# THE PHYSICAL MEANING OF THE "BOOST-ROTATION SYMMETRIC" SOLUTIONS WITHIN THE GENERAL INTERPRETATION OF EINSTEIN'S THEORY OF GRAVITATION

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Abstract. The answer to the question, what physical meaning should be attributed to the so-called boost-rotation symmetric exact solutions to the field equations of general relativity, is provided within the general interpretation scheme for the "theories of relativity", based on group theoretical arguments, and set forth by Erich Kretschmann already in the year 1917.

## 1. Introduction

In the same year 1915, when Einstein and Hilbert [1, 2] gave the final mathematical expression to the long efforts done by Einstein for finding a generally covariant theory of gravitation based on the absolute differential calculus of Ricci and Levi Civita [3], Erich Kretschmann published a long article [4], entitled "Über die prinzipielle Bestimmbarkeit der berechtigten Bezugssysteme beliebiger Relativitätstheorien", in which a minute analysis of the relation between observation and mathematical structure in a theory possessing a generic postulate of relativity is developed. No wonder then, if two years later, with the paper [5] entitled "Über den physikalischen Sinn der Relativitätspostulate; A. Einsteins neue und seine ursprüngliche Relativitätstheorie", the same author produced an analysis of the relation between the "special" and the "general" theory of relativity that had to become a source of permanent enlightenment for the relativists. The analysis relies on a fundamental distinction between the group of invariance and the group of covariance of a theory, that appears to have escaped the attention both of Einstein and of Hilbert. A faithful account of Kretschmann's result was given by Philipp Frank's review [6] of the paper, that reads, in English translation:

"Einstein understands, under his general principle of relativity, the injunction that the laws of nature must be expressed through equations that are covariant with respect to arbitrary coordinate transformations. The Author shows now that any natural phenomenon obeying any law can be described by generally covariant equations. Therefore the existence of such equations does not express any physical property. For instance the uniform propagation of light in a space free from gravitation can be expressed also in a covariant way. However, there is a representation of the same phenomena that admits only a more restricted group (the Lorentz transformations). This group, that cannot be further restricted by any representation of the phenomena, is characteristic of the system under question. The invariance with respect to it is a physical property of the system and, in the sense of the Author, it represents the postulate of relativity for the corresponding domain of phenomena.

In Einstein's general theory of relativity, through appropriate choice of the coordinates, the field equations can be converted in a form that is no longer covariant under the group of coordinate transformations. The Author provides a series of examples of such conversions. But the equations converted in this way in general no longer admit any group, and in this sense Einstein's theory of general relativity is an "absolute theory", while the special theory of relativity satisfies the postulate of relativity for the Lorentz transformations also in the sense of the Author."

Kretschmann's viewpoint, that deprives the coordinates and the covariance under general transformations of physical meaning in a nearly complete way[1], was recognised correct by Einstein [8], and has become part and parcel of the present day understanding of "general relativity": coordinates, and the values that the components of tensorial entities may assume with respect to a given chart, do not matter; the objective physical content of the theory is written in the geometry of the manifold, and it can be read only through the invariant quantities associated with the latter. The same acceptance was met with by Kretschmann's way of assessing the "relativity content" of a given theory. For him, it should not be ascertained through the group of covariance allowed by the particular expression adopted for writing the equations of that theory, but through its group of invariance, meant to be "a physical property of the system", directly inscribed by the Killing vectors in the intrinsic, geometric structure of the manifold.

Kretschmann's analysis [5], however, only considered the group of invariance of a general solution to the field equations of "general relativity", that contains only the identity, and the group of invariance for the particular solution of the same theory that occurs when $R_{iklm} = 0$, namely, the inhomogeneous Lorentz group. Since that time, solutions of Einstein's theory of 1915 whose groups of invariance correspond to a "relativity content" intermediate between the above mentioned extremes have been found, and investigated at length by the relativists. To these solutions belong the so called "boost rotation symmetric" solutions. Scope of the present paper is

[1]A residual physical meaning is however left. In facts Kretschmann embraces Einstein's view [7] that the description of the whole physical experience can be reduced to accounting for spatiotemporal coincidences. Therefore coordinates have no physical meaning in themselves, but of course a restriction of physical origin on the admissible coordinate transformations is mandatory: since a coordinate system must faithfully absolve the physical function of reckoning the spacetime coincidences, it must preserve the individuality of the single event. To this end, only one to one coordinate transformations can be allowed for.

the assessment of the physical meaning of these solutions as dictated, in keeping with Kretschmann's idea, by the geometric structure of their manifolds.

## 2. The "boost-rotation symmetric" solutions

The perusal of the literature dealing with the "boost-rotation symmetric" solutions, spanning a time interval of four decades, shows that all the vacuum solutions associated with nonspinning sources [9], [10], [11], [12], [13, 14, 15], [16, 17], [18], (see also [19]) can be generated in one and the same way by starting from some solution belonging to the class found long ago by Weyl [20] and by Levi-Civita [21]. For the convenience of the reader, the definition of the latter class of solutions in the canonical coordinates introduced by Weyl is reported in Appendix A. For instance, the solution like the one reported in [17] can be obtained by choosing the function $\psi$, that fulfils the "potential" equation (A.2), in such a way that

$$\psi = \frac{1}{2}\ln\left[(r^2+z^2)^{\frac{1}{2}}+z\right]+\frac{1}{2}\ln\frac{r_1+r_2-2l}{r_1+r_2+2l}, \tag{2.1}$$

where $r_i = [(z-z_i)^2+r^2]^{\frac{1}{2}}$, and the positive constants $z_1$ and $z_2$ are so chosen that $z_2 - z_1 = 2l > 0$. If $\psi$ were a Newtonian potential, its particular expression (2.1) would correspond to the sum of the potential of two rods both endowed with linear mass density $\sigma = 1/2$, and lying on the $z$-axis. One of the rods extends itself from $z = 0$ to $z = -\infty$, while the other one covers the segment between $z_1$ and $z_2$. But this imagery is just a "Bildraum" deception for, if only the semi-infinite rod were present, the metric generated by the Weyl method would be such that $R_{iklm} = 0$, while, if only the finite rod with $z_1 < z < z_2$ were present, the solution would be in one to one correspondence with the original [22] Schwarzschild solution[2] for a mass $m = l$.

In order to obtain the "boost-rotation symmetric" solution corresponding to this Weyl field, one goes over to the primed cylindrical polar coordinates $x'^1 = z'$, $x'^2 = r'$, $x'^3 = \varphi'$, $x'^4 = t'$ from the unprimed, canonical coordinates specified in Appendix A, through the coordinate transformation

$$z' = \pm[(r^2+z^2)^{\frac{1}{2}}+z]^{\frac{1}{2}}\cosh t, \tag{2.2}$$

$$r' = [(r^2+z^2)^{\frac{1}{2}}-z]^{\frac{1}{2}}, \tag{2.3}$$

$$t' = [(r^2+z^2)^{\frac{1}{2}}+z]^{\frac{1}{2}}\sinh t, \tag{2.4}$$

$$\varphi' = \varphi. \tag{2.5}$$

We note in passing that this transformation neither conforms to Einstein's mentioned injunction that coordinate transformations should be one-to-one, in order to preserve the identity of the events, nor obeys the prescriptions

[2] whose manifold, at variance with the "Schwarzschild" solution referred to in the literature, that was actually proposed by Hilbert [23], does not cover the "inner region" of the latter.

by Hilbert and Lichnerowicz about the admissible transformations of coordinates [23, 24]. In fact, besides the obvious doubling of the Weyl manifold due to the $\pm$ sign of (2.2), one notes that the events of the original manifold for which $t$ is finite and otherwise arbitrary, $r = 0$, $-\infty < z < 0$, in the primed coordinates all end up in the coordinate plane for which $z' = t' = 0$ in a way that only depends on $z$, but not on $t$. Therefore the transformation of Eqs. (2.2)-(2.5) loses track of the individuality of events as it is specified within the Weyl manifold. A glance to the drawings (a) and (b) of Figure 1

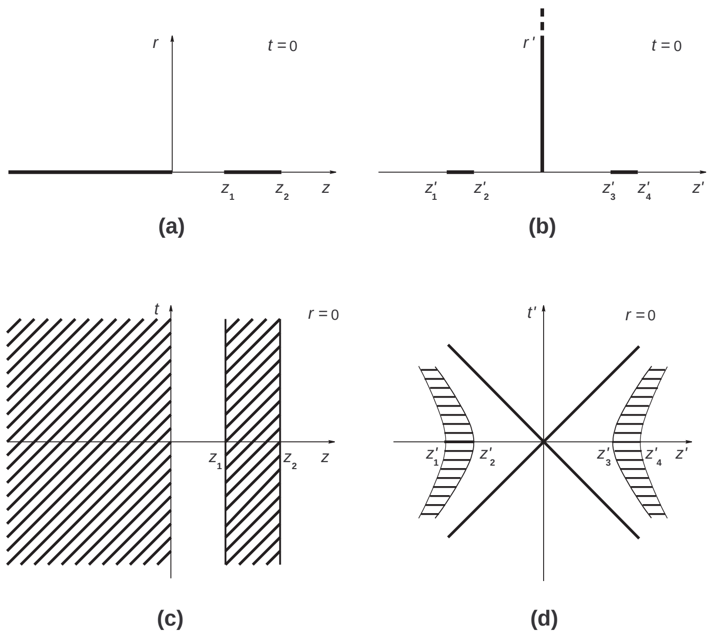


FIGURE 1. Left side: sketch, in canonical coordinates, of the "Newtonian sources" corresponding to the "potential" $\psi$ of Eq. (2.1). a) $z$, $r$ diagram for $t = 0$. c) $z$, $t$ diagram for $r = 0$. Right side: representation in the primed coordinates corresponding to the transformation of Eqs. (2.2)-(2.5). b) $z'$, $r'$ diagram for $t = 0$. d) $z'$, $t'$ diagram for $r = 0$.

shows that the semi-infinite rod should go in the entire plane $z' = 0$, while the finite rod $z_1 z_2$ is doubled in the mirror images $z'_1 z'_2$ and $z'_3 z'_4$. This is true for $t = 0$, and also for any value of $t$, but with a different scale along $z'$. When $t$ is finite, the shaded area in the left part of (c) goes in the origin of the $z'$, $t'$ diagram (d), while, when $t = \pm\infty$, it is spread on the diagonals of the same diagram. The strip on the right part of (c) goes instead in the two shaded areas of (d), delimited by two hyperbolae that, in the primed representation, asymptotically approach the two diagonals. Diagram (d) shows how the transformation (2.2)-(2.5) produces a doubling in the representation, since the whole $z$, $t$ plane, cut along the negative part of the $z$ axis, appears twice in the left and right quadrants of (d), in a way akin to the duplication of the original Schwarzschild manifold in the left and right quadrants of the Kruskal manifold [25, 26, 27]. Moreover, like in the latter case, the interval, when written in the primed coordinates, happens to be a solution of Einstein's equations not only within the left and right quadrants, but also in the upper and lower ones, i.e. the full diagram yields both a duplication and an extension of the original Weyl manifold. It is remarkable that the solution of the upper and of the lower quadrants could be obtained [30] also from a Weyl solution, by first subjecting it to the formal change

$$t \to iz, \ \ z \to it, \quad i = \sqrt{-1}, \tag{2.6}$$

that preserves the reality of the interval.

## 3. A matter of interpretation

When confronted with the diagrams of the left and of the right sides of Figure 1, one is awestruck by the mathematical beauty of the coordinate transformation that has brought the two standing rods of the Weyl solution, apparently two standing masses that no doubt need a strut to be held forever at rest despite their mutual gravitational pull [28, 29], into a bifurcate horizon and two masses executing hyperbolic motions independent of each other, of course thanks to struts providing the necessary push. This transformation is not a peculiarity that only applies to the Weyl solution defined by Eq. (2.1); it provides the cornerstone upon which all the "boost-rotation symmetric" solutions of Refs. [10]-[18] are built. However, both the static character of the originating Weyl metrics, and the particular time dependent behaviour in the left and right quadrants seen in the primed coordinates of diagram (d), are just a coordinate imagery, possibly a "Bildraum" deception, because, as taught long ago [5] by Kretschmann and Einstein [7], since the coordinates are nearly devoid of physical meaning, such is also the case for the expressions that a solution takes in a certain chart. We have to search for the physical meaning of a solution by studying its invariant features, in particular its group of invariance.

The Weyl-Levi Civita solutions are particular examples, endowed with axial symmetry, of the general class of static solutions. It is generally said in the textbooks that these solutions are invariantly defined by the existence of a timelike Killing vector $\xi^i$ that is also hypersurface orthogonal:

$$\xi_i\xi^i > 0, \quad \xi_{i;k} + \xi_{k;i} = 0, \quad \xi_{[i}\xi_{k,l]} = 0. \tag{3.1}$$

It must be noticed, however, that this definition is not stringent enough: a manifold for which $R_{iklm} = 0$ of course possesses a vector that fulfils (3.1), because, since the group of invariance of that manifold is the inhomogeneous Lorentz group of special relativity, it possesses an infinity of them. But, when $R_{iklm} \neq 0$, it generally happens that at each event equations (3.1) allow only for a unique way of defining the direction of the timelike, hypersurface orthogonal Killing vector. This uniqueness is crucial for the physical interpretation. When it occurs, the "relativity content" of the manifold is the following: the Killing vectors fulfilling (3.1) provide a one parameter group of invariance, and their hypersurface orthogonality yields a unique, intrinsic, absolute distinction between space and time, namely, provides a gravitational aether in which absolute space, absolute time, absolute rest are meaningful physical notions, since they are invariantly inscribed in the geometry of the manifold. In general relativity, only solutions endowed with this intrinsic structure can be properly named static. Weyl-Levi Civita solutions with a nonvanishing Riemann tensor are static in the sense defined above; the manifolds associated to them possess however a further symmetry, since their group of invariance is constituted by the two Killing vectors that define respectively the translation along absolute time and the spatial rotation around a given axis.

As a consequence, the physical reading of diagrams (a) and (c), and the physical reading of (b) and of the left and right quadrants of (d) cannot be but one and the same: in an absolute, invariant sense, we have to do with bodies at rest with respect to the manifold; despite their mutual gravitational pull, they are kept in such a condition by the existence of a well investigated [28, 29] strut between them.

## 4. The bifurcate horizon is singular in an invariant sense

One can object that, although the left and right quadrants of diagram (d) are no doubt static in the absolute sense explained above, hence cannot provide an idealised model for the process of emission and absorption of gravitational radiation by material bodies, the upper and lower quadrants are indeed time dependent in an absolute sense. In fact, on crossing the horizon by going from the left and right quadrants to the upper and the lower ones, the hypersurface orthogonal, timelike Killing vector becomes null and then spacelike. The upper and lower quadrants provide in fact two distorted copies of a time dependent solution endowed with cylindrical symmetry belonging to the class that Beck found [30] in 1925 from the Weyl-Levi Civita solutions through the formal change (2.6), and their intrinsic reading is completely different from the one that applies to the left and right quadrants.

It has been remarked above that the extension of diagram (d) is reminiscent of the duplication and extension of the original Schwarzschild solution that goes under the name of Kruskal [25, 26, 27]; for that extension, it has been proved already [31, 32] that a local, invariant, intrinsic singularity occurs when approaching the horizon. The same thing occurs with the extension of diagram (d). Since the singularity is defined in an invariant way, its existence can be conveniently ascertained by using Weyl's canonical coordinates.

The quantity under question is the norm $\alpha$ of the four acceleration

$$a^i \equiv \frac{\mathrm{d}u^i}{\mathrm{d}s} + \Gamma^i_{kl}u^k u^l \tag{4.1}$$

of a test particle whose world line is a line of absolute rest in the above explained sense. Besides being invariant, this quantity is intrinsic to the manifold, like it is the world line of absolute rest.

The nonvanishing components of $a^i$ for a worldline of rest in a Weyl-Levi Civita solution are

$$a^1 = \exp{(2\psi - 2\gamma)}\frac{\partial\psi}{\partial z},\ a^2 = \exp{(2\psi - 2\gamma)}\frac{\partial\psi}{\partial r}. \tag{4.2}$$

When calculating $\alpha \equiv (-a_i a^i)^{1/2}$ in the near proximity of the semi-infinite rod ($r \ll |z|$) we can neglect the contribution to $\psi$ coming from the finite rod $z_1 z_2$, since its presence cannot give rise to a divergence of $\alpha$. Then

$$e^{2\psi} \approx (r^2 + z^2)^{\frac{1}{2}} + z,\ e^{2\psi - 2\gamma} \approx 2(r^2 + z^2)^{\frac{1}{2}}, \tag{4.3}$$

and the relevant term of the squared norm of the acceleration defined above reads

$$\alpha^2 = e^{2\psi - 2\gamma}\left[\left(\frac{\partial\psi}{\partial z}\right)^2 + \left(\frac{\partial\psi}{\partial r}\right)^2\right] \approx \frac{1}{2\left[(r^2 + z^2)^{\frac{1}{2}} + z\right]}. \tag{4.4}$$

When $z$ is negative $\alpha$ diverges when the limit $r \to 0$ is taken, i.e. when the world line of absolute rest is drawn closer and closer to the horizon produced by the semiinfinite rod.

In the Kruskal manifold, a similar intrinsic singularity occurs when considering the norm of the four acceleration along a line of absolute rest located, in the left and right quadrants, at positions closer and closer to the bifurcate horizon, possibly to warn that it is not a good idea to envisage joining manifolds of different "relativity content", and that Schwarzschild's original manifold [22] is all what is allowed to provide a model for the spherically symmetric gravitational field of a particle.

The same occurrence happens with the "boost-rotation symmetric" manifolds. In this case too, one inclines to think that the singular behaviour of $\alpha$ under analogous circumstances is again there to spell the same kind of warning.

## 5. Conclusion

The standard view about the vacuum C-metric [16] and its relatives, as discussed e.g. in [9], [10], [11], [12], [13, 14, 15], [17], [18], assumes that the singularities representing the nonspinning masses of these vacuum solutions exhibit a uniformly accelerating motion relative to an inertial frame at infinity. This interpretation is problematic, since it relies on approximate, asymptotic group symmetries of the corresponding manifolds, while the exact Killing group symmetry that prevails everywhere in the submanifolds where the world lines of the masses are located shows that the nonspinning masses are at rest with respect to the latter, intrinsically static submanifolds in the invariant, absolute sense explained in Section 3. Moreover the submanifolds that contain the world lines of the masses are joined to the remaining parts of the manifolds at hypersurfaces that are singular in the invariant, local, intrinsic sense expounded in Section 4.

## Appendix A. Weyl's method of solution

In the static, axially symmetric case, despite the nonlinear structure of Einstein's field equations, Weyl succeeded in reducing the problem to quadratures through the introduction of his "canonical cylindrical coordinates". Let $x^0 = t$ be the time coordinate, while $x^1 = z$, $x^2 = r$ are the coordinates in a meridian half-plane, and $x^3 = \varphi$ is the azimuth of such a half-plane; then the line element of a static, axially symmetric field *in vacuo* can be tentatively written as:

$$\mathrm{d}s^2 = e^{2\psi}\mathrm{d}t^2 - \mathrm{d}\sigma^2,\ e^{2\psi}\mathrm{d}\sigma^2 = r^2\mathrm{d}\varphi^2 + e^{2\gamma}(\mathrm{d}r^2 + \mathrm{d}z^2); \tag{A.1}$$

the two functions $\psi$ and $\gamma$ depend only on $z$ and $r$. Remarkably enough, in the "Bildraum" introduced by Weyl $\psi$ fulfils the potential equation

$$\Delta\psi = \frac{1}{r}\left\{\frac{\partial(r\psi_z)}{\partial z} + \frac{\partial(r\psi_r)}{\partial r}\right\} = 0 \tag{A.2}$$

($\psi_z$, $\psi_r$ are the derivatives with respect to $z$ and to $r$ respectively), while $\gamma$ is obtained by solving the system

$$\gamma_z = 2r\psi_z\psi_r,\ \gamma_r = r(\psi_r^2 - \psi_z^2); \tag{A.3}$$

due to the potential equation (A.2)

$$\mathrm{d}\gamma = 2r\psi_z\psi_r\mathrm{d}z + r(\psi_r^2 - \psi_z^2)\mathrm{d}r \tag{A.4}$$

happens to be an exact differential.

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