The Physical Principles of the Quantum Theory.

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(Communicated by S. Chapman, F.R.S.—Received June 25, 1932.)

The object of this paper is to reformulate the principles of the quantum theory as a sequence of propositions which shall be either summary statements of standard experimental procedure or hypotheses concerning the results of experiment and having an immediate physical interpretation. It is shown that the standard process in micro-physics is a generalised spectral analysis, whose properties are simply expressible in symbolic form by means of projective or "idempotent" operators (Einzeloperatoren). It appears that only two hypotheses need be made and that these relate to the existence and properties of transition probabilities. From these fundamental principles, which have a direct physical significance, it is possible to deduce the subsidiary principles which form the accepted basis of the mathematical analysis of the quantum theory and which deal with the representation of quantum states and physical quantities by vectors and linear operators respectively.

In this paper the emphasis is laid on the experimental process determining a state of a system and on the associated operators rather than on the state itself or the vector representing it in the system space. Projective operators, which represent actual processes of measurement, and unitary operators, which represent actual transformations of systems of measurement, are given priority over the (statistical) operators which represent physical variables. This method of representation makes the physical meaning of the theory fundamental, instead of leaving it to be extracted from a purely mathematical system of non-commutative algebra or differential equations.

§ 1. *The General Principles of the Quantum Theory.*

The first step in the rational analysis of the principles of the quantum theory is the distinction between the "general principles," which are valid for any physical system, and the "special principles," which are characteristic of particular physical systems. The present paper is concerned only with the general principles. The special principles will be considered in a subsequent communication.

The fundamental concepts, in terms of which the general principles are formulated, are the concepts of a *system*, of the *states* of a system, of the *tran-*

sitions of a system from one state to another, and of the *probability* of these transitions. The general principles themselves relate exclusively to the mathematical representation of these physical concepts. They assert that a system is represented by a certain Hilbertian space $\mathfrak{H}$, the states of the system by rays or unit vectors, α, β, ..., in the system space $\mathfrak{H}$, and the transitions of the system by unitary transformations, U, V, ..., in $\mathfrak{H}$. The probability of a transition between two states is taken to be equal to $|(\alpha, \beta)|^2$, the squared modulus of the scalar product of the two rays α, β representing the states.

In this scheme a physical quantity or a dynamical variable is a derivative concept. It is associated with a group of unitary transformations, $\{U(s)\}$, depending upon one parameter s, and it is represented by the infinitesimal operator of this group,

$$P = \lim_{s \to 0} \partial U(s)/\partial s,$$

more precisely, by the operator $(h/2\pi i)$ P where h is subsequently identified with Planck's constant. The effective or average value of a physical quantity represented by an operator X in a state represented by a unit vector α is taken to be equal to $(X\alpha, \alpha)$, the scalar product of $X\alpha$ by α.

The two modes of exposition of the quantum theory, represented by the books of Dirac and Weyl, emphasise respectively the representation of states by vectors and the representation of quantities by operators. When the exposition is restricted to that part of the quantum theory which depends exclusively upon the general principles, Weyl's method is inevitable and it is this method—the representation of operators by algebras, groups and matrices —which is mainly employed here.

Both modes of exposition are purely deductive systems in which only the remote conclusions can be physically interpreted and experimentally proved. The principles themselves, as is clear, from the summary above, have no immediate physical significance; the object of this investigation is to analyse them into simple assumptions free from this defect. Until this problem is solved the quantum theory will present the anomaly of a physical theory incapable of giving a physical interpretation of its own principles.

§ 2. *Selective Operators.*

The general interpretation of the quantum theory given by Bohr, Heisenberg and Dirac has stressed the uncertain and unpredictable character of experimental observations. It is clearly as impossible to base the quantum theory on these *negative* qualities alone as to ground the theory of relativity solely on the negative results of the Michelson-Morley experiment. On the other hand, a precise analysis of the *positive* characteristics of micro-physical measurement does furnish an entirely adequate physical basis for the quantum theory. It is argued here that the main experimental process of microscopic physics is a generalised form of spectral analysis. The essence of this type of process can be expressed quite simply in a symbolic form (§ 3), from which the accepted "general principles" of the quantum theory can be rigorously deduced with the aid of two auxiliary assumptions which are wholly physical in content.

Any physical experiment is an interaction between the system observed and the apparatus of observation. The action of the system on the apparatus in producing an observable record has its counterpart in the reaction of the apparatus on the system in producing an unobserved change of state. Hence only two types of experiment can yield unambiguous results—that type in which the action of the system on the apparatus is completely determined by the *initial* state of the system and that in which it is determined by the *final* state of the system. The characters of the system which can be inferred in these two cases are distinguished by Eddington† as "retrospective" and "contemporaneous" respectively. In an historical study of individual systems only the second type of experiment is efficacious.

In this special type of experiment the inference regarding the character of the observed system is valid precisely because the experiment *impresses* this character upon the system. Hence, if the same system is immediately subjected again to the same process, it will suffer no further change. If the experimental process is represented by the operator P this property of the process is symbolised by the equation

$$P^2 \equiv P \,.\, P = P.$$

When the system is subjected to some other process, represented by the operator Q, the character impressed by Q may be incompatible with the character impressed by P, in which case the process Q will cause a transition of the system from one state to another. Symbolically, the compatibility or incompatibility of the characters impressed by P and Q is represented by the equations

$$PQ = QP \quad \text{or} \quad PQ \neq QP$$

respectively.

† "The Decline of Determinism," Presidential Address to the Mathematical Association, January 4, 1932.

Furthermore, two characters may be exclusive in the sense that no system possessing either character can receive the impress of the other character, *i.e.*, the process yields a null result. In symbols, exclusiveness implies that

$$PQ = O = QP,$$

where O is the null operator.

The preceding concepts have analogies in genetic biology. Here the process which impresses a character is selective generation or breeding. The propagation of a " pure line " which " breeds true " is analogous to the reproduction of the same character by repeated similar processes. In a pure line the determining character may be dominant or recessive, but these two kinds of characters are mutually exclusive, *i.e.*, recessive offspring cannot be bred from dominant parents in the pure line and *vice versa*. The incompatibility of dominant and recessive characters in the " mixed line " is illustrated by the generation of pure recessives from hybrid dominants.

In view of this analogy it is convenient to describe a process and its representative operator as " selective " if $P^2 = P$. (This avoids the awkward adjective " idempotent," introduced by Sylvester, and the untranslateable term " Einzeloperator " due to J. v. Neumann.) The state of a system is specified by the set of selective processes which produce no change in the system. These processes correspond to the totality of (compatible) characters possessed by the system in some state, and they completely describe the observable properties of that state.

§ 3. *Spectral Sets of Operators.*

The principles of the preceding section are of wide application. They are recognised in biology, and appear to be applicable to the psychology of conditioned reflexes. In physics, however, these general principles require specialisation in view of the quantitative nature of physical characters. Moreover, the theory has to be framed to include physical quantities which may vary either continuously or discretely. Hence the typical physical character is taken to be either that some variable ξ (such as a positional co-ordinate) has a numerical value not exceeding some prescribed number x, or that it has a value greater than x. The corresponding selective operators are written S_x or S_x'. We have to consider the properties of the sets of selective operators $\{S_x\}$, $\{S_x'\}$ where x has all possible values.

These properties can only be known by abstraction from the concrete selective processes actually employed in micro-physics. Now the type of process which is of primary importance in this domain is exemplified in Stern and Gerlach's analysis of metallic vapours by an inhomogeneous magnetic field, in the analysis of positive rays by Aston's mass-spectrograph, and in the magnetic analysis of β-rays. All these methods present analogies with the spectral analysis of radiation. Their essential characteristic is the resolution of an inhomogeneous aggregate into (relatively) homogeneous parts.

In our symbolism the process of separation of a partial relatively homogeneous aggregate in which $x < \xi \leqslant y$ can be represented only by the operator

$$S_x'S_y = S_yS_x'.$$

Of course, such an aggregate may contain no members, in which case the operator $S_x'S_y$ is equivalent to the null operator O and we say that the region $x < \xi \leqslant y$ is absent from the "spectrum" of the variable for the particular aggregate subjected to analysis. The spectrum itself is then defined negatively as the set of values not excluded as "absent."

In this paper the term spectral analysis will be applied not only to the type of process illustrated above but will also be given a still wider significance. In the processes just cited, the aggregate to be analysed is composed of systems which are *simultaneously* passed through the analysing field but it is clear that no essential feature of the method of analysis would be varied if the individual systems were *separately* and successively subjected to the analytical process. Under these circumstances the complete process would consist of a multitude of separate experiments upon individual systems. The aggregate of these systems would then cease to be an actual collection and would become a mental fiction similar to a Gibbsian "ensemble."

From consideration of particular examples we can see that the necessary and sufficient conditions that two sets of complementary selective operators, $\{S_x\}$ and $\{S_x'\}$ should represent a process of spectral analysis are expressed by the following equations :—

$$S_xS_y = S_yS_x = S_x, \quad \text{if} \quad x \leqslant y;$$

$$\left.\begin{aligned} S_x'S_y &= S_yS_x', \\ S_y'S_x &= 0 = S_xS_y' \end{aligned}\right\} \text{if} \quad x \leqslant y;$$

$$S_a = I, \quad S_b = 0,$$

$$S_x + S_x' = I,$$

if a, b are the upper and lower bounds of the variable ξ and I is the identical operator. Two complementary sets of spectral operators are called "spectral sets"† if they satisfy these conditions.

The mathematical analysis is considerably simplified if the continuous set of operators $\{S_x\}$ is replaced by the finite spectral set $\{P_n\}$ defined by $P_n = S_{x_n}{}' S_{x_{n+1}}$, where $\{x_n\}$ is a finite set of values of the variable ξ, dividing the complete domain of ξ into intervals of equal content, $x_{n+1} - x_n$, (or *a priori* probability‡); and if the character impressed by P_n is taken to be that ξ has the value x_n. The approximation involved in this substitution can be indefinitely sharpened by a subsequent passage to the limit in which the upper bound of the length of the intervals $x_{n+1} - x_n$ is made to tend to zero. The properties of the operators P_n are easily deduced from those of the set $\{S_n\}$. They are

$$P_n^2 = P_n,$$

$$P_m P_n = 0, \quad \text{if} \quad m \neq n.$$

There is an obvious analogy between the set of operators $\{P_n\}$ and the geometrical operators which project a vector on to a set of orthogonal axes in multi-dimensional space, and this analogy is the guide to the subsequent theory of the representation of these operators. In anticipation of the results of this theory the operators will be described as "projective," and two projective operators P, Q such that $PQ = 0 = QP$, will be described as "orthogonal."

This analogy also suggests that the transition probability from the state determined by a projective operator P_m to the state determined by a projective operator Q_n should be the square of the cosine of the angle between the axes of P_m and Q_n, *i.e.*, the characteristic§ (Spur) of the matrix representing the product $P_m Q_n$, since these quantities satisfy the obvious requirement that

$$\sum_n \mathrm{Sp}\,(P_m Q_n) = 1.$$

Futhermore it suggests that the process represented by a numerical operator λ should be a change in the number of systems in the aggregate affected in the ratio $\lambda^2 : 1$. We shall adopt this last suggestion as a pure convention, obviously

† This term is due to Reisz. v. Neumann's expression "die Zerlegung der Einheit," which is translated by M. H. Stone as "the canonical resolution of the identity."

‡ Two intervals of a variable have equal *a priori* probability when they can be transformed into one another by a congruent transformation.

§ The characteristic of a matrix is the numerical sum of the elements in its principal diagonal.

not inconsistent with any physical fact, or the agreed properties of selective operators.

§ 4. *Transition Probabilities.*

The preceding section outlines the "descriptive" theory of spectral analysis, the axioms of which appear to be an immediate induction from the customary methods of physical measurement. The "metrical" theory of spectral analysis depends upon two additional assumptions regarding the probabilities of transition from one state to another. These assumptions are as follows :—

(*a*) There is a definite probability p that a system in the state defined by a projective operator A ($= P_m$, say) will pass over into the state defined by a projective operator B ($= Q_n$, say) as a result of the process of spectral analysis defined by the spectral set, $\{Q_n\}$ to which B belongs.

(*b*) This probability is completely determined by the operators A and B, and is the same for the transitions $B \rightarrow A$ and $A \rightarrow B$.

It follows from these assumptions that the two "partial" aggregates which are separated from any given aggregate by the processes represented by A and ABA will consist of systems in the same state, *i.e.*, the state determined by A, and will differ only in the number of systems which they contain. The convention introduced at the end of the last section allows this result to be expressed in the form

$$ABA = \lambda A,$$

where λ is a numerical operator. Moreover, λ is evidently the probability p and is completely determined by the operator ABA, so that it may be rewritten in the form

$$\lambda = p\,(ABA).$$

Hence

$$ABA = p\,(ABA)\,.\,A \quad \text{and} \quad p\,(ABA) = p\,(BAB).$$

The theory of the representation of projective operators is an immediate corollary to these two assumptions.† If $\{E_n\}$ is a complete set of orthogonal, projective operators, the matrix operator with character (E_j, E_k) of any projective operator P is defined to be

$$P_{jk} = E_j P E_k.$$

† A purely mathematical investigation is sufficient to deduce the possible irreducible representations of a *single* set of projective operators from the relations $P_n{}^2 = P_n$, $P_m P_n = 0$. The two physical assumptions made above are necessary to determine the representation of *two* distinct sets of projective operators $\{E_n\}$ and $\{P_n\}$. They ensure that these two sets have the *same* field of operation, *i.e.*, that their representative matrices act upon the same set of vectors.

The two fundamental properties of matrix operators are :—

$$P_{ij}P_{kl} = E_iPE_jE_kPE_l = O, \quad \text{if} \quad j \neq k\,;$$

and

$$\begin{aligned} P_{ij}P_{jk} &= E_iPE_jPE_k \\ &= p\,(PE_jP)\,E_iPE_k \\ &= p\,(P_{jj})\,P_{ik}. \end{aligned}$$

In particular the matrix operators of the operator E_n are all null operators except the one with character (E_n, E_n) for which

$$E_{n,\,nn} = E_n.$$

§ 5. *The Matrix Representation of Operators.*

The fact that the matrix operators of P and of the set $\{E_n\}$, together with the numerical operators, form a field which is closed with respect to multiplication implies that these operators can be represented by matrices—in fact the operators $\{P_{ij}\}$ form the basis of a "complete matrix ring."†

All the irreducible representations of the operators $\{P_{ij}\}$ are equivalent to a representation in which the only non-zero element in the matrix representing an operator of character (E_j, E_k) is in the row j and the column k. Also it is possible to choose the representation so that this element has the form $x_j{}^*x_k$ where $\{x_k\}$ is a set of complex numbers and $\{x_k{}^*\}$ their conjugates. It is clear that

$$x_k{}^*x_k = p\,(P_{kk}),$$

so that only the moduli of the numbers $\{x_k\}$ are determinate ; their phases can be prescribed arbitrarily.

Moreover, it can be shown that the representations of the matrix operators of two distinct projective operators P and Q are simultaneously reducible to the canonical form defined above. For,

$$P_{ij}Q_{kl} = O, \quad \text{if} \quad j \neq k,$$

and

$$\begin{aligned} P_{ij}Q_{jk} &= E_i\,.\,PE_jQ\,.\,E_k \\ &= \text{a matrix operator of character } (E_i, E_k), \end{aligned}$$

hence the result follows at once.

It remains to consider the representation of the projective operators themselves. Although the general concept of the *sum* of two or more operators

† Van der Waerden, "Moderne Algebra" (1931).

cannot yet be defined in physical language (see § 9) there is an obvious sense in which a projective operator P is the *sum* of its matrix operators $\{P_{ij}\}$, *i.e.*, if ϕ denotes any aggregate of systems, the aggregate of systems $P\phi$ is the sum of the aggregates $P_{ij}\phi$. This result follows from the fact that the system of projective operators $\{E_j\}$ is "complete," *i.e.*, any aggregate is exhaustively analysed by the corresponding process of spectral analysis. This is the physical significance of the "double Peirce reduction" of P,

$$P = \sum_{j,k} P_{jk}.$$

This relation shows that a projective operator P can be represented by a matrix which is the sum of the matrices representing its matrix operators P_{jk}.

Again, if A and B are two projective operators the definition of the *sum* $\Sigma\, AE_jB$ as equal to AB can be similarly justified. Hence, if $C = AB$,

$$C_{jk} = E_jABE_k = \sum_l E_jAE_l \,.\, E_lBE_k$$
$$= \sum_l A_{jl}B_{lk}.$$

Therefore the matrix representing AB is the ordinary matrix product of the matrices representing A and B.

The representation of the operators by matrices implies a simultaneous representation of the states of a system by vectors in the space $\mathfrak{P}$. The action of the physical processes denoted by the operators upon the states of the system is then represented by the transformation of the vectors by the matrices. The state which is determined by the projective operator P_n is represented by the vector σ_n, which is invariant under the transformation by the matrix representing P_n. It is convenient to use the same symbols to denote the physical process, the corresponding operator and its representative matrix; also to use the same symbols for the state and its representative vector. With this convention

$$P_n\sigma_n = \sigma_n,$$
$$P_m\sigma_n = 0, \quad \text{if} \quad m \neq n.$$

With the convention that the states determined by the projective operators A and B are represented by vectors α and β, of unit length, it follows that the probability of the transitions $\alpha \to \beta$ or $\beta \to \alpha$ is $|(\alpha, \beta)|^2$.

§ 6. *Average Values of Individual Variables.*

It is now possible to deduce the result anticipated at the end of § 3, *i.e.*, that the probability of the transitions $A \to B$ or $B \to A$ equals the characteristic of the matrix representing the operator ABA. The diagonal matrix elements of A are the numbers $p\,(A_{kk})$, *i.e.*, the numbers $p\,(E_k A E_k)$ which are the probabilities of the transition $A \rightarrow E_k$. Since the set of operators $\{E_k\}$ is complete, the sum of these numbers is unity, *i.e.*,

$$\mathrm{Sp}A = 1.$$

Now

$$ABA = p\,(ABA) \,.\, A,$$

whence

$$\mathrm{Sp}\,(ABA) = p\,(ABA) \,.\, \mathrm{Sp}A$$
$$= p\,(ABA),$$

—the required result. This expression for the transition probability is exact, whereas the first expression deduced below for the average value of a variable is only an approximation, subsequently made exact by a passage to a limit.

We can now obtain a simple expression for the average value of a variable ξ in the state specified by the projective operator A. An average value can be defined only in terms of the experimental process by which it is actually determined. To determine the average value it is necessary to analyse an aggregate of systems in the specified state by means of the set of operators $\{P_n\} = \{S'_{x_n} S_{x_{n+1}}\}$. The probability of the transition $A \rightarrow P_n$ is simply $\mathrm{Sp}\,(AP_n)$. Hence the average value of ξ is approximately $\sum_n x_n \mathrm{Sp}\,(AP_n)$.

If we introduce the matrix S defined by the equation,

$$S = \sum_n x_n P_n,$$

the average value of the variable ξ in the state determined by A is concisely expressible as

$$E\,(\xi) = \mathrm{Sp}\,(AS).†$$

This approximation to the average value of ξ can be sharpened by increasing the number of operators in the set $\{P_j\}$. The exact value obtained on proceeding to the limit is given by the Stieltjes' integral

$$\int_{-\infty}^{+\infty} x\, d\omega\,(x),$$

where

$$\omega\,(x) = \mathrm{Sp}\,(AS_x).$$

The main problem of quantum theory is the determination of average values,

† E( ) is the symbol used by v. Neumann for the " Erwartungswert."

and for this problem the variable ξ is represented by the operator $S = \sum_n x_n P_n$, or, accurately by

$$\int_{-\infty}^{+\infty} x dS_x.$$

Hence S may be called the statistical operator of ξ. Conversely, any complete set of projective operators $\{Q_n\}$ together with a set of numbers $\{y_n\}$ determines a statistical operator, $T = \sum_n y_n Q_n$ and thus represents some variable η.

Finally, we note that the average value of $f(\xi)$, where f is any polynomial function of ξ or the limit of a sequence of such function, is given by

$$E(f) = \sum_n f(x_n) \, Sp(AP_n),$$

i.e., by

$$Sp(Af(S)),$$

where

$$f(S) = \sum_n f(x_n) \, P_n.$$

§ 7. *Congruent Transformations and Unitary Operators.*

Further progress in the theory of the representation of operators requires the powerful methods of group theory introduced by Weyl.† The argument is that a group of congruent transformations of a physical quantity ξ induces a corresponding group of unitary transformations in the system space $\mathfrak{H}$ and that the infinitesimal operator of this group corresponds to the variable dynamically conjugate to ξ. The sum of two operators can then be defined in terms of the product of the finite operators which they generate when regarded as infinitesimal operators.

Congruent transformations arise from the comparison of different methods of determining the physical characteristics of the same system. Hence a congruent transformation of a variable is simply a permutation of the proper states of this variable, *i.e.*, the states determined by the projective operators of ξ. Hence the corresponding transformation U in the system space $\mathfrak{H}$,

$$\sigma_n \rightarrow U\sigma_n = \sigma'_n,$$

is unitary, so that

$$(U\sigma_n, U\sigma_n) = 1.$$

The matrix operators and matrix elements of a unitary operator U are defined as in the case of projective operators. If $\{E_n\}$ is the spectral set of

† "The Theory of Groups and Quantum Mechanics" (Eng. trans., 1931), p. 185.

projective operators taken as the basis, the matrix operator of character (E_j, E_k) is

$$U_{jk} = E_j U E_k.$$

From the unitary property of U it follows, by the usual argument, that the j, k-matrix element of U and the k, j-matrix element of its inverse U^{-1} are conjugate complex numbers.

When the variable which is the subject of a congruent transformation is the time, the corresponding unitary operator U transforms the state of a system at any given time into its state at some subsequent time and thus determines the historical development of the system. If the initial state of the system is a proper state, α_j, of some projective operator, A_j, belonging to a spectral set $\{A_j\}$, the final state, β_j, of the system is not necessarily a proper state of any of these operators. Nevertheless the final state is completely determinate, and its properties may be specified by subjecting the systems in this state to the process of spectral analysis represented by the set of operators $\{A_j\}$. The double transition, $\alpha_j \rightarrow \beta_j \rightarrow A_k\beta_j$, is determined by the operator A_kU. Hence the probability of the transition from a proper state of A_j to a proper state of A_k under the influence of U will be zero if

$$A_k U \alpha_j = 0,$$

i.e., if

$$U_{kj} = A_k U A_j = 0.$$

Thus the evanescent matrix elements of U with basis $\{A_j\}$ determine what transitions the operator U cannot produce.

In general, the probability of the transition $\alpha_j \rightarrow A_k\beta_j$ is determined by A_k and B_j—the projective operator specifying the state β_j. In the notation of § 4 this transition probability is $p\,(A_kB_jA_k)$.

Now the transformation

$$\alpha_j \rightarrow U\alpha_j = \beta_j,$$

implies the transformation

$$A_j \rightarrow UA_jU^{-1} = B_j.$$

Hence

$$p\,(A_kB_jA_k) = p\,(A_kUA_jU^{-1}A_k)$$

$$= p\,(A_kUA_j \,.\, A_jU^{-1}A_k)$$

$$= p\,(U_{kj} \,.\, U_{jk}{}^{-1}) = |u_{kj}|^2,$$

where u_{kj} is the only surviving matrix element of U_{kj}.

§ 8. *Conjugate Variables.*

The set of congruent transformations of a single variable ξ form a group, and, since there is isomorphic correspondence between these transformations and the unitary transformations which they induce in the system space, these latter must also form a group. An individual transformation of this group, $V(s)$ will be distinguished by a parameter s which is to be chosen (as is always possible) so that

$$V(s)\,V(t) = V(s+t).$$

It will be shown in this section that the group of unitary operators $\{V(s)\}$ represents the variable dynamically conjugate to ξ.

Let Q be the projective operator determining a state ψ which is unchanged by the group of unitary operators $\{V(s)\}$. Then

$$Q = V^{-1}(s)\,QV(s),$$

for all s.

Let P_j, P_k be two projective operators determining states σ_j, σ_k which are transformed into one another by the unitary operators $V(t)$, $V^{-1}(t)$. Then

$$P_j = V^{-1}(t)\,P_kV(t).$$

Hence

$$\begin{aligned}\mathrm{Sp}\,(P_jQ) &= \mathrm{Sp}V^{-1}(t)\,P_kV(t)\,.\,V^{-1}(t)\,QV(t)\\ &= \mathrm{Sp}\,V^{-1}(t)\,.\,P_kQ\,.\,V(t)\\ &= \mathrm{Sp}\,(P_kQ).\end{aligned}$$

Accordingly there is the same probability of transition from the state determined by Q to any proper state of the variable ξ. Also, since $\sum_j \mathrm{Sp}\,(P_jQ) = 1$, these transition probabilities all have the same value, N^{-1}, where N is the number of projective operators in the complete set $\{P_j\}$.†

Accordingly if $\{Q_k\}$ is the set of projective operators which are invariant under the group $\{V(s)\}$, then all the transition probabilities $P_j \rightleftarrows Q_k$ are equal. Under these circumstances we say that the two sets of operators and the corresponding sets of states, are *conjugate*.

It follows from standard matrix theory that the totality of projective operators $\{Q_k\}$ which are invariant under $\{V(s)\}$ form a complete spectral set, and that if those operators are taken as a basis, the unitary operator $V(s)$ is represented by a diagonal matrix, in which the diagonal element v_{kk} has the

† This assumes that the domain of ξ is finite, *i.e.*, ξ is a cyclic variable. Non-cyclic variables (if such exist) must be treated as limiting cases.

form exp (isy_k) where y_k is a real number. Hence, for matrices, we have the result,

$$V(s) = \sum_k \exp(isy_k) \cdot Q_k.$$

To interpret this matricial equation, we note that, in accordance with the result of § 6 the set of real numbers $\{y_k\}$ and the spectral set $\{Q_k\}$ are associated with some variable η which is represented by the statistical operator

$$T = \sum_k y_k Q_k,$$

whence

$$V(s) = \exp(isT).$$

Thus the same variable η is represented by the statistical operator Q in the theory of average values and by the unitary operators $\{V(s)\}$ in the theory of congruent transformations.

The concept of two dynamically conjugate variables cannot be bodily transferred from classical theory to quantum theory, as an exact determination of the simultaneous values of a co-ordinate and its momentum is experimentally impossible. But the rôle of the variable conjugate to ξ is played in the quantum theory by the variable η defined indirectly above. This identification of the nature of η is strengthened by the following analogy :—in classical theory a definite constant value of a momentum implies that it is equally probable that the value of the co-ordinate lies in any two intervals of equal length. Similarly in quantum theory, for a system in the state defined by Q_k (for which η has the definite value y_k) there is equal probability for all the transitions $Q_k \rightarrow P_j$, $(j = 1, 2, \ldots)$.

§ 9. *Average Values of the Sum or Difference of Two Variables.*

In quantum theory the concept of average values is subject to severe limitations, which are imposed by the very nature of the experiments which have to be made in order to measure them. Thus, if ξ and η are any two variables, although we can experimentally define the average values of ξ, of η and of $\xi + \eta$ for any prescribed state, we cannot define the average value of $\xi\eta$ unless the statistical operators S and T commute, and even the relation

$$E(\xi) + E(\eta) = E(\xi + \eta)$$

is not self-evidently true.

To establish this relation for variables which are conjugate to co-ordinates susceptible of *continuous* variation it is sufficient to utilise the double representation of such variables as statistical operators and as unitary operators. The variable ξ is represented either by the statistical operator

$$S = \Sigma x_n P_n,$$

or by the group of unitary operators

$$U(\lambda) = \exp(i\lambda S).$$

It is clear that if the variable conjugate to ξ is continuously variable, the group $\{U(\lambda)\}$ is also continuous and that its infinitesimal operator is

$$[\partial U(\lambda)/\partial\lambda]_{\lambda=0} = iS.$$

Now let η be represented by

$$T = \Sigma y_n Q_n,$$

and by the group

$$V(\lambda) = \exp(i\lambda T).$$

Although it is not immediately evident what *statistical* operator will represent the sum $\xi + \eta$ it is clear that the representative *group* will be $\{U(\lambda) V(\lambda)\}$ for this represents the joint effect of $V(\lambda)$ followed by $U(\lambda)$. Hence we can deduce that the statistical operator of $\xi + \eta$ is the infinitesimal operator of the group,

$$-i\{U(\lambda) V(\lambda)\},$$

i.e.,

$$S + T.$$

Hence, if S and T represent the statistical operators of the variables ξ and η, then $S + T$ represents $\xi + \eta$. Similarly $\xi - \eta$ is represented by the group $\{U(\lambda) . V^{-1}(\lambda)\}$ and by the statistical operator $S - T$, and, in general, the statistical operator of $a\xi + b\eta$ is $aS + bT$, where a, b are any two numbers. This result completes the theory of the representation of physical quantities by operators. From this point the "general" quantum theory can be developed as in Weyl's treatise. The exchange relations for conjugate variables follow from their representation by unitary operators and the theory of the angular momentum and polarisation operators is deducible from the isomorphism of the congruence groups and their associated unitary groups in system space. The problems of "special" quantum theory, and, in particular, the theory of the Hamiltonian operator will be discussed in a future paper.

Summary.

An analysis of the general nature of physical measurement shows that quantum states are defined by certain "selective" processes, represented

by operators which satisfy the condition, $P^2 = P$. The general principles of the representation of quantum states and physical variables by sectors and matrices in Hilbertian space can then be deduced from two physical hypotheses :—

(*a*) that (under specified conditions) there is a definite probability for the transition from the state represented by P to the state represented by Q.

(*b*) that this probability is the same for the transitions $P \rightarrow Q$ and $Q \rightarrow P$.

The association of congruent transformations with unitary operators leads to the theory of dynamically conjugate variables, and completes the theory of the "effective" value of a variable, averaged over all systems in a definite state.

A Unique Electrode Potential Characteristic of a Metal, and a Theory for the Mechanism of Electrode Potential.

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(Communicated by T. M. Lowry, F.R.S.—Received June 29, 1932.)

1. *General.*

Metal electrodes immersed in air free solutions of KCl were found to come to a definite reproducible potential. From the Nernst expression, this potential corresponds to a certain concentration in the solution of the ion of the electrode metal. Chemical analysis showed that the concentration that actually exists in the bulk of the solution is much smaller than that deduced from the potential, even after electrode and solution have been in contact for weeks.

It was felt that a knowledge of the cause of this definite potential was fundamental to the general investigation of the potentials of metals in solutions of varying aeration, p_H, and salt concentration, a subject of great practical importance which is at present in rather an indefinite and unsatisfactory condition.

Work was undertaken with a view to obtaining information about the potentials of metals immersed in air-free solutions containing few of their own ions. In most of the experiments cadmium was used as an electrode, but check experiments have been made using other metals.

The results disclose a single potential to which a metal tends to come, and leads to a theoıy for the mechanism which produces this. Further work was done on solutions which had been aerated.