For any 7-subgroup of G, there are 7 fixed-points in $\mathbf{P}^6$ under its action, and three lines in the $\mathbf{P}^2$ parameter space such that these fixed points lie in the corresponding varieties. We are free to choose any such subgroup, and any of the three lines, without loss of generality, as the normalizer of G ([8]) acts transitively on the eight triplets of lines.

Let H be the subgroup generated by τ, and choose the line $y_3 = 0$. We may then use the coordinate $y = y_2/y_1$ to parametrize our $\mathbf{P}^1$-family. We then have a matrix N_y whose rank 4 locus defines a degree 14 dimension 3 variety $X_y \subset \mathbf{P}^6$. In addition to the 49 double-points, the 7 fixed-points under τ are also double-points. In general,

⋆ For convenience, k-forms should contain the coefficient $(2\pi i)^{-k}$. This places the closed forms in the integral cohomology.

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these are the only singular points. (This has been checked using Macaulay ([17]) for the case $y = 1$.)

For $y = 0$ and $y = \infty$, the variety X_y decomposes into 14 distinct 3-planes intersecting on the coordinate planes.

In addition to the line-triplet we have chosen, there are seven other equivalent line-triplets. These intersect our chosen line in 21 points: $y^{21} - 289y^{14} - 58y^7 + 1 = 0$. For these values of y, the variety gains seven further double-points.

Using a construction similar to that of Candelas *et al.* ([3]), let $M_y = \widetilde{X_y/H}$ by a minimal (canonical) resolution of the quotient ([14]).

The map $x_i \mapsto x_i w^{5i^2}$ in the normalizer has the same effect as $y \mapsto yw$.

Hence, the natural parameter is $\phi = y^7$, and the manifold is denoted M_ϕ.

To give a brief review of the definition (in matrix notation):

DEFINITION 4. Let N be the skew-symmetric matrix $[x_{i+j}y_{i-j}]_{i,j \in \mathbf{Z}_7}$ where $y_i + y_{-i} = 0$, and $P = [p_i]$ the Pfaffians of the 6×6 diagonal minors; denote by X_Y, $Y = [y_i]$, the zero locus of P.

For $y_3 = 0$, let $y = y_2/y_1$ and denote the variety X_y. Let $H = \langle \tau \rangle$ be the group acting on X_y by $\tau : x_i \mapsto wx_i$. We take a minimal resolution of X_y/H, parametrize this family by $\phi = y^7$, and denote the resulting family of threefolds M_ϕ.

Gaining and resolving double-points corresponds to collapsing an S^3 to a point and then blowing it up to a $\mathbf{P}^1$ ([4]): e.g., by blowing up along $S_i = \{x_i = x_{i-3} = x_{i+3} = x_{i-2}x_{i+2} - y^2 x_{i-1}x_{i+1} = 0\}$, $i \in \mathbf{Z}_7$. This increases the Euler-characteristic by 2, either by increasing $h^{1,1}$ and $h^{2,2}$ by one each or by reducing $h^{1,2}$ and $h^{2,1}$ by one each. The blow-ups are along codimension 1 surfaces going through the double-points, and each such blow-up provides us with an extra (1, 1)-form. Neither of these processes, the collapsing and the blowing up, affect the dualizing sheaf as both processes are local and contained in a set containing no codimension 1 subvariety.

The creation and resolving of the $49 + 7$ double-points thus increases the Euler-characteristic to 14. The action of H has 14 fixed points: two on each $\mathbf{P}^1$ from the blowing up of the initial fixed-points. These quotient singularities can be resolved without affecting the dualizing sheaf ([14]). The Euler characteristic of the resolved quotient is given by Roan in [13] to be 98 using

$$\chi(\widetilde{V/H}) = \sum_{g,h \in H} \frac{\chi(V^g \cap V^h)}{|H|} \tag{8}$$

for any smooth V, V^g the fixed-point set in V of g, H an Abelian group. Determining the Betti numbers may now be done by finding the dimension of the deformation space.

For Calabi–Yau varieties with double-points, the moduli space is smooth ([12], [15]) with $\mathrm{Def}\,\widetilde{X} \hookrightarrow \mathrm{Def}\,X$ ([5]). This factors through $\mathrm{Def}\,\widetilde{X} \cong \mathrm{Def}$

$(X; P_1, \dots, P_k)$ which is the deformations of X with marked double-points at P_i. We have $\dim \operatorname{Def} X_A = h^1(X_A, \Theta_{X_A}) = h^{1,2} = 50$; for X_Y and X_y, the dimension is the same. On X_Y, we have the action of G and may decompose the inclusion $\operatorname{Def} \widetilde{X_Y} \hookrightarrow \operatorname{Def} X_Y$ into sums of G-eigenspaces. These give local systems on the $\mathbf{P}^2$ parameter space. Deformations along the $\mathbf{P}^2$ parameter space give G-fixed subspaces of the deformation space, thus giving lower bounds of the dimensions of these: 2 and 1 for $\widetilde{X_Y}$ and $\widetilde{X_y}$ respectively. The action of the normalizer of G ([8]), acting on the $\mathbf{P}^2$ parameter space and permuting the non-identity elements of G, ensures that the G-eigenspaces on which G is not the identity all have the same dimension. Knowing that the inclusions $\operatorname{Def} \widetilde{X_y} \hookrightarrow \operatorname{Def} \widetilde{X_Y} \hookrightarrow \operatorname{Def} X_A$ are all proper inclusions, we find $\dim \operatorname{Def} \widetilde{X_y} = 1$ and $\dim \operatorname{Def} \widetilde{X_Y} = 2$ which gives the Betti number $h^{1,2}$ of the respective varieties.

The X_y (resp. X_Y) should be resolved so as to make the group H act on $\widetilde{X_y}$ (resp. $\widetilde{X_Y}$); this may be done by first resolving the 7 double-points of X_Y/H and use this to determine $\widetilde{X_Y}$. The betti numbers of the resolution of the 14 quotient singularities is given in [1]: $h^{1,1}$ and $h^{2,2}$ increases by 3 for each fixed-point, thus making $h^{1,1} = h^{2,2} = 50$.

The variety now being smooth, the trivial dualizing sheaf is again identical to the canonical sheaf, which must therefore be trivial too. It should be pointed out that the resolution may not be unique. However, different resolutions will merely correspond to different parts of the A-model (Kähler structure related) moduli space: e.g., a flop corresponds to changing the sign of one component of $H^{1,1}$, thereby moving the Kähler cone ([10]).

PROPOSITION 5. *For general $y \in \mathbf{P}^1$, the manifolds $M_y = \widetilde{X_y/H}$ are Calabi–Yau manifolds having $\chi(M_y) = 98$, $h^{1,1} = h^{2,2} = 50$, and $h^{1,2} = h^{2,1} = 1$; the global section of the canonical sheaf inherited from X_A as given by 3. At the points $\phi = 0$ and $\phi = \infty$, the variety decomposes into 14 3-planes, and for $1 - 57\phi - 289\phi^2 + \phi^3 = 0$, where y lies on an intersection between two special lines in the $\mathbf{P}^2$ parameter space, there is an extra double-point.*

The families M_y and X_A thus look like good mirror candidates.

4. Mirror Symmetry

We now have a 1-parameter family of Calabi–Yau manifolds M_ϕ with a global section $\Omega(\phi)$ of the canonical bundle given. By the mirror symmetry conjecture, there is a special point in our moduli space corresponding to the 'large radius limit'. Around this point, H^3 should have maximally unipotent monodromy. As M_ϕ degenerates into 14 3-planes for $\phi = 0$ (and for $\phi = \infty$) we will start off with this as the assumed special point.

Following Morrison ([9]), there should be Gauss–Manin flat families of 3-cycles γ_0, γ_1, i.e. sections of $R_3\pi_*\mathbf{C}$, defined in a punctured neighborhood of $\phi = 0$ with

$f_i(\phi) = \int_{\gamma_i(\phi)} \Omega(\phi)$, such that f_0 extends across $\phi = 0$ and $f_1/f_0 = g + \log\phi$ where g extends across $\phi = 0$. The natural coordinate $t = t(\phi)$ is then given by $t = f_1/f_0$: ie., the complexified Kähler structure on the mirror is $\omega = t\omega_0$ where ω_0 is a fixed Kähler form (the dual of a line). As this enters only as $\exp\int_\eta \omega$, we may use the coordinate $q = e^t = \phi e^g$.

The curve count on the mirror is arrived at using the mirror symmetry assumption: that the B-model Yukawa coupling derived from the variation of complex structure (Hodge-structure) should be equal to the A-model Yukawa coupling on the mirror. The A-model Yukawa coupling is expressed in terms of the corresponding Kähler structure given by the natural coordinate and the number of rational curves in any curve class (ie., of any given degree) by

$$\kappa_{ttt} = n_0 + \sum_{d=1}^{\infty} n_d \tfrac{d^3 q^d}{1-q^d}. \tag{9}$$

The B-model Yukawa-coupling κ_{ttt} may be defined as in [9], by

$$\kappa_{ttt} = \kappa_{\frac{d}{dt}\frac{d}{dt}\frac{d}{dt}} = \kappa \cdot \tfrac{d}{dt} \otimes \tfrac{d}{dt} \otimes \tfrac{d}{dt} = \int_{M_\phi} \hat{\Omega} \wedge \nabla^3_{\frac{d}{dt}} \hat{\Omega} \tag{10}$$

with t the parameter on the moduli-space, (d/dt) seen as a tangent vector on the moduli space, ∇ the Gauss–Manin connection, and $\hat{\Omega} = \Omega/f_0$ the normalized canonical form. (In the following, I will write $\nabla_u = \nabla_{\frac{d}{du}}$ for any parameter u.)

All of this can be determined from knowing the Picard–Fuchs equation ([9]). The Picard–Fuchs equation is a differential equation on the parameter space whose solutions are $\int_{\gamma(\phi)} \Omega(\phi)$ for γ Gauss–Manin flat sections on $R_3\pi_*\mathbf{C}$: ie., $\gamma(\phi) = \sum_i u_i v_i(\phi)$ where $v_i(\phi) \in H_3(M_\phi, \mathbf{Z})$, $u_i \in \mathbf{C}$. This equation has order 4: ie., for $f = \int_\gamma \Omega$, where $\gamma = \gamma(\phi) \in \Gamma(R_3\pi_*\mathbf{C})$ is any ∇-flat section of 3-cycles, we have

$$\int_{\gamma(\phi)} \sum_{i=0}^{4} A_i(\phi)(\nabla_{\phi\frac{d}{d\phi}})^i \Omega = \sum_{i=0}^{4} A_i(\phi) D^i_\phi f(\phi) = 0 \tag{11}$$

for $D_\phi = \phi(d/d\phi) = d/d\log\phi$ the logarithmic derivative. Maximally unipotent monodromy around $\phi = 0$ is equivalent to having $A_i(0) = 0$ for $i < 4$ and $A_4(0) \neq 0$.

First, I will find γ_0 and calculate f_0. From this, I will determine the Picard–Fuchs equation. Knowing the Picard–Fuchs equation, f_1 can be found as another special solution. Furthermore, the Yukawa coupling, κ, satisfies a differential equation expressed in terms of the A-coefficients.

5. The Pfaffian Quotient Near $\phi = 0$

For simplicity, all calculations are pulled back from the manifold M_ϕ to the variety $X_y \subset \mathbf{P}^6$. At $y = 0$, the variety $X_y \subset \mathbf{P}^6$ degenerates into 14 3-planes intersecting along coordinate axes, the group H acting on each 3-plane. One of these planes is given by $x_4 = x_5 = x_6 = 0$. Let $\gamma_0(0)$ be the cycle given on this 3-plane (minus the axes) by $|x_i/x_0| = \varepsilon$ for $i = 1, 2, 3$. We may extend this definition by continuity to a neighborhood of $y = 0$.

Rather than working with γ_0, it is more convenient to work with the 6-cycle Γ on $\mathbf{P}^6 \setminus X_y$ given by $|x_i/x_0| = \varepsilon$ for $i = 1, 2, 3$ and $|x_i/x_0| = \delta$ for $i = 4, 5, 6$, and view Ω as the residue of

$$\Psi = \Omega \wedge \frac{dp_{v_0}}{2\pi i p_{v_0}} \wedge \frac{dp_{v_1}}{2\pi i p_{v_1}} \wedge \frac{dp_{v_2}}{2\pi i p_{v_2}} = \frac{(-1)^v (N_{v_3 v_4 v_5 v_6})}{(2\pi i)^6 p_{v_0} p_{v_1} p_{v_2}} \cdot x_0^7 \bigwedge_{i=1}^{6} d\left(\frac{x_i}{x_0}\right) \tag{12}$$

for any permutation v of $0, \ldots, 6$. We now get

$$f_0(\phi) = \int_{\gamma_0(\phi)} \Omega(\phi) = \int_\Gamma \Psi(\phi), \tag{13}$$

where the last integral is over a cycle which is independent of ϕ.

In order to make the numerator as simple as possible, choose $v_1, v_2, v_3 = 0, 3, 4$. This makes $(N_{v_1 v_2 v_5 v_6}) = x_3 x_4$. Setting $x_0 = 1$ for simplicity (or writing x_i for x_i/x_0), the integral becomes

$$\int_\Gamma \frac{x_3 x_4}{p_0 p_3 p_4} \cdot \bigwedge_{i=1}^{6} \frac{dx_i}{2\pi i} = \int_\Gamma \frac{1}{\prod_{i=0,3,4}(1 - \sum_{j=1}^{4} v_{i,j})} \cdot \bigwedge_{i=1}^{6} \frac{dx_i}{2\pi i x_i}, \tag{14}$$

where

$$[v_{i,j}]_{\substack{i=0,3,4\\ j=1,\ldots,4}} = \begin{bmatrix} \frac{x_2 x_5}{x_3 x_4} \cdot y & \frac{x_4 x_6}{x_3} \cdot y^2 & \frac{x_1 x_3}{x_4} \cdot y^2 & -\frac{x_1 x_6}{x_3 x_4} \cdot y^3 \\ \frac{x_1 x_4}{x_2 x_3} \cdot y & \frac{x_2}{x_3 x_6} \cdot y^2 & \frac{x_3 x_5}{x_2 x_6} \cdot y^2 & -\frac{x_5}{x_2 x_3} \cdot y^3 \\ \frac{x_3 x_6}{x_4 x_5} \cdot y & \frac{x_5}{x_1 x_4} \cdot y^2 & \frac{x_2 x_4}{x_1 x_5} \cdot y^2 & -\frac{x_2}{x_4 x_5} \cdot y^3 \end{bmatrix}. \tag{15}$$

Taking the power expansion of the right hand fraction in terms of $v_{i,j}$, the only terms that give a contribution are products $v^n = \prod_{i,j} v_{i,j}^{n_{i,j}}$ that are independent of the x_i. The ring of products of $v_{i,j}$ which do not contain x_i is $\mathbf{C}[r_i]$ where (see appendix for description of method for finding the r_k)

$$\begin{aligned} r_1 &= v_{1,4} v_{2,3} v_{3,3} = -y^7 = -\phi, \\ r_2 &= v_{1,2} v_{2,3} v_{3,4} = -y^7, \\ r_3 &= v_{1,3} v_{2,4} v_{3,3} = -y^7, \\ r_4 &= v_{1,2} v_{2,2} v_{2,3} v_{3,1} = y^7 = \phi, \\ r_5 &= v_{1,3} v_{2,1} v_{3,2} v_{3,3} = y^7 = \phi, \\ r_6 &= v_{1,1} v_{2,1} v_{2,3} v_{3,1} v_{3,3} = y^7 = \phi. \end{aligned} \tag{16}$$

Instead of evaluating the sum over v^n, we may now evaluate the sum over r^m including as weights the number of times the term $r^m = v^n$ occurs. This makes

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the integral, using the appropriate correspondence between m and n,

$$\begin{aligned}
\int_\Gamma \Psi(\phi) &= \sum_{\substack{(m_i)\in \mathbf{N}_0^6 \\ m=\sum_i m_i}} (-1)^{m_1+m_2+m_3}\phi^m \prod_i \binom{n_i}{n_{i,1}, n_{i,2}, n_{i,3}, n_{i,4}} \\
&= \sum_{\substack{m_1,m_6,u_1,u_2\in \mathbf{N}_0 \\ m=m_1+m_6+u_1+u_2}} (-1)^{m_1}\phi^m \cdot \frac{m!}{m_1!m_6!u_1!u_2!(m-u_1)!(m-u_2)!} \times \\
&\quad \times \sum_{m_2+m_4=u_1} (-1)^{m_2} \frac{(m+m_4+m_6)!}{m_2!m_4!(m_4+m_6)!} \cdot \sum_{m_3+m_5=u_2} (-1)^{m_3} \frac{(m+m_5m_6)!}{m_3!m_5!(m_5+m_6)!} \\
&= \sum_{\substack{m_1,m_6,u_1,u_2\in \mathbf{N}_0 \\ m=m_1+m_6+u_1+u_2}} (-1)^{m_1}\phi^m \cdot \binom{m}{u_1}^2 \binom{m}{u_2}^2 \binom{m+m_6}{m} \binom{m+m_6}{m_1, u_1+m_6, u_2+m_6} \\
&= 1 + 5\phi + 109\phi^2 + 3317\phi^3 + 121501\phi^4 + \cdots
\end{aligned} \tag{17}$$

This function, f_0, should be a solution to a Picard–Fuchs equation given by $\sum_{i=0}^4 A_i D_\phi f_0(\phi) = 0$, where $D_\phi = \phi(d/d\phi)$ and A_i are polynomials in ϕ with $A_i(0) = 0$ for $i < 4$. Entering general polynomials for A_i, we find a solution for $\deg A_i = 5$:

$$\begin{aligned}
\sum_{i=0}^{4} A_i D_\phi^i &= (1 - 57\phi - 289\phi^2 + \phi^3)(\phi - 3)^2 D_\phi^4 \\
&\quad + 4\phi(\phi - 3)(85 + 867\phi - 149\phi^2 + \phi^3)D_\phi^3 \\
&\quad + 2\phi(-408 - 7597\phi + 2353\phi^2 - 239\phi^3 + 3\phi^4)D_\phi^2 \\
&\quad + 2\phi(-153 - 4773\phi + 675\phi^2 - 87\phi^3 + 2\phi^4)D_\phi \\
&\quad + \phi(-45 - 2166\phi + 12\phi^2 - 26\phi^3 + \phi^4).
\end{aligned} \tag{18}$$

This is the so called Picard–Fuchs operator.

Solving for $f_1(\phi) = f_0(\phi) \cdot (g(\phi) + \log\phi)$, we get $g(\phi) = \alpha + 14\phi + 287\phi^2 + \cdots$, where α is a constant. The natural coordinate is $t = g(\phi) + \log\phi$ or $q = e^t = c_2(\phi + 14\phi^2 + 385\phi^3 + \cdots)$ where $c_2 = e^\alpha$.

We then calculate the Yukawa coupling. This is a symmetric 3-tensor on the parameter space, $\mathbf{P}^1$, which will be globally defined but with poles. The Yukawa coupling is given by ([9],[3])

$$\begin{aligned}\kappa_{ttt} &= \left(\frac{d\log\phi}{dt}\right)^3 \kappa_{\log\phi\log\phi\log\phi} \\ &= \left(\frac{d\log\phi}{dt}\right)^3 \int_{M_\phi} \hat{\Omega}\wedge\nabla^3_{\phi\frac{d}{d\phi}}\hat{\Omega} \\ &= \left(\frac{d\log\phi}{dt}\right)^3 1 f_0(\phi)^2 \int_{M_\phi} \Omega\wedge\nabla^3_{\phi\frac{d}{d\phi}}\Omega.\end{aligned} \tag{19}$$

To move f_0 to outside the differential, we use Griffiths transversality property which implies that $\Omega\wedge\nabla^i\Omega = 0$ for $i < 3$.

The term $\int_{M_\phi}\Omega\wedge\nabla^3_{\phi dd\phi}\Omega$ satisfies a differential equation ([9]):

$$\phi\frac{d}{d\phi}\log\left(\int_{M_\phi}\Omega\wedge\nabla^3_{\phi\frac{d}{d\phi}}\Omega\right) = -A_3 2A_4. \tag{20}$$

This gives us

$$\int_{M_\phi}\Omega\wedge\nabla^3_{\phi\frac{d}{d\phi}}\Omega = \frac{c_1(3-\phi)}{1-57\phi-289\phi^2+\phi^3} \tag{21}$$

for some constant c_1. The denominator may be seen to have zeros at three points in the parameter space. These are the points where the manifold has singularities: where our particular special line in the bigger parameter space $\mathbf{P}^2$ intersects other special lines, and, hence, has an additional double point coming from the seven extra double points on X_y.

The final step is to express κ_{ttt} in terms of q. Using the power series expansion $q = q(\phi)$ and its inverse series giving $\phi = \phi(q)$, and $\frac{d\log\phi}{dt} = \frac{q}{\phi}\frac{d\phi}{dq}$, we may express κ_{ttt} as

$$\begin{aligned}\kappa_{ttt} &= \left(\frac{q}{\phi(q)}\frac{d}{dq}\phi(q)\right)^3 \frac{1}{f_0(\phi(q))^2}\cdot\frac{c_1(3-\phi)}{1-57\phi-289\phi^2+\phi^3} \\ &= c_1\left(3+14\frac{q}{c_2}+714\left(\frac{q}{c_2}\right)^2+24584\left(\frac{q}{c_2}\right)^3+906122\left(\frac{q}{c_2}\right)^4+\cdots\right) \\ &= c_1\left(3+14\frac{1^3(\frac{q}{c_2})^1}{1-(\frac{q}{c_2})^1}+\frac{714-14c_2}{8}\frac{2^3(\frac{q}{c_2})^2}{1-(\frac{q}{c_2})^2}+\cdots\right).\end{aligned} \tag{22}$$

In order that there be only non-negative integer coefficients in the last line, we set $c_2 = 1$. Putting $c_1 = 2m$, we get

$$\kappa_{ttt} = m\cdot\left(6+28\tfrac{q}{1-q}+175\tfrac{2^3q^2}{1-q^2}+1820\tfrac{3^3q^3}{1-q^3}+28294\tfrac{4^3q^4}{1-q^4}+\cdots\right). \tag{23}$$

The actual value of m cannot be seen from this series alone. However, m is supposed to have a fixed value as determined by the value of the Yukawa coupling.

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PROPOSITION 6. *The manifold M_ϕ has maximally unipotent monodromy around $\phi = 0$, the Picard–Fuchs equation is given by (18). Assuming $c_2 = 1$ and $c_1 = 2m$, the mirror has degree $6m$, and the rational curve count is $28m$ lines, $175m$ conics, $1820m$ cubics, etc.*

As the general X_A that was initially assumed to be the mirror, has degree 14, and the first term of the q-series of κ_{ttt} gives the degree of the mirror to be a multiple of 6, this cannot be the case. However, the point $\phi = \infty$ remains to be checked. There is another striking observation:⋆ the Picard–Fuchs equation is exactly the same as for the A-model of $G(2, 7) \subset \mathbf{P}^{20}$ intersected by a general $\mathbf{P}^{13}$ ([2]). In this case, $m = 7$.

6. The Pfaffian Quotient Near $\phi = \infty$

Initially, the Picard–Fuchs equation seems to be regular at infinity, which would be most surprising as M_∞ degenerates into 14 3-planes just like M_0. However, global sections of the canonical bundle $\Gamma(\omega_{M_\phi})$ may be viewed as a line-bundle on the parameter space, and as such it is isomorphic to $\mathcal{O}_{\mathbf{P}^1}(1)$. To see this, recall that the global section Ω was of degree -7 in y, hence, degree -1 in ϕ. In order to get a global section of the canonical bundle near $\phi = \infty$, one should use $\tilde{\Omega} = \phi \cdot \Omega$. This modification and changing coordinate to $\tilde{\phi} = 1/\phi$ amounts to the change $D_\phi \mapsto -D_{\tilde{\phi}} - 1$ in the Picard–Fuchs operator, making it

$$\begin{aligned}\sum_{i=0}^{4} \tilde{A}_i D_{\tilde{\phi}}^i = {} & (1 - 289\tilde{\phi} - 57\tilde{\phi}^2 + \tilde{\phi}^3)(1 - 3\tilde{\phi})^2 D_{\tilde{\phi}}^4 \\ & + 4\tilde{\phi}(3\tilde{\phi} - 1)(143 + 57\tilde{\phi} - 87\tilde{\phi}^2 + 3\tilde{\phi}^3) D_{\tilde{\phi}}^3 \\ & + 2\tilde{\phi}(-212 - 473\tilde{\phi} + 725\tilde{\phi}^2 - 435\tilde{\phi}^3 + 27\tilde{\phi}^4) D_{\tilde{\phi}}^2 \\ & + 2\tilde{\phi}(-69 - 481\tilde{\phi} + 159\tilde{\phi}^2 - 171\tilde{\phi}^3 + 18\tilde{\phi}^4) D_{\tilde{\phi}} \\ & + \tilde{\phi}(-17 - 202\tilde{\phi} - 8\tilde{\phi}^2 - 54\tilde{\phi}^3 + 9\tilde{\phi}^4).\end{aligned} \tag{24}$$

We now see that the monodromy is maximally unipotent around $\phi = \infty$.

We may now proceed as for the previous case, but calculating $\tilde{f}_0$ from the Picard–Fuchs equation rather than the opposite. This gives a Yukawa-coupling in terms of $\tilde{q}$:

$$\kappa_{\tilde{t}\tilde{t}\tilde{t}} = \tilde{c}_1 \left(1 + 42\frac{\tilde{q}}{\tilde{c}_2} + 6958\left(\frac{\tilde{q}}{\tilde{c}_2}\right)^2 + \cdots\right), \tag{25}$$

where $\tilde{c}_1 = c_1 = 2m$: just enter $\phi = \infty$ into the Yukawa-coupling 21 after multiplying

⋆ This observation was made by Duco van Straten.

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with ϕ^2 owing to the transition to $\tilde{\Omega} = \phi \cdot \Omega$. Putting $\tilde{c}_2 = 1$, we get

$$\kappa_{\tilde{t}\tilde{t}\tilde{t}} = m\Big(2 + 84\tfrac{\tilde{q}}{1-\tilde{q}} + 1729\tfrac{2^3\tilde{q}^2}{1-\tilde{q}^2} + 83412\tfrac{3^3\tilde{q}^3}{1-\tilde{q}^3} + 5908448\tfrac{4^3\tilde{q}^4}{1-\tilde{q}^4} + \cdots\Big). \tag{26}$$

PROPOSITION 7. *The manifold M_ϕ has maximally unipotent monodromy around $\phi = \infty$, the Picard–Fuchs equation is given by (24). Assuming $\tilde{c}_2 = 1$ and $m = 7$ to give the mirror degree* 14, *the rational curve count is* 588 *lines,* 12103 *conics,* 583884 *cubics, etc.*

The lines on the general Pfaffian have been counted by Ellingsrud and Strømme and is 588 (private communication).

7. The Grassmannian G(2,7) Quotient

Due to the equality between the B-model Picard–Fuchs operator at $\phi = 0$ for the Pfaffian quotient and the A-model Picard–Fuchs operator for an intersection of $G(2,7) \subset \mathbf{P}^{20}$ with a general $\mathbf{P}^{13}$, it is natural to take a closer look at $G(2,7)$. In particular, it is possible to perform an orbifold construction on this which is 'dual' to that on the Pfaffian.

The Pfaffian quotient was constructed from an intersection between the general Pfaffian in $\mathbf{P}^{20}$ and a special family of 6-planes: $\mathbf{P}^6_y$. We may take the family $\mathbf{P}^{13}_y$ of 13-planes in $P(E^\vee \wedge E^\vee) \cong \mathbf{P}^{20}$ dual to $\mathbf{P}^6_y$, and take $Y_y = G(2,7) \cap \mathbf{P}^{13}_y \subset \mathbf{P}^{20}$. Again, we have a group action by $\tau : x_{i,j} \mapsto x_{i,j}w^{i+j}$ which restricts to this intersection, and the natural coordinate being $\phi = y^7$. The τ-fixed points, $e_i \wedge e_{i+3}$, are double-point singularities, as are the images under τ of $(e_{i+1} - e_{i-1}) \wedge ((e_{i+3} - e_{i-3}) + y \cdot (e_{i-2} - e_{i+2}))$. Let W_y be the resolved quotient $\widetilde{Y_y/\tau}$. This is a Calabi–Yau manifold ([2]). I will proceed without going into the resolution as this has no impact on the B-model.

To summarize the definition (in matrix notation):

DEFINITION 8. *Let $U_y = [x_{i,j}]_{i,j \in \mathbf{Z}_7}$, $x_{i,j} + xj$, $i = 0$, be the skew-symmetric matrix with $x_{i+4,i-4} = -yx_{i+1,i-1}$. (This amounts to specializing to the 13-planes $\mathbf{P}^{13}_y$ dual to the $\mathbf{P}^6_y$ used for the Pfaffians, and giving a specific coordinate system.) Let Y_y denote the rank 2 locus of U_y in $\mathbf{P}^{13}$. Divide this out with the group action generated by $\tau : x_{i,j} \mapsto x_{i,j}w^{i+j}$, take a minimal resolution of this, parametrize the resulting family of threefolds by $\phi = y^7$ denoting it W_ϕ.*

In order to get an expression for the canonical form, we may look at an affine piece of $G(2,7)$ given by $u_1 \wedge u_2$ where $u_i = [u_{i,j}]$, $i = 1, 2$, $j = 0, \ldots, 6$, and where $u_{1,0} = u_{2,2} = 1$, $u_{1,2} = u_{2,0} = 0$. The defining equations then become

$$u_{1,i}u_{2,i+1} - u_{1,i+1}u_{2,i} = y \cdot (u_{1,i-2}u_{2,i+3} - u_{1,i+3}u_{2,i-2}), \quad i \in \mathbf{Z}_7. \tag{27}$$

Now, as we have a complete intersection, we may define the canonical form Ω as the

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residue of

$$\Psi = \frac{\bigwedge_{i=1,3,4,5,6} du_{1,i} \wedge du_{2,i}}{(2\pi i)^{10} \prod_{i\in\mathbf{Z}_7}(u_{1,i}u_{2,i+1} - u_{1,i+1}u_{2,i} - y\cdot(u_{1,i-2}u_{2,i+3} - u_{1,i+3}u_{2,i-2}))}. \tag{28}$$

For $y = 0$, the variety decomposes. One of the components may be given in affine coordinates by $u_1 \wedge u_2$ where $u_1 = [1, 0, 0, 0, 0, 0, 0]$, $u_2 = [0, 0, 1, u_{2,3}, u_{2,4}, u_{2,5}, 0]$. We may define the 3-cycle $\gamma_0(0)$ by $|u_{2,j}| = \varepsilon$ for $j = 3, 4, 5$, and extend this to a neighborhood: say, $|y| < \delta$. As for the Pfaffian, we will rather use the 10-cycle Γ in $\mathbf{P}^{20}$ defined by $|u_{i,j}| = \varepsilon_{i,j}$, where again $u_i = [u_{i,j}]$ with $u_{1,0} = u_{2,2} = 1$, $u_{1,2} = u_{2,0} = 0$. The actual choices of δ the $\varepsilon_{i,j}$ will be made so as to make the quotients $v_{i,j}$ defined below sufficiently small, but will otherwise be of no importance.

We may now rewrite the residual form so as to suite our purpose of evaluating it as a power series in y:

$$\Psi = \frac{1}{\prod_i(1 - \sum_j v_{i,j})} \cdot \bigwedge_{i=1,3,4,5,6} \frac{du_{1,i} \wedge du_{2,i}}{(2\pi i)^2 u_{1,i}u_{2,i}}, \tag{29}$$

where

$$\begin{aligned}
&v_{1,1} = -y\cdot\frac{u_{1,5}u_{2,3}}{u_{2,1}}, \quad v_{1,2} = y\cdot\frac{u_{1,3}u_{2,5}}{u_{2,1}}\\
&v_{2,1} = -y\cdot\frac{u_{1,6}u_{2,4}}{u_{1,1}}, \quad v_{2,2} = y\cdot\frac{u_{1,4}u_{2,6}}{u_{1,1}}\\
&v_{3,1} = y\cdot\frac{u_{2,5}}{u_{1,3}}\\
&v_{4,1} = \frac{u_{1,3}u_{2,4}}{u_{1,4}u_{2,3}}, \quad v_{4,2} = y\cdot\frac{u_{1,1}u_{2,6}}{u_{1,4}u_{2,3}}, \quad v_{4,3} = -y\cdot\frac{u_{1,6}u_{2,1}}{u_{1,4}u_{2,3}}\\
&v_{5,1} = \frac{u_{1,4}u_{2,5}}{u_{1,5}u_{2,4}}, \quad v_{5,2} = -y\cdot\frac{1}{u_{1,5}u_{2,4}}\\
&v_{6,1} = \frac{u_{1,5}u_{2,6}}{u_{1,6}u_{2,5}}, \quad v_{6,2} = y\cdot\frac{u_{1,3}u_{2,1}}{u_{1,6}u_{2,5}}, \quad v_{6,3} = -y\cdot\frac{u_{1,1}u_{2,3}}{u_{1,6}u_{2,5}}\\
&v_{7,1} = y\cdot\frac{u_{1,4}}{u_{2,6}}.
\end{aligned} \tag{30}$$

In order that the power series expansion converge, we need $\sum_j |v_{i,j}| < 1$. In order to obtain this, set $\varepsilon_{1,i}/\varepsilon_{2,i} < \varepsilon_{1,j}/\varepsilon_{2,j}$ for $3 \leqslant i < j \leqslant 6$, and δ sufficiently small.

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If we look at the ring generated by the $v_{i,j}$, the subring of elements that do not contain terms $u_{i,j}$ is $\mathbf{C}[r_i]$, where

$$
\begin{aligned}
r_1 &= v_{1,1}v_{2,1}v_{3,1}v_{4,2}v_{5,2}v_{6,2}v_{7,1} = y^7 = \phi \\
r_2 &= v_{1,2}v_{2,1}v_{3,1}v_{4,3}v_{5,2}v_{6,1}v_{6,3}v_{7,1} = -y^7 = -\phi \\
r_3 &= v_{1,1}v_{2,1}v_{3,1}v_{4,1}v_{4,3}v_{5,1}v_{5,2}v_{6,1}v_{6,3}v_{7,1} = y^7 = \phi \\
r_4 &= v_{1,1}v_{2,2}v_{3,1}v_{4,1}v_{4,3}v_{5,2}v_{6,3}v_{7,1} = -y^7 = -\phi.
\end{aligned}
\tag{31}
$$

For any monomial $r^m = \prod_i r_i^{m_i}$, the corresponding $u^n = \prod_{i,j} u_{i,j}^{n_{i,j}}$ appears $\prod_i \binom{n_i}{n_{i,1},\dots}$ number of times, $n_i = \sum_j n_{i,j}$. The power series expansion for $f_0 = \int_{\gamma_0} \Omega$ will then be given by

$$
\begin{aligned}
\int_\Gamma \Psi &= \sum_{\substack{(m_i)\in\mathbf{N}_0^4 \\ m=\sum_i m_i}} (-1)^{m_2+m_4}\phi^m \cdot \prod_{i=1}^{7} \binom{n_i}{n_{i,1},\dots} \\
&= \sum_{\substack{(m_i)\in\mathbf{N}_0^4 \\ m=\sum_i m_i}} (-1)^{m_2+m_4}\phi^m \cdot \binom{m}{m_2}\binom{m}{m4}\binom{m+m_3}{m} \\
&\quad \cdot \binom{m+m_2+m_3}{m_1, m_2+m_3, m_2+m_3+m_4}\binom{m+m_3+m_4}{m_1, m_3+m_4, m_2+m_3+m_4} \\
&= 1 + 5\phi + 109\phi^2 + 3317\phi^3 + 121501\phi^4 + 4954505\phi^5 + \cdots
\end{aligned}
\tag{32}
$$

which may be recognized as exactly the same series as for the Pfaffian quotient. Hence, the Picard–Fuchs operator etc. all become the same as for the Pfaffian quotient.

The global sections of the canonical sheaf again forms a $\mathcal{O}_{\mathbf{P}^1}(1)$ line-bundle on the $\mathbf{P}^1$ parameter space. Hence, this grassmannian quotient has the same Picard–Fuchs operator at $\phi = \infty$ as the Pfaffian quotient.

PROPOSITION 9. *The B-models of M_y and W_y have the same Picard–Fuchs operator. Hence, the Yukawa-coupling may at most differ by a factor.*

Of course, it is natural to conjecture that the Yukawa-couplings are equal, making the B-models isomorphic.

8. Comments on the Results

Apparently, there is a strong relation between the varieties defined by the Pfaffians and the Grassmannian $G(2, 7)$. The B-models of the M_y and W_y are isomorphic, and according to mirror symmetry and assuming that we actually have the mirrors, the A-models of the general Pfaffian and general $G(2,7)$ sections should also be isomorphic, and vice versa. It may of course be possible that we have found models with the same B-model but different A-models, in which case they would not be mirrors. Assuming that we actually have mirror symmetry, it would appear that varying the complex structure on M_y or W_y leads to a transition from the Kähler structure on the Pfaffian section X_A to that of the Grassmannian section Y_A.

CONJECTURE 10. *The pairs $M_y + W_y$ is the mirror family of $X_A + Y_A$ where M_y and W_y (resp. X_A and Y_A) form different parts of the A-model (Kähler) moduli space.*

Such transitions are known using Landau–Ginzburg models to model the analytic continuation to 'negative' Kähler structures ([6], [16]).

It is worth noting that the smooth varieties X_A and Y_A cannot be birationally equivalent. As $h^{1,1} = 1$, this has a unique positive integral generator (the dual of a line); if birational, these two must correspond up to a rational factor. Integrating the third power of this over the variety gives the degree; the ratio of the degrees would then be the third power of a rational number, which is not possible for $42/14 = 3$.

For the $\mathbf{P}^2$ families of varieties X_Y and Y_Y, however, a birational map has been found ([11]).

Appendix: Finding Generators of Subring

Assume that we have a list of variables x_i, $i = 1, \dots, n$, and Laurent-monomials $v_j = \alpha_j \prod_i x_i^{a_{j,i}}$, $j = 1, \dots, m$, with $a_{j,i} \in \mathbf{Z}$. We wish to find $r_k = \prod_j v_j^{b_{k,j}}$ such that $r_k(x)$ is independent of x_i and generates the ring of polynomials in $v_j(x)$ independent of x_i (or some extension of this ring).

An optimistic approach is simply the find a set of linearly independent vectors with integer coefficients generating the kernel of the matrix $A = [a_{j,i}] : \mathbf{C}^m \to \mathbf{C}^n$. In the nicest cases, in particular in the two cases that we are treating, one may even find such vectors with non-negative integer coefficients. If these vectors are $b_k = [b_{k,j}]_j$, $k = 1, \dots, m-n$, define $r_k = \prod_j v_j^{b_{k,j}}$.

More generally, there is a risk that some of the r_k will not be monomials, but Laurent monomials: some $b_{k,j}$ will be negative. These can still be used as generators, but in the sum over monomials in r_k, only those which are monomials in v_j, ie. without negative powers of v_j, are considered.

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References

1. Batyrev, V. V. and Dais, D. I.: Strong McKay correspondence, string-theoretic Hodge numbers and mirror symmetry, *Topology* **35**, (4) (1996), 901–929.
2. Batyrev, V. V., Ciocan-Fontanine, I., Kim, B. and van Straten, D.: Conifold Transitions and Mirror Symmetry for Calabi–Yau Complete Intersections in Grassmannians, *Nucl. Phys. B* **514**, (3), (1998), 640–666.
3. Candelas, P., de la Ossa, X. C., Green, P. S. and Parkes, L.: A pair of Calabi–Yau manifolds as an exactly soluble superconformal theory, In: *Essays on Mirror Manifolds*, International Press, Cambridge, MA, 1992, pp. 31–95.
4. Clemence, C. H.: Double solids, *Adv. Math.* **47**, (1983), 107–230.
5. Friedman, R.: Simultaneous resolution of threefold double points, *Math. Ann.* **274**, (1986), 671–689.
6. Greene, B. R.: String theory on Calabi–Yau manifolds, hep-th/9702155 (23 February 1997).
7. Hübsch, T.: Calabi–Yau manifolds, a bestiary for physicists, World Scientific, Singapore, 1992.
8. Miele, M.: Klassifikation der Durchschnitte Heisenberg-invarianter Systeme von Quadriken in P_6, Erlangen: Univ. Erlangen-Nuernberg, iii, 95 S. (1993).
9. Morrison, D. R.: Picard–Fuchs equations and mirror maps for hypersurfaces, In: Essays on Mirror Manifolds; Cambridge, MA: International Press, 1992, pp. 241–264.
10. Morrison, D. R.: Mathematical aspects of mirror symmetry, In: *Complex Algebraic Geometry*, Lectures of a summer program, Park City, UT, 1993, Amer. Math. Soc., Providence, RI. IAS/Park City Math. Ser. 3, 1997, pp. 267–340.
11. Popescu, S. and Gross, M.: Calabi–Yau 3-folds and moduli of abelian surfaces I, to appear.
12. Ran, Z.: Deformations of Calabi–Yau kleinfolds, In: *Essays on Mirror Manifolds*, International Press, Cambridge, MA, 1992, pp. 241–264.
13. Roan, S.-S.: On the generalization of Kummer surfaces, *J. Differential Geom.* **30** (1989), 523–537.
14. Roan, S.-S.: Minimal resolutions of Gorenstein orbifolds in dimension three, *Topology* **35** (2) (1996).
15. Tian, G.: Smoothing 3-folds with trivial canonical bundle and ordinary double points, In: *Essays on Mirror Manifolds*, International Press, Cambridge, MA, 1992, pp. 241–264.
16. Witten, E.: Two-dimensional gravity an intersection theory on moduli space, *Surveys in Differential Geom.* **1** (1991), 243–310.
17. Macaulay 2, computer program by D. Grayson, M. Stillman, http://www.math.uiuc.edu/Macaulay2/.