

The pion mass and decay constant at three loops in two-flavour chiral perturbation theory

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ABSTRACT: A calculation of the pion mass and decay constant at NNNLO in two-flavour chiral perturbation theory is presented. The results are cross-checked by using both the exponential and square root parametrizations of the Goldstone matrix field, as well as by comparing to the known leading log coefficients of the two quantities. A small numerical study of the quark mass dependence is performed, and for a physical quark mass there is good agreement with lower order results.

KEYWORDS: Chiral Lagrangians, Effective Field Theories

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1 Introduction

Chiral perturbation theory (ChPT) [1, 2] is a low energy effective field theory of QCD. It is built using the approximate chiral symmetry $SU(N_f)_L \times SU(N_f)_R$ of QCD, where N_f is the number of quark flavours, which is spontaneously broken to $SU(N_f)_V$ by a non-vanishing quark condensate $\langle \bar{q}q \rangle \neq 0$. The $N_f^2 - 1$ broken generators yield as many pseudo-Goldstone bosons. These are identified with the lightest pseudoscalar mesons living in the coset space $SU(N_f)_L \times SU(N_f)_R / SU(N_f)_V$. For $N_f = 2$ or $SU(2)$ only pions appear, whereas for $N_f = 3$ or $SU(3)$ there are the pions, kaons and the eta.

The masses and decay constants of these composite particles can be calculated within ChPT to a given order in the chiral expansion, i.e., to order $(p^2)^n$ where the integer $n \geq 1$. These were known at next-to-next-to-leading order (NNLO) for the pions in both $SU(2)$ [3–5] and $SU(3)$ [6]. In this paper we extend the two-flavour or $SU(2)$ case to the next order, p^8 or NNNLO. All relevant three-loop integrals are known [7, 8]. As a consistency check, we also calculate the mass at three-loop order in $O(N)$ ϕ^4 theory. The

general method to NNLO is described in detail in [5]. We extend it to one order higher in the expansion.

The motivation behind this work is twofold. The expressions themselves are of intrinsic interest but in so-called hard-pion ChPT it was argued that mass logarithms could be calculated also for pions with hard momenta. This was checked at two-loop order in ChPT [9]. At three-loop order it was found that the chiral mass logarithm does not agree with the prediction of [9] in [10]. This work is a first step towards checking the results of [10] and possibly being able to correct and extend the arguments of [9] towards a full proof.

2 Mass in the $O(N)$ ϕ^4 model

The Lagrangian for the $O(N)$ ϕ^4 model is given by

$$\begin{aligned} \mathcal{L} = & (1 + l_1\lambda + c_1\lambda^2 + d_1\lambda^3) \frac{1}{2} \partial_\mu \phi^T \partial^\mu \phi - (1 + l_2\lambda + c_2\lambda^2 + d_2\lambda^3) \frac{1}{2} M^2 \phi^T \phi \\ & - (1 + l_3\lambda + c_3\lambda^2 + d_3\lambda^3) \frac{\lambda}{4} (\phi^T \phi)^2 - \phi^T f. \end{aligned} \quad (2.1)$$

ϕ is a vector of N real fields ϕ_a and f is the external current. We have indicated here the higher order terms with l_i, c_i and d_i as well and have used the equations of motion (or field redefinitions) to discard the higher order terms in the coupling to the external current f . Up to two-loop order the counterterms are given by¹

$$\begin{aligned} l_1 &= (c\mu)^{-2\epsilon} (l_1^r), \\ l_2 &= (c\mu)^{-2\epsilon} \left(l_2^r + \frac{N+2}{16\pi^2\epsilon} \right), \\ l_3 &= (c\mu)^{-2\epsilon} \left(l_3^r + \frac{N+8}{16\pi^2\epsilon} \right), \\ c_1 &= (c\mu)^{-4\epsilon} \left(c_1^r + \frac{1}{(16\pi^2)^2\epsilon} \left(-\frac{N+2}{2} \right) \right), \\ c_2 &= (c\mu)^{-4\epsilon} \left(c_2^r + \frac{(N+5)(N+2)}{(16\pi^2\epsilon)^2} - \frac{3(N+2)}{(16\pi^2)^2\epsilon} + \frac{2(N+2)}{16\pi^2\epsilon} (-2l_1^r + l_2^r + l_3^r) \right), \\ c_3 &= (c\mu)^{-4\epsilon} \left(c_3^r + \frac{(N+8)^2}{(16\pi^2\epsilon)^2} - \frac{2(5N+22)}{(16\pi^2)^2\epsilon} + \frac{2(N+8)}{16\pi^2\epsilon} (-l_1^r + l_3^r) \right). \end{aligned} \quad (2.2)$$

We use dimensional regularization with $d = 4 - 2\epsilon$ and modified minimal subtraction (\overline{MS}). The choice of c determines which version of \overline{MS} is used. In this manuscript we use the usual ChPT [2] version with

$$c = -\frac{1}{2} [\log(4\pi) + \Gamma'(1) + 1]. \quad (2.3)$$

The mass is defined as the pole of the two point function, see e.g. the discussion in [6], as

$$\frac{i}{p^2 - M^2 - \Sigma(p^2)} \quad (2.4)$$

¹These are slightly different from [5] since we have chosen not to put $l_1^r = 0$.

where Σ is the sum of one-particle irreducible diagrams. The relevant diagrams are shown in figure 1 when neglecting those involving vertices with more than four legs. p^2 corresponds to terms with 1, p^4 with l_i , p^6 with c_i and p^8 with d_i in (2.1).

The physical mass M_ϕ is given by the solution of

$$M_\phi^2 - M^2 - \Sigma(M_\phi^2) = 0. \quad (2.5)$$

We write the mass as

$$M_\phi^2 = M^2 (1 + M_4^2 + M_6^2 + M_8^2) \quad (2.6)$$

and the self-energy as

$$\Sigma(p^2) = \Sigma_4(p^2) + \Sigma_6(p^2) + \Sigma_8(p^2). \quad (2.7)$$

Where we used the subscript 4,6,8 for NLO, NNLO and NNNLO respectively. With this expansion we can solve for the mass perturbatively. We evaluate diagrams at $p^2 = M_\phi^2$ as a perturbative expansion away from M^2 . This is why derivatives of the self-energy show up. Taking into account that here $\partial^2 \Sigma_4 / (\partial p^2)^2 = 0$ we get

$$\begin{aligned} M^2 M_4 &= \Sigma_4(M^2), \\ M^2 M_6 &= \Sigma_6(M^2) + M^2 M_4^2 \frac{\partial \Sigma_4}{\partial p^2}, \\ M^2 M_8 &= \Sigma_8(M^2) + M^2 M_6^2 \frac{\partial \Sigma_4}{\partial p^2} + M^2 M_4^2 \frac{\partial \Sigma_6}{\partial p^2}(M^2). \end{aligned} \quad (2.8)$$

All nonlocal divergences cancel as they should and we get a finite result by setting

$$\begin{aligned} d_2 - d_1 &= (c\mu)^{-6\epsilon} \left\{ d_2^r - d_1^r - \frac{\pi_{16}^3}{\epsilon^3} (-60 - 52N - 13N^2 - N^3) \right. \\ &\quad - \frac{\pi_{16}^3}{\epsilon^2} \left(\frac{284}{3} + \frac{206}{3}N + \frac{32}{3}N^2 \right) - \frac{\pi_{16}^3}{\epsilon} (-74 - 47N - 5N^2) \\ &\quad - \frac{\pi_{16}^2}{\epsilon^2} (-20l_3^r - 10l_2^r - 14Nl_3^r - 7Nl_2^r - 2N^2l_3^r - N^2l_2^r + 40l_1^r + 28Nl_1^r + 4N^2l_1^r) \\ &\quad - \frac{\pi_{16}^2}{\epsilon} \left(10l_3^r + 6l_2^r + 5Nl_3^r + 3Nl_2^r - 21l_1^r - \frac{21}{2}Nl_1^r \right) \\ &\quad - \frac{\pi_{16}}{\epsilon} (-2c_3^r - 2c_2^r + 4c_1^r - 2l_2^r l_3^r - Nc_3^r - Nc_2^r + 2Nc_1^r - Nl_2^r l_3^r + 4l_1^r l_3^r + 4l_1^r l_2^r \\ &\quad \left. - 6l_1^{r2} + 2Nl_1^r l_3^r + 2Nl_1^r l_2^r - 3Nl_1^{r2}) \right\}. \end{aligned} \quad (2.9)$$

Here we introduced the shorthand $\pi_{16} = 1/(16\pi^2)$. We express the result for the mass in terms of the logarithm

$$L_M = \log \frac{M^2}{\mu^2}. \quad (2.10)$$

The full result for the mass at three-loop order is

$$\begin{aligned}
 M_4^2/\lambda &= (N+2)\pi_{16}L_M + l_2^r - l_1^r, \\
 M_6^2/\lambda^2 &= c_2^r - c_1^r - l_1^r l_2^r + l_1^{r2} + \pi_{16}(2l_2^r - 2l_1^r + Nl_2^r - Nl_1^r) \\
 &\quad + \pi_{16}L_M(2l_3^r + 2l_2^r - 6l_1^r + Nl_3^r + Nl_2^r - 3Nl_1^r) + \pi_{16}^2(3/2 + 3/4N) \\
 &\quad + \pi_{16}^2L_M(-6 - N + N^2) + \pi_{16}^2L_M^2(10 + 7N + N^2), \\
 M_8^2/\lambda^3 &= d_2^r - d_1^r - l_2^r c_1^r - l_1^r c_2^r + 2l_1^r c_1^r + l_1^{r2} l_2^r - l_1^{r3} \\
 &\quad + \pi_{16}(N+2)(c_2^r - c_1^r + l_2^r l_3^r + (1/2)l_2^{r2} - l_1^r l_3^r - 4l_1^r l_2^r + (7/2)l_1^{r2}) \\
 &\quad + \pi_{16}L_M(N+2)(c_3^r + c_2^r - 3c_1^r + l_2^r l_3^r - 3l_1^r l_3^r - 3l_1^r l_2^r + 6l_1^{r2}) \\
 &\quad + \pi_{16}^2(3l_3^r - (9/2)l_2^r - (3/2)l_1^r + (3/2)Nl_3^r - (1/4)Nl_2^r - (11/4)Nl_1^r + N^2l_2^r - N^2l_1^r) \\
 &\quad + \pi_{16}^2L_M(-12l_3^r + 14l_2^r + 10l_1^r - 2Nl_3^r + 13Nl_2^r - 9Nl_1^r + 2N^2l_3^r + 3N^2l_2^r - 7N^2l_1^r) \\
 &\quad + \pi_{16}^2L_M^2(20l_3^r + 10l_2^r - 50l_1^r + 14Nl_3^r + 7Nl_2^r - 35Nl_1^r + 2N^2l_3^r + N^2l_2^r - 5N^2l_1^r) \\
 &\quad + \pi_{16}^3(124/3 + 64\zeta_3 + (239/6)N + 40N\zeta_3 + (115/12)N^2 + 4N^2\zeta_3) \\
 &\quad + \pi_{16}^3L_M(217 + 131N + (53/4)N^2 + N^3) \\
 &\quad + \pi_{16}^3L_M^2(-128 - 64N + 5N^2 + (5/2)N^3) + \pi_{16}^3L_M^3(60 + 52N + 13N^2 + N^3). \quad (2.11)
 \end{aligned}$$

This result can be checked in a number of ways. The nonlocal divergences cancelled as they should. The terms leading in N can be derived using a gap equation similar to what was done for the nonlinear sigma model in [11, 12]. The renormalization group equations are known to five loop order [13], these can be used to check the $1/\varepsilon$ terms in (2.9). All checks are satisfied.

3 Chiral perturbation theory

The effective Lagrangian in ChPT is expanded in powers of p^2 as

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_4 + \mathcal{L}_6 + \mathcal{L}_8 + \dots \quad (3.1)$$

The relevant degrees of freedom are the Goldstone Bosons from the spontaneous breakdown of $SU(N_f)_L \times SU(N_f)$ to $SU(N_f)_V$. These can be described by a special unitary $N_f \times N_f$ matrix u . For two flavours the lowest order Lagrangian is

$$\mathcal{L}_0 = \frac{F^2}{4} \langle u_\mu u^\mu + \chi_+ \rangle \quad (3.2)$$

with $u_\mu = i(u^\dagger(\partial_\mu - ir_\mu)u - u(\partial_\mu - il_\mu)u^\dagger)$, $\chi_+ = u^\dagger \chi u^\dagger + u \chi^\dagger u$, $\chi = B(s + ip)$. The fields l_μ, r_μ, s and p are the usual $N_f \times N_f$ external fields of ChPT. F and B are the two low-energy-constants (LECs) at leading order for the two-flavour case. The next-order Lagrangian was classified in [2]. The NNLO Lagrangian can be found in [14]. The NNNLO Lagrangian \mathcal{L}_8 is at present not known, but there will be one combination of p^8 LECs contributing to the mass and another to the decay constant, we will call these combinations r_{M8} and r_{F8} , respectively.

The NLO and NNLO low-energy-constants (LECs) are conventionally denoted as l_i and c_i , respectively. The divergent parts needed to one- [2] and two-loop order [15] are known

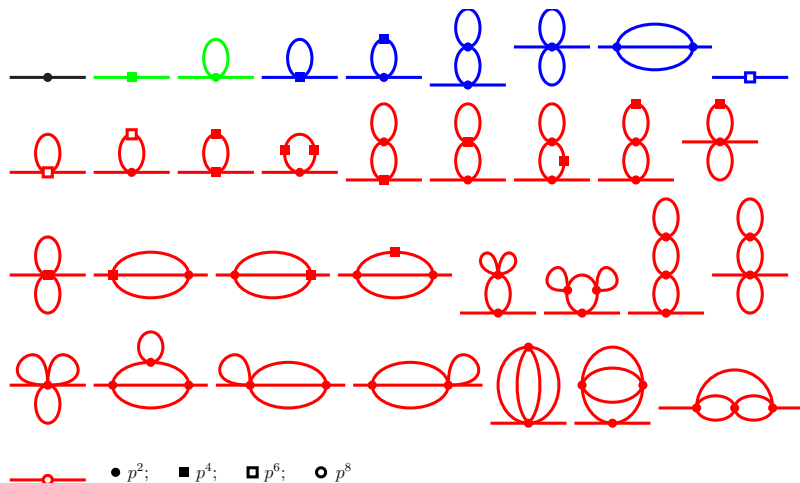


Figure 1. The 34 diagrams contributing to Σ . The first one is LO, the next two NLO, the next 6 NNLO, all in the first line. The remaining diagrams are those needed at NNNLO.

in general and the equivalent formulas to (2.2) can be found there. For later convenience we introduce the lowest order pion mass

$$M^2 = 2B\hat{m} \tag{3.3}$$

where $2\hat{m} = m_u + m_d$. In the remainder we will work in the isospin limit with $m_u = m_d$.

4 The calculation and checks

The diagrams contributing are shown in figure 1. The Feynman diagrams are programmed in FORM [16]. The derivatives w.r.t. p^2 needed are obtained by taking the derivative diagram by diagram at this stage. Then the expressions are rewritten in integrals. These are reduced to a set of master integrals. This is done using integration-by-parts and Lorentz invariance identities through a Laporta algorithm. We have used the program REDUZE [17] for this. The resulting master integrals are all known to the order in ϵ required and we quote them in appendix A.

A three-loop calculation needs a large number of checks. We have checked that the nonlocal divergences cancel, that we reproduce the known two-loop and leading logarithm results. As a final check we use two different parametrizations for u in terms of a traceless Hermitian 2×2 matrix Φ , namely the exponential, $u = \exp \frac{i}{\sqrt{2}F} \Phi$, and a square root, $u = \sqrt{1 - \frac{1}{2F^2} \Phi^2} + \frac{i}{\sqrt{2}F} \Phi$, parametrization. The diagrams are quite different in these two parametrizations but the final result must of course be the same. All checks are satisfied by our results. Since essentially the same programs were used for ϕ^4 the checks discussed in section 2 are another partial check on our main results.

5 The pion mass and decay constant

The physical mass is defined as the pole of the two-point function (2.4) with $\Sigma(p^2)$ the self-energy. The physical pion mass M_π^2 is then found as the solution of

$$M_\pi^2 - M^2 - \Sigma(M_\pi^2) = 0. \quad (5.1)$$

The decay constant is defined through the relation

$$\langle 0 | A_\mu(0) | \pi(p) \rangle = i\sqrt{2}p_\mu F_\pi \quad (5.2)$$

and is calculated using diagrams of the same topology as those in Σ (the only difference is that one of the external legs corresponds to the axial current). For the decay constant one also needs to calculate the wave function renormalization factor Z defined as the residue of the propagator in (2.4), i.e.

$$Z = \frac{1}{1 - \frac{\partial \Sigma}{\partial p^2}} \quad (5.3)$$

at $p^2 = M_\pi^2$.

The physical pion mass and decay constant can be written in expanded form

$$\begin{aligned} M_\pi^2 &= M^2 (1 + M_4^2 + M_6^2 + M_8^2), \\ F_\pi &= F (1 + F_4 + F_6 + F_8). \end{aligned} \quad (5.4)$$

5.1 Mass

We can solve (5.1) perturbatively and obtain

$$\begin{aligned} M^2 M_4^2 &= \Sigma_4(M^2), \\ M^2 M_6^2 &= \Sigma_6(M^2) + M^2 M_4^2 \frac{\partial \Sigma_4}{\partial p^2}(M^2), \\ M^2 M_8^2 &= \Sigma_8(M^2) + M^2 M_6^2 \frac{\partial \Sigma_4}{\partial p^2}(M^2) + M^2 M_4^2 \frac{\partial \Sigma_6}{\partial p^2}(M^2), \end{aligned} \quad (5.5)$$

here we used the fact that $\partial^2 \Sigma_4 / (\partial p^2)^2 = 0$.

In order to obtain a finite result we need the subtraction

$$\begin{aligned} r_{M8} &= (c\mu)^{-6\epsilon} \left\{ r_{M8}^r - \frac{\pi_{16}^3}{\epsilon^3} (125/72) - \frac{\pi_{16}^3}{\epsilon^2} (71/24) - \frac{\pi_{16}^3}{\epsilon} (28223/12960) \right. \\ &\quad - \frac{\pi_{16}^2}{\epsilon^2} (-(7/2)l_4^r + (263/18)l_3^r - (7/3)l_2^r + (49/3)l_1^r) \\ &\quad - \frac{\pi_{16}^2}{\epsilon} ((302/27)l_3^r + (433/60)l_2^r + (3/20)l_1^r) \\ &\quad - \frac{\pi_{16}}{\epsilon} (416c_{18}^r + 208c_{17}^r + 32c_{16}^r - 96c_{14}^r - 8c_{13}^r + 48c_{12}^r + 384c_{11}^r + 192c_{10}^r - 80c_9^r \\ &\quad - 160c_8^r - 80c_7^r - 96c_6^r + 8c_5^r + 56c_4^r - 16c_3^r - 32c_2^r + 96c_1^r - 14l_3^r l_4^r + 16l_3^r{}^2 \\ &\quad \left. - (16/3)l_2^r l_3^r - (8/3)l_1^r l_3^r) \right\}. \end{aligned} \quad (5.6)$$

r_{M8} is the combination of p^8 LECs that contributes to the mass.

This is a single scale problem and only logarithms of the mass scale show up, the expression is thus fairly compact. We use the abbreviations

$$x = \frac{M^2}{16\pi^2 F^2}, \quad L_M = \log \frac{M^2}{\mu^2}, \quad l_i^q = 16\pi^2 l_i^r, \quad c_i^q = (16\pi^2)^2 c_i^r, \quad r_{M8}^q = (16\pi^2)^3 r_{M8}^r. \quad (5.7)$$

The results can be written in the form

$$\begin{aligned} M_4^2 &= x (a_{10}^M + a_{11}^M L_M), \\ M_6^2 &= x^2 (a_{20}^M + a_{21}^M L_M + a_{22}^M L_M^2), \\ M_8^2 &= x^3 (a_{30}^M + a_{31}^M L_M + a_{32}^M L_M^2 + a_{33}^M L_M^3). \end{aligned} \quad (5.8)$$

The coefficients are

$$\begin{aligned} a_{10}^M &= 2l_3^q, \\ a_{11}^M &= 1/2, \\ a_{20}^M &= 64c_{18}^q + 32c_{17}^q + 96c_{11}^q + 48c_{10}^q - 16c_9^q - 32c_8^q - 16c_7^q - 32c_6^q + l_3^q + 2l_2^q + l_1^q + (163/96), \\ a_{21}^M &= -3l_3^q - 8l_2^q - 14l_1^q - (49/12), \\ a_{22}^M &= 17/8, \\ a_{30}^M &= r_{M8}^q - 3l_3^q c_9^q - 64l_3^q c_8^q - 32l_3^q c_7^q - 128l_3^q c_6^q + 32c_{18}^q + 16c_{17}^q - 4c_{13}^q + 24c_{12}^q + 48c_{11}^q \\ &\quad + 24c_{10}^q - 8c_9^q - 16c_8^q - 8c_7^q - 40c_6^q + 12c_5^q + 4c_4^q - 8c_3^q - 7(l_3^q)^2 - 22l_1^q l_3^q - 4l_2^q l_3^q \\ &\quad + \frac{157}{48} l_3^q + \frac{8651}{1200} l_2^q + \frac{3823}{1200} l_1^q + \frac{4869659}{777600} - \frac{13}{6} \zeta_3, \\ a_{31}^M &= -416c_{18}^q - 208c_{17}^q - 32c_{16}^q + 96c_{14}^q + 8c_{13}^q - 48c_{12}^q - 384c_{11}^q - 192c_{10}^q + 72c_9^q + 144c_8^q \\ &\quad + 72c_7^q + 64c_6^q - 8c_5^q - 56c_4^q + 16c_3^q + 32c_2^q - 96c_1^q - 8(l_3^q)^2 - 48l_2^q l_3^q - 84l_1^q l_3^q - \frac{88}{3} l_3^q \\ &\quad - \frac{231}{10} l_2^q - \frac{69}{5} l_1^q - \frac{74971}{8640}, \\ a_{32}^M &= \frac{23}{2} l_3^q - 11l_2^q - 38l_1^q - \frac{91}{24}, \\ a_{33}^M &= \frac{103}{24}, \end{aligned} \quad (5.9)$$

where ζ_k is the Riemann-Zeta function. The leading log coefficient a_{33}^M agrees with [11, 12].

The LECs l_i^r are well known but the c_i^r less well. In [5, 15] combinations of the p^6 LECs appearing at p^6 in $\pi\pi$ -scattering, the mass and decay constant were defined, $r_1, \dots, r_6, r_M, r_F$, and numerical estimates using resonance saturation were done in [5]. The expressions in terms of the c_i^r are given in appendix B. We can check whether the c_i^r dependence can be rewritten in terms of those. This can be done for a_{20}^M by definition and also for a_{31}^M . However not completely for a_{30}^M which in any case contains the free p^8 LEC combination r_{M8}^r . Defining $r_i^q = 16\pi^2 r_i^r$ for $i = 1, \dots, 6, M, F$, we obtain

$$\begin{aligned} a_{30}^M &= r_{M8}^q - 4l_3^q r_F^q - 128l_3^q c_6^q - r_6^q + 3r_5^q + \frac{7}{2} r_4^q + \frac{1}{2} r_3^q + \frac{1}{2} r_M^q - 7(l_3^q)^2 - 22l_1^q l_3^q - 4l_2^q l_3^q \\ &\quad + \frac{157}{48} l_3^q + \frac{8651}{1200} l_2^q + \frac{3823}{1200} l_1^q + \frac{4869659}{777600} - \frac{13}{6} \zeta_3, \\ a_{31}^M &= -6r_6^q - 14r_5^q - 11r_4^q - 5r_3^q - 2r_2^q - \frac{5}{2} r_1^q + r_M^q - r_F^q - 8(l_3^q)^2 - 48l_2^q l_3^q - 84l_1^q l_3^q - \frac{88}{3} l_3^q \\ &\quad - \frac{231}{10} l_2^q - \frac{69}{5} l_1^q - \frac{74971}{8640}. \end{aligned} \quad (5.10)$$

5.2 Decay constant

For the decay constant everything is analogous except that we need to evaluate the diagrams with one leg replaced by the axial current and take into account the wave function renormalization factor Z . Denoting the sum of one-particle-irreducible diagrams of the axial current as $A(p^2 = M_\pi^2) = A_4(p^2) + A_6(p^2) + A_8(p^2)$ the expression for the decay constant is (normalized to 1 at lowest order)

$$F_\pi = F \sqrt{Z(M_\pi^2)} A(M_\pi^2). \quad (5.11)$$

Putting in the expanded expressions for Σ and A and using $\partial^2 \Sigma_4 / (\partial p^2)^2 = \partial A_4 / \partial p^2 = 0$, we obtain

$$\begin{aligned} F_4 &= \frac{1}{2} \frac{\partial \Sigma_4}{\partial p^2} + A_4, \\ F_6 &= \frac{1}{2} \frac{\partial \Sigma_6}{\partial p^2} + \frac{3}{8} \left(\frac{\partial \Sigma_4}{\partial p^2} \right)^2 + \frac{1}{2} \frac{\partial \Sigma_4}{\partial p^2} A_4 + A_6, \\ F_8 &= A_8 + \Sigma_4 \frac{\partial A_6}{\partial p^2} + \frac{5}{16} \left(\frac{\partial \Sigma_4}{\partial p^2} \right)^3 + \frac{3}{8} \left[A_4 \left(\frac{\partial \Sigma_4}{\partial p^2} \right)^2 + 2 \frac{\partial \Sigma_4}{\partial p^2} \frac{\partial \Sigma_6}{\partial p^2} \right] \\ &\quad + \frac{1}{2} \left[\frac{\partial \Sigma_4}{\partial p^2} A_6 + \frac{\partial \Sigma_6}{\partial p^2} A_4 + \Sigma_4 \frac{\partial^2 \Sigma_6}{(\partial p^2)^2} + \frac{\partial \Sigma_8}{\partial p^2} \right] \end{aligned} \quad (5.12)$$

with all right hand sides evaluated at $p^2 = M^2$.

In order to obtain a finite result we need the subtraction

$$\begin{aligned} r_{F8} &= (c\mu)^{-6\epsilon} \left\{ r_{F8}^r + \frac{\pi_{16}^3}{\epsilon^3} (185/72) + \frac{\pi_{16}^3}{\epsilon^2} (2117/432) - \frac{\pi_{16}^3}{\epsilon} (20183/12960) \right. \\ &\quad - \frac{\pi_{16}^2}{\epsilon^2} ((659/72)l_4^r - 2l_3^r - (1/6)l_2^r - (34/3)l_1^r) \\ &\quad - \frac{\pi_{16}^2}{\epsilon} (-(13/108)l_4^r - (53/12)l_3^r + (27/40)l_2^r + (61/15)l_1^r) \\ &\quad - \frac{\pi_{16}}{\epsilon} (16c_{20}^r + 64c_{18}^r + 32c_{17}^r - 8c_{16}^r + 24c_{14}^r + 2c_{13}^r - 12c_{12}^r + 96c_{11}^r + 48c_{10}^r + 8c_9^r \\ &\quad + 16c_8^r + 8c_7^r - 48c_6^r - 4c_5^r - 28c_4^r + 4c_3^r + 8c_2^r - 24c_1^r - (7/2)l_4^r + 3l_3^r l_4^r - (28/3)l_2^r l_4^r \\ &\quad \left. - 16l_2^r l_3^r - (44/3)l_1^r l_4^r - 28l_1^r l_3^r) \right\}. \end{aligned} \quad (5.13)$$

The results can be written in the form

$$\begin{aligned} F_4 &= x (a_{10}^F + a_{11}^F L_M), \\ F_6 &= x^2 (a_{20}^F + a_{21}^F L_M + a_{22}^F L_M^2), \\ F_8 &= x^3 (a_{30}^F + a_{31}^F L_M + a_{32}^F L_M^2 + a_{33}^F L_M^3). \end{aligned} \quad (5.14)$$

The full results for the coefficients are, with $r_{F8}^q = (16\pi^2)^3 r_{F8}^r$,

$$\begin{aligned}
a_{10}^F &= l_4^q, \\
a_{11}^F &= -1, \\
a_{20}^F &= 8c_9^q + 16c_8^q + 8c_7^q - 2l_3^q - l_2^q - (1/2)l_1^q - (13/192), \\
a_{21}^F &= -(1/2)l_4^q - 2l_3^q + 4l_2^q + 7l_1^q + (23/12), \\
a_{22}^F &= -(5/4), \\
a_{30}^F &= r_{F8}^q - 8l_4^q c_9^q - 16l_4^q c_8^q - 8l_4^q c_7^q - 32l_4^q c_6^q - 64c_{18}^q - 32c_{17}^q + c_{13}^q - 6c_{12}^q - 96c_{11}^q - 48c_{10}^q \\
&\quad + 16c_9^q + 32c_8^q + 16c_7^q + 40c_6^q - 6c_5^q - 2c_4^q + 2c_3^q - l_3^q l_4^q - 2(l_3^q)^2 + l_2^q l_4^q + 4l_2^q l_3^q + \frac{1}{2}l_1^q l_4^q \\
&\quad + 12l_1^q l_3^q + \frac{313}{192}l_4^q + \frac{241}{48}l_3^q + \frac{1469}{800}l_2^q + \frac{2359}{600}l_1^q - \frac{383293667}{1555200} + \frac{8}{9}\zeta_3, \\
a_{31}^F &= -16c_{20}^q - 64c_{18}^q - 32c_{17}^q + 8c_{16}^q - 24c_{14}^q - 2c_{13}^q + 12c_{12}^q - 96c_{11}^q - 48c_{10}^q + 80c_6^q + 4c_5^q \\
&\quad + 28c_4^q - 4c_3^q - 8c_2^q + 24c_1^q - l_3^q l_4^q - 4l_2^q l_4^q + 16l_2^q l_3^q - 7l_1^q l_4^q + 28l_1^q l_3^q - \frac{13}{6}l_4^q + \frac{17}{2}l_3^q \\
&\quad + \frac{569}{60}l_2^q + \frac{77}{10}l_1^q - \frac{7499}{2160}, \\
a_{32}^F &= \frac{3}{8}l_4^q + \frac{27}{2}l_2^q + 33l_1^q + \frac{1037}{144}, \\
a_{33}^F &= -\frac{83}{24}. \tag{5.15}
\end{aligned}$$

Also here the leading log coefficient, a_{33}^F , agrees with [12].

We can similarly to the previous subsection rewrite the results in terms of the combinations r_i^q $i = 1, \dots, 6, M, F$. The rewriting is not fully possible here.

$$\begin{aligned}
a_{30}^F &= r_{F8}^q - l_4^q r_F^q - 32l_4^q c_6^q + (1/2)r_6^q - (3/2)r_5^q - (3/2)r_4^q - (1/4)r_3^q - r_M^q + 4c_{12}^q - 2c_6^q - c_5^q \\
&\quad - l_3^q l_4^q - 2(l_3^q)^2 + l_2^q l_4^q + 4l_2^q l_3^q + \frac{1}{2}l_1^q l_4^q + 12l_1^q l_3^q + \frac{313}{192}l_4^q + \frac{241}{48}l_3^q + \frac{1469}{800}l_2^q + \frac{2359}{600}l_1^q \\
&\quad - \frac{383293667}{1555200} + \frac{8}{9}\zeta_3, \\
a_{31}^F &= -(17/6)r_6^q + (33/2)r_5^q + (33/2)r_4^q + 3r_3^q - (1/8)r_2^q - r_M^q - (3/2)r_F^q - 16c_{20}^q - 20c_{14}^q \\
&\quad - 96c_{12}^q + 148c_6^q - 46c_5^q + (80/3)c_3^q - l_3^q l_4^q - 4l_2^q l_4^q + 16l_2^q l_3^q - 7l_1^q l_4^q + 28l_1^q l_3^q - \frac{13}{6}l_4^q \\
&\quad + \frac{17}{2}l_3^q + \frac{569}{60}l_2^q + \frac{77}{10}l_1^q - \frac{7499}{2160}. \tag{5.16}
\end{aligned}$$

6 Numerical study: mass dependence

Now that the analytic forms of the mass and decay constant have been obtained, we can do a first numerical analysis of the mass dependence. We present only results for one choice of input parameters to give an impression of the size of the NNNLO correction.

The expansions given in the previous section correspond to an expansion expressed in terms of the lowest order mass M and decay constant F in the form

$$\begin{aligned}
 M_\pi^2 &= M^2 \left\{ 1 + x (a_{10}^M + a_{11}^M L_M) + x^2 (a_{20}^M + a_{21}^M L_M + a_{22}^M L_M^2) \right. \\
 &\quad \left. + x^3 (a_{30}^M + a_{31}^M L_M + a_{32}^M L_M^2 + a_{33}^M L_M^3) \right\}, \\
 F_\pi &= F \left\{ 1 + x (a_{10}^F + a_{11}^F L_M) + x^2 (a_{20}^F + a_{21}^F L_M + a_{22}^F L_M^2) \right. \\
 &\quad \left. + x^3 (a_{30}^F + a_{31}^F L_M + a_{32}^F L_M^2 + a_{33}^F L_M^3) \right\}
 \end{aligned} \tag{6.1}$$

with $x = M^2/(16\pi^2 F^2)$ and $L_M = \log(M^2/\mu^2)$.

There are many ways to rewrite this expansion but the second most standard version is the inverse, namely

$$\begin{aligned}
 M^2 &= M_\pi^2 \left\{ 1 + \xi (b_{10}^M + b_{11}^M L_\pi) + \xi^2 (b_{20}^M + b_{21}^M L_\pi + b_{22}^M L_\pi^2) \right. \\
 &\quad \left. + \xi^3 (b_{30}^M + b_{31}^M L_\pi + b_{32}^M L_\pi^2 + b_{33}^M L_\pi^3) \right\}, \\
 F &= F_\pi \left\{ 1 + \xi (b_{10}^F + b_{11}^F L_\pi) + \xi^2 (b_{20}^F + b_{21}^F L_\pi + b_{22}^F L_\pi^2) \right. \\
 &\quad \left. + \xi^3 (b_{30}^F + b_{31}^F L_\pi + b_{32}^F L_\pi^2 + b_{33}^F L_\pi^3) \right\}
 \end{aligned} \tag{6.2}$$

where $\xi = M_\pi^2/(16\pi^2 F_\pi^2)$ and $L_\pi = \log(M_\pi^2/\mu^2)$. The analytic expressions of the $a_{ij}^{F,M}$ are in the previous section and the $b_{ij}^{F,M}$ can be found in appendix C. These are often referred to as the x - and ξ -expansion, see e.g. [18].

As input we use $\mu = 0.77$ GeV, $\bar{l}_1 = -0.4$, $\bar{l}_2 = 4.3$ from [19], $\bar{l}_3 = 3.41$, $\bar{l}_4 = 4.51$ from the $N_f = 2$ estimates of [18]. The numerical values for the r_i^r are taken from [5]

$$\begin{aligned}
 10^4 r_1^r &= -0.6, & 10^4 r_2^r &= 1.3, & 10^4 r_3^r &= -1.7, & 10^4 r_4^r &= -1.0, \\
 10^4 r_5^r &= 1.1, & 10^4 r_6^r &= 0.3, & r_M^r &= 0, & r_F^r &= 0.
 \end{aligned} \tag{6.3}$$

The remaining c_i^r and r_{M8}^r, r_{F8}^r have been set to zero. The resulting numerical values of the $a_{ij}^{F,M}$ and $b_{ij}^{F,M}$ can be found in table 1. Note that the numerical values of a_{30}^F and b_{30}^F are rather large. This is due to the very large numerical coefficient $383293667/1555200 \approx 246.5$ appearing there, the remaining coefficients are of more natural size.

The quantities in (6.1)–(6.2) are plotted in figure 2(a)–(d), with the same inputs as above. For the ξ -expansion we kept $F_\pi = 92.2$ MeV constant while varying M_π and for the x -expansion we kept $F = 92.2/1.073$ MeV constant while varying M . The convergence around the physical value $M_\pi^2 \approx 0.02$ GeV² is excellent. For the mass, the ξ -expansion converges much better, for the decay constant it is somewhat better. The effect of the very large constants a_{30}^F and b_{30}^F is clearly visible in the results for the decay constant.

7 Conclusions

In this paper we presented the calculation of the NNNLO contributions to the pion mass and decay constant in the isospin limit of two-flavour ChPT. We also calculated the mass in the $O(N)$ ϕ^4 case to show the principle. This required the evaluation of 2×34 diagrams

ij	a_{ij}^M	b_{ij}^M	a_{ij}^F	b_{ij}^F
10	0.0028	-0.0028	1.0944	-1.0944
11	0.5	-0.5	-1.0	1.0
20	1.6530	-1.6577	-0.0473	-1.1500
21	2.4573	-3.2904	-1.9058	4.1388
22	2.125	-0.625	-1.25	-0.25
30	0.4133	-6.8035	-244.5350	242.2724
31	-3.7044	4.2718	-15.4989	28.5703
32	17.1476	0.6204	-9.3946	-6.7751
33	4.2917	5.1458	-3.4583	-0.4167

Table 1. Numerical values of the $a_{ij}^{M,F}$ and $b_{ij}^{M,F}$ for the input parameters given in the text.

and their derivatives w.r.t. the momentum. The master integrals needed for the calculation were known. The tree level contributions from the NNNLO Lagrangian are unknown, but were here parameterized as free renormalized parameters. We reproduced the known NNLO results and the known leading logarithms.

A small numerical study of the two quantities was performed, and there was continuing good convergence at the physical pion mass.

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A Master integrals

Below, the master integrals needed for the calculation² are listed. To simplify the expressions of the master integrals below, we define

$$C(\varepsilon, M^2) = (4\pi)^\varepsilon \Gamma(1 + \varepsilon) M^{-2\varepsilon}. \tag{A.1}$$

Also, we denote the external momentum as p with $p^2 = M^2$, as is relevant for the quantities considered here.

A.1 One loop

We only need one one-loop integral, the well-known tadpole:

$$\frac{1}{i} \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 - M^2} = \frac{C(\varepsilon, M^2)}{16\pi^2} M^2 \left(\frac{1}{\varepsilon} + 1 + \varepsilon + \varepsilon^2 \right). \tag{A.2}$$

²In this section, everything has been truncated so as to only include terms relevant for this particular calculation.

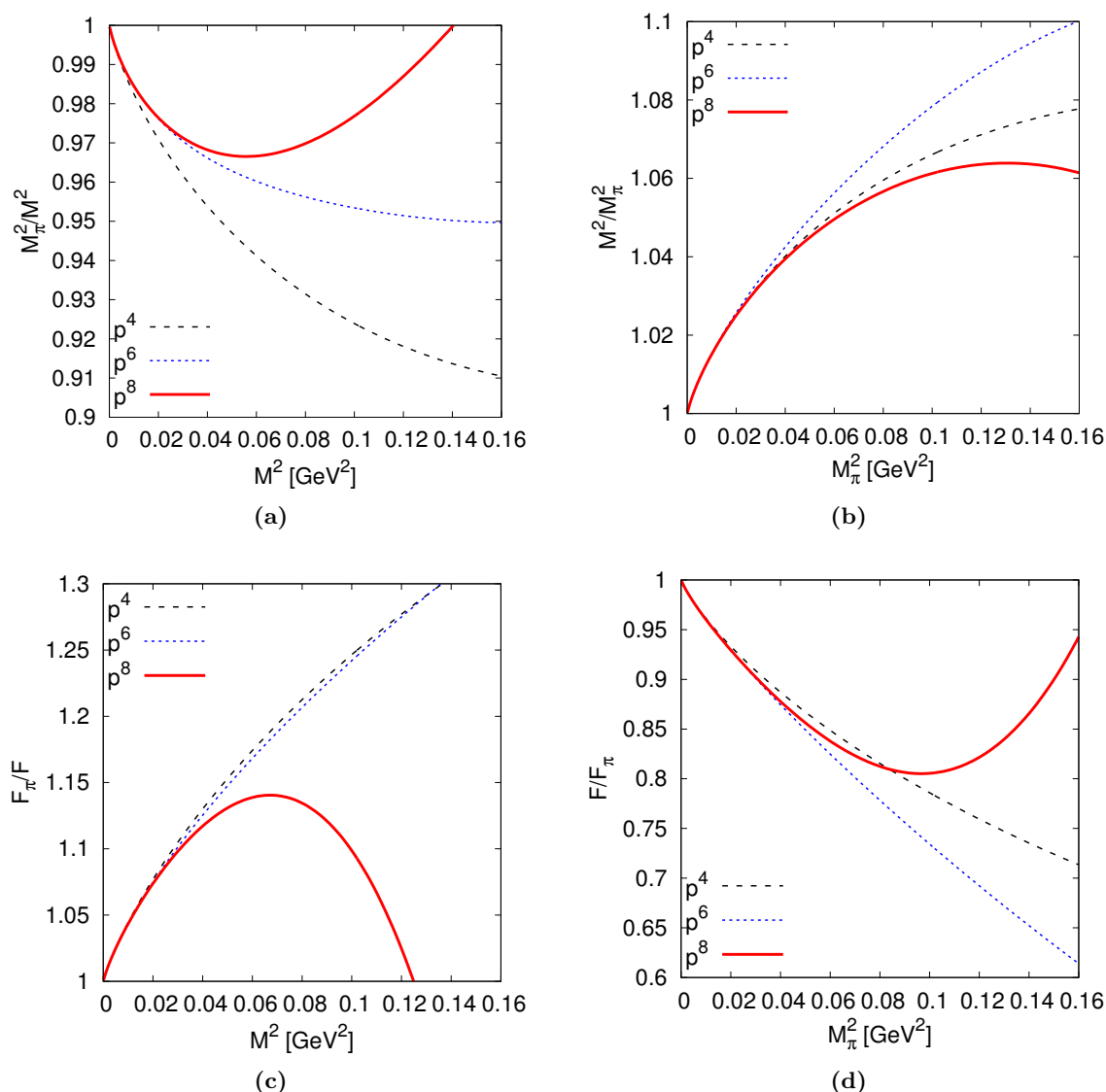


Figure 2. (a) The x -expansion for the mass, (b) the x -expansion for the decay constant, (c) the ξ -expansion for the mass, (d) the ξ -expansion for the decay constant at NLO, NNLO and NNNLO. LO is constant at 1 for all four plots.

A.2 Two loops

All needed two-loop integrals can be reduced to the equal mass on-shell sunset or products of tadpoles. The sunset result is, see e.g. [20]

$$\frac{1}{i^2} \int \frac{d^d k}{(2\pi)^d} \frac{d^d l}{(2\pi)^d} \frac{1}{(k^2 - M^2)(l^2 - M^2)((p - k - l)^2 - M^2)} = \left(\frac{C(\varepsilon, M^2)}{16\pi^2} \right)^2 M^2 \left[\frac{3}{2} \frac{1}{\varepsilon^2} + \frac{17}{4} \frac{1}{\varepsilon} + \frac{59}{8} + \left(\frac{65}{16} + 8\zeta_2 \right) \varepsilon \right]. \quad (\text{A.3})$$

A.3 Three loops

We only need two more master integrals at three-loop order. The other combinations are products of the one- and two-loop integrals. The first one is a vacuum integral:

$$\frac{1}{i^3} \int \frac{d^d k}{(2\pi)^d} \frac{d^d l}{(2\pi)^d} \frac{d^d q}{(2\pi)^d} \frac{1}{(k^2 - M^2)((k+l)^2 - M^2)((l+q)^2 - M^2)(q^2 - M^2)} = \left(\frac{C(\varepsilon, M^2)}{16\pi^2} \right)^3 M^4 \left[2 \frac{1}{\varepsilon^3} + \frac{23}{3} \frac{1}{\varepsilon^2} + \frac{35}{2} \frac{1}{\varepsilon} + \frac{275}{12} \right]. \quad (\text{A.4})$$

The second needed three-loop integral has external momentum running through it

$$\frac{1}{i^3} \int \frac{d^d k}{(2\pi)^d} \frac{d^d l}{(2\pi)^d} \frac{d^d q}{(2\pi)^d} \frac{1}{(k^2 - M^2)((k+l)^2 - M^2)((p-l)^2 - M^2)((l+q)^2 - M^2)(q^2 - M^2)} = \left(\frac{C(\varepsilon, M^2)}{16\pi^2} \right)^3 M^2 \left[\frac{1}{\varepsilon^3} + \frac{16}{3} \frac{1}{\varepsilon^2} + 16 \frac{1}{\varepsilon} + 20 - 2\zeta_3 + 16\zeta_2 \right]. \quad (\text{A.5})$$

These two are denoted as I_{12} and I_8 , respectively, in [8].

B The expressions for the r_i^r

These are reproduced from [15].

$$\begin{aligned} r_F^r &= 8c_7^r + 16c_8^r + 8c_9^r, \\ r_M^r &= -32c_6^r - 16c_7^r - 32c_8^r - 16c_9^r + 48c_{10}^r + 96c_{11}^r + 32c_{17}^r + 64c_{18}^r, \\ r_1^r &= 64c_1^r - 64c_2^r + 32c_4^r - 32c_5^r + 32c_6^r - 64c_7^r - 128c_8^r - 64c_9^r + 96c_{10}^r + 192c_{11}^r - 64c_{14}^r \\ &\quad + 64c_{16}^r + 96c_{17}^r + 192c_{18}^r, \\ r_2^r &= -96c_1^r + 96c_2^r + 32c_3^r - 32c_4^r + 32c_5^r - 64c_6^r + 32c_7^r + 64c_8^r + 32c_9^r - 32c_{13}^r + 32c_{14}^r \\ &\quad - 64c_{16}^r, \\ r_3^r &= 48c_1^r - 48c_2^r - 40c_3^r + 8c_4^r - 4c_5^r + 8c_6^r - 8c_{12}^r + 20c_{13}^r, \\ r_4^r &= -8c_3^r + 4c_5^r - 8c_6^r + 8c_{12}^r - 4c_{13}^r, \\ r_5^r &= -8c_1^r + 10c_2^r + 14c_3^r, \\ r_6^r &= 6c_2^r + 2c_3^r. \end{aligned} \quad (\text{B.1})$$

C Analytic expressions for $b_{ij}^{M,F}$

Below, the analytic forms of the coefficients $b_{ij}^{M,F}$ are given. The notation is the same as in section 5. We only give here the result in terms of the r_i^r .

The mass coefficients are

$$\begin{aligned}
 b_{10}^M &= -2l_3^r, \\
 b_{11}^M &= -\frac{1}{2}, \\
 b_{20}^M &= -\frac{163}{96} - r_M^q - 4l_3^q l_4^q + 8(l_3^q)^2 - 2l_2^q - l_1^q, \\
 b_{21}^M &= \frac{13}{3} - l_4^q + 11l_3^q + 8l_2^q + 14l_1^q, \\
 b_{22}^M &= -\frac{5}{8}, \\
 b_{30}^M &= -\frac{4209509}{777600} + r_q^6 - 3r_q^5 - \frac{7}{2}r_q^4 - \frac{1}{2}r_q^3 + \frac{13}{6}\zeta_3 - \frac{163}{24}l_4^q - 4l_4^q r_q^M + \frac{93}{16}l_3^q + 10l_3^q r_q^M \\
 &\quad + 128l_3^q c_6^q - 10l_3^q (l_4^q)^2 + 40(l_3^q)^2 l_4^q - 40(l_3^q)^3 - \frac{7451}{1200}l_2^q - 8l_2^q l_4^q + 12l_2^q l_3^q - \frac{3223}{1200}l_1^q \\
 &\quad - 4l_1^q l_4^q + 6l_1^q l_3^q, \\
 b_{31}^M &= \frac{134551}{8640} + 6r_q^6 + 14r_q^5 + 11r_q^4 + 5r_q^3 + 2r_q^2 + \frac{5}{2}r_q^1 + \frac{11}{2}r_q^M + \frac{52}{3}l_4^q - \frac{5}{2}(l_4^q)^2 - \frac{56}{3}l_3^q \\
 &\quad + 54l_3^q l_4^q - 84(l_3^q)^2 + \frac{291}{10}l_2^q + 32l_2^q l_4^q - 48l_2^q l_3^q + \frac{34}{5}l_1^q + 56l_1^q l_4^q - 84l_1^q l_3^q, \\
 b_{32}^M &= -\frac{71}{3} - \frac{1}{2}l_4^q - \frac{161}{4}l_3^q - 45l_2^q - 60l_1^q, \\
 b_{33}^M &= \frac{247}{48}. \tag{C.1}
 \end{aligned}$$

The coefficients for the decay constant are

$$\begin{aligned}
 b_{10}^F &= -l_4^q, \\
 b_{11}^F &= 1, \\
 b_{20}^F &= \frac{13}{192} - r_q^F - (l_4^q)^2 + 2l_3^q l_4^q + l_2^q + \frac{1}{2}l_1^q, \\
 b_{21}^F &= -\frac{29}{12} + 3l_4^q - 4l_2^q - 7l_1^q, \\
 b_{22}^F &= -\frac{1}{4}, \\
 b_{30}^F &= \frac{380653067}{1555200} - \frac{1}{2}r_q^6 + \frac{3}{2}r_q^5 + \frac{3}{2}r_q^4 + \frac{1}{4}r_q^3 - \frac{8}{9}\zeta_3 - 4c_{12}^q + 2c_6^q + c_5^q + \frac{65}{192}l_4^q + l_4^q r_q^M \\
 &\quad - 3l_4^q r_q^F + 32l_4^q c_6^q - 2(l_4^q)^3 - \frac{35}{24}l_3^q + 4l_3^q r_q^F + 8l_3^q (l_4^q)^2 - 8(l_3^q)^2 l_4^q - \frac{3069}{800}l_2^q + 5l_2^q l_4^q \\
 &\quad - \frac{2959}{600}l_1^q + \frac{5}{2}l_1^q l_4^q, \\
 b_{31}^F &= \frac{58121}{8640} + \frac{17}{6}r_q^6 - \frac{33}{2}r_q^5 - \frac{33}{2}r_q^4 - 3r_q^3 + \frac{1}{8}r_q^2 + \frac{13}{2}r_q^F + 16c_{20}^q + 20c_{14}^q + 96c_{12}^q \\
 &\quad - 148c_6^q + 46c_5^q - \frac{80}{3}c_3^q - \frac{145}{12}l_4^q + 10(l_4^q)^2 - \frac{5}{6}l_3^q - 16l_3^q l_4^q - \frac{389}{60}l_2^q - 20l_2^q l_4^q \\
 &\quad + \frac{63}{10}l_1^q - 35l_1^q l_4^q, \\
 b_{32}^F &= \frac{859}{144} - \frac{25}{4}l_4^q + \frac{29}{2}l_2^q + 16l_1^q, \\
 b_{33}^F &= -\frac{5}{12}. \tag{C.2}
 \end{aligned}$$

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