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# The Plancherel transform on $SL_2(k)$ and its application to the decomposition of tensor products of irreducible representations

By

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**Introduction.**

Let $k$ be a locally compact, non-discrete, totally disconnected topological field, with an odd residual characteristic. Let $G=SL_2(k)$ be the group of two by two unimodular matrices with entries in $k$. Let $\mathcal{S}(G)$ be the space of complex valued functions which are locally constant and compactly supported. We define and study the Plancherel transform of $f\in\mathcal{S}(G)$, and next define the Plancherel transform of a distribution on $G$, applying the Plancherel formula. We discuss tensor products of irreducible unitary representations of $G$, of the principal series, of the supplementary series or the special representation. Calculating the Plancherel transform of certain distributions, we give their explicit decomposition formulas into irreducibles.

To decompose tensor products is one of the fundamental problems in group representation theory, and many authors have been studying this problem. Historically, as to semisimple Lie groups and their related groups, there are works of L. Pukanszky [12], R. P. Martin [5] and J. Repka [13] for $SL_2(\boldsymbol{R})$, and G. Mackey [7] and M. A. Naimark [9], [10] and [11] for $SL_2(\boldsymbol{C})$, and N. Tatsuuma [18] for inhomogeneous Lorentz group. For principal series representations, the problem was studied by I. Gel'fand-M. Graev [2] and N. Anh [1] for $SL_n(C)$, and by F. Williams [19] for general complex semisimple Lie groups. For the present group $G=SL_2(k)$, Martin [6] studied the tensor products of a principal series representation with any irreducible one. He gave the decomposition formulas by an approach analogous to that of [19], that is, by using Mackey's subgroup theorem, tensor product theorem and Mackey-Anh's reciprocity theorem. The decompositions are expressed as a direct integral on the subset of the unitary dual $\hat{G}$ of $G$ with respect to the Plancherel measure.

However, the harmonic analysis on a semisimple Lie group is now much studied. So, it seems desirable to establish the decomposition formula in more explicit manner. Here, we give the decompositions directly, naturally obtaining the intertwining projections of the product spaces to each irreducible component. Our method is an extension of Naimark's idea and available for other groups.

We sketch the contents of this paper. W denoted by $p$ a fixed prime element in $k$, $q=|p|^{-1}$, and by $\varepsilon$ a fixed $(q-1)$-st primitive root of unity in $k$. In the first three sections, we summarize results concerning the Fourier analysis over $k$, given in [4] and [14], reconstructing some of them to fit on our purpose. In § 4, summarize results on the irreducible unitary representation of $G$. Most of them are well known ([4], [16] etc.). In this paper, we realize, for instance, the principal series representation $\mathcal{R}_\pi$ on the space $\mathcal{S}_\pi$, and its $\chi$-realization $\hat{\mathcal{R}}_\pi$ is on the space $\hat{\mathcal{S}}_\pi$. The operator for $g\in G$ of the representation $\hat{\mathcal{R}}_\pi$ is given by means of a kernel $K_\pi(g|u, v)$. The $\chi$-realization is useful for our decomposition.

In § 5, we define and study the Plancherel transform on $G$: for $f\in\mathcal{S}(G)$, we make correspond the function $K_\pi(f|u, v)$ of $u, v\in k^\times$ and $\pi$, where

$$K_\pi(f|u, v)=\int_G f(g)K_\pi(g|u, v)dg\,.$$

In § 6, we describe the tensor products $\mathcal{R}_{\pi_1}\otimes\mathcal{R}_{\pi_2}$ of

Case (I): principal series $\otimes$ principal series,

Case (II): supplementary series $\otimes$ principal series,

Case (III): supplementary series $\otimes$ supplementary series.

As the limiting cases of Case (II) and Case (III), we consider tensor products

Case (IV): the special representation $\otimes$ principal series,

Case (V): the special representation $\otimes$ supplementary series,

Case (VI): the special representation $\otimes$ the special representation.

For the tensor product in Case (I), we define an intertwining operator $U$ of $R=\{R_g, \mathcal{S}(G)\}$, the right regular representation, into $\mathcal{R}_{\pi_1}\otimes\mathcal{R}_{\pi_2}$, whose image is dense in $L^2\otimes L^2$. Let $\langle\,,\rangle$ be the inner product in $L^2\otimes L^2$. Put $B(f, f)=\langle\varphi, \varphi\rangle$ for $f\in\mathcal{S}(G)$, where $\varphi=Uf$. Then

$$B(f, f)=\int_G\int_G H(g_1g_1^{-1})f(g_1)\overline{f(g_2)}dg_1dg_2\,, \tag{1}$$

where $H(g)$ is a certain distribution on $G$. We give $H$ explicitly.

We define in § 7 the Plancherel transform $\hat{D}$ of a distribution $D$ on $G$. $\hat{D}(u, v, \pi)$ is formally given by $\int_G D(g)K_\pi(g^{-1}|v, u)dg$. We prove in Theorem 7.1 that for $H$ above $\hat{H}$ vanishes outside of $\Pi$, where $\Pi$ is the set of $\pi$ determined by the value of $\pi_1\pi_2(-1)$. The right hand side of (1) is rewritten as

$$\int_\Omega\int_k\int_k \hat{H}(u, v, \pi)K_\pi(\check{f}|t, u)\overline{K}_\pi(\check{f}|t, v)dtdudvm(\pi)d\pi\,, \tag{2}$$

where $m(\pi)d\pi$ is the Plancherel measure on $G$. After computing explicitly in § 8 the Plancherel transform $\hat{H}$, we have in § 9, the decomposition formula for Case (I). In more detail, rewriting (2) we obtain

$$\langle\varphi, \varphi\rangle=\sum_s\int_\Pi\langle\Phi, \Phi\rangle_\pi m(\pi)d\pi\,. \tag{3}$$

Here $\Phi=\Phi(t;\pi,s)$ is a function on $k\times\Pi\times\{1,\varepsilon,p,\varepsilon p\}$, and as a function in $t$, $\Phi$ is in the spaces of irreducible representations. A linear mapping $W:\varphi\to\Phi$ is extended to a unitary $G$-morphism of the space $L^2\otimes L^2$ onto the Hilbert space $\mathfrak{H}$ (in § 9.2). To prove that $W$ is *onto*, we use "continuous analogue of the Schur lemma" in [7]. The $G$-morphism $W$ display the decomposition for this case (Theorems 9.4 and 9.5).

In § 10, we can compute easily $\hat{H}$ for Case (II) by using results in Case (I) and establish the formula (3) and finally get the decomposition (Theorem 10.3). In § 11, 12, we treat the tensor product for Case (III). This case is further devided into two cases: for $\pi_i(x)=|x|^{\alpha_i}$, $-1<\alpha_i<0$ $(i=1,2)$,

Case (III. A): $0<1+\alpha_1+\alpha_2$; Case (III. B): $-1<1+\alpha_1+\alpha_2\leqq 0$.

We again compute $\hat{H}$ for $H$ in (III. A) by using results in Case (II) and the Hankel transform of a "homogeneous distribution" (Proposition 3.7). Then we get the decomposition (Theorem 11.4). For Case (III. B), $\hat{H}$ can not be computed directly, so we do it by analytic continuation of $\hat{H}$ in (III. A). In the decomposition formula, there appears a supplementary series representation as a new discrete component (Theorem 12.3).

In the last section, § 13, we give the decomposition formulas of the limiting cases. For Case (VI), it is obtained by taking the limit $\alpha_1\to-1$ in the formula (3) for Case (II) (Theorem 13.4). For Cases (V) and (VI), decompositions are obtained from the formula (3) in Case (III), but the supplementary series component disappears here (Theorems 13.5, 13.6).

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## 1. Preliminaries.

Let $k$ be a locally compact, totally disconnected, non-discrete topological field, $k^+$ the additive group, $k^\times$ the multiplicative group, $O$ the ring of integers in $k$, $P$ the maximal ideal in $O$ and $p$ a generator of $P$. $O/P$ is a finite field with $q$ elements, $q$ a prime power. Throughout this paper we assume that $q$ is odd. Let $dx$ denote the Haar measure on $k^+$, normalized that $\int_O dx=1$. The valuation (non-archimedean) $|\circ|$ on $k$ is determined by $d(ax)=|a|dx$, $a\in k^\times$, and $|0|=0$ for which $|p|^{-1}=q$, $O=\{x;|x|\leqq 1\}$ and $P=\{x;|x|<1\}$. Let $O^\times=O\backslash P$ be the group of units in $O$. Let $\varepsilon$ be a primitive $(q-1)$-st root of unity in $O^\times$, then the collection $\{0,1,2,\cdots,\varepsilon^{q-2}\}$ is a complete set of coset representatives for $O/P$. The set $A_1=1+P=\{x;|1-x|<1\}$ is a compact subgroup of $O^\times$, and every element $x$ of $k^\times$ can be uniquely written as $x=p^n y$, $y=\varepsilon^m a$, $(n\in\boldsymbol{Z}$, $0\leqq m\leqq q-2$, $y\in O^\times$ and $a\in A_1)$. Thus $|x|=q^{-n}$ and $k\simeq\boldsymbol{Z}\times O^\times=\boldsymbol{Z}\times\boldsymbol{Z}_{q-1}\times A_1$, $\boldsymbol{Z}_{q-1}\simeq\boldsymbol{Z}/(q-1)$.

We denote by $(k^\times)^2$ the set of square elements in $k^\times$, then it is known that in our case, $q$ an odd number, $(k^\times)^2$ is a subgroup of $k^\times$ of index four, and a complete set of coset representatives of $k^\times/(k^\times)^2$ is given by $E=\{1,\varepsilon,p,\varepsilon p\}$:

$k^\times=(k^\times)^2\cup\varepsilon(k^\times)^2\cup p(k^\times)^2\cup\varepsilon p(k^\times)^2$.

Any quadratic extension of $k$ is, up to isomorphism, given by $L_\tau=k(\sqrt{\tau})$, $\tau\in E'=\{\varepsilon, p, \varepsilon p\}$. The norm $N_\tau$ and the trace $S_\tau$ for $z=x+\sqrt{\tau}y$ are defined by $N_\tau(z)=z\bar{z}=x^2-\tau y^2$ and $S_\tau(z)=z+\bar{z}=2x$ respectively.

The subgroup $k_\tau^\times=N_\tau(L_\tau^\times)$ of $k^\times$ includes $(k^\times)^2$ and $[k^\times : k_\tau^\times]=[k_\tau^\times : (k^\times)^2]=2$. Furthermore if $-1\in(k^\times)^2$, $k_\tau^\times=(k^\times)^2\cup\tau(k^\times)^2$ for each $\tau\in E'$, and if $-1\notin(k^\times)^2$, $k_\varepsilon^\times=(k^\times)^2\cup\varepsilon(k^\times)^2$, $k_p^\times=(k^\times)^2\cup\varepsilon p(k^\times)^2$ and $k_{\varepsilon p}^\times=(k^\times)^2\cup p(k^\times)^2$.

The collection $\{P^n\}_{n=1,2,\dots}$ of ideals $P^n=\{x\,;\,|x|\leqq q^{-1}\}$ in $k$, gives a neighborhood basis of 0 in $k^+$. There is a unitary character $\chi(x)$ on $k^+$ which is trivial on $O=P^0$ but non-trivial on $P^{-1}$. Every unitary character on $k^+$ has the form $\chi(ux)$ for some $u\in k$. The Fourier transform on $k^+$ is defined for $f\in L^1$ as $\hat{f}(u)=\mathcal{F}f(u)=\int_k f(x)\chi(ux)dx$, and its inverse transform as $f(u)=\mathcal{F}^{-1}\hat{f}(u)=\int_k \hat{f}(x)\chi(-ux)dx$.

Let $\mathcal{S}$ be the space of testing functions on $k$, that is, the space of complex valued functions which are locally constant and compactly supported. The space $\mathcal{S}$ is topologized by defining a null sequence to be $\{\varphi_n\}$ where $\{\varphi_n\}$ all vanish outside a fixed compact set, and are constant on each cosets of a fixed $P^m$, and tend uniformly to zero. The Fourier transform $\hat{\varphi}$ of $\varphi\in\mathcal{S}$ again belongs to $\mathcal{S}$ and, if $\varphi$ is constant on the cosets of $P^n$ and supported by $P^{-m}$, then $\hat{\varphi}$ is constant on the cosets of $P^m$ and supported in $P^{-n}$. Thus the Fourier transform is a topological isomorphism of $\mathcal{S}$ onto itself. Each element in $\mathcal{S}'$, the topological dual of $\mathcal{S}$, is called a distribution on $k$. The Fourier transform $\hat{f}$ of a distribution $f$ is defined by $(\hat{f}, \varphi)=(f, \hat{\varphi})$ for $\varphi\in\mathcal{S}$.

The principal value integral is defined for a locally summable function $f$ by

$$P\text{-}\int_k f(x)dx=\lim_{n\to\infty}\int_k [f]_n(x)dx\,, \tag{1.1}$$

where $[f]_n$ is a function equal to $f$ on the set $\{x\,;\,q^{-n}\leqq|x|\leqq q^n\}$ and zero outside. The principal value integral Fourier transform $P\text{-}\int_k f(x)\chi(ux)dx=(P\text{-}\mathcal{F})f(u)$ coincides with the usual transform for $f\in L^1$, and if $(P\text{-}\mathcal{F})f(u)$ exists for $f\in L^2$, it coincides for almost all $u\in k$ with the usual $\hat{f}(u)$ by the Plancherel theorem.

We set $A_n=1+P^n$, $n\geqq 1$ and $A_0=O$. The collection $A_n$ is a neighborhoods basis of 1 in $k^\times$. The character group $\tilde{k}^\times$ of $k^\times$ is expressed as $k^\times=T\times\tilde{O}^\times$, $\tilde{O}^\times=\boldsymbol{Z}_{q-1}\times\tilde{A}_1$, where $T=[-\pi/\log q,\ \pi/\log q)$ is one dimensional torus and $\tilde{O}^\times$ a countable set, Each element $\pi$ of $\tilde{k}^\times$ is written as $\pi(x)=|x|^{i\gamma}\theta(x)$ where $\gamma\in T$, and $\theta$ is determined by $\theta(p)=1$ and $\theta|O^\times$. The measure $d\pi$ on $\tilde{k}^\times$ is given by $\sum_{\theta\in\tilde{O}^\times}d\gamma$, $\int_T d\gamma=1$.

Following [14], we say that, when $\theta\equiv 1$, $\pi$ is unramified or has ramification degree 0 and that, when $\theta$ is trivial on $A_h$ and non-trivial on $A_{h-1}(h\geqq 1)$, $\pi$ is ramified of degree $h$.

Non-unitary characters on $k^\times$ are obtained by replacing the pure imaginary

$i\gamma$ by a complex number $\alpha$ with non-zero real part. Non-unitary characters in which we are specially interested are of the form $\pi(x)=|x|^{\alpha}$, $\alpha$ real and $-1<\alpha<0$. The following character is called the signature of $k^{\times}$ with respect to $\tau$:

$$\text{(1.2)} \qquad \operatorname{sgn}_{\tau}(x)=\begin{cases} 1 & x\in k_{\tau}^{\times}, \\ -1 & x\in k^{\times}\backslash k_{\tau}^{\times}. \end{cases}$$

The character $\operatorname{sgn}_{\varepsilon}(x)=|x|^{-\pi i/\log q}$ is unramified, and $\operatorname{sgn}_{p}(x)$ and $\operatorname{sgn}_{\varepsilon p}(x)$ are both ramified of degree 1. In the following, we will denote $|x|$ by $\rho(x)$, and $|x|^{-1}$ by $\pi_{sp}(x)$. The latter relates with the special representation.

Let $\mathcal{S}^{\times}$ be the space of functions $\varphi$ in $\mathcal{S}$, satisfying $\varphi(0)=0$. It is the space of testing functions on $k^{\times}$. On this space the Mellin transform is defined by $\tilde{\varphi}(\pi)=\int_{k}\varphi(x)\pi(x)d^{\times}x$ where $d^{\times}x=\rho^{-1}(x)dx$ (the Haar measure on $k^{\times}$).

The image $\tilde{\mathcal{S}}^{\times}$ under the Mellin transform of $\mathcal{S}^{\times}$ is a space of functions on $\tilde{k}^{\times}$. It is proved that, for $\varphi$ supported by the set $\{x\,;\,q^{-n}\leqq|x|\leqq q^{n}\}$, the Mellin transform $\tilde{\varphi}(\pi)=\tilde{\varphi}(\alpha,\theta)$ is characterized by

$$\text{(1.3)} \qquad \tilde{\varphi}(\alpha,\theta)=\sum_{k=-n}^{n}a_{k}(\theta)q^{ik\alpha}, \qquad \pi(x)=|x|^{\alpha}\theta(x).$$

Here, $a_{k}(\theta)$ vanish except for only a finite number of $\theta$.

The gamma function is defined for all characters $\pi$ of $k^{\times}$ (not necessarily unitary) except $\pi\equiv 1$. If $\pi(x)=|x|^{\alpha}\theta(x)$ is ramified of degree $h\geqq 1$,

$$\text{(1.4)} \qquad \Gamma(\pi)=\Gamma_{\theta}(\alpha)=P\text{-}\int_{k}\pi(x)\chi(x)d^{\times}x=C_{\theta}q^{h(\alpha-1/2)},$$

where $C_{\theta}$ is a complex number such that $|C_{\theta}|=1$ and $C_{\theta}C_{\theta^{-1}}=\theta(-1)$. If $\pi(x)=|x|^{\alpha}$, $\operatorname{Re}(\alpha)>0$,

$$\text{(1.5)} \qquad \Gamma(\pi)=\Gamma(\alpha)=P\text{-}\int_{k}\pi(x)\chi(x)d^{\times}x=\frac{1-q^{\alpha-1}}{1-q^{-\alpha}}.$$

For $\operatorname{Re}(\alpha)\leqq 0$, $\alpha\neq 0$, $\Gamma(\alpha)$ is defined as the analytic continuation of (1.5) and is meromorphic, zero at $\alpha=1$ and has a pole at $\alpha=0$.

The gamma function on $k$ is closely related to the Fourier analysis on $k$ as in the case of the usual gamma function on $R$. For instance, $f(x)=\Gamma(\pi)^{-1}\pi(x)\rho^{-1}(x)=\Gamma(\pi)^{-1}\pi\rho^{-1}(x)$ is a homogeneous distribution on $k$, and if $\pi\equiv 1$, it denotes the delta function $\Delta$ on $k$ supported at 0. The Fourier transform $f$ of this distribution is given by $\hat{f}=\pi^{-1}$:

$$\text{(1.6)} \qquad (f,\hat{\varphi})=\frac{1}{\Gamma(\pi)}(\pi\rho^{-1},\hat{\varphi})=(\pi^{-1},\varphi),$$

For $\pi(x)=|x|^{\alpha}\theta(x)$, this is proved first in case $(0<\operatorname{Re}\alpha<1)$ by changing the order of integration, then by analitic continuation to any $\alpha$.

## § 2. The spaces $\mathcal{S}_\pi$ and $\hat{\mathcal{S}}_\pi$.

Let $\pi=|\cdot|^\alpha\theta$ be a non unitary character of $k^\times$, $\rho(x)=|x|$, $T_w^\pi$ a mapping of $\mathcal{S}$ such that $T_w^\pi\varphi(x)=\pi\rho^{-1}(x)\varphi(-1/x)$, and $\mathcal{S}_\pi$ the linear span of $\mathcal{S}$ and $T_w^\pi\mathcal{S}$. In this section we study the Fourier transform $\hat{\mathcal{S}}_\pi$ of the space $\mathcal{S}_\pi$, in the sense of principal value integral, and in the next section we study the Fourier transform of $T_w^\pi$.

**2.1.** For $\varphi\in\mathcal{S}$, $\pi\rho^{-1}(x)\varphi(-1/x)$ is locally constant, zero on a neighborhood of 0 in $k$, and $\pi\rho^{-1}(x)\varphi(0)$ for large $|x|$. Therefore every function $f$ in $\mathcal{S}_\pi$ is canonically expressed as

$$f=\varphi+a\phi_\pi\,, \tag{2.1}$$

where $\varphi\in\mathcal{S}$, $a$ a complex number and

$$\phi_\pi(x)=\begin{cases} 0\,, & |x|\leqq 1\,,\\ \pi\rho^{-1}(x)\,, & |x|>1\,. \end{cases} \tag{2.1a}$$

The topology in $\mathcal{S}_\pi$ is defined in such a way that $\{\varphi_n+a_n\phi_\pi\}$ is a null sequence, if $\{\varphi_n\}$ is a null sequence in $\mathcal{S}$ and $a_n\to 0$. Then $T_w^\pi$ is an isomorphic mapping of $\mathcal{S}_\pi$ onto itself.

**Lemma 2.1.** ([14], Lemmas 1 and 2)

$$\int_{|x|=q^k}\chi(x)dx=\begin{cases} q^k\left(1-\frac{1}{q}\right), & k<0\,,\\ -1\,, & k=1\,,\\ 0\,, & k>1\,. \end{cases} \tag{A}$$

*If $\pi$ is ramified of degree $h\geqq 1$,*

$$\int_{|x|=q^k}\pi(x)\chi(x)d^\times x=\begin{cases} \Gamma(\pi)\,, & k=h\,,\\ 0\,, & k\neq h\,. \end{cases} \tag{B}$$

We set

$$G_n(u\,;\pi)=P\text{-}\int_{q^n|u|\leq|x|}\pi\rho^{-1}(x)\chi(x)dx\,,\qquad u\in k^\times,\ n>0\,. \tag{2.2}$$

Then the following holds.

**Lemma 2.2.** *For $\pi=|\cdot|^\alpha$, $\alpha\neq 0$,*

$$G_n(u\,;\pi)=\begin{cases} 0\,, & |u|>q^{-n+1}\,,\\ \Gamma(\pi)-\frac{q^{(n-1)\alpha}}{1-q^{-\alpha}}\left(1-\frac{1}{q}\right)\pi(u)\,, & 0<|u|\leqq q^{-n+1}\,, \end{cases} \tag{A}$$

*and for $\pi\equiv 1$ $(\alpha=0)$,*

$$G_n(u\,;\pi)=\begin{cases} 0\,, & |u|>q^{-n+1}\,,\\ \left(\frac{-\log|u|}{\log q}-u+1\right)\left(1-\frac{1}{q}\right)-\frac{1}{q}\,, & 0<|n|\leqq q^{-n+1}\,. \end{cases} \tag{B}$$

*For $\pi$ is ramified of degree $h\geqq 1$,*

(C) $$G_n(u\,;\,\pi)=\begin{cases}0\,, & |u|>q^{-n+h}\,,\\ \Gamma(\pi)\,, & 0<|u|\leqq q^{-n+h}\,.\end{cases}$$

*Proof.* It is easy to see, from Lemma 2.1, that the values of $G_n(u\,;\,\pi)$ are zero for $|u|>q^{-n+1}$ in (A) and (B), and for $|u|>q^{-n+h}$ in (C). Let $\pi(x)=|x|^\alpha$ and $|u|=q^{-m}\leqq q^{-n+1}$, then

$$G_n(u\,;\,\pi)=\int_{q^n|u|\leq|x|}\pi\rho^{-1}(x)\chi(x)dx=\sum_{k=n-m}^{0}q^{\alpha k}(1-q^{-1})-q^{\alpha-1}\,.$$

By the direct calculation we obtain the required formulas in (A) and (B). Similarly we obtain the formula in (C). Q. E. D.

**Remark.** For $\pi\equiv 1$, $G_n(u\,;\,\pi)=\lim_{\alpha\to 0}G_n(u\,;\,|\cdot|^\alpha)$.

**2.2.** We consider the Fourier transform of $f\in\mathcal{S}_\pi$. Let $f=\varphi+a\psi_\pi$ be as in (2.1). We consider

$$\hat{f}(u)=(P\text{-}\mathcal{F})f(u)=P\text{-}\int_k f(x)\chi(xu)dx\,,\qquad u\in k^\times\,.$$

Since $(P\text{-}\mathcal{F})\psi_\pi$ is given in Lemma 2.2 as $(P\text{-}\mathcal{F})\psi_\pi=P\text{-}\int_{1<|x|}\pi\rho^{-1}(x)\chi(xu)\,dx$ $=\pi^{-1}(u)G_1(u\,;\,\pi)$, the principal value integral Fourier transform $\hat{f}$ of $f\in\mathcal{S}_\pi$ always exists and $\hat{f}=\hat{\varphi}+a\pi^{-1}(\cdot)G_1(\cdot\,;\,\pi)$ for every $\pi=|\cdot|^\alpha\theta$. In particular, take the constant function $1=\varphi_0(x)+\psi_\pi(x)$ in $\mathcal{S}_\pi$, where $\varphi_0$ is the characteristic function of $O$ and $\pi(x)=|x|$. Then $\hat{1}=\hat{\varphi}_0(u)+|u|^{-1}G_1(u\,;\,\pi)=0$ for $u\in k^\times$. Note that if $\mathrm{Re}(\alpha)<1/2$, then $\mathcal{S}_\pi\subset L^2$ and $\hat{f}\in\mathcal{S}_\pi$ coincides for almost all $u$ with the Fourier transform in $L^2$-sense, and moreover if $\mathrm{Re}(\alpha)<0$, $\mathcal{S}_\pi\subset L^1$ and then $\hat{f}$ coincides with the usual one,

As to the inverse transform, we consider the integral $P\text{-}\int_k\hat{\psi}_\pi(u)\chi(-xu)du$. This integral converges only for $\pi$ in $\mathrm{Re}(\alpha)<1$ and coincides with $\psi_\pi(x)$. Thus we have the following proposition,

**Prooposition 2.3.** *The principal value integral Fourier transform $\hat{\mathcal{S}}_\pi$ of $\mathcal{S}_\pi$ is the space of the functions on $k^\times$ of the form $\hat{f}=\hat{\varphi}+a\pi^{-1}(\cdot)G_1(\cdot\,;\,\pi)$ where $\varphi\in\mathcal{S}$, $a\in\boldsymbol{C}$ and $G_1(u\,;\,\pi)$ is in (2.2). For $\pi$ in $\mathrm{Re}(\alpha)<1$, the inverse transform $\hat{\mathcal{S}}_\pi\to\mathcal{S}_\pi$ is given by the usual inverse Fourier transform $\mathcal{F}^{-1}$.*

The space $\hat{\mathcal{S}}_\pi$ is topologized by null sequences $\{\varphi_n+a_n\pi^{-1}(\cdot)G_1(\cdot\,;\,\pi)\}$ where $\{\varphi_n\}$ are null sequences in $\mathcal{S}$ and $a_n\to 0$, then the mapping $P\text{-}\mathcal{F}:\mathcal{S}_\pi\to\hat{\mathcal{S}}_\pi$ is continuous and moreover for $\pi$ with $\mathrm{Re}(\alpha)<1$, it is topological.

For the case $\pi=\pi_{sp}=|\cdot|^{-1}$, there exists a $T_w^\pi$-invariant subspace $\mathcal{S}_{sp}$ in $\mathcal{S}_\pi$, consisting of functions $f$ such that $\int_k f(x)dx=0$. Since $\mathcal{S}_\pi\subset L^1$, every function

$f$ in $\mathcal{S}_\pi$ has the usual Fourier transform $\hat{f}(u)=\varphi(u)+a|u|G_1(u\,;|\cdot|^{-1})$ with $\varphi\in\mathcal{S}$. The condition "$f\in\mathcal{S}_{sp}$" is equivalent to "$\varphi\in\mathcal{S}^\times$". Therefore $\hat{\mathcal{S}}_{sp}$ is the space of functions of the form $\varphi+a|u|G_1(u\,;|\cdot|^{-1})$, $\varphi\in\mathcal{S}^\times$.

**2.3.** Let $\lambda$ be a non-unitary character of $k^\times$, then for $f\in\mathcal{S}_\pi$ the integrals $\int_k\lambda(x)f(x)dx$ and $\int_k\lambda(u)\hat{f}(u)du$ converge under suitable conditions on $\lambda$ and $\pi$, and they give linear forms, "distributions" on $\mathcal{S}_\pi$ and $\hat{\mathcal{S}}_\pi$ respectively. The following is on the Fourier transform of distributions.

**Proposition 2.4.** *Let* $\pi=|\cdot|^\alpha\theta$ *and* $\lambda=|\cdot|^\beta\tau$, $\theta,\tau\in\tilde{O}^\times$. *Assume that* $0<\mathrm{Re}(\beta)<1$ *and* $\mathrm{Re}(\beta-\alpha)>0$. *Then for* $f\in\mathcal{S}_\pi$,

$$\int_k\lambda\rho^{-1}(u)\hat{f}(u)du=\Gamma(\lambda)\int_k\lambda^{-1}(x)f(x)dx$$

To prove this, we need the following:

**Lemma 2.5.** *Let* $\lambda$ *and* $\pi$ *be as above. Then the function*

$$\Phi(u)=\sum_{k=1}^{\infty}|\lambda\pi^{-1}(u)\rho^{-1}(u)\{G_k(u\,;\pi)-G_{k+1}(u:\pi)\}|$$

*is zero if* $|u|\geqq q^s$, $s=\max.(1,h)$, *and* $h$ *the ramified degree of* $\pi$. *Moreover* $\Phi$ *is summable.*

*Proof.* This is proved by concrete forms on $G_k(u\,;\pi)-G_{k+1}(u\,;\pi)$ which we can calculate from Lemma 2.2.

*Proof of Proposition* 2.4. Let $f=\varphi+a\psi_\pi$ be in $\mathcal{S}_\pi$. For $f=\varphi$, we have already the desired equality in (1.6).

Now for $\psi_\pi$,

$$\int_k\lambda\rho^{-1}(u)(P\text{-}\mathcal{F})\psi_\pi(u)du=\int_k\lambda\rho^{-1}(u)\lim_{n\to\infty}\int_{1<|x|\leq q^n}\pi\rho^{-1}(x)\chi(ux)dxdu$$

$$=\int_k\lim_{n\to\infty}\Phi_n(u)du$$

where $\Phi_n(u)=\lambda\pi^{-1}\rho^{-1}(u)\{G_1(u\,;\pi)-G_{n+1}(u\,;\pi)\}$. We have

$$|\Phi_n(u)|=|\lambda\pi^{-1}\rho^{-1}(u)\{G_1(u\,;\pi)-G_{n+1}(u\,;\pi)\}|$$

$$\leqq\sum_{k=1}^{n}|\lambda\pi^{-1}\rho^{-1}(u)\{G_k(u\,;\pi)-G_{k+1}(u\,;\pi)\}|\leqq\Phi(u)\,,$$

where $\Phi$ is the function in Lemma 2.5 which is zero if $|u|>q^s$, $s$ large enough, and is summable. So, by Lebesgue's theorem,

$$\int_k\lim_{n\to\infty}\Phi_n(u)du=\lim_{n\to\infty}\int_{|u|\leq q^s}\Phi_n(u)du$$

$$=\lim_{n\to\infty}\int_{|u|\leq q^s}\lambda\rho^{-1}(u)\left\{\int_{1<|x|\leq q^n}\pi\rho^{-1}(x)\chi(ux)dx\right\}du\,,$$

and by Fubini's theorem, we can change the order of integration and finally come to

$$\lim_{n\to\infty}\int_{1<|x|\leq q^n}\pi\rho^{-1}(x)\left\{\int_{|u|\leq q^s}\lambda\rho^{-1}(u)\chi(ux)du\right\}dx$$

$$=\Gamma(\lambda)\lim_{n\to\infty}\int_{1<|x|\leq q^n}\lambda^{-1}\pi\rho^{-1}(x)dx=\Gamma(\lambda)\int_k\lambda^{-1}(x)\psi_\pi(x)dx\,. \qquad \text{Q.E.D.}$$

**Corollary 2.6.** *Let* $\lambda$ *and* $\pi$ *be as in Proposition* 2.4, *and moreover we assume* $\mathrm{Re}(\alpha)<1$. *Then it holds*

$$\int_k\lambda\rho^{-1}(x)\mathcal{F}^{-1}\hat{f}(x)dx=\Gamma(\lambda)\lambda(-1)\int_k\lambda^{-1}(u)\hat{f}(u)du\,, \qquad f\in\mathcal{S}_\pi\,.$$

*Proof.* Replace $\lambda$ by $\lambda^{-1}\rho$ in the formula of Proposition 2.4, and used $\Gamma(\lambda^{-1})\Gamma(\lambda\rho)=\lambda(-1)$. Q.E.D.

## §3. The Hankel transform.

**3.1.** The Bessel function of order $\pi$ is defined as follows: for $u, v\in k^\times$

$$J_\pi(u, v)=P\text{-}\int_k\chi(ux+vx^{-1})\pi(x)d^\times x \tag{3.1}$$

$$=\lim_{n\to\infty}\sum_{k=-n}^{n}\int_{|x|=q^k}\chi(ux+vx^{-1})\pi(x)d^\times x\,.$$

This principal value integral converges. In fact, for fixed $u, v\in k^\times$, except only a finite number of $k$ in (3.1), integration terms vanish. Because, from Lemma 2.1, for large $k>0$,

$$\int_{|x|=q^{-k}}\chi(ux+vx^{-1})\pi(x)d^\times x=\int_{|x|=q^{-k}}\chi(vx^{-1})\pi(x)d^\times x$$

$$=\pi(v)\int_{|x|=|v|q^k}\chi(x)\pi^{-1}(x)d^\times x=0\,,$$

and

$$\int_{|x|=q^k}\chi(ux+vx^{-1})\pi(x)d^\times x=\pi^{-1}(u)\int_{|x|=|u|^{-1}q^k}\chi(x)\pi(x)d^\times x=0\,.$$

Thus we remark that for every compact subset $A\subset k^\times$, there exist large $n>0$ such that

$$J_\pi(u, v)=\sum_{k=-n}^{n}\int_{|x|=q^{-k}}\chi(ux+vx^{-1})\pi(x)d^\times x\,, \qquad u, v\in A\,.$$

The Bessel functions have the following properties:

(B.1) $J_\pi(-u, -v)=\pi(-1)J_\pi(u, v)$,

(B.2) $J_\pi(u, v)=J_{\pi^{-1}}(v, u)$,

(B.3) $\pi(u)J_\pi(u, v)=\pi(v)J_{\pi^{-1}}(u, v)$,

(B.4) $J_\pi(u, v)=\overline{J_{\bar\pi}(-u, -v)}$, where $\bar\pi(x)=\overline{\pi(x)}$.

(B.5) If $\pi$ is ramified of degree $h\geqq 0$, $\alpha\neq 1$ and $|uv|<q^l$, $l=\max.(1, h)$,

$$J_\pi(u, v)=\pi(v)\Gamma(\pi^{-1})+\pi^{-1}(u)\Gamma(\pi),$$

and if $\pi\equiv 1$ and $|uv|<q$, $J_\pi(u, v)$ is obtained by the limit of $J_\pi(u, v)$, $\pi=|\cdot|^\alpha$, as $\alpha\to 0$. It equals $(1-q^{-1})\left(\frac{-\log|uv|}{\log q}+1\right)-2q^{-1}$. (see [14]).

**3.2.** The Hankel transform of order $\pi$ is defined for $\hat f\in\hat{\mathcal S}_\pi$ by

$$H_\pi\hat f(u)=\int_k J_\pi(u, v)\hat f(v)dv, \qquad u\in k^\times. \tag{3.2}$$

**Proposition 3.1.** *Let $\pi=|\cdot|^\alpha\theta$ such that $-1<\mathrm{Re}(\alpha)<1$, and $\hat f\in\hat{\mathcal S}_\pi$. For $u\in k^\times$,*

$$H_\pi\hat f(u)=P\text{-}\int_k \pi\rho^{-1}(x)f\left(\frac{-1}{x}\right)\chi(ux)dx. \tag{3.3}$$

*Proof.* Since $f=\mathcal F^{-1}(P\text{-}\mathcal F)f=\mathcal F^{-1}\hat f$ from Proposition 2.3, then it holds

$$\int_{q^{-n}\leq|x|\leq q^n}\int_k \chi(ux+vx^{-1})\pi\rho^{-1}(x)\hat f(v)dvdx$$

$$=\int_{q^{-n}\leq|x|\leq q^n}\pi\rho^{-1}(x)f(-x^{-1})\chi(ux)\,dx.$$

The right hand side tends, as $n\to\infty$, to that of (3.3). Since the integrand on the left hand side of above is summable, we can change the order of integration, and then it equals

$$\int_k \hat f(v)\int_{q^{-n}\leq|x|\leq q^n}\chi(ux+vx^{-1})\pi\rho^{-1}(x)dxdv. \tag{3.4}$$

We prove this tends to the left hand side of (3.3). Let $|u|=q^m$ and $n>m$. Choose an integer $l$ such that $\hat f$ is zero for $|v|>q^l$. Then, (3.4) equals

$$\int_{q^{-m}\leq|v|\leq q^l}\hat f(v)\int_{q^{-n}\leq|x|\leq q^n}\cdots\,dxdv+\int_{|v|<q^{-m}}\hat f(v)\int_{q^{-n}\leq|x|\leq q^n}\cdots\,dxdv \tag{3.5}$$

$$=\text{(i)}+\text{(ii)}.$$

Since $\int_{q^{-n}\leq|x|\leq q^n}\chi(ux+vx^{-1})\pi\rho^{-1}(x)dx=J_\pi(u, v)$, for $n$ large enough and for $|v|$ such that $q^{-m}\leqq|v|\leqq q^l$, we have

$$\text{(i)}=\int_{q^{-m}\leq|v|\leq q^l}\hat f(v)J_\pi(u, v)dv. \tag{3.6}$$

On the other hand,

$$\text{(3.7)}\qquad \text{(ii)}=\int_{q^{-n}\le|v|\le q^{-m}}\hat{f}(v)\left\{\int_{q^{-n}\le|x|\le q^{-m}}\chi(vx^{-1})\pi\rho^{-1}(x)dx\right\}dv$$

$$+\int_{|v|<q^{-n}}\hat{f}(v)\left\{\int_{q^{-n}\le|x|\le q^{-m}}\chi(vx^{-1})\pi\rho^{-1}(x)dx\right\}dv$$

$$+\int_{|v|<q^{-m}}\hat{f}(v)\left\{\int_{q^{-m}<|x|\le q^{n}}\chi(ux)\pi\rho^{-1}(x)dx\right\}dv\,.$$

We denote the inner integrals in (3.7) by $A_n(v)$, $B_n(v)$ and $C_n(u)$ respectively.

$$A_n(v)=\pi(v)\int_{q^m|v|\le|x|\le q^n|v|}\chi(x)\pi^{-1}\rho^{-1}(x)dx=\pi(v)\{G_m(v\,;\,\pi^{-1})-G_{n+1}(u\,;\,\pi^{-1})\}.$$

For $B_n(v)$,

$$B_n(v)=B_n=\int_{q^{-n}\le|x|\le q^{-m}}\chi(vx^{-1})\pi\rho^{-1}(x)dx=\int_{q^m\le|x|\le q^n}\pi^{-1}\rho^{-1}(x)dx$$

$$=\begin{cases}(1-q^{-1})q^{-m\alpha}\{q^{-(n-m+1)\alpha}-1\}(q^{-\alpha}-1)^{-1}, & \pi=|\cdot|^{\alpha},\ \alpha\neq 0,\\ (1-q^{-1})(n-m+1), & \pi\equiv 1,\\ 0, & \pi \text{ ramified}.\end{cases}$$

For $C_n(u)$ with $n$ large enough,

$$C(u)=C_n(u)=\int_{q^{-m}\le|x|\le q^n}\chi(ux)\pi^{-1}(x)dx$$

$$=\pi^{-1}(u)\int_{1<|x|\le q^n|u|}\chi(x)\pi\rho^{-1}(x)dx=\begin{cases}-q^{\alpha-1}\pi^{-1}(u), & \pi \text{ unramified},\\ \pi^{-1}(u)\Gamma(\pi), & \pi \text{ ramified}.\end{cases}$$

Thus

$$\text{(ii)}=\int_{q^{-n}\le|v|\le q^{-m}}\hat{f}(v)\pi(v)A_n(v)dv+B_n\int_{|v|<q^{-m}}\hat{f}(v)dv+C(u)\int_{|v|<q^{-m}}\hat{f}(v)dv$$

$$=\int_{q^{-n}\le|v|<q^{-m}}\hat{f}(v)\{\pi(v)G_m(v\,;\,\pi^{-1})+C(u)\}\,dv-\int_{q^{-n}\le|v|<q^{-m}}\hat{f}(v)\pi(v)G_{n+1}(v\,;\,\pi^{-1})dv$$

$$+B_n\int_{|v|<q^{-n}}\hat{f}(v)dv+C(u)\int_{|v|<q^{-n}}\hat{f}(v)dv$$

$$=\text{(a)}+\text{(b)}+\text{(c)}+\text{(d)}\,.$$

We show that, as $n\to\infty$,

$$\text{(3.8)}\qquad \text{(a)}\longrightarrow\int_{|v|<q^{-m}}\hat{f}(v)J_\pi(u,v)dv\,,\qquad \text{and (b), (c), (d)}\to 0\,.$$

For (a), since $|v|<q^{-m}$, the direct calculation and (B.5) leads us to

$$\pi(v)G_m(v\,;\,\pi^{-1})+C(u)=\pi(v)\Gamma(\pi^{-1})+\pi^{-1}(u)\Gamma(\pi)=J_\pi(u,v)\,.$$

Considering the function of the form $\hat{f}=\varphi+a\pi^{-1}(\cdot)G_m(\cdot\,;\,\pi)$, we get

$$(\mathrm{a}) \longrightarrow \int_{|v|<q^{-m}} \hat{f}(v) J_\pi(u, v) dv .$$

Next we show the other integral terms (b), (c) and (d) tend to zero. For (b), $G_{n+1}(v; \pi^{-1})$ is zero if $|v|>q^{-n}$, then

$$(\mathrm{b})=\int_{|v|=q^{-n}} \hat{f}(v)\pi(v)G_{n+1}(v; \pi^{-1})dv$$

$$=\int_{|v|=q^{-n}} \varphi(v)\pi(v)G_{n+1}(v; \pi^{-1})dv+\int_{|v|=q^{-n}} G_1(v; \pi)G_{n+1}(v; \pi^{-1})dv .$$

These integrals converge for $\pi$, $-1<\mathrm{Re}(\alpha)<1$, and tend to zero as $n\to\infty$ with order of $q^{-n\alpha}$ and $q^{(\alpha-1)n}$. For (c), if $\pi$ is unramifed and $\neq 1$, $B_n$ is bounded as $n\to\infty$, and from the summability of $\hat{f}$, we get (c) tends to zero. If $\pi\equiv 1$, (c) equals

$$(1-q^{-1})^{-1}(n-m+1)\int_{|v|<q^{-n}} \{\varphi(v)+G_1(v; \pi)\} dv .$$

It is easy to see that this tends to zero as $n\to\infty$. If $\pi$ is ramified, the integral (c) already equals zero. Thus (c) tends to zero.

It is also easy to see that (d) tends to zero. Thus we proved (ii) tends to $\int_{|v|<q^{-m}} \hat{f}(v)J_\pi(u, v)dv$. Combining this with (3.6), we get the proof. Q. E. D.

From this proposition, for $\pi$ in $-1<\mathrm{Re}(\alpha)<1$ and $u\in k^\times$, the integral in (3.2) converges, and $H_\pi=(P\text{-}\mathcal{F})T_w^\pi\mathcal{F}^{-1}$. Note that for $\hat{f}\in\hat{\mathcal{S}}_\pi$, $H_\pi\hat{f}$ is again in $\hat{\mathcal{S}}_\pi$, and moreover $H_\pi$ gives an isomorphism of $\hat{\mathcal{S}}_\pi$ onto itself.

**Corollary 3.2.** *For $\pi$ in $-1<\mathrm{Re}(\alpha)<1$, $H_\pi^2=\pi(-1)I$.*

*Proof.* It is a consequence of the fact that $(T_w^\pi)^2=\pi(-1)I$, and Propositions 2.3 and 3.1.

**3.3.** We have the following propositions:

**Proposition 3.3.** *Let $\pi=|\cdot|^\alpha\theta$ in $-1<\mathrm{Re}(\alpha)<1$, then*

(I) $$\int_k H_\pi\hat{f}(u)H_{\pi^{-1}}\hat{h}(u)du=\pi(-1)\int_k \hat{f}(u)\hat{h}(u)du , \qquad \hat{f}\in\hat{\mathcal{S}}_\pi,\ \hat{h}\in\hat{\mathcal{S}}_{\pi^{-1}} .$$

(II) $$\int_k \pi(u)H_\pi\hat{f}(u)H_\pi\hat{h}(u)du=\pi(-1)\int_k \pi(u)\hat{f}(u)\hat{h}(u)du , \qquad \hat{f},\ \hat{h}\in\hat{\mathcal{S}}_\pi .$$

*Proof.* (I) Let $\pi$ be as above. We can assume that $-1<\mathrm{Re}(\alpha)\leqq 0$. In case $\mathrm{Re}(\alpha)=0$ ($\pi$ is unitary), $H_\pi=(P\text{-}\mathcal{F})T_w^\pi\mathcal{F}^{-1}$ by Proposition 3.1 and each operator in the right hand side is defined on $L^2$, then the usual Plancherel transform for $L^2$ gives the formula (I).

In case $-1<\mathrm{Re}(\alpha)<0$, $\hat{f}\in\hat{\mathcal{S}}_\pi\subset L^2$ and $H_\pi\hat{f}\in\hat{\mathcal{S}}_\pi$ is a bounded function on $k$.

On the other hand, $H_{\pi^{-1}}\hat{h}\in\hat{\mathcal{S}}_{\pi^{-1}}$ is in $L^1$ and is the limit of $\hat{g}_n(u)=\int_k g_n(x)\chi(ux)dx$, where $g_n(x)\in\mathcal{S}$ is a function equal to $\pi^{-1}\rho^{-1}(x)h(-x^{-1})$ if $|x|\leqq q^n$ and zero otherwise. We have an expression $\pi^{-1}\rho^{-1}(x)h(-x^{-1})=\phi+c\psi_{\pi^{-1}}$, $\varphi\in\mathcal{S}$, $\psi_{\pi^{-1}}$ as (2, 1a) and $c\in\boldsymbol{C}$. Then

$$\begin{aligned}\hat{g}_n(u)&=\hat{\phi}+c\int_{1\leq|x|\leq q^n}\pi^{-1}\rho^{-1}(x)\chi(ux)dx\\&=\hat{\phi}+c\pi(u)\{G_1(u\,;\,\pi^{-1})-G_{n+1}(u\,;\,\pi^{-1})\},\end{aligned}$$

and

$$\begin{aligned}|\hat{g}_n(u)|&\leqq|\hat{\phi}(u)|+|c|\,|\pi(u)\{G_1(u\,;\,\pi^{-1})-G_{n+1}(u\,;\,\pi^{-1})\}|\\&\leqq|\hat{\phi}(u)|+|c|\,|\sum_{k=1}^{n}\pi(u)\{G_k(u\,;\,\pi^{-1})-G_{k+1}(u\,;\,\pi^{-1})\}|\\&\leqq|\hat{\phi}(u)|+|c|\Phi(u),\end{aligned}$$

where $\Phi$ is the summable function given by $\Phi(u)=\sum_{k=1}^{n}|\pi(u)\{G_k(u\,;\,\pi^{-1})-G_{k+1}(u\,;\,\pi^{-1})\}|$. By the Lebesgue's theorem,

$$\int_k H_\pi\hat{f}(u)H_{\pi^{-1}}\hat{h}(u)du=\int_k\lim_{n\to\infty}H_\pi\hat{f}(u)\hat{g}_n(u)du=\lim_{n\to\infty}\int_k H_\pi\hat{f}(u)\hat{g}_n(u)du\,.$$

Since $\pi\rho^{-1}(x)f(-x^{-1})\in L^2$, the above equals

$$\begin{aligned}\lim_{n\to\infty}\int_k\pi\rho^{-1}(-x)f(-x^{-1})g_n(-x)dx&=\pi(-1)\int_k f(x)h(-x)dx\\&=\pi(-1)\int_k\hat{f}(u)\hat{h}(u)du\,.\end{aligned}$$

(II) $\pi(u)\hat{h}(u)$ is in $\hat{\mathcal{S}}_{\pi^{-1}}$, therefore from (I),

$$\int_k H_\pi\hat{f}(u)H_{\pi^{-1}}(\pi\hat{h})(u)du=\pi(-1)\int_k\pi(u)\hat{f}(u)\hat{h}(u)du\,.$$

On the other hand, $H_{\pi^{-1}}(\pi\hat{h})(u)=\int_k J_{\pi^{-1}}(u,v)\pi(v)h(v)dv$, and (B.3) shows that $H_{\pi^{-1}}(\pi\hat{h})(u)=\pi(u)H_\pi\hat{h}(u)$. This gives the formula (II). Q. E. D.

**Corollary 3.4.** *Let $\hat{f},\hat{h}\in\hat{\mathcal{S}}_\pi$. If $\pi$ is unitary* (Re $(\alpha)=0$),

$$\int_k H_\pi\hat{f}(u)\overline{H_\pi\hat{h}(u)}du=\int_k\hat{f}(u)\overline{\hat{h}(u)}du\,.$$

*If $\pi=|\cdot|^\alpha$, $\alpha$ real and $-1<\alpha<1$,*

$$\int_k\pi(u)H_\pi\hat{f}(u)\overline{H_\pi\hat{h}(u)}du=\int_k\pi(u)\hat{f}(u)\overline{\hat{h}(u)}du\,.$$

*Proof.* First equality is obtained from the fact $\bar{\pi}=\pi^{-1}$, (B.4) and (I). The second one from (II). Q. E. D.

By the corollary, there hold that, for $\pi$ unitary, $\|H_\pi \hat{f}\|=\|\hat{f}\|$, where $\|\hat{f}\|^2=\int |\hat{f}(u)|^2 du$, and that for $\pi(x)=|x|^\alpha$, $\alpha$ real and $-1<\alpha<1$, $\|H_\pi \hat{f}\|_\pi=\|\hat{f}\|_\pi$, where $\|f\|_\pi^2=\int_k \pi(u)|\hat{f}(u)|^2 du$. If $\pi$ is unitary, $\hat{\mathcal{S}}_\pi$ is dense in $L^2$. If $\pi(x)=|x|^\alpha$, $-1<\alpha<1$, $\hat{\mathcal{S}}_\pi$ is dense in $L_\pi^2$, the space of square summable functions with respect to the measure $\pi(u)du$. Thus $H_\pi$ can be extended as a unitary operator of $L^2$ and of $L_\pi^2$ respectively.

**3.4.** Proposition 3.1 is extended to the cases $\pi=\pi_{sp}=|\cdot|^{-1}$ and $\pi=|\cdot|$ as follows.

**Proposition 3.5.** (1) *For* $\pi=|\cdot|$ *and* $f\in\mathcal{S}$ *it holds*

$$H_\pi \hat{f}(u)=P\text{-}\int_k f\Big(\frac{-1}{x}\Big)\chi(ux)dx=(P\text{-}\mathcal{F})T_w^\pi\mathcal{F}^{-1}\hat{f}(u)\,.$$

(2) *For* $\pi=\pi_{sp}$, *let* $H_{sp}=H_{\pi_{sp}}|\hat{\mathcal{S}}_{sp}$, $\mathcal{S}_{sp}$ *the space of functions* $f$ *in* $\mathcal{S}_{\pi_{sp}}$ *such that* $\int_k f(x)dx=0$. *Then it holds for* $\hat{f}\in\hat{\mathcal{S}}_{sp}$.

$$H_{sp}\hat{f}=P\text{-}\int_k |x|^2 f\Big(\frac{-1}{x}\Big)\chi(ux)dx=\mathcal{F}T_w^\pi\mathcal{F}^{-1}f(u)\,.$$

*Proof.* The proofs are similar to that of Proposition 3.1 but the convergence of integrations (a), (b), (c) and (d) in (ii) in this proposition should be checked. Q. E. D.

**Proposition 3.6.** *For* $\hat{f}$, $\hat{h}\in\hat{\mathcal{S}}_{sp}$, *it holds*

$$\int_k |u|^{-1}H_{sp}\hat{f}(u)H_{sp}\hat{h}(u)du=\int_k |u|^{-1}\hat{f}(u)\hat{h}(u)du\,.$$

*Proof.* Since for $\pi=\pi_{sp}$, $\pi H_{sp}\hat{h}=H_{\pi^{-1}}(\pi\hat{h})$ and $\pi\hat{h}\in\mathcal{S}$, it is enough to prove the equality

$$\int_k H_\pi\hat{f}(u)H_{\pi^{-1}}\hat{h}(u)du=\int_k \hat{f}(u)\hat{h}(u)du\,,\qquad \hat{f}\in\hat{\mathcal{S}}_{sp}\text{ and }\hat{h}\in\mathcal{S}\,.$$

In case $h\in\mathcal{S}^\times$, $h(-x^{-1})$ is also in $\mathcal{S}^\times$ and then $H_{\pi^{-1}}\hat{h}(u)=P\text{-}\int_k h(-x^{-1})\cdot\chi(ux)dx\in\mathcal{S}$. Thus $H_\pi\hat{f}$, $H_{\pi^{-1}}\hat{h}\in L^2$. This leads to the equality by the Plancherel theorem. In case $h\notin\mathcal{S}^\times$, $h(-x^{-1})$ is expressed as $\varphi_1(x)+c$, $\varphi_1\in\mathcal{S}$ and $c\in\boldsymbol{C}$. Since $P\text{-}\int_k 1\,\chi(ux)dx=0$ for $u\in k^\times$, we get $H_{\pi^{-1}}\hat{h}(u)=\hat{\varphi}_1(u)$ by Proposition 3.5, and

$$\int_k H_{sp}\hat{f}(u)H_{\pi^{-1}}\hat{h}(u)du=\int_k \rho^{-2}(x)f(-x^{-1})\varphi_1(x)dx$$

$$=\int_k f(x)\{h(-x)-c\}dx=\int_k f(x)h(-x)dx=\int_k \hat{f}(u)\hat{h}(u)du\,,\qquad \text{Q. E. D.}$$

By Proposition 3.5 (2), $H_{sp}$ gives an isomorphism of $\hat{S}_{sp}$ onto it self. Again, by Proposition 3.6, it holds for $\hat{f}\in\mathcal{S}_{sp}$, $\|H_{sp}\hat{f}\|_{sp}=\|\hat{f}\|_{sp}$, where $\|\hat{f}\|_{sp}^2=\int_k|u|^{-1}|f(u)|^2du$, and $H_{sp}$ can be extended as a unitary operator of $L_{sp}^2$. Here $L_{sp}^2$ the space of square summable functions with respect to the measure $|u|^{-1}du$.

**3.5.** The Hankel transform $H_\pi$, $\pi=|\cdot|^\alpha\theta$ and $-1<\mathrm{Re}(\alpha)<1$, is an isomorphism of $\hat{S}_\pi$ onto itself, and so is $H_{sp}$ for $\hat{S}_{sp}$. The following proposition is on the Hankel transform of the distribution $\lambda=|\cdot|^\beta\tau$, $\tau\in\tilde{O}^\times$.

**Proposition 3.7.** *Let* $\pi=|\cdot|^\alpha\theta$, $-1<\mathrm{Re}(\alpha)<1$ (resp. $\pi=\pi_{sp}$), *and* $\lambda=|\cdot|^\beta\tau$ *such that* $0<\mathrm{Re}(\beta)<1$ *and* $0<\mathrm{Re}(\beta-\alpha)$. *Then for* $\hat{f}\in\hat{S}_\pi$ (resp. $\hat{f}\in\hat{S}_{sp}$),

$$\int_k\lambda\rho^{-1}(u)H_\pi\hat{f}(u)du=\Gamma(\lambda)\Gamma(\lambda\pi^{-1})\lambda\pi(-1)\int_k\lambda^{-1}\pi(u)\hat{f}(u)du\,.$$

*Proof.* This equality is a consquence of Proposition 3.1 (resp. Proposition 3.6), Proposition 2.4 and Corollary 2.6. In fact,

$$\int_k\lambda\rho^{-1}(u)(P\text{-}\mathcal{F})(\pi\rho^{-1}(\cdot)f(-x^{-1}))(u)du=\Gamma(\lambda)\int_k\lambda^{-1}\pi\rho^{-1}(x)f(-x^{-1})dx$$

$$=\Gamma(\lambda)\lambda\pi(-1)\int_k\lambda\pi^{-1}\rho^{-1}(x)f(x)dx=\Gamma(\lambda)\Gamma(\lambda\pi^{-1})\lambda\pi(-1)\int_k\lambda^{-1}\pi(u)\hat{f}(u)du\,.\quad \text{Q.E.D.}$$

## §4. Unitary representations of $SL_2(k)$.

In this section we describe unitary representations of $G=SL_2(k)$.

Let $G$ be the group of matrices $g=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$, $\alpha\delta-\beta\gamma=1$, with elements $\alpha$, $\beta$, $\gamma$ and $\delta$ in $k$. We consider the subgroups of $G$ as follows:

$$D=\left\{d(a)=\begin{pmatrix}a^{-1} & 0\\ 0 & a\end{pmatrix};\ a\in k^\times\right\}\simeq k^\times\,, \tag{4.1}$$

$$N^+=\left\{n^+(y)=\begin{pmatrix}1 & y\\ 0 & 1\end{pmatrix};\ y\in k\right\},\quad N=\left\{n(x)=\begin{pmatrix}1 & 0\\ x & 1\end{pmatrix};\ x\in k\right\}\simeq k^+\,.$$

Put

$$w=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}. \tag{4.2}$$

Let $G^0$ be the dense subset in $G$ of elements $g$ such that $\delta\neq0$. Every element $g\in G^0$ can be decomposed as follows:

$$g=d(a)n^+(y)n(x)\,,\qquad \text{with } a=\delta,\ y=\delta\beta \text{ and } x=\gamma\delta^{-1}\,. \tag{4.3}$$

**4.1.** Let $\pi$ be a (not necessary unitary) character of $k^\times$. It is extended to that of the subgroup $B=DN^+$ by $\pi(b)=\pi(a)$ for $b=d(a)n^+(y)\in B$. The induced representation $\mathrm{Ind}_B^G\pi$ is realized on $\mathcal{S}_\pi$, and for which the operator $T_g^\pi$ is

$$T_g^\pi\varphi(x)=\pi(\beta x+\delta)|\beta x+\delta|^{-1}\varphi\left(\frac{\alpha x+\gamma}{\beta x+\delta}\right), \qquad \varphi\in\mathcal{S}_\pi . \tag{4.4}$$

In particular, $T_w^\pi\varphi(x)=\pi\rho^{-1}(x)\varphi(-1/x)$. We denote this representation by $\mathfrak{R}_\pi=\{T^\pi, \mathcal{S}_\pi\}$.

When $\pi\in\tilde{k}^\times$, $\mathfrak{R}_\pi$ is extended to a unitary representation with respect to the norm

$$\|\varphi\|^2=\int_k|\varphi(x)|^2dx , \tag{4.5}$$

known as a representation of principal series, and is irreducible except in the cases $\pi(x)=\mathrm{sgn}_\tau x$ with $\tau=\varepsilon,\ p,\ \varepsilon p$.

When $\pi(x)=|x|^\alpha$, $-1<\alpha<1$ and $\alpha\neq 0$, $\mathfrak{R}_\pi$ is extended to a unitary one with respect to the norm

$$\|\varphi\|_\pi^2=\frac{1}{\Gamma(\pi^{-1})}\int_k\int_k\pi^{-1}\rho^{-1}(x_1-x_2)\varphi(x_1)\overline{\varphi(x_2)}dx_1dx_2 , \tag{4.6}$$

known as an irreducible representation of supplementary series.

Representations $\mathfrak{R}_\pi$ and $\mathfrak{R}_{\pi^{-1}}$ of principl series or of supplementary series, are equivalent and the intertwining operator $E_\pi$: $\mathfrak{R}_\pi\to\mathfrak{R}_{\pi^{-1}}$ is given by

$$E_\pi\varphi(x)=\frac{1}{\Gamma(\pi^{-1})}\int_k\pi^{-1}\rho^{-1}(x-x')\varphi(x')dx' , \tag{4.7}$$

The special representation $\mathfrak{R}_{sp}$ arises as the limiting case of supplementary series $\mathfrak{R}_\pi$, $\pi(x)=|x|^\alpha$, as $\alpha\to-1$. $\mathfrak{R}_{sp}$ is defined as $\mathfrak{R}_{\pi_{sp}}|\mathcal{S}_{sp}$, and is extended to a unitary one with respect to the norm

$$\|\varphi\|_{sp}^2=c\int_k\int_k\log|x_1-x_2|\varphi(x_1)\overline{\varphi(x_2)}dx_1dx_2 , \tag{4.8}$$

where $c=(1-q^{-1})(\log q)^{-1}$, and is irreducible. As to this norm, $\|\varphi\|_{sp}=\lim_{\alpha\to-1}\|\varphi\|_\alpha$ for a compactly supported function $\varphi$ in $\mathcal{S}_{sp}$.

Representations $\mathfrak{R}_\pi$ given above are realized in another way called the $\chi$-realization. It is the Fourier transform $\hat{\mathfrak{R}}_\pi=\{\hat{T}^\pi, \hat{\mathcal{S}}_\pi\}$ of the representation $\mathfrak{R}_\pi$. We already discussed the space $\hat{\mathcal{S}}_\pi$. The transformed operator $\hat{T}_g^\pi=(P\text{-}\mathcal{F})T_g^\pi\mathcal{F}^{-1}$ are expressed by means of a kernel $K_\pi(g|u, v)$ which is a distribution for every $u\in k$ given as follows:

for $\varphi\in\hat{\mathcal{S}}_\pi$, $\hat{T}_g^\pi\varphi(u)=\int_k K_\pi(g|u, v)\varphi(v)dv$

$$=\int_k\pi(a)|a|\Delta(v-a^2u)\varphi(v)dv=\pi(a)|a|\varphi(a^2u) , \qquad g=d(a) , \tag{4.9}$$

$$=\int_k\chi(-ux)\Delta(v-u)\varphi(v)dv=\chi(-ux)\varphi(u) , \qquad g=n(x) , \tag{4.10}$$

$$=H_\pi\varphi(u)=\int_k J_\pi(u, v)\varphi(v)dv\,, \qquad g=w\,, \tag{4.11}$$

where $\Delta$ is the delta distribution on $k$ supported at 0. The operator $\hat{T}_g^\pi$ for other elements $g=g_1g_2$ is given as

$$\int_k K_\pi(g_1g_2|u, v)\varphi(v)dv=\int_k\int_k K_\pi(g_1|u, t)K_\pi(g_2|t, v)\varphi(v)dvdt\,. \tag{4.12}$$

The relation $T_g^{\pi^{-1}}E_\pi=E_\pi T_g^\pi$ is transformed into

$$K_{\pi^{-1}}(g|u, v)\pi(v)=\pi(u)K_\pi(g|u, v)\,. \tag{4.13}$$

In the Fourier transform $\hat{\mathfrak{R}}_{sp}=\{\hat{T}^\pi, \hat{\mathcal{S}}_{sp}\}$ of the special representation, the transformed operators $\hat{T}^\pi$ for $d(a)$, $n(x)$ and $w$ have the same expression as (4.9), (4.10) and (4.11) for $\varphi\in\hat{\mathcal{S}}_{sp}$ respectively.

**4.2.** Let $\tau$ be a fixed element in $\{\varepsilon, p, \varepsilon p\}$, $L_\tau=k(\sqrt{\tau})$ the quadratic extension. $L_\tau$ is a local field of the same kind as $k$ with the valuation $|z|_\tau=N_\tau(z)$ for $z=x+\sqrt{\tau}\,y\in L_\tau$. The Haar measure on $L_\tau$ and $L_\tau^\times$ are given by $dz=dxdy$ and $d^\times z=dxdy/|z|_\tau$ respectively. A set of elements $t$ in $L_\tau$ satisfying $t\bar{t}=a$ for an $a\in k^\times$ is called a circle in $L_\tau$. The circle $C_\tau$ with $a=1$ is the (compact) kernel of the homomorphism $N_\tau$: $L_\tau^\times\to k_\tau^\times$. On a circle $C_\tau$ we denote a measure $d^\times t$, invariant under multiplication of element in $C_\tau$, normalized as $\int_{C_\tau}d^\times t=1$.

Fix $\nu\in L_\tau^\times$ such that $\nu\bar{\nu}\notin(k^\times)^2$. If $N_\tau(z)=r^2\in(k^\times)^2$, $z$ is written as $rt$ for some $t\in C_\tau$. If $N_\tau(z)\notin(k^\times)^2$, then $N_\tau(z)=\nu\bar{\nu}r^2\in\nu\bar{\nu}(k^\times)^2$ and $z$ is written as $\nu rt$. $(r, t)$ or $(\nu r, t)$ is the polar coordinate of $z$, but $(r, t)$ and $(-r, -t)$, or $(\nu r, t)$ and $(-\nu r, -t)$ give the same elements.

If a function $f(z)$ on $L_\tau$ satisfies $f(tz)=f(z)$ for $t\in C_\tau$, then $f(z)=\varphi(N_\tau(z))$, where $\varphi$ is a function on $k$. For a summable function $f$, we have

$$\int_{L_\tau} f(z)dz=a_\tau\int_k\varphi(x)dx\,, \tag{4.14}$$

where

$$a_\tau=\frac{2(1+q^{-1})}{1+|\tau|} \quad\text{and}\quad \varphi(N_\tau(z))=\int_{C_\tau} f(tz)d^\times t\,.$$

Representations of the discrete series are obtained as invariant components of the Weil representation. The latter is defined as follows. Let $\mathcal{S}(L_\tau)$ be the space of complex valued, compactly supported, locally constant functions $\Phi$ on $L_\tau$. For $\Phi\in\mathcal{S}(L_\tau)$,

$$W_g\Phi(z)=\begin{cases} \operatorname{sgn}_\tau(a)|a|\Phi(az)\,, & g=d(a)\,,\\ \chi(xN_\tau(z))\Phi(z)\,, & g=n(x)\,,\\ c_\tau\hat{\Phi}(z)\,, & g=w\,,\end{cases} \tag{4.15}$$

where the coefficient $c_\tau$ is determined by $c_\tau=\frac{a_\tau}{2}\int_k\chi(x)\operatorname{sgn}_\tau(x)dx$, and

(4.16) $$\hat{\Phi}(z)=\int_{L_\tau}\chi(S_\tau(z\bar{z}'))\Phi(z')dz'\,,\qquad \text{with } S_\tau(z)=z+\bar{z}\,.$$

For $t\in C_\tau$, we define the opertor $R_t$ in $\mathcal{S}(L_\tau)$ by $R_t\Phi(z)=\Phi(tz)$, then $R_t$ commutes with $W_g$. Let $\pi$ be a unitary character of $C_\tau$, and $\mathcal{S}(L_\tau, \pi)$ be the subspace of functions $\Phi$ in $\mathcal{S}(L_\tau)$ satisfying $R_t\Phi=\pi(t)\Phi$. Then, $\mathcal{S}(L_\tau, \pi)$ is an invariant subspace. Putting $T_g^\pi=W_g|\mathcal{S}(L_\tau, \pi)$, we define a representation $\mathfrak{R}_\pi^+=\{T_g^\pi, \mathcal{S}(L_\tau, \pi)\}$.

We set

(4.17) $$\Phi_\pi(z)=\int_{C_\tau}\Phi(tz)\overline{\pi(t)}d^\times t$$

for $\Phi\in\mathcal{S}(L_\tau)$. Then $\Phi_\pi$ is in $\mathcal{S}(L, \pi)$ and we have the inversion formula $\Phi(z)=\sum_{\pi\in\tilde{C}_\tau}\Phi_\pi(z)$, where $\tilde{C}_\tau$ is the character group of $C_\tau$, and the Plancherel formula $\int_{L_\tau}|\Phi(z)|^2dz=\sum_{\pi\in\tilde{C}_\tau}\int_{L_\tau}|\Phi_\pi(z)|^2dz$. So we get the decomposition of $\{W_g, \mathcal{S}(L_\pi)\}$ into $\{T_g^\pi, \mathcal{S}(L_\tau, \pi)\}$.

**Lemma 4.1.** *For every $\pi\not\equiv 1$ and $\Phi\in\mathcal{S}(L_\tau)$, $\Phi_\pi$ vanishes on a neighborhood of 0 in $L_\tau$. Moreover, $\Phi_\pi\equiv 0$ except for a finite number of $\pi\in\tilde{C}_\tau$.*

*Proof.* Let $\mathfrak{P}_\tau$ be the maximal ideal in $\mathfrak{O}_\tau$, the ring of integers in $L_\tau$. Suppose that $\Phi$ is supported by $\mathfrak{P}_\tau^{-n}$ and constant on the cosets of $\mathfrak{P}_\tau^n$ for some positive integer $n$. We set $\Phi=\Phi_1+\Phi_2$, $\Phi_1$ equal to $\Phi$ if $z\in\mathfrak{P}_\tau^n$ and zero otherwise. Then $\Phi_\pi=(\Phi_1)_\pi+(\Phi_2)_\pi$. Clearly $(\Phi_1)_\pi=0$ for $\pi\not\equiv 1$. $\Phi_2$ is supported by $\mathfrak{P}_\tau^{-n}\cap(\mathfrak{P}_\tau^n)^c$ and constant on the cosets of $\mathfrak{P}_\tau^n$, then $\Phi_2(tz)=\Phi_2(z)$ for $t\in(1+\mathfrak{P}_\tau^{2n})\cap C_\tau$ and $z\in L_\tau$. Therefore, if $\pi$ is not trivial on $(1+\mathfrak{P}_\tau^{2n})\cap C_\tau$, $(\Phi_2)_\pi\equiv 0$. The number of characters which are trivial on $(1+\mathfrak{P}_\tau^{2n})\cap C_\tau$ is finite. Thus the lemma. Q.E.D.

The following is known. If $\pi$ is not of order two, the representation $\mathfrak{R}_\pi^+$ is irreducible, $\mathfrak{R}_\pi^+$ and $\mathfrak{R}_{\pi^{-1}}{}^+$ are equivalent, and the interwining operator $E_\pi$ between $\mathfrak{R}_\pi^+$ and $\mathfrak{R}_{\pi^{-1}}{}^+$ is given by the form

(4.18) $$E_\pi:\ \Phi_\pi(z)\longrightarrow\Phi_\pi(\bar{z})\,.$$

**4.3.** If $\pi$ is of order two, the intertwining operator $E_\pi$ maps $\mathcal{S}(L_\pi, \pi)$ into itself. In order to study the reducibility of $\mathfrak{R}_\pi^+$, we should discuss in detail the character $\pi$. We confine ourselves $\pi=\pi_0$, the character of order two in $\tilde{C}_\varepsilon$. Let $C_\varepsilon'$ be the subgroup $(1+\mathfrak{P}_\varepsilon)\cap C_\varepsilon$ of $C_\varepsilon$. The index of $C_\varepsilon'$ in $C_\varepsilon$ is $q+1$. Since $\pi_0$ is of order two. $\pi_0$ is trivial on $C_\varepsilon'$, and $\pi_0(t)=1$ or $-1$ according as $t$ is a square element in $C_\varepsilon$ or not. We set $S^1=\{z\in L_\varepsilon;\ N_\varepsilon(z)\in(k^\times)^2\}$ and $S^2=\{z\in L_\varepsilon;\ N_\varepsilon(z)\notin(k^\times)^2\}$. The following proposition holds.

**Proposition 4.2.** *The representation of discrete series $\mathfrak{R}_{\pi_0}^+$, $\pi_0$ the character of $C_\varepsilon$ of order two, splits into two irreducible components $\mathfrak{R}_0^1=\{T_g^{\pi_0}, \mathcal{S}(L_\varepsilon, \pi_0)|S^1\}$ and $\mathfrak{R}_0^\varepsilon=\{T_g^{\pi_0}, \mathcal{S}(L, \pi_0)|S^2\}$, where $\mathcal{S}(L_\varepsilon, \pi_0)|S^1$ is the space of functions in*

$\mathcal{S}(L_\varepsilon, \pi_0)$ *supported in* $S^1$.

To prove this, we need the following lemmas.

**Lemma 4.3.** *If* $-1\in(k^\times)^2$, *then* $\pi_0(-1)=-1$, *and if* $-1\notin(k^\times)^2$, *then* $\pi_0(-1)=1$.

*Proof.* First we show that, if $-1\in(k^\times)^2$, $-1$ is not a square element in $C_\varepsilon$. Assume that $-1=z^2=(x+\sqrt{\varepsilon}\,y)^2$. Since $z\bar{z}=1$, we have $zx=0$, and then $x=0$ and $-1=\varepsilon y^2\in\varepsilon(k^\times)^2$, which is a contradiction. Second, it is easy to see that, if $-1\notin(k^\times)^2$, $-1$ is a square in $C_\varepsilon$, and hence $\pi_0(-1)=1$. Q.E.D.

**Lemma 4.4.** **4.4.** *Put* $A(z)=\pi_0(\bar{z}z^{-1})$, *then*

$$A(z)=\begin{cases} 1, & z\in S^1, \\ -1, & z\in S^2. \end{cases}$$

*Proof.* The proof is obtained by using the polar coordinate of $z$. Since every $z\in S^1$ is expressed as $z=rt$, $r\in k^\times$ and $t\in C_\tau$, $A(z)=\pi_0(\bar{t}/t)=\pi_0(\bar{t}^2)=1$. Next, let $z$ be in $S^2$. If $-1\in(k^\times)^2$, $z$ is expressed as $z=\sqrt{\varepsilon}\,rt$, and then by the above lemma $A(z)=\pi_0(-\bar{t}^2)=\pi_0(-1)=-1$. If $-1\notin(k^\times)^2$, take an element $\nu$ such that $\nu\bar{\nu}=-1$. Then $z$ can be expressed as $z=\nu rt$ and hence $A(z)=\pi_0((\bar{\nu}/\nu)\bar{t}^2)=\pi_0(-\bar{\nu}^2)=\pi_0(\bar{\nu}^2)$ by the above lemma. It holds $\bar{\nu}^2\in C_\varepsilon$ but $\nu\notin C_\varepsilon$, and hence $A(z)=-1$. Q.E.D.

*Proof of Proposition* 3.2. On the space $\mathcal{S}(L_\tau, \pi_0)$, the operator $E_{\pi_0}$ in (4.18) is a non-trivial intertwining operator of $\mathcal{R}^+_{\pi_0}$ onto itself, and

$$E_{\pi_0}\Phi_{\pi_0}(z)=\Phi_{\pi_0}(\bar{z})=\pi_0(\bar{z}z^{-1})\Phi_{\pi_0}(z)=A(z)\Phi_{\pi_0}(z)\,,$$

The space of intertwining operators is at most two dimensional, and therefore we have the proposition. Q.E.D.

**4.4.** Fix $\tau$ in $E'=\{\varepsilon, p, \varepsilon p\}$. Let $\pi\in\tilde{C}_\tau$, $\pi\not\equiv1$ and extend it to a unitary character of $L_\tau^\times$. Put $\Phi'(z)=\Phi_\pi(z)\pi^{-1}(z)$. Then $\Phi'(tz)=\Phi'(z)$ for all $t\in C_\tau$, so $\Phi'(z)=\varphi(N_\tau(z))$, and $\varphi$ is a locally constant function on $k_\tau^\times$, vanishing near 0. By (4.14), $\int_{L_\tau}|\Phi'(z)|^2dz=a_\tau\int_k|\varphi(u)|^2du$. Thus the mapping $U:\Phi'_\pi\to\varphi$ is an isometry, up to the multiple by $a_\tau$, of $\mathcal{S}(L_\tau, \pi)$ onto $\mathcal{S}^\times(k_\tau^\times)$, the space of functions in $\mathcal{S}^\times$ supported in $k_\tau^\times$. The operator $UT_g^\pi U^{-1}$, denoted again by $T_g^\pi$, is given by the kernel as follows:

$$T_g^\pi\varphi(u)=\int_k K_\pi^+(g\,|\,u, v)\varphi(v)dv$$

$$=(\mathrm{sgn}_\tau a)\,|a|\,\pi(a)\varphi(a^2u)\,,\qquad g=d(a)\,, \tag{4.19}$$

$$=\chi(-xu)\varphi(u)\,,\qquad g=n(x)\,, \tag{4.20}$$

(4.21) $$=H_{\pi}^{d}\varphi(u)=a_{\pi}c_{\tau}\int_{k}J_{\pi}^{d}(u, v)\varphi(v)dv\,, \qquad g=w\,,$$

where

(4.22) $$J_{\pi}^{d}(u, v)=\int_{t\bar{t}=u^{-1}v}\chi(ut+vt^{-1})\pi(t)d^{\times}t\,.$$

(4.19), (4.20) and (4.21) are analogous to (4.9), (4.10) and (4.11) respectively.

For the later discussion, we deduce (4.21) in detail.

$$T_{w}^{\pi}\varphi(u)=c_{\tau}\int_{L_{\tau}}\chi(S_{\tau}(z\bar{z}')\Phi'(z')\pi(z')dz'\pi^{-1}(z)$$

$$=c_{\tau}\int_{L_{\tau}}\left\{\int_{C_{\tau}}\chi(S_{\tau}(z\bar{z}'\bar{t}))\pi((z'/z)t)d^{\times}t\right\}\Phi'(z')dz'\,.$$

On the other hand, the inner integral is

$$\int_{C_{\tau}}\chi(S_{\tau}(z\bar{z}'\bar{t}))\pi((z'/z)t)d^{\times}t=\int_{C_{\tau}}\chi(z\bar{z}'t^{-1}+\bar{z}z't)\pi((z'/z)t)d^{\times}t\,,$$

and changing the variable $(z'/z)t$ by $t$, then it equals

$$J_{\pi}^{d}(u, v)=\int_{t\bar{t}=u^{-1}v}\chi(ut+vt^{-1})\pi(t)d^{\times}t\,,$$

where $u=N_{\tau}(z)$ and $v=N_{\tau}(z')$. So we have, for $g=w$

$$T_{g}^{\pi}\varphi(u)=\int_{L_{\tau}}J_{\pi}^{d}(u, v)\Phi'(z')dz'=a_{\tau}c_{\tau}\int_{k}J_{\pi}^{d}(u, v)\varphi(v)dv\,.$$

**4.5.** The intertwining operator $E_{\pi}: \mathfrak{R}_{\pi}^{+}\to\mathfrak{R}_{\pi^{-1}}^{+}$, in (4.18), is transformed on the space $\mathcal{S}^{\times}(k_{\tau}^{\times})$ as $E_{\pi}\varphi(u)=\pi(u)\varphi(u)$, because $\varphi(u)=\Phi'(z)=\Phi_{\pi}(z)\pi^{-1}(z)$ and $\Phi_{\pi}(\bar{z})\pi(z)=\Phi'(z)\pi(z\bar{z})=\varphi(u)\pi(u)$. In particular, in case $\mathfrak{R}_{\pi_0}^{+}$, $E_{\pi_0}\varphi(u)=A(u)\varphi(u)$, where $A(u)=A(z)$ in Lemma 4.4 and $u=N_{\tau}(z)$. Since $A(u)=1$ for $u\in(k^{\times})^{2}$ and $=-1$ for $u\in\varepsilon(k^{\times})^{2}$, the representations $\mathfrak{R}_{0}^{1}$ and $\mathfrak{R}_{0}^{\varepsilon}$ in Proposition 4.2 are realized on $\mathcal{S}^{\times}((k^{\times})^{2})=\mathcal{S}^{\times}|(k^{\times})^{2}$ and $\mathcal{S}^{\times}(\varepsilon(k^{\times})^{2})=\mathcal{S}^{\times}|\varepsilon(k^{\times})^{2}$ respectively.

**4.6.** For $\pi\in\tilde{C}_{\tau}$, another discrete series representation comming from $\mathfrak{R}_{\pi}^{+}$ is given on $\mathcal{S}^{\times}|(k^{\times}\backslash k_{\tau}^{\times})$ as

$$T_{g}^{\pi}\psi(u)=\int_{k}K_{\pi}^{-}(g|u, v)\psi(v)dv\,, \qquad \psi\in\mathcal{S}^{\times}(k^{\times}\backslash k_{\tau}^{\times})\,.$$

The kernel $K_{\pi}^{-}$ is obtained from (4.18), (4.19) and (4.20) by replacing "$u, v\in k_{\tau}^{\times}$" by "$u, v\in k_{\tau}^{\times}\backslash k_{\tau}^{\times}$". We denote this representation by $\mathfrak{R}_{\pi}^{-}=\{T_{g}^{\pi}, \mathcal{S}^{\times}(k^{\times}\backslash k_{\tau}^{\times})\}$. If $\pi$ is not of order two, $\mathfrak{R}_{\pi}^{-}$ is again irreducible, equivalent to $\mathfrak{R}_{\pi^{-1}}^{-}$, but inequivalent to any $\mathfrak{R}_{\pi}^{+}$.

In the following, $\mathfrak{R}_{\pi}^{+}$ and $\mathfrak{R}_{\pi}^{-}$ appear in the form of their direct sum $\mathfrak{R}_{\pi}=\mathfrak{R}_{\pi}^{+}\oplus\mathfrak{R}_{\pi}^{-}$. The kernel $K_{\pi}(g|u, v)$ for $\mathfrak{R}_{\pi}$ is defined on $k^{\times}\times k^{\times}$, equal to $K_{\pi}^{+}(g|u,v)$ on $k_{\tau}^{\times}\times k_{\tau}^{\times}$, equal to $K_{\pi}^{-}(g|u, v)$ on $(k_{\tau}^{\times})^{c}\times(k_{\tau}^{\times})^{c}$, and zero if $u^{-1}v\notin k_{\tau}^{\times}$.

For $\pi_0$ of order two in $\tilde{C}_\varepsilon$, $\mathcal{R}_{\pi_0}^-$ again splits into $\mathcal{R}_0^p=\mathcal{R}_{\pi_0}^-|\mathcal{S}^\times(p(k^\times)^2)$ and $\mathcal{R}_0^{\varepsilon p}=\mathcal{R}_{\pi_0}^-|\mathcal{S}^\times(\varepsilon p(k^\times)^2)$. The representations $\mathcal{R}_0^1$, $\mathcal{R}_0^\varepsilon$, $\mathcal{R}_0^p$ and $\mathcal{R}_0^{\varepsilon p}$ are all irreducible and mutually inequivalent. The kernel for $\mathcal{R}_{\pi_0}=\mathcal{R}_0^1\oplus\mathcal{R}_0^\varepsilon\oplus\mathcal{R}_0^p\oplus\mathcal{R}_0^{\varepsilon p}$ is defined similarly as above, and is zero if $u^{-1}v\notin(k^\times)^2$. As to an other character of order two in $\tilde{C}_p$ or in $\tilde{C}_{\varepsilon p}$, reducible representations is constructed similarly but it is equivalent to $\mathcal{R}_{\pi_0}$.

The representation $\mathcal{R}^+$ is extended to a unitary one $\bar{\mathcal{R}}_\pi^+=\{T_g^\pi, L^2(k_\tau^\times)\}$, and $\mathcal{R}_\pi^-$ is also to $\bar{\mathcal{R}}_\pi^-=\{T_g^\pi, L^2((k_\tau^\times)^c)\}$ and $\mathcal{R}_0^s$, $s\in E$, to $\bar{\mathcal{R}}_0^s=\{T_g^{\pi_0}, L^2(s(k^\times)^2)\}$.

We denote by $\Omega_s$ the set of characters of the form $\pi=|\cdot|^\alpha$, $-1<\alpha<1$, and by $\Omega_d$ the set of all elements in $\tilde{C}_\tau$ with $\tau\in E'=\{\varepsilon, p, \varepsilon p\}$, except of order two. Put $\Omega=\tilde{k}^\times\cup\{\pi_{sp}\}\cup\Omega_d\cup\{\pi_0\}$ and $\Omega_u=\Omega\cup\Omega_s$. We have seen that any irreducible unitary representation appears as a completion of a subrepresentation in one of $\mathcal{R}_\pi$, $\pi\in\Omega_u$. Moreover the "support" of the Plancherel measure is $\Omega$.

## § 5. The Plancherel transform.

In this section, reviewing the Plancherel formula, we define and discuss the Plancherel transform.

**5.1.** Let $\mathcal{S}(G)$ be the space of locally constant, compactly supported functions on $G$. For every $f\in\mathcal{S}(G)$ and $\pi\in\Omega=\tilde{k}^\times\cup\{\pi_{sp}\}\cup\Omega_d\cup\{\pi_0\}$, the operator $T^\pi(f)=\int_G f(g)\mathcal{T}_g^\pi dg$, $\mathcal{T}^\pi=\hat{T}^\pi$ if $\pi\in\tilde{k}^\times\cup\{\pi_{sp}\}$ and $\mathcal{T}^\pi=T^\pi$ if $\pi\in\Omega_d\cup\{\pi_0\}$, has an integral kernel $K_\pi(f|u, v)$ given by

$$K_\pi(f|u, v)=\int_G f(g)K_\pi(g|u, v)dg\,, \tag{5.1}$$

where $K_\pi(g|u, v)$ is in (4.9), (4.10) and (4.11) for $\pi\in\tilde{k}^\times\cup\{\pi_{sp}\}$, and in (4.19), (4.20) and (4.21) for $\pi\in\Omega_d\cup\{\pi_0\}$. As will be proved soon, the kernel is a function on $k\times k$ and of trace class with $\mathrm{tr}\, T^\pi(f)=\int_k K_\pi(f|u, u)du$.

The inversion formula is proved in [4] and [15]: for $f\in\mathcal{S}(G)$,

$$\begin{aligned} f(e) &=\int_\Omega \mathrm{tr}\, T^\pi(f)m(\pi)d\pi \\ &=\int_{\tilde{k}^\times}\mathrm{tr}\, T^\pi(f)m(\pi)d\pi+m(\pi_{sp})\,\mathrm{tr}\, T^{\pi_{sp}}(f) \\ &\quad+\sum_{\pi\in\Omega_d} m(\pi)\,\mathrm{tr}\, T^\pi(f)+m(\pi_0)\,\mathrm{tr}\, T^{\pi_0}(f)\,, \end{aligned} \tag{5.2}$$

where $m(\pi)=1/(2|\Gamma(\pi)|^2)$ for $\pi\in\tilde{k}^\times$ but as to $m(\pi_{sp})$, $m(\pi)$ for $\pi\in\Omega_d$ and $m(\pi_0)$, see [15].

The inversion formula and Proposition 5.1 lead us immediately to the Plancherel formula:

$$(5.3)\qquad \int_G |f(g)|^2 dg = \int_\Omega \mathrm{tr}\,(T^\pi(f*f^*))m(\pi)d\pi$$

$$= \int_{\tilde{k}^\times}\int_k\int_k |K_\pi(f\,|\,u,v)|^2 du dv m(\pi)d\pi$$

$$+m(\pi_{sp})\int_k\int_k |K_{\pi_{sp}}(f\,|\,u,v)|^2 \pi_{sp}(uv^{-1})du dv$$

$$+\sum_{\pi\in\Omega_d} m(\pi)\int_k\int_k |K_\pi(f\,|\,u,v)|^2 du dv$$

$$+m(\pi_0)\int_k\int_k |K_{\pi_0}(f\,|\,u,v)|^2 du dv\,.$$

The above equality implies the map $f\to K_\pi(f\,|\,u,v)$ is an isometry of $\mathcal{S}(G)$ into $L^2_{dudvm(\pi)d\pi}(k\times k\times\Omega)$ and by the general theorem of the Plancherel formula on a locally compact group the image of $\mathcal{S}(G)$ is dense in the latter space. We call $K_\pi(f\,|\,u,v)$ the Plancherel transform of $f$.

**5.2.** Again let $f\in\mathcal{S}(G)$ and consider the kernel $K_\pi(f\,|\,u,v)$ of the operator $T^\pi(f)$.

**Proposition 5.1.** *The following equalities hold*:

$$(1)\qquad \begin{aligned} K_\pi(R_g f\,|\,u,v) &= \int_k K_\pi(f\,|\,u,t)K_\pi(g^{-1}|\,t,v)dt\,,\\ K_\pi(L_g f\,|\,u,v) &= \int_k K_\pi(g\,|\,u,t)K_\pi(f\,|\,t,v)dt\,,\end{aligned}$$

*where $R_g$ is the right regular representation of $G$ and $L_g$ the left regular one.*

(2) $K_\pi(g^{-1}|\,u,v)=\check{K}_{\pi^{-1}}(g\,|\,v,u)$, *where* $\check{K}_\pi(g\,|\,u,v)=K_\pi(g\,|-u,-v)$.

(3) $K_{\pi^{-1}}(f\,|\,u,v)=K_\pi(f\,|\,u,v)\pi(uv^{-1})$,

(4) $K_\pi(\check{f}\,|\,u,v)=\check{K}_\pi(f\,|\,v,u)\pi(u^{-1}v)$, *where* $\check{f}(g)=f(g^{-1})$.

(5) $K_\pi(\bar{f}\,|\,u,v)=\overline{\check{K}_{\bar{\pi}}(f\,|\,u,v)}$, *where* $\bar{f}(g)=\overline{f(g)}$.

(6) $K_\pi(f^*\,|\,u,v)=\overline{K_{\bar{\pi}}(f\,|\,v,u)}\pi(u^{-1}v)$, *where* $f^*=\bar{\check{f}}$.

(7) $K_\pi(f_1*f_2\,|\,u,v)=\int_k K_\pi(f_1\,|\,u,t)K_\pi(f_2\,|\,t,v)dt$,

*where* $f_1*f_2(g)=\int_G f_1(g_1)f_2(g_1^{-1}g)dg_1$.

(8) $K_\pi(f_1*f_2^*\,|\,u,v)=\int_k K_\pi(f_1\,|\,u,t)\overline{K_{\bar{\pi}}(f_2\,|\,v,t)}\pi(t^{-1}v)dt$,

*and especially if $\pi$ is unitary,*

$$K_\pi(f_1*f_2^*\,|\,u,v)=\int_k K_\pi(f_1\,|\,u,t)\overline{K_\pi(f_2\,|\,v,t)}dt\,.$$

*Proof.* The proof is routine. Take (2) for instance, it is easily to proved for $g=d(a)$ and $n(x)$. For $g=w$, it is proved by the Bessel functions properties (§3). Q. E. D.

**5.3.** Now, we express the kernel $K_\pi(f\,|\,u,v)$, $f\in\mathcal{S}(G)$, explicitly as a function on $k^\times\times k^\times$.

Let $G^0$ be the set of elements $g=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$ in $G$ such that $\delta\neq 0$, then $G=G^0\cup wG^0$, and every function $f$ in $\mathcal{S}(G)$ is expressed as $f=f_1+f_2$ where $f_1$, $f_2\in\mathcal{S}(G)$ are supported in $G^0$ and $wG^0$ respectively. We discuss the Plancherel transform $K_\pi(f\,|\,u,v)$ for $f$ supported in $wG^0$. For the function $f_1$ supported in $G^0$, it is expressed as $f_1=L_{w^{-1}}f$, $f$ as above. Then the Plancherel transform of $f_1$ is given by Proposition 5.1 (1). Since each element in $wG^0$ is given as $wn^+(x)d(a^{-1})n(y)=n(x)d(a)wn(y)$, $f(g)=f(n(x)d(a)wn(y))=f(x,y,a)$ is locally constant with respect to parameters $x$, $y\in k$ and $a\in k^\times$. So, $f(x,y,a)$ is expressed as a finite linear combination of functions of the form $\xi(x)\eta(y)\kappa(a)$, $\xi$, $\eta\in\mathcal{S}$ and $\kappa\in\mathcal{S}^\times$. The Haar measure on $G$ is given by $dg=|a|^{-2}d^\times a dx dy$ on $wG^0$, $g=n(x)d(a)wn(y)$. Note that

$$\int_k\xi(x)\mathcal{T}^\pi_{n(x)}\varphi(u)dx=\int_k\xi(x)K_\pi(n(x)\,|\,u,v)\varphi(v)dvdx=\hat{\xi}(u)\varphi(u)\,.$$

Let $f(g)=\xi(x)\eta(y)\kappa(a)$ as above, then

$$\text{(5.4)}\qquad \int_k K_\pi(f\,|\,u,v)\varphi(v)dv=\int_{k^\times}\int_k\int_k\xi(x)\eta(y)\kappa(a)\mathcal{T}^\pi_{n(x)d(a)wn(y)}\varphi(u)|a|^{-2}d^\times a dx dy$$

$$=\int_k\hat{\xi}(u)\kappa(a)[\mathcal{T}^\pi_{d(a)w}\hat{\eta}\varphi](u)|a|^{-2}d^\times a\,.$$

Further we discuss the forms of kernel $K_\pi(f\,|\,u,v)$ dividing into two cases: (A) $\pi\in\tilde{k}^\times\cup\{\pi_{sp}\}$ and (B) $\pi\in\Omega_d\cup\{\pi_0\}$.

Case (A). In (5.4), rewrite $\hat{\xi}$ and $\hat{\eta}$ by $\xi$ and $\eta$ respectively.

$$\int_k K_\pi(f\,|\,u,v)\varphi(v)dv=\int_k\xi(u)\kappa(a)\pi(a)|a|(\hat{T}^\pi_w\eta\varphi)(a^2u)|a|^{-2}d^\times a$$

$$=\int_k\int_k\xi(u)\kappa(a)J_\pi(au,av)\eta(v)\varphi(v)dv|a|^{-1}d^\times a\,.$$

because $J_\pi(a^2u,v)=\pi^{-1}(a)J_\pi(au,av)$. As we see in §3. $J_\pi(u,v)$ is a function on $k^\times\times k^\times$. Hence the kernel $K_\pi(f\,|\,u,v)$ is a function on $k^\times\times k^\times$:

$$\text{(5.5)}\qquad K_\pi(f\,|\,u,v)=\xi(u)\eta(v)M_\pi(u,v)$$

where $M_\pi(u,v)=\int_k\kappa(a)J_\pi(au,av)|a|^{-1}d^\times a$

Suppose $\xi$, $\eta$ are supported by $P^{-m}$ $(m>0)$. Take an integer $k$ $(k>0)$, and set $\xi=\xi_1+\xi_2$, where $\xi_1$ is equal to $\xi$ on $P^k$ and zero outside and $\xi_2$ is equal to $\xi$ on $(P^k)^c$ and zero outside. Set $\eta=\eta_1+\eta_2$ similarly. Then

$$\xi(u)\eta(v)=\xi_1(u)\eta_1(v)+\xi_1(u)\eta_2(v)+\xi_2(u)\eta_1(v)+\xi_2(u)\eta_2(v)\,. \tag{5.6}$$

The first three terms on the right hand side are zero outside of the set $\{(u, v);\ |uv|\leqq q^{m-k}\}$ and the last is zero for $|u|$, $|v|<q^{-k}$.

Let $\xi(u)\eta(v)$ of $f$ be one of the first three terms. Suppose $\kappa(a)$ is supported by $\{a\,;\,q^{-n}<|a|<q^n\}$. If we take $k$ as $k\geqq m+2n-1$, it holds that $|a^2uv|\leqq q^{m+2n-k}\leqq q$ for $u, v\in \mathrm{Supp}\,[\eta]$ and $a\in\mathrm{Supp}\,[\kappa]$. By the Bessel function property (B.4),

$$\begin{aligned}M_\pi(u, v)&=\int_k\{\Gamma(\pi^{-1})\pi(av)+\Gamma(\pi)\pi^{-1}(au)\}\kappa(a)|a|^{-1}d^\times a\\ &=\pi(v)\Gamma(\pi^{-1})\tilde{\kappa}_1(\pi)+\pi^{-1}(u)\Gamma(\pi)\tilde{\kappa}_1(\pi^{-1})\,.\end{aligned}$$

where for $\pi=|\cdot|^\alpha\theta$, $\tilde{\kappa}_1(\pi)=\int_k\kappa(a)|a|\pi(a)d^\times a=\sum_n c_n(\theta)q^{\alpha n}$ (finite sum). Thus we have

$$K_\pi(f\,|\,u, v)=\xi(u)\eta(v)\{\pi(v)\Gamma(\pi^{-1})\tilde{\kappa}_1(\pi)+\pi^{-1}(u)\Gamma(\pi)\tilde{\kappa}_1(\pi^{-1})\}\,. \tag{5.7}$$

Let $\xi(u)\eta(v)$ of $f$ be the last term in (5.6). Then for all $u\in\mathrm{Supp}\,[\xi]\subset k^\times$, $v\in\mathrm{Supp}\,[\eta]\subset k^\times$,

$$\begin{aligned}M_\pi(u, v)&=\int_k\kappa(a)\Big\{P-\int_k\chi(aux+av/x)\pi(x)\,d^\times x\Big\}|a|^{-1}d^\times a\\ &=\int_k\kappa(a)\Big\{\int_{q^{-l}\leq|x|\leq q^l}\chi(aux+avx^{-1})\pi(x)d^\times x\Big\}|a|^{-1}d^\times a\,,\end{aligned}$$

for an integer $l$ large enough. We chang the order of integration, then

$$M_\pi(u, v)=\int_k\tilde{\kappa}_2(ux+vx^{-1})X_l(x)\pi(x)d^\times x\,,$$

where $\tilde{\kappa}_2(ux+vx^{-1})=\int_k\chi(a(ux+vx^{-1}))\kappa(a)|a|^{-2}da$ and $X_l$ is the characteristic function of $\{x\,;\,q^{-l}\leqq|x|\leqq q^l\}$. The function $G(u, v, x)=\xi(u)\eta(v)\tilde{\kappa}_2(ux+vx^{-1})X_l(x)$ is locally constant and supported on $q^{-k}\leqq|u|$, $|v|\leqq q^{-m}$ and on $q^{-l}\leqq|x|\leqq q^l$, and therefore $G$ is written as $\sum_i a_i(u)b_i(v)c_i(x)$ (finite sum), $a_i$, $b_i$ and $c_i\in\mathcal{S}^\times$. Thus we have

$$K_\pi(f\,|\,u, v)=\sum_i a_i(u)b_i(v)\tilde{c}_i(\pi)\ \text{(finite sum)}\,. \tag{5.8}$$

Now, as to $f_1\in\mathcal{S}(G)$ supported in $G^0$, set $f_1(wg)=f(g)$. Then $f$ is in $wG^0$, $f_1=L_{w^{-1}}f$ and

$$K_\pi(f_1\,|\,u, v)=\int_k K_\pi(w\,|\,u, t)K_\pi(f\,|\,t, v)dt\,. \tag{5.9}$$

On the right hand side, for a fixed $v\in k^\times$, a function $K_\pi(f\,|\,u, v)$ in $u$ is operated by $H_\pi$.

Case (B). We treat the kernel for discrete series representations. Let $\pi\in\Omega_d\cup\{\pi_0\}$. From (5.4),

$$\int_k K_\pi(f\,|\,u,v)\varphi(v)dv=\int_k \xi(u)\kappa(a)(T^{\pi}_{d(a)w}\eta\varphi)(u)|a|^{-2}d^\times a$$

$$=a_\tau c_\tau\xi(u)\int_k\Big(\int_k\eta(v)\kappa(a)\pi(a)(\mathrm{sgn}_\tau a)|a|^{-1}J^d_\pi(a^2u,v)\varphi(v)dv\Big)d^\times a\,,$$

where $a_\tau$ and $c_\tau$ are in (4.14) and (4.15) respectively. As in (4.22), $J^d_\pi(u,v)$ is a function on $k^\times\times k^\times$, and then

$$K_\pi(f\,|\,u,v)=a_\tau c_\tau\xi(u)\eta(v)M_\pi(u,v)\,,$$

where $M_\pi(u,v)=\int_k\kappa(a)\pi(a)(\mathrm{sgn}_\tau a)|a|^{-1}J^d_\pi(a^2u,v)d^\times a$. Note that

$$J^d_\pi(a^2u,v)=\int_{C_\tau}\chi(S_\tau(az\bar z'\bar t))\pi(a(z'/z)t)d^\times t$$

where $u=N_\tau(z)$, $v=N_\tau(z')$. Then we have

$$M_\pi(u,v)=\int_k\int_{C_\tau}\kappa(a)(\mathrm{sgn}_\tau a)|a|^{-1}\chi(aS_\tau(z\bar z'\bar t))\pi(a(z'/z)t)d^\times t d^\times a \tag{5.10}$$

$$=\int_{C_\tau}\tilde\kappa_2(S_\tau(z\bar z'\bar t))\pi((z'/z)t)d^\times t\,,$$

where $\tilde\kappa_2(x)=\int_k\kappa(a)(\mathrm{sgn}_\tau a)|a|^{-1}\chi(ax)d^\times a$. $\tilde\kappa_2$ is in $\mathcal{S}$. Since $\tilde\kappa_2$ is constant on the neighborhood of 0, the last side of (5.10) is zero for small $|uv|$. Thus $M_\pi(u,v)$ is locally constant, supported in the set $\{(u,v)\,;\,s<|uv|,\ s$ a small number$\}$ and except a finite number of $\pi\in\Omega_d$, $M_\pi=0$. So, $\xi(u)\eta(v)M_\pi(u,v)$ is in $\mathcal{S}^\times\times\mathcal{S}^\times$ and we obtain $K_\pi(f\,|\,u,v)=\sum_i\alpha^i(u)\beta^i(v)$ (finite sum), $\alpha^i$, $\beta^i\in\mathcal{S}^\times$. From §4.6, it is easy to see that for $\pi\in\Omega_d\cap\tilde C_\tau$, $\alpha^i(u)\beta^i(v)=0$ if $uv^{-1}\notin k^\times_\tau$, and moreover for $\pi=\pi_0$, $\alpha^i(u)\beta^i(v)=0$ if $uv^{-1}\notin(k^\times)^2$.

**Theorem 5.2.** *The Plancherel transform $K_\pi(f\,|\,u,v)$ of $f\in\mathcal{S}(G)$ is expressed as a finite linear combination of the functions on $k^\times\times k^\times\times\Omega$ of the following form:*

(A) *For $\pi\in\tilde k^\times\cup\{\pi_{sp}\}$, the functions*

$$\Gamma(\pi^{-1})\xi(u)\eta(v)\pi(v)\tilde k(\pi)+\Gamma(\pi)\pi^{-1}(u)\xi(u)\eta(v)\tilde k(\pi^{-1})\,.$$

$$\Gamma(\pi^{-1})(H_\pi\xi)(u)\eta(v)\pi(v)\tilde k(\pi)+\Gamma(\pi)(H_\pi\pi^{-1}\xi)(u)\eta(v)\tilde\kappa(\pi^{-1})\,,$$

$$a(u)b(v)\tilde c(\pi)\,,\qquad (H_\pi a)(u)b(v)\tilde c(\pi)\,,$$

*where $\xi$, $\eta\in\mathcal{S}$ and $\kappa$, $a$, $b$, $c\in\mathcal{S}^\times$.*

(B) *For $\pi\in\Omega_d\cup\{\pi_0\}$, the functions $\alpha_\pi(u)\beta_\pi(v)$, where $\alpha_\pi$ and $\beta_\pi\in\mathcal{S}^\times$ vanishing except for only a finite number of $\pi$. Moreover $\alpha_\pi(u)\beta_\pi(v)=0$ if $uv\notin k^\times_\tau$, and*

$\alpha_{\pi_0}(u)\beta_{\pi_0}(v)=0$ *if* $uv\notin(k^{\times})^2$.

**Corollary 5.3.** *Fix* $\pi\in\tilde{k}^{\times}\cup\{\pi_{sp}\}$, *and consider* $K_\pi(f|u, v)$ *as a function in* $u, v$. *Then it is a linear combination of* $\varphi(u)\psi(v)$, *where if* $\pi\in\tilde{k}^{\times}$, $\varphi\in\hat{\mathcal{S}}_\pi$ *and* $\psi\in\hat{\mathcal{S}}_{\pi^{-1}}$, *and if* $\pi=\pi_{sp}$, $\varphi\in\hat{\mathcal{S}}_{sp}$ *and* $\psi\in\hat{\mathcal{S}}$. *Fix* $\pi\in\Omega_d\cap\tilde{C}_\tau$, *then* $K_\pi^+(f|u, v)$ *is a linear combination of* $\varphi(u)\psi(v)$ *where* $\varphi, \psi\in\mathcal{S}^{\times}(k_\tau^{\times})$, *and* $K_\pi^-(f|u, v)$ *is a linear combination of* $\varphi(u)\psi(v)$ *where* $\varphi, \psi\in\mathcal{S}^{\times}((k_\tau^{\times})^c)$. *For* $\pi=\pi_0$, $K_{\pi_0}^s(f|u, v)$ *is expressed as of the functions* $\varphi(u)\psi(v)$ *where* $\varphi, \psi\in\mathcal{S}^{\times}(s(k^{\times})^2)$, $\mathcal{S}\in E$.

## § 6. Tensor products of irreducible unitary representations.

**6.1.** Let $\mathfrak{R}_{\pi_i}=\{T^{\pi_i}, \mathcal{S}_{\pi_i}\}$ $(i=1, 2)$ be representations of principal series or of supplementary series. Let $\mathcal{S}_{\pi_1}\otimes\mathcal{S}_{\pi_2}$ denote the tensor product of $\mathcal{S}_{\pi_1}$ with $\mathcal{S}_{\pi_2}$, that is, the space of finite linear combinations of $\xi(x_1)\eta(x_2)$, $\xi\in\mathcal{S}_{\pi_1}$, $\eta\in\mathcal{S}_{\pi_2}$. The topology is defined in such a way that a sequence of functions $\{\xi_n\eta_n\}$ converges to $\xi\eta$ if and only if $\xi_n\to\xi$ in $\mathcal{S}_{\pi_1}$ and $\xi_n\to\eta$ in $\mathcal{S}_{\pi_2}$. The operator $T_g$ of the tensor product $\mathfrak{R}_{\pi_1}\otimes\mathfrak{R}_{\pi_2}$ of $\mathfrak{R}_{\pi_1}$ and $\mathfrak{R}_{\pi_2}$ is given as follows: for $\varphi\in\mathcal{S}_{\pi_1}\otimes\mathcal{S}_{\pi_2}$

$$T_g\varphi(x_1, x_2)=\pi_1\rho^{-1}(\beta x_1+\delta)\pi_2\rho^{-1}(\beta x_2+\delta)\varphi\Big(\frac{\alpha x_1+\gamma}{\beta x_1+\delta}, \frac{\alpha x_2+\gamma}{\beta x_2+\delta}\Big). \tag{6.1}$$

$\mathfrak{R}_{\pi_1}\otimes\mathfrak{R}_{\pi_2}$ is extended to a unitary representation with respect to the inner products corresponding to the following norms:

(I) If $\pi_1, \pi_2\in\tilde{k}^{\times}$ ($\mathfrak{R}_{\pi_1}$, $\mathfrak{R}_{\pi_2}$ are of principal series),

$$\|\varphi\|_{\mathrm{I}}^2=\int_k\int_k\varphi(x_1, x_2)\overline{\varphi(x_1, x_2)}dx_1dx_2.$$

(II) If $\pi_1(x)=|x|^{\alpha_1}$, $-1<\alpha_1<0$ and $\pi_2\in\tilde{k}^{\times}$ ($\mathfrak{R}_{\pi_1}$ is of supplementary series)

$$\|\varphi\|_{\mathrm{II}}^2=\frac{1}{\Gamma(\pi_1^{-1})}\int_k\int_k\int_k\pi_1^{-1}\rho^{-1}(x_1-x_1')\varphi(x_1, x_2)\overline{\varphi(x_1', x_2)}dx_1dx_1'dx_2.$$

(III) If $\pi_1(x)=|x|^{\alpha_1}$ and $\pi_2(x)=|x|^{\alpha_2}$, $-1<\alpha_1, \alpha_2<0$ ($\mathfrak{R}_{\pi_1}$, $\mathfrak{R}_{\pi_2}$ are of supplementary series),

$$\|\varphi\|_{\mathrm{III}}^2=\frac{1}{\Gamma(\pi_1^{-1})\Gamma(\pi_2^{-1})}\int_k\int_k\int_k\int_k\pi_1^{-1}\rho^{-1}(x_1-x_1')\pi_2^{-1}\rho^{-1}(x_2-x_2')\times\varphi(x_1, x_2)\overline{\varphi(x_1', x_2')}dx_1dx_1'dx_2dx_2'.$$

As limiting cases of (II) and (III), we have tensor products with the special representation as follows:

(IV) $\mathfrak{R}_{sp}\otimes\mathfrak{R}_{\pi_2}$, ($\pi_1=|x|^{-1}$ and $\pi_2\in\tilde{k}^{\times}$), for $\varphi\in\mathcal{S}_{sp}\otimes\mathcal{S}_{\pi_2}$,

$$\|\varphi\|_{\mathrm{IV}}^2=c\int_k\int_k\int_k\log|x_1-x_1'|\varphi(x_1, x_2)\overline{\varphi(x_1', x_2)}dx_1dx_1'dx_2,$$

where $c=(1-q^{-1})(\log q)^{-1}$.

(V) $\mathfrak{R}_{sp}\otimes\mathfrak{R}_{\pi_2}(\pi_1(x)=|x|^{-1}$ and $\pi_2(x)=|x|^{\alpha_2}$, $-1<\alpha_2<0)$, for $\varphi\in\mathcal{S}_{sp}\otimes\mathcal{S}_{\pi_2}$,

$$\|\varphi\|_{V}^{2}=c\int_{k}\int_{k}\int_{k}\int_{k}\log|x_1-x_1'|\pi_2^{-1}\rho^{-1}(x_2-x_2')\varphi(x_1, x_2)\overline{\varphi(x_1', x_2')}dx_1dx_1'dx_2dx_2'.$$

(VI) $\mathcal{R}_{sp}\otimes\mathcal{R}_{sp}(\pi_1$ and $\pi_2=|x|^{-1})$, for $\varphi\in\mathcal{S}_{sp}\otimes\mathcal{S}_{sp}$,

$$\|\varphi\|_{VI}^{2}=c^2\int_{k}\int_{k}\int_{k}\int_{k}\log|x_1-x_1'|\log|x_2-x_2'|\varphi(x_1, x_2)\overline{\varphi(x_1', x_2')}dx_1dx_1'dx_2dx_2'.$$

Let $(\pi_1, \pi_2)$ be one of the pairs of characters in (I), (II) and (III), and $\mathcal{H}$ be the space of $\varphi\in\mathcal{S}_{\pi_1}\otimes\mathcal{S}_{\pi_2}$ satisfying $\varphi(x_1, x_2)=0$ on a neighborhood of the diagonal "$x_1=x_2$". $\mathcal{H}$ is $G$-invariant and has the same completion $\bar{\mathcal{H}}$ as $\mathcal{S}_{\pi_1}\otimes\mathcal{S}_{\pi_2}$. We denote the representations on $\bar{\mathcal{H}}$ by $\mathcal{R}_{\pi_1}\bar{\otimes}\mathcal{R}_{\pi_2}$. Our problem is to decompose these tensor products into irreducibles.

**6.2.** We consider a linear mapping $U$ of $\mathcal{S}(G)$. For $f\in\mathcal{S}(G)$ and $g=d(a)n^+(y_1)n(x_1)$, put

$$(Uf)(x_1, x_2)=\pi_2^{-1}\rho(y_1)\int_{k}\pi_1^{-1}\pi_2(a)f(d(a)n^+(y_1)n(x_1))d^{\times}a, \tag{6.2}$$

where $x_2=x_1+1/y_1$. In other words,

$$(Uf)(x_1, x_2)=\pi_2\rho^{-1}(x_2-x_1)(Sf)\left(x_1, \frac{1}{x_2-x_1}\right),$$

where

$$(Sf)(x_1, y_1)=\int_{k}\pi_1^{-1}\pi_2(a)f(d(a)n^+(y_1)n(x_1))d^{\times}a.$$

**Proposition 6.1.** *For $f\in\mathcal{S}(G)$, $Uf=\varphi\in\mathcal{H}$ and $UR_g=T_gU$ where $g\rightarrow R_g$ is the right regular representation of $G$.*

*Proof.* Let $G^0$ be the open subset in $G$ as in §4.

(1) Let $f$ be supported in $G^0$. The function $(Sf)(x_1, y_1)=\int_{k}\pi_1^{-1}\pi_2(a)$ $f(d(a)n^+(y_1)n(x_1))d^{\times}a$ is a finite linear combination of $\xi(x_1)\eta(y_1)$, $\xi, \eta\in\mathcal{S}$. Then the function $Uf$ is a linear combination of

$$\psi(x_1, x_2)=\pi_2\rho^{-1}(x_2-x_1)\xi(x_1)\eta\left(\frac{1}{x_2-x_1}\right),$$

and $\psi$ is locally constant, compactly supported with respect to $x_1$, zero on a neighborhood of the diagonal "$x_1=x_2$", and for large $|x_2|$, $\psi(x_1, x_2)=d\pi_2^{-1}\rho(x_2)\xi(x_1)$ with $d=\eta(0)$. Thus we get $Uf\in\mathcal{H}$.

(2) Let $f$ be supported in $G^0w$, put $f_1=R_w^{-1}f$. Then $f_1$ is supported in $G^0$ and from (1), $Uf_1=\varphi_1\in\mathcal{H}$. It holds that

$$(Uf)(x_1, x_2)=(UR_wf_1)(x_1, x_2)=\pi_2^{-1}\rho(y_1)\int_{k}\pi_1^{-1}\pi_2(a)f_1(d(a)n^+(y_1)n(x_1)w)d^{\times}a,$$

$n^+(y_1)n(x_1)w=d(x_1)n^+(x_1(x_1y_1+1))n(-x_1^{-1})$ by (4.3), and $-x_1^{-1}+x_1^{-1}(x_1y_1+1)^{-1}$ $=-y_1(x_1y_1+1)^{-1}=-x_2^{-1}$,

$$=\pi_2^{-1}\rho(y_1)\int_k \pi_1^{-1}\pi_2(a)f(d(a)n^+(x_1(x_1y_1+1))n(-x_1^{-1}))d^\times a$$

$$=\pi_2^{-1}\rho(y_1)\pi_1\pi_2^{-1}(x_1)\pi_2\rho^{-1}(x_1(x_1y_1+1))\varphi_1(-x_1^{-1}, -x_1^{-1}+x_1^{-1}(x_1y_1+1)^{-1})$$

$$=\pi_1\rho^{-1}(x_1)\pi_2\rho^{-1}(x_2)\varphi_1(-x_1^{-1}, -x_2^{-1})=T_w\varphi_1(x_1, x_2),$$

where $T_w$ is in (6.1) for $g=w$. Thus we get $Uf=UR_wf_1=T_w\varphi_1\in\mathcal{H}$.

To show that $UR_g=T_gU$, it is enough to check it for $g=d(a)$ and $n(x)$, because for $g=w$, it is already over. This is easy. Q. E. D.

**Proposition 6.2.** *The linear G-morphism U of $\mathcal{S}(G)$ into $\mathcal{H}$ in* (6.2) *is continuous and surjective.*

*Proof.* The continuity is clear from the definition of $U$. Let us prove the surjectivity. Suppose $\varphi(x_1, x_2)=\xi(x_1)\eta(x_2)\in\mathcal{H}$ and $\xi$ be compactly supported. Put

$$f(d(a)n^+(y_1)n(x_1))=\pi_1\pi_2^{-1}(a)\kappa(a)\pi_2\rho^{-1}(y_1)\varphi(x_1, x_1+y_1^{-1}) \tag{6.3}$$

where $\kappa(a)\in\mathcal{S}^\times$ such that $\int_k\kappa(a)d^\times a=1$. Then $f$ is a preimage of $\varphi$ under $U$. In fact, $f$ is locally constant in $(x_1, y_1)$ and compactly supported with repect to $x_1$, and for large $|y_1|$, $\varphi(x_1, x_1+y_1^{-1})=0$, and for small $|y_1|$, $\varphi(x_1, x_1+y_1^{-1})$ is expressed as $d\xi(x_1)\pi_2\rho^{-1}(y_1^{-1})$, $d\in\boldsymbol{C}$. Then $f$ is compactly supported with respect to $y_1$, and $Uf=\varphi$.

If $\varphi(x_1, x_2)=\xi(x_1)\eta(x_2)\in\mathcal{H}$ and $\xi$ is not compactly supported, we can assume that $\xi$ is zero on a neighborhood of $x_1=0$. Then $T_w\varphi(x_1, x_2)=\pi_1\rho^{-1}(x_1)\pi_2\rho^{-1}(x_2)$ $\varphi(-x_1^{-1}, -x_2^{-1})\in\mathcal{H}$ is compactly supported with respect to $x_1$, and there exists $h\in\mathcal{S}(G)$ such that $Uh=T_w\varphi$. So, $U(R_w^{-1}h)=T_w^{-1}(Uh)=\varphi$. Q. E. D.

**6.3.** Let $\langle\varphi, \psi\rangle$ be one of the inner products in (I), (II) and (III). For $f, h\in\mathcal{S}(G)$ we define $B(f, h)$ as

$$B(f, h)=\langle Uf, Uh\rangle. \tag{6.4}$$

$B$ is a continuous sesquilinear form on $\mathcal{S}(G)\times\mathcal{S}(G)$ by Proposition 6.2, and there exists a distribution $H_1(g_1, g_2)$ on $G\times G$ such that

$$B(f, h)=\int_G\int_G H_1(g_1, g_2)f(g_1)\bar{h}(g_2)dg_1dg_2.$$

Put $\varphi=Uf$, $\psi=Uh$. Then by Proposition 6.1,

$$B(R_gf, R_gh)=\langle T_g\varphi, T_g\psi\rangle=\langle\varphi, \psi\rangle=B(f, h),$$

that is, $H_1(g_1g, g_2g)=H_1(g_1, g_2)$ for all $g\in G$. Hence there exists a distribution $H(g)$ acting on $\mathcal{S}(G)$ such that $H_1(g_1, g_2)=H(g_1g_2^{-1})$. So we have

$$B(f, h)=\int_G\int_G H(g)f(gg_1)\overline{h(g_1)}dgdg_1=\int_G H(g)f_1(g)dg, \tag{6.5}$$

where $f_1(g)=\int_G f(g_1)\overline{h(g^{-1}g_1)}dg_1=f*h^*(g)$.

**Proposition 6.3.** *Corresponding to the tensor products in* (I), (II) *and* (III), *the kernel distributions in* (6.5) *are written as follows: for* $g=d(a)n^+(y)n(x)$,

(H. I) $$H(g)=\pi_1^{-1}\pi_2(a)\Delta(x)\Delta(y)\,,$$

(H. II) $$H(g)=\frac{1}{\Gamma(\pi_1^{-1})}\pi_1^{-1}\pi_2(a)\pi_1^{-1}\rho^{-1}(x)\Delta(y)\,,$$

(H. III) $$H(g)=\frac{1}{\Gamma(\pi_1^{-1})\Gamma(\pi_2^{-1})}\pi_1^{-1}\pi_2(a)\pi_1^{-1}\rho^{-1}(x)\pi_2^{-1}\rho^{-1}(y)$$

To prove this proposition, we apply the following:

**Lemma 6.4.** *Let* $\pi_1\in\tilde{k}^\times$ *or* $\pi_1(x)=|x|^{\alpha_1}$, $-1<\alpha_1<0$, *and* $\pi_2$ *similar. Let* $f\in\mathcal{S}(G)$ *and put* $Uf=\varphi$. *Then, for* $g=d(a_1)n^+(y_1)n(x_1)$,

(A) $$\int_k \pi_1^{-1}\pi_2(a)f(d(a)g)d^\times a=\pi_1\pi_2^{-1}(a_1)\pi_2\rho^{-1}(y_1)\varphi(x_1,\ x_2)\,,$$

(B) $$\int_k\int_k \pi_1^{-1}\pi_2(a)\pi_1^{-1}\rho^{-1}(x)f(d(a)n(x)g)d^\times a\,dx$$
$$=\pi_1^{-1}\pi_2^{-1}(a_1)\pi_2\rho^{-1}(y_1)\int_k \pi_1^{-1}\rho^{-1}(x)\varphi(x+x_1,\ x_2)dx\,,$$

(C) $$\int_k\int_k \pi_1^{-1}\pi_2(a)\pi_2^{-1}\rho^{-1}(-y)f(d(a)n^+(y)g)d^\times a\,dy$$
$$=\pi_1\pi_2(a_1)\pi_2\rho^{-1}(y_1)\int_k \pi_2^{-1}\rho^{-1}(x)\varphi(x_1,\ x+x_2)dx\,,$$

*where* $x_2=x_1+y_1^{-1}$.

*Proof.* We prove this by using (6.2) and by changing variables. (A) is easy. (B) Remarking $n(x)d(a_1)=d(a_1)n(a_1^{-2}x)$ and replacing $x$ by $a_1^2x$, we have

$$M=\int_k\int_k \pi_1^{-1}\pi_2(a)\pi_1^{-1}\rho^{-1}(x)f(d(a)n(x)g)d^\times a\,dx\,.$$

$$=\int_k\int_k \pi_1^{-1}\pi_2(a)\pi_1^{-2}(a_1)\pi_1^{-1}\rho^{-1}(x)f(d(aa_1)n(x)n^+(y_1)n(x_1))d^\times a\,dx\,.$$

Since $n(x)n^+(y_1)=d(xy_1+1)n^+(y_1(xy_1+1))n(x(xy_1+1)^{-1})$ by (4.3), we replace $a$ by $aa_1^{-1}(xy+1)^{-1}$. Then we have

$$M=\int_k\int_k \pi_1^{-1}\pi_2^{-1}(a_1)\pi_1^{-1}\pi_2(a)\pi_1\pi_2^{-1}(xy_1+1)\pi_1^{-1}\rho^{-1}(x)$$
$$\times f(d(a)n^+(y_1(xy_1+1))n(x(xy_1+1)^{-1})n(x_1))d^\times a\,dx$$
$$=\pi_1^{-1}\pi_2^{-1}(a_1)\pi_2\rho^{-1}(y_1)\int_k \pi_1\rho^{-1}(xy_1+1)\pi_1^{-1}\rho^{-1}(x)\varphi(x(xy_1+1)^{-1}+x_1,\ x_2)dx\,,$$

because $x(xy_1+1)^{-1}+x_1+y_1^{-1}(xy_1+1)^{-1}=x_1+y_1^{-1}=x_2$. We change the variable $x$ by $x(xy_1+1)^{-1}=x'$, then $x=x'(-x'y_1+1)^{-1}$, $xy_1+1=(-x'y_1+1)^{-1}$ and $dx=\rho^{-2}(-x'y_1+1)dx'$. Thus we obtain

$$M=\pi_1^{-1}\pi_2^{-1}(a)\pi_2\rho^{-1}(y_1)\int_k \pi_1^{-1}\rho^{-1}(x')\varphi(x'+x_1, x_2)dx'.$$

(C) is similar as (B). Q.E.D.

*Proof of Proposition* 6.3. The formula (H.I) follows from Lemma 6.4 (A), and (H.II) from (B). The formula (H.III) follows from (B) and (C). Q.E.D.

## §7. The Plancherel transform of a distribution.

Let $\mathcal{M}$ be the imge of $\mathcal{S}(G)$ under the Plancherel transform. We consider the induced topology on $\mathcal{M}$ from $\mathcal{S}(G)$. Let $D$ be a distribution on $G$. We define the Plancherel transform $\hat{D}$ of $D$ as follows: for $F\in\mathcal{M}$. take $f\in\mathcal{S}(G)$ such that $F(u, v, \pi)=K_\pi(f|u, v)$ and put

$$\int_\Omega\int_k\int_k \hat{D}(u, v, \pi)F(u, v, \pi)dudvm(\pi)d\pi=\int_G D(g)f(g)dg. \tag{7.1}$$

Then, $\hat{D}\in\mathcal{M}'$, the dual of $\mathcal{M}$. We call $\hat{D}$ the Plancherel transform of $D$. From the inversion formula (5.2), we obtain

$$\begin{aligned}\int_G D(g)f(g)dg&=\int_G\left\{\int_\Omega\int_k D(g)K_\pi(L_{g^{-1}}f|v, v)dvm(\pi)d(\pi)\right\}dg\\ &=\int_G\left\{\int_\Omega\int_k\int_k D(g)K_\pi(g^{-1}|v, u)K_\pi(f|u, v)dudvm(\pi)d(\pi)\right\}dg.\end{aligned} \tag{7.2}$$

Thus $\hat{D}$ can be formally expressed as $\hat{D}(u, v, \pi)=\int_k D(g)K_\pi(g^{-1}|v, u)dg$.

According to (5.3), (7.1) is written as

$$\begin{aligned}\int_G D(g)f(g)dg&=\int_{\widetilde{k}^\times}\int_k\int_k \hat{D}(u, v, \pi)K_\pi(f|u, v)dudvm(\pi)d\pi\\ &+m(\pi_{sp})\int_k\int_k \hat{D}(u, v, \pi_{sp})K_{\pi_{sp}}(f|u, v)dudv\\ &+\sum_{\pi\in\Omega_d} m(\pi)\int_k\int_k \hat{D}(u, v, \pi)K_\pi(f|u, v)dudv\\ &+m(\pi_0)\int_k\int_k \hat{D}(u, v, \pi_0)K_{\pi_0}(f|u, v)dudv.\end{aligned} \tag{7.3}$$

Here the notations $m(\pi)$ are described in (5.2).

We recall the abbreviation of notations: $\pi_1\pi_2(x)=\pi_1(x)\pi_2(x)$, $\pi\,\mathrm{sgn}_\tau(x)=\pi(x)\,\mathrm{sgn}_\tau(x)$, and so on. We prove the following:

**Theorem 7.1.** *Let $H(g)$ be one of the distributions in Proposition* 6.3, *and* $\hat{H}$

*the Plancherel transform of H. Then* $\hat{H}(u, v, \pi)=0$ *if* $\pi\in\tilde{k}^{\times}\cup\{\pi_{sp}\}$ *and* $\pi_1\pi_2\pi(-1)\neq 1$, *and* $\hat{H}(u, v, \pi)=0$ *if* $\pi\in(\Omega_d\cap\tilde{C}_\tau)\cup\{\pi_0\}$ *and* $\pi_1\pi_2\pi\,\mathrm{sgn}_\tau(-1)\neq 1$.

*Proof.* In the equality

$$\int_G H(g)f(g)dg=\int_\Omega\int_k\int_k \hat{H}(u, v, \pi)K_\pi(f\,|\,u, v)dudvm(\pi)d(\pi)\,.$$

We replace $f$ by $L_{d(a^{-1})}f(g)=f(d(a)g)$, $a\in k^{\times}$. Then it is easy to see from the explicit form of $H(g)$ that

$$\int_G H(g)f(d(a)g)dg=\pi_1\pi_2^{-1}(a)\int_G H(g)f(g)dg\,. \tag{7.4}$$

On the other hand, for $\pi\in\tilde{k}^{\times}\cup\{\pi_{sp}\}$, $K_\pi(L_{d(a)}f\,|\,u, v)=\pi\rho(a)K_\pi(f\,|\,a^2u, v)$, and for $\pi\in(\Omega_d\cap\tilde{C}_\tau)\cup\{\pi_0\}$, $K_\pi(L_{d(a)}f\,|\,u, v)=\pi\rho\,\mathrm{sgn}_\tau(a)K_\pi(f\,|\,a^2u, v)$. From these equalities, (7.4) and Proposition 5.1 (1),

$$\begin{aligned}
\int_G H(g)f(g)dg&=\int_G H(g)(L_{d(a)}f)(d(a)g)dg \\
&=\pi_1\pi_2^{-1}\pi\rho(a)\int_{\tilde{k}^{\times}}\int_k\int_k \hat{H}(u, v, \pi)K_\pi(f\,|\,a^2u, v)dudvm(\pi)d\pi \\
&\quad+\pi_1\pi_2^{-1}\pi_{sp}\rho(a)m(\pi_{sp})\int_k\int_k \hat{H}(u, v, \pi_{sp})K_{\pi_{sp}}(f\,|\,a^2u, v)dudv \\
&\quad+\pi_1\pi_2^{-1}\pi\rho\,\mathrm{sgn}_\tau(a)\sum_{\pi\in\Omega_d} m(\pi)\int_k\int_k \hat{H}(u, v, \pi)K_\pi(f\,|\,a^2u, v)dudv \\
&\quad+\pi_1\pi_2^{-1}\pi_0\rho\,\mathrm{sgn}_\varepsilon(a)m(\pi_0)\int_k\int_k \hat{H}(u, v, \pi_0)K_{\pi_0}(f\,|\,a^2u, v)dudv\,.
\end{aligned} \tag{7.5}$$

Now, put $a=-1$ and compare (7.5) with (7.3) for $D=H$. $\pi_1\pi_2^{-1}\pi\rho(-1)$ and $\pi_1\pi_2^{-1}\pi\rho\,\mathrm{sgn}_\tau(-1)$ equal always 1 or $-1$. So, we easily see that the integral with respect to $\pi$ on the set of $\Omega$, consisted of elements $\pi\in\tilde{k}^{\times}\cup\{\pi_{sp}\}$ such that $\pi_1\pi_2\pi(-1)=-1$ and $\pi\in\Omega_d\cup\{\pi_0\}$ such that $\pi_1\pi_2\pi\,\mathrm{sgn}_\tau(-1)=-1$, is zero. Thus we obtain the theorem. Q. E. D.

To simplify the notations on integration domains, we set

$$\Pi_{pr}=\Pi_{pr}(\pi_1\pi_2(-1))=\{\pi\in\tilde{k}^{\times}\,;\ \pi(-1)=\pi_1\pi_2(-1)\}\,,$$

$$\Pi_d=\Pi_d(\pi_1\pi_2(-1))=\bigcup_{\tau\in E'}\{\pi\in(\Omega_d\cap\tilde{C}_\tau)\,;\ \pi\,\mathrm{sgn}_\tau(-1)=\pi_1\pi_2(-1)\}\,,$$

$$Q_{sp}=\begin{cases}\{\pi_{sp}\}, & \text{if } \pi_1\pi_2(-1)=1\,,\\ \varnothing\,, & \text{if } \pi_1\pi_2(-1)=-1\,,\end{cases} \tag{7.6}$$

$$Q_d=\begin{cases}\{\pi_0\}, & \text{if } \pi_0(-1)=\pi_1\pi_2(-1)\,,\\ \varnothing\,, & \text{if } \pi_0(-1)\neq\pi_1\pi_2(-1)\,,\end{cases}$$

and put

(7.7) $$\Pi=\Pi(\pi_1\pi_2(-1))=\Pi_{pr}\cup Q_{sp}\cup\Pi_d\cup Q_d\,.$$

In the succeeding sections, we shall explicitly calculate the Plancherel transform of the distributions $H(g)$ in Proposition 6.3, and after obtaining it, we can get the decomposition formulas for the tensor products of representations. Note the following. In (7.3) we replace $D$ by $H$ and $f$ by $f_1=f*h^*$, $f,\ h\in\mathcal{S}(G)$. From Proposition 5.1 (4) and (8), for $\pi\in\widetilde{k^\times}$ or $\pi\in\Omega_d\cup\{\pi_0\}$,

$$K_\pi(f_1|u,v)=\int_k K_\pi(f|u,t)K_\pi(h^*|t,v)dt$$

$$=\int_k K_\pi(\check{f}|t,-u)\bar{K}_\pi(\check{h}|t,-v)\pi^{-1}(uv^{-1})dt\,,$$

where $\check{f}(g)=f(g^{-1})$ and $\bar{K}_\pi(f|u,v)=\overline{K_\pi(f|u,v)}$, and for $\pi=\pi_{sp}$

$$K_{\pi_{sp}}(f_1|u,v)=\int_k K_{\pi_{sp}}(\check{f}|t,-u)\bar{K}_{\pi_{sp}}(\check{h}|t,-v)\pi_{sp}(tu^{-1})dt\,.$$

Thus from Theorem 7.1 we have for $\varphi=Uf$, $\psi=Uh\in\mathcal{H}\subset\mathcal{S}_{\pi_1}\otimes\mathcal{S}_{\pi_2}$,

(7.8) $$\langle\varphi,\psi\rangle=\int_G H(g)f_1(g)dg$$

$$=\int_{\Pi_{pr}}\int_k\int_k\int_k\hat{H}(-u,-v,\pi)\pi^{-1}(uv^{-1})K_\pi(\check{f}|t,u)\bar{K}_\pi(\check{h}|t,v)dtdudv\,m(\pi)d\pi$$

$$+m(\pi_{sp})\int_k\int_k\int_k\hat{H}(-u,-v,\pi_{sp})\pi_{sp}(u^{-1})K_{\pi_{sp}}(\check{f}|t,u)\bar{K}_{\pi_{sp}}(\check{h}|t,v)\pi_{sp}(t)\,dtdudv$$

$$+\sum_{\pi\in\Pi_d}m(\pi)\int_k\int_k\int_k\hat{H}(-u,-v,\pi)\pi^{-1}(uv^{-1})K_\pi(\check{f}|t,u)\bar{K}_\pi(\check{h}|t,v)dtdudv$$

$$+m(\pi_0)\int_k\int_k\int_k\hat{H}(-u,-v,\pi_0)\pi_0^{-1}(uv^{-1})K_{\pi_0}(\check{f}|t,u),\ \bar{K}_{\pi_0}(\check{h}|t,v)dtdudv\,.$$

## § 8. The Plancherel transform of $H(g)$ in (H.I).

In this section, we calculate the Plancherel transform $\hat{H}$ of the kernel distribution $H$ in (H.I) in Proposition 6.3.

First let $\pi(x)=|x|^\alpha\theta(x)(-\pi/\log q<\mathrm{Im}(\alpha)\leqq\pi/\log q)$ be a character of $k^\times$ and suppose that it is satisfies $\pi(-1)=\theta(-1)=1$. Then, as in § 1, $\theta=\theta'\theta_1$ where $\theta'$ is a character of the group $\{1,\varepsilon,\cdots,\varepsilon^{q-2}\}\simeq\boldsymbol{Z}_{q-1}$ satisfying $\theta'^{(q-1)/2}\equiv1$ and $\theta_1$ is a character of $A_1=1+P=(1+P)^2$. So, we can determine $\theta'(\varepsilon)^{1/2}$ for all $\theta'$. Then we define the square roots of $\pi$ as $\pi^{1/2}(x)=|x|^{\alpha/2}\theta'^{m/2}(\varepsilon)\theta_1(a_1)$ for $x=p^n\varepsilon^m a=p^n\varepsilon^m a_1^2$, $a,\ a_1\in1+P$. Thus, since $\pi$ in $\Pi_{pr}\cup Q_{sp}$ (resp. $(\pi_d\cap\tilde{C}_\tau)\cup Q_d$) satisfies the condition $\pi_1\pi_2^{-1}\pi(-1)=1$ (resp. $\pi_1\pi_2^{-1}\pi\,\mathrm{sgn}_\tau(-1)=1$), we can take the square root of $\pi_1\pi_2^{-1}\pi\rho$ (resp. $\pi_1\pi_2^{-1}\pi\rho\,\mathrm{sgn}_\tau$).

Let $\pi_1,\ \pi_2$ fix in $\tilde{k}^\times$. We define the functions $A(\pi,s)(u)$, $s\in E=\{1,\varepsilon,p,\varepsilon p\}$, on $k$ as follows: for $\pi\in\Pi_{pr}\cup Q_{sp}$,

$$(8.1)\qquad A(\pi,\ s)(u)=\begin{cases}(\pi_1\pi_2^{-1}\pi\rho)^{1/2}\rho^{-1}(u), & u\in s(k^{\times})^2,\\ 0, & \text{otherwise,}\end{cases}$$

and for $\pi\in(\Pi_d\cap\tilde{C}_\tau)\cup Q_d$,

$$(8.2)\qquad A(\pi,\ s)(u)=\begin{cases}(\pi_1\pi_2^{-1}\pi\rho\ \mathrm{sgn}_\tau)^{1/2}\rho^{-1}(u), & u\in s(k^{\times})^2,\\ 0, & \text{otherwise.}\end{cases}$$

Now we have the Plancherel transform of $\hat{H}$ in (H. I).

**Theorem 8.1.** *Let $H$ be in* (H. I), $H(g)=\pi_1^{-1}\pi_2(a)\Delta(x)\Delta(y)$ *for* $g=d(a)n^+(y)n(x)$. *Then it's Plancherel transform $\hat{H}$ is given as follows*:

$$\hat{H}(u,\ v,\ \pi)=2\sum_{s\in E}A(\pi,\ s)(u)\bar{A}(\pi,\ s)(v)\ ,\qquad \textit{for }\ \pi\in\Pi_{pr}\cup\Pi_d\cup Q_d\ ,$$

$$\hat{H}(u,\ v,\ \pi)=2\sum_{s\in E}A(\pi,\ s)(u)\bar{A}(\pi,\ s)(v)\pi_{sp}^{-1}(v)\ ,\qquad \textit{for }\ \pi\in Q_{sp}\ ,$$

*where* $\bar{A}(\pi,\ s)(v)=\overline{A(\pi,\ s)(v)}$.

For the proof we remark the following. For any $s\in E$, $\mathrm{sgn}_s$ is a character of $E\simeq k^{\times}/(k^{\times})^2$ and $\mathrm{sgn}_r s=\mathrm{sgn}_s r$. Therefore, for $u\in s'(k^{\times})^2$, $\sum_{r\in E}\mathrm{sgn}_s r\ \mathrm{sgn}_r u=\sum_{r\in E}\mathrm{sgn}_s r\ \mathrm{sgn}_{s'}r=4\delta_{ss'}$, $\delta$ the Kroncker's delta. Hence we have: for $\pi\in\Pi_{pr}\cup Q_{sp}$,

$$(8.3)\qquad A(\pi,\ s)(u)=\frac{1}{4}\sum_{r\in E}\mathrm{sgn}_s r\ (\pi_1\pi_2^{-1}\pi\rho)^{1/2}\rho^{-1}\ \mathrm{sgn}_r(u)\ ,$$

and for $\pi\in(\Pi_d\cap\tilde{C}_\tau)\cup Q_d$,

$$(8.4)\qquad A(\pi,\ s)(u)=\frac{1}{4}\sum_{r\in E}\mathrm{sgn}_s r\ (\pi_1\pi_2^{-1}\pi\rho\ \mathrm{sgn}_\tau)^{1/2}\rho^{-1}\ \mathrm{sgn}_r(u)\ .$$

Thus, the right hand sides in the formulas in Theorem 8.1 are rewritten as follows: for $\pi\in\Pi_{pr}$,

$$2\sum_{s\in E}A(\pi,\ s)(u)\bar{A}(\pi,\ s)(v)=\frac{1}{2}\sum_{r\in E}(\pi_1\pi_2^{-1}\pi\rho)^{1/2}\rho^{-1}\ \mathrm{sgn}_r(u)(\pi_1^{-1}\pi_2\pi^{-1}\rho)^{1/2}\rho^{-1}\ \mathrm{sgn}_r(v)\ ,$$

and for $\pi\in Q_{sp}$,

$$2\sum_{s\in E}A(\pi,\ s)(u)\bar{A}(\pi,\ s)(v)\pi_{sp}^{-1}(v)$$

$$=\frac{1}{2}\sum_{r\in E}(\pi_1\pi_2^{-1}\pi_{sp}\rho)^{1/2}\rho^{-1}\ \mathrm{sgn}_r(u)(\pi_1^{-1}\pi_2\pi_{sp}^{-1}\rho)^{1/2}\rho^{-1}\ \mathrm{sgn}_r(v)\ .$$

For $\pi\in(\Pi_d\cap\tilde{C}_\tau)\cup Q_d$,

$$2\sum_{s\in E}A(\pi,\ s)(u)\bar{A}(\pi,\ s)(v)$$

$$=\frac{1}{2}\sum_{r\in E}(\pi_1\pi_2^{-1}\pi\rho\ \mathrm{sgn}_\tau)^{1/2}\rho^{-1}\ \mathrm{sgn}_r(u)\ (\pi_1^{-1}\pi_2\pi^{-1}\rho\ \mathrm{sgn}_\tau)^{1/2}\rho^{-1}\ \mathrm{sgn}_r(v)\ .$$

So, Theorem 8.1 is reduced to the following.

**Proposition 8.2.** *The Plancherel transform $\hat{H}$ of $H$ in* (H.I) *is given as follows: for $\pi\in\Pi_{pr}\cup Q_{sp}$,*

$$\hat{H}(u, v, \pi)=\frac{1}{2}\sum_{\tau\in E}(\pi_1\pi_2^{-1}\pi\rho)^{1/2}\rho^{-1}\operatorname{sgn}_\tau(u)\ (\pi_1^{-1}\pi_2\pi^{-1}\rho)^{1/2}\rho^{-1}\operatorname{sgn}_\tau(v),$$

*and for $\pi\in(\Pi_d\cap\tilde{C}_\tau)\cup Q_d$,*

$$\hat{H}(u, v, \pi)==\frac{1}{2}\sum_{\tau\in E}(\pi_1\pi_2^{-1}\pi\rho\operatorname{sgn}_\tau)^{1/2}\rho^{-1}\operatorname{sgn}_\tau(u)\ (\pi_1^{-1}\pi_2\pi^{-1}\rho\operatorname{sgn}_\tau)^{1/2}\rho^{-1}\operatorname{sgn}_\tau(v).$$

*Proof.* Let $H(g)=\pi_1^{-1}\pi_2(a)\Delta(x)\Delta(y)$, where $\pi_1, \pi_2\in\tilde{k}^\times$. Then, using Proposition 5.1 (1), (7.3) and Theorem 7.1, we have for $f\in\mathcal{S}(G)$,

$$\int_G H(g)f(g)dg=\int_k\left[\int_\Pi\int_k\pi_1^{-1}\pi_2(a)K_\pi(L_{d(a^{-1})}f\mid u, u)dum(\pi)d\pi\right]d^\times a$$

$$=\int_k\left[\int_{\Pi_{pr}}\int_k\pi_1^{-1}\pi_2\pi^{-1}\rho^{-1}(a)K_\pi(f\mid a^{-2}u, u)dum(\pi)\,d\pi\right]d^\times a \tag{8.5}$$

$$+[Q_{sp}]m(\pi_{sp})\int_k\left[\int_k\pi_1^{-1}\pi_2\pi_{sp}^{-1}\rho^{-1}(a)K_{\pi_{sp}}(f\mid a^{-2}u, u)du\right]d^\times a \tag{8.6}$$

$$+\int_k\left[\sum_{\pi\in\Pi_d}m(\pi)\int_k\pi_1^{-1}\pi_2\pi^{-1}\rho^{-1}\operatorname{sgn}_\tau(a)K_\pi(f\mid a^{-2}u, u)du\right]d^\times a \tag{8.7}$$

$$+[Q_d]m(\pi_0)\int_k\left[\int_k\pi_1^{-1}\pi_2\pi_0^{-1}\rho^{-1}\operatorname{sgn}_\varepsilon(a)K_{\pi_0}(f\mid a^{-2}u, u)\,du\right]d^\times a, \tag{8.8}$$

where $[Q_{sp}]$ (resp. $[Q_d]$) means that if $Q_{sp}=\emptyset$ (resp. $Q_d=\emptyset$) the term just following it does not exist. (cf. Theorem 7.1). We will study each of these terms separately. First we prove the following lemma.

**Lemma 8.3.** *Let $f\in L^1_{d^\times x}((k^\times)^2)$, then it holds*

$$\int_k f(x^2)d^\times x=2\int_{(k^\times)^2}f(x)d^\times x. \tag{8.9}$$

*Proof.* Since the space $\mathcal{S}^\times((k^\times)^2)$ is dense in $L^1_{d^\times x}((k^\times)^2)$, it is enough to prove for the characteristic function $f$ of the set $S=p^{2n}\varepsilon^{2i}(1+P^m)$ $(m>0)$. In the correspondence $x\to x^2$, there exist two preimages $S_1=p^n\varepsilon^i(1+P^m)$ and $-S_1$ of $S$. Then the left hand side $=\int_{S_1}d^\times x+\int_{-S_1}d^\times x=2\int_{1+P^m}dx=2q^{-m}$ and also the right hand side $=2\int_S d^\times x=2q^{-m}$. Thus we get the lemma. Q.E.D.

Now, let us continue the proof of Proposition 8.2. First take the term (8.5), and denote it by $A$. Change the integration order with respect to $d^\times a$ and $dum(\pi)d\pi$ and put $\lambda=\pi_1\pi_2^{-1}$, then by Corollary 5.3 and Lemma 8.3

$$A=\int_{\Pi_{pr}}\int_k\left\{\int_k(\lambda\pi\rho)^{1/2}(a^2)K_\pi(f\mid a^2u,\, u)d^\times a\right\}du\, m(\pi)d\pi$$

$$=2\int_{\Pi_{pr}}\int_k\int_{(k^\times)^2}(\lambda\pi\rho)^{1/2}\rho^{-1}(a)K_\pi(f\mid au,\, u)da\,du\,m(\pi)d\pi$$

$$=2\int_{\Pi_{pr}}\int_k\int_{(k^\times)^2 v}(\lambda\pi\rho)^{1/2}\rho^{-1}(u)(\lambda^{-1}\pi^{-1}\rho)^{1/2}\rho^{-1}(v)K_\pi(f\mid u,\, v)du\,dv\,m(\pi)d\pi\,.$$

As to the integration with respect to $du\,dv$, it holds

$$\int_k\int_{(k^\times)^2 v}\cdots du\,dv=\int_k\int_{k,\, uv\in(k^\times)^2}\cdots du\,dv=\frac{1}{4}\sum_{\tau\in E}\int_k\int_k \mathrm{sgn}_\tau uv\cdots du\,dv\,.$$

Thus

(8.10) $$A=\frac{1}{2}\sum_{\tau\in E}\sum_{\theta\in\widetilde{O}^\times}\int_T\int_k\int_k(\pi_1\pi_2^{-1}\pi\rho)^{1/2}\rho^{-1}\,\mathrm{sgn}_\tau(u)$$

$$(\pi_1^{-1}\pi_2\pi^{-1}\rho)^{1/2}\rho^{-1}\,\mathrm{sgn}_\tau(v)K_\pi(f\mid u,\, v)du\,dv\,m(\pi)d\gamma,\quad (\pi=|\cdot|^{i\gamma}\theta)\,.$$

This gives the formula $\hat{H}$ for $\pi\in\Pi_{pr}$ in Proposition 8.2.

To justify the change of integration orders, we check that the integral (8.10) is absolutely convergent. This can be done using the explicit form of $K_\pi(f\mid u,\, v)$ given in Theorem 5.2, Proposition 3.7 and $m(\pi)=\pi(-1)/(2\Gamma(\pi)\Gamma(\pi^{-1}))$.

Next we treat the term (8.6). It holds that

(8.11) $$\int_k\left\{\int_k\pi_1^{-1}\pi_2\pi_{sp}^{-1}\rho^{-1}(a)K_\pi(f\mid a^{-2}u,\, u)du\right\}d^\times a$$

$$=\frac{1}{2}\sum_{\tau\in E}\int_k\int_k(\pi_1\pi_2^{-1}\pi_{sp}\rho)^{1/2}\rho^{-1}\,\mathrm{sgn}_\tau(u)(\pi_1^{-1}\pi_2\pi_{sp}^{-1}\rho)^{1/2}\rho^{-1}\,\mathrm{sgn}_\tau(v)K_{\pi_{sp}}(f\mid u,\, v)du\,dv$$

The equality (8.11) is given under the condition that integrals of the right hand side are absolutely convergent, and the absolute convergency is similarly proved. Thus we have the formula $\hat{H}$ for $\pi\in Q_{sp}$ in the Proposition 8.2.

For the term (8.7) and (8.8), again it holds, for $\pi\in(\Pi_d\cap\tilde{C}_\tau)\cup Q_d$,

(8.12) $$\int_k\int_k\pi_1^{-1}\pi_2\pi^{-1}\rho^{-1}\,\mathrm{sgn}_\tau(a)K_\pi(f\mid a^{-2}u,\, u)du\,d^\times a$$

$$=\frac{1}{2}\sum_{\tau\in E}\int_k\int_k(\pi_1\pi_2^{-1}\pi\,\mathrm{sgn}_\tau)^{1/2}\rho^{-1}\,\mathrm{sgn}_\tau(u)\ (\pi_1^{-1}\pi_2\pi^{-1}\rho\,\mathrm{sgn}_\tau)^{1/2}\rho^{-1}\,\mathrm{sgn}_\tau(v)$$
$$\times K_\pi(f\mid u,\, v)du\,dv\,,$$

under the condition that integals in right hand side are absolutely convergent, and it is more easy to check this, because of the form $K_\pi(f\mid u,\, v)$ in Theorem 5.2 (B). Q. E. D.

## § 9. The decomposition formula in Case (I).

**9.1.** Let $\pi_1,\, \pi_2\in\tilde{k}^\times$, and $\mathfrak{R}_{\pi_1}\otimes\mathfrak{R}_{\pi_2}=\{T^{\pi_1}\otimes T^{\pi_2},\ \mathcal{S}_{\pi_1}\otimes\mathcal{S}_{\pi_2}\}$ be the tensor product of two principal series representations. The inner product corresponding to $\| \|_I$ in (I) is § 6.1 in $L^2 \otimes L^2$,

$$\|\varphi\|_I^2 = \int_k \int_k |\varphi(x_1, x_2)|^2 dx_1 dx_2 .$$

Take $f \in \mathcal{S}(G)$ such that $Uf = \varphi$ and put $f_1 = f * f^*$, then

$$\|\varphi\|_I^2 = \int_G H(g) f_1(g) dg \tag{9.1}$$

$$= \int_\Pi \int_k \int_k \hat{H}(u, v, \pi) K_\pi(f_1 | u, v) du dv m(\pi) d\pi ,$$

where $H$ is in (H. I). From (7.8), Theorem 8.1 and the fact that $\hat{H}(-u, -v, \pi) = \hat{H}(u, v, \pi)$, we get

$$\begin{aligned} \|\varphi\|_I^2 = & \sum_{s \in E} \int_{\Pi_{pr}} \int_k \int_k \int_k 2A(\pi, s)(u) \pi^{-1}(u) \bar{A}(\pi, s)(v) \overline{\pi^{-1}}(v) \\ & \times K_\pi(\check{f} | t, u) \bar{K}_\pi(\check{f} | t, v) dt du dv m(\pi) d\pi \\ & + [Q_{sp}] m(\pi_{sp}) \sum_{s \in E} \int_k \int_k \int_k 2A(\pi_{sp}, s)(u) \pi_{sp}^{-1}(u) \bar{A}(\pi_{sp}, s)(v) \overline{\pi_{sp}^{-1}}(v) \\ & \times K_{\pi_{sp}}(\check{f} | t, u) \bar{K}_{\pi_{sp}}(\check{f} | t, v) \pi_{sp}(t) dt du dv \\ & + \sum_{\pi \in \Pi_d} m(\pi) \sum_{s \in E} \int_k \int_k \int_k 2A(\pi, s)(u) \pi^{-1}(u) \bar{A}(\pi, s)(v) \overline{\pi^{-1}}(v) \\ & \times K_\pi(\check{f} | t, u) \bar{K}_\pi(\check{f} | t, v) dt du dv \\ & + [Q_d] m(\pi_0) \sum_{s \in E} \int_k \int_k \int_k 2A(\pi_0, s)(u) \pi_0^{-1}(u) \bar{A}(\pi_0, s)(v) \overline{\pi_0^{-1}}(v) \\ & \times K_{\pi_0}(\check{f} | t, u) \bar{K}_{\pi_0}(\check{f} | t, v) dt du dv . \end{aligned} \tag{9.2}$$

where $[Q_{sp}]$ and $[Q_d]$ are as in (8.6) and (8.8) respectively.

We put for $\pi \in \Pi$,

$$\begin{aligned} \Phi(t; \pi, s) &= \sqrt{2} \int_k A(\pi, s)(u) \pi^{-1}(u) K_\pi(\check{f} | t, u) du \\ &= \begin{cases} \sqrt{2} \int_{s(k^\times)^2} (\pi_1 \pi_2^{-1} \pi^{-1} \rho)^{1/2} \rho^{-1}(u) K_\pi(\check{f} | t, u) du , & \text{for } \pi \in \Pi_{pr} \cup Q_{sp} , \\ \sqrt{2} \int_{s(k^\times)^2} (\pi_1 \pi_2^{-1} \pi^{-1} \rho \operatorname{sgn}_\tau)^{1/2} \rho^{-1}(u) K_\pi(\check{f} | t, u) du , & \text{for } \pi \in (\Pi_d \cap \tilde{C}_\tau) \cup Q_d . \end{cases} \end{aligned} \tag{9.3}$$

By Theorem 5.2, this integral converges and the function $\Phi(t; \pi, s)$ in $t$ is in $\hat{\mathcal{S}}_\pi$ if $\pi \in \Pi_{pr}$, and $\Phi(t; \pi_{sp}, s)$ is in $\hat{\mathcal{S}}_{sp}$. Let $\pi \in \Pi_d \cap \tilde{C}_\tau$. By the definition, $s(k^\times)^2 \subset k_\tau^\times$ if and only if $\operatorname{sgn}_\tau s = 1$. Again Theorem 5.2, we see that $\Phi(t; \pi, s) \in \mathcal{S}^\times(k_\tau^\times)$ if $\operatorname{sgn}_\tau s = 1$, and $\in \mathcal{S}^\times((k_\tau^\times)^c)$ if $\operatorname{sgn}_\tau s = -1$. For every $s \in E$, $\Phi(t; \pi_0, s)$ is in $\mathcal{S}^\times(s(k^\times)^2)$. In addition, we have the following identity:

$$\Phi(t; \pi^{-1}, s) = \Phi(t; \pi, s) \pi(t) \qquad \text{for } \pi \in \Pi_{pr} \cup \Pi_d , \tag{9.4}$$

which follows from Proposition 5.1 (3) and (9.3).

We define a linear mapping:

(9.5)
$$V: f \longrightarrow \Phi=\Phi(t;\pi, s).$$

Here $\Phi$ is a function defined on $k\times\Pi\times E$. From (9.2), we obtain for $\varphi=Uf$,

(9.6)
$$\|\varphi\|_1^2=\sum_{s\in E}\int_{\Pi_{pr}}\int_k|\Phi(t;\pi, s)|^2 dt m(\pi)d\pi+[Q_{sp}]m(\pi_{sp})\sum_s\int_k|\Phi(t;\pi_{sp}, s)|^2\pi_{sp}(t)dt$$
$$+\sum_{\pi\in\Pi_d} m(\pi)\sum_s\int_k|\Phi(t;\pi, s)|^2 dt+[Q_d]m(\pi_0)\sum_s\int_k|\Phi(t;\pi_0, s(|^2 dt=\|\Phi\|^2 (put).$$

Now we have the commutative diagram:

(9.7)
$$\begin{array}{ccccccc}
\varphi(x_1, x_2) & \xleftarrow{\ U\ } & f & \longrightarrow & K_\pi(\check{f}|t, u) & \longrightarrow & \Phi \\
\downarrow g & & \downarrow g & & \downarrow g & & \downarrow g \\
(T_g^{\pi_1}\otimes T_g^{\pi_2})\varphi & \xleftarrow{\ U\ } & R_g f & \longrightarrow & K_\pi(L_g(\check{f})|t, u) & \longrightarrow & \boldsymbol{T}_g\Phi,
\end{array}$$

where $\boldsymbol{T}_g\Phi=\mathcal{T}_g^\pi\Phi(t;\pi, s)$. For $\pi\in\Pi_{pr}$, $\mathcal{T}_g^\pi=\hat{T}_g^\pi$ acts in $t$ as the principal series representation $\mathfrak{R}_\pi$ in § 4.1. For $\pi=\pi_{sp}$, $\mathcal{T}_g^\pi=\hat{T}_g^{\pi_{sp}}$ as the special representation $\mathfrak{R}_{sp}$ in § 4.1. For $\pi\in\Pi_d\cup Q_d$, $\mathcal{T}_g^\pi=T_g^\pi$ as the discrete series representation: if $\pi\in\Pi_d\cap\tilde{C}_\tau$ and $\mathrm{sgn}_\tau s=1$, $\mathcal{T}_g^\pi=T_g^\pi$ as $\mathfrak{R}_\pi^+$ in § 4.4, and if $\mathrm{sgn}_\tau s=-1$, as $\mathfrak{R}_\pi^-$ in § 4.6, and for $\pi\in Q_d$, $\mathcal{T}_g^\pi=T_g^\pi$ as $\mathfrak{R}_0^\pm$ in § 4.6.

Note that if $\pi_1\pi_2(-1)=1$ the special representation terms appear, and if $\pi_1\pi_2(-1)=-1$ they disappear. From Lemma 4.2, $\pi_0(-1)=1$ if $-1\notin(k^\times)^2$, and $\pi_0(-1)=-1$ if $-1\in(k^\times)^2$. Then again note that in case $\pi_1\pi_2(-1)=1$, split discrete series representation terms all appear if $-1\notin(k^\times)^2$ and disappear if $-1\in(k^\times)^2$, and in case $\pi_1\pi_2(-1)=-1$, they appear if $-1\in(k^\times)^2$ and disappear if $-1\notin(k^\times)^2$.

**9.2.** To give the decomposition formula, we construct a Hilbert spaces $\mathfrak{H}^{(+)}$ and $\mathfrak{H}^{(-)}$. Let

(9.8)
$$\Pi_{pr}=\Pi_{pr}(+1)=\{\pi\in\tilde{k}^\times;\ \pi(-1)=1\},$$
$$\Pi_d=\Pi_d(+1)=\bigcup_{\tau\in E'}\{\pi\in\Omega_d\cap\tilde{C}_\tau;\ \pi\,\mathrm{sgn}_\tau(-1)=1\},$$
$$Q_d=Q_d(+1)=\begin{cases}\{\pi_0\}, & \text{if } -1\notin(k^\times)^2,\\ \emptyset, & \text{if } -1\in(k^\times)^2,\end{cases}$$

These sets are in (7.6) for $\pi_1\pi_2(-1)=1$. Also put

(9.9)
$$\Pi=\Pi(+1)=\Pi_{pr}\cup\{\pi_{sp}\}\cup\Pi_d\cup Q_d.$$

Let $\mathfrak{H}^{(+)}$ be a space of complex valued measurable functions $\Lambda=\Lambda(t;\pi, s)$

on $k\times\Pi\times E$ satisfying the following conditions:

($\mathfrak{H}$. 1) For $\pi\in\Pi_{pr}\cup\Pi_{d}$, $\Lambda(t;\pi^{-1},s)=\Lambda(t;\pi,s)\pi(t)$.

($\mathfrak{H}$. 2) Let $\pi\in\Pi_{d}\cap\tilde{C}_{\tau}$, then if $\mathrm{sgn}_{\tau}s=1$, $\Lambda(t;\pi,s)=0$ for almost all $t\in(k_{\tau}^{\times})^{c}$, and if $\mathrm{sgn}_{\tau}s=-1$, $\Lambda(t;\pi,s)=0$ for almost all $t\in k_{\tau}^{\times}$. Let $\pi\in Q_{d}$, then $\Lambda(t;\pi,s)=0$ for almost all $t\notin s(k_{\tau}^{\times})^{2}$.

($\mathfrak{H}^{(+)}$. 3) $\Lambda=\Lambda(t;\pi,s)$ is square integrable in the following sense:

$$\|\Lambda\|^{2}=\sum_{s\in E}\int_{\Pi_{pr}}\int_{k}|\Lambda(t;\pi,s)|^{2}dtm(\pi)d\pi+m(\pi_{sp})\sum_{s}\int_{k}|\Lambda(t;\pi_{sp},s)|^{2}\pi_{sp}(t)dt$$
$$+\sum_{\pi\in\Pi_{d}}m(\pi)\sum_{s}\int_{k}|\Lambda(t;\pi,s)|^{2}dt+[Q_{d}]m(\pi_{0})\sum_{s}\int_{k}|\Lambda(t;\pi_{0},s)|^{2}dt<\infty, \tag{9.10}$$

where $[Q_{d}]=[Q_{d}(+1)]$ means that if $-1\in(k^{\times})^{2}$ the term just following it vanishes.

$\mathfrak{H}^{(+)}$ is a separable Hilbert space with the inner product corresponding to (9.10). We define a representation $\mathfrak{R}^{(+)}=\{T^{(+)},\mathfrak{H}^{(+)}\}$ of $G$ by

$$T_{g}^{(+)}\Lambda=\mathfrak{T}_{g}^{\pi}\Lambda(t;\pi,s), \tag{9.11}$$

where $\mathfrak{T}_{g}^{\pi}$ is the irreducible unitary representation corresponding to $\pi$ or $(\pi,s)$ as is explained for the diagram (9.7).

The unitary representations obtained by completion from $\hat{\mathfrak{R}}_{\pi}$ etc. are denoted as follows: (a) $\bar{\mathfrak{R}}_{\pi}$ for $\hat{\mathfrak{R}}_{\pi}$ with $\pi\in\Pi_{pr}$, (b) $\bar{\mathfrak{R}}_{sp}$ for $\hat{\mathfrak{R}}_{sp}$, (c) $\bar{\mathfrak{R}}_{\pi}^{+}$ and $\bar{\mathfrak{R}}_{\pi}^{-}$ for $\mathfrak{R}_{\pi}^{+}$ and $\mathfrak{R}_{\pi}^{-}$ with $\pi\in\Pi_{d}$ respectively, and (c) $\bar{\mathfrak{R}}_{0}^{s}$, $s\in E$, for $\mathfrak{R}_{0}^{s}$. Let $\Pi_{pr}'$ be the set of the equivalence classes with a relation $\pi\sim\pi^{-1}$ on $\Pi_{pr}$ and $\Pi_{d}'$ similar. Then the representation $\mathfrak{R}^{(+)}$ is expressed as a direct integral

$$\mathfrak{R}^{(+)}\simeq[4]\int_{\Pi_{pr}'}\bar{\mathfrak{R}}_{\pi}m(\pi)d\pi\oplus[4]\bar{\mathfrak{R}}_{sp}$$
$$\oplus[2]\sum_{\pi\in\Pi_{d}'}(\bar{\mathfrak{R}}_{\pi}^{+}\oplus\bar{\mathfrak{R}}_{\pi}^{-})\oplus[Q_{d}](\bar{\mathfrak{R}}_{0}^{1}\oplus\bar{\mathfrak{R}}_{0}^{\varepsilon}\oplus\bar{\mathfrak{R}}_{0}^{p}\oplus\bar{\mathfrak{R}}_{0}^{\varepsilon p}). \tag{9.12}$$

where [4] and [2] are the multiplicities of the representations.

The Hilbert space $\mathfrak{H}^{(-)}$ is defined similarly as $\mathfrak{H}^{(+)}$. Let

$$\Pi_{pr}=\Pi_{pr}(-1)=\{\pi\in\tilde{k}_{\times};\ \pi(-1)=-1\},$$
$$\Pi_{d}=\Pi_{d}(-1)=\bigcup_{\tau\in E'}\{\pi\in\Omega_{d}\cap\tilde{C}_{\tau});\ \pi\,\mathrm{sgn}_{\tau}(-1)=-1\}, \tag{9.13}$$
$$Q_{d}=Q_{d}(-1)=\begin{cases}\{\pi_{0}\}, & \text{if } -1\in(k^{\times})^{2},\\ \emptyset, & \text{if } -1\notin(k^{\times})^{2}.\end{cases}$$

Put

$$\Pi=\Pi(-1)=\Pi_{pr}\cup\Pi_{d}\cup Q_{d}. \tag{9.14}$$

$\mathfrak{H}^{(-)}$ is a Hilbert space of functions $\Lambda=\Lambda(t;\pi,s)$ on $k\times\Pi\times E$, $\Pi$ in (9.14), satisfying ($\mathfrak{H}$. 1), ($\mathfrak{H}$. 2) and the condition:

($\mathfrak{H}^{(-)}$. 3) $\Lambda=\Lambda(t;\eta,s)$ is square integrable in the following sense:

(9.15) $$\|A\|^2=\sum_{s\in E}\int_{\Pi_{pr}}\int_k |A(t\,;\,\pi,\,s)|^2 dt m(\pi)d\pi$$

$$+\sum_{\pi\in\Pi_d} m(\pi)\sum_s\int_k |A(t\,;\,\pi,\,s)|^2 dt+[Q_d]m(\pi_0)\sum_s\int_k |A(t\,;\,\pi_0,\,s)|^2 dt<\infty\,,$$

where $[Q_d]=[Q_d(-1)]$ means that if $-1\notin(k^\times)^2$ the term just following it vanishes.

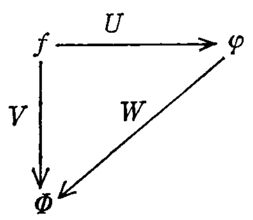
On $\mathfrak{H}^{(-)}$, we define a representation $\mathfrak{R}^{(-)}$ of $G$ and it is expressed as a direct integral

(9.16) $$\mathfrak{R}^{(-)}\simeq[4]\int_{\Pi'_{pr}}\mathfrak{K}_\pi m(\pi)d\pi\oplus[2]\sum_{\pi\in\Pi_d}{}'(\mathfrak{K}_\pi^+\oplus\mathfrak{K}_\pi^-)$$

$$\oplus[Q_d](\mathfrak{K}_0^1\oplus\mathfrak{K}_0^\varepsilon\oplus\mathfrak{K}_0^p\oplus\mathfrak{K}_0^{\varepsilon p})\,.$$

**9.3.** Suppose $\pi_1\pi_2(-1)=1$. We show the tensor product $\mathfrak{K}_{\pi_1}\bar{\otimes}\mathfrak{K}_{\pi_2}$ in § 6.1 equals $\mathfrak{R}^{(+)}$. Similarly, in case $\pi_1\pi_2(-1)=-1$ the tensor product equals $\mathfrak{R}^{(-)}$. In this subsection, $\mathfrak{R}$ means $\mathfrak{R}^{(+)}$ and so on.

Every element $\Phi$ in (9.5) is in $\mathfrak{H}$. So, we get a linear isomorphic $G$-morphism $W:\varphi\to\Phi$ of $\mathcal{H}$ into $\mathfrak{H}$ such that $WU=V$, and it is extended to an isomorphic mapping from $L^2\otimes L^2$ into $\mathfrak{H}$, denoted again by $W$.




**Proposition 9.1.** *The image of $L^2\otimes L^2$ under $W$ is the whole space $\mathfrak{H}$.*

*Proof.* For each $s\in E$, let $\mathfrak{H}_s$ be the subspace of the functions $A=A(t\,;\,\pi,s)$ in $\mathfrak{H}$ such that $A(t\,;\,\pi,\,s')=0$ if $s'\neq s$. Then

(9.17) $$\mathfrak{R}=\mathfrak{R}_1\oplus\mathfrak{R}_\varepsilon\oplus\mathfrak{R}_p\oplus\mathfrak{R}_{\varepsilon p}\,,$$

where for $s\in E$, $\mathfrak{R}_s=\{\boldsymbol{T}^s,\,\mathfrak{H}_s\}$, $\boldsymbol{T}^s$ the restriction of $\boldsymbol{T}$ to $\mathfrak{H}_s$. All the irreducible component in $\mathfrak{R}_s$ appears with multiplicity one. Take $\mathfrak{R}_1$. It is denoted by

(9.18) $$\mathfrak{R}_1=\int_{\Pi'}\{\mathfrak{T}^\pi,\,\mathfrak{H}(\pi)\}\,m(\pi)d\pi$$

where $\mathfrak{H}(\pi)=L^2$ for $\pi\in\Pi'_{pr}$, $=L^2_{sp}$ (in § 3.4) for $\pi=\pi_{sp}$, $=L^2(k_\tau^\times)$ for $\pi\in\Pi'_d\cap\tilde{C}_\tau$ and $=L^2((k^\times)^2)$ for $\pi\in Q_d$.

Let $\mathfrak{M}$ be the image of $L^2\otimes L^2$ under $W$, and $P_s$ the orthogonal projections of $\mathfrak{H}$ onto $\mathfrak{H}_s$. Then $\mathfrak{M}_s=P_s\mathfrak{M}$ is $G$-invariant.

We shall prove the proposition by two steps: (1) $\mathfrak{M}_s=\mathfrak{H}_s$ for every $s\in E$, and (2) $\mathfrak{M}=\mathfrak{H}$.

Step 1. We prove $\mathfrak{M}_1=\mathfrak{H}_1$, and the other cases are proved similarly. On $\mathfrak{H}_1$, we consider the representation $\int_{\Pi'}\{\mathfrak{T}_f^{\pi}, \mathfrak{H}(\pi)\}m(\pi)d\pi$ of the group algebre $L^1(G)$, corresponding to $\mathfrak{N}_1$. Note that $\mathfrak{M}_1$ is closed and $L^1(G)$-innvariant.

**Lemma 9.2.** *The representations $f\to\mathfrak{T}_f^{\pi}$ of $L^1(G)$ satisfy the following properties:*

1. *$\mathfrak{T}_f^{\pi}$, as an operator valued function on the locally compact space $\Pi'$, is continuous in the sense of the operator norm and is zero at infinity.*
2. *The operator $\mathfrak{T}_f^{\pi}$ is compact.*
3. *Every representation $f\to\mathfrak{T}_f^{\pi}$ is irreducible.*
4. *For arbitrary $\pi_1, \pi_2\in\Pi'$, $\pi_1\neq\pi_2$, the representations are not equivalent.*

*Proof.* For given $f\in L^1(G)$ and $\varepsilon>0$, we have $h\in\mathcal{S}(G)$ such that $\|f-h\|_1<\varepsilon$ where $\|\ \|_1$ is $L^1$-norm. Since $\mathfrak{T}_h^{\pi}$ is given as an integral operator with $K_\pi(h|u,v)$ as in Theorem 5.2, we see easily that 1 and 2 hold for $\mathfrak{T}_h^{\pi}$. This lead us immediately to 1 and 2 for $\mathfrak{T}_f^{\pi}$. 3 and 4 are obvious. Q. E. D.

**Lemma 9.3.** *Let $\mathfrak{N}$ be a $\mathfrak{T}_f^{\pi}$-invariant subspace, $f\in L^1(G)$, in $\mathfrak{H}_1=\int_{\Pi'}\mathfrak{H}(\pi)m(\pi)d\pi$. Then $\mathfrak{N}$ is the set of all vectors $\Phi=\Phi(\pi)\in\mathfrak{H}_1$ which satisfy the condition $\Phi(\pi)=0$ for almost all $\pi\in N$, where $N$ is a fixed $d\pi$-measurable set in $\Pi'$.*

Under the properties in Lemma 9.2, Lemma 9.3 holds and it is obtained by modifying a little Corollary 1 of Theorem 8, "Continuous Analogue of the Schur Lemma", in [8, p. 358, p. 356]. Thus $\mathfrak{M}_1=\mathfrak{H}_1$ will be proved if we show that $N$ is a set of measure zero. For this, it suffices to prove that for each $\pi\in\Pi$, there exists $\varphi\in\mathcal{H}\subset\mathcal{S}_{\pi_1}\otimes\mathcal{S}_{\pi_2}$ such that $\Phi(t;\pi,1)\neq 0$ in $\mathfrak{H}(\pi)$ where $\Phi(t;\pi,1)$ is the component of $\Phi=W\varphi$.

We give $\varphi$ as $\varphi=Uf$, $f\in\mathcal{S}(G)$ supported in $wG^0$. Take $f$ as

$$f(g)=\xi(-x)\eta(-y)\kappa(a^{-1}),\qquad \text{for } g=n(x)d(a)wn(y), \tag{9.19}$$

where $\xi, \eta\in\mathcal{S}$ and $\kappa\in\mathcal{S}^\times$. Then, $\check{f}(g)=f(g^{-1})=\eta(y)\xi(x)\kappa(a)$, and from (5.5), $K_\pi(\check{f}|t, u)=\hat{\eta}(t)\hat{\xi}(u)M_\pi(t, u)$, where

$$M_\pi(t, u)=\begin{cases}\displaystyle\int_k\kappa(a)J_\pi(at, au)\pi^{-1}(a)d^\times a, & \text{for } \pi\in\Pi_{pr}\cup\{\pi_{sp}\},\\[2mm] \displaystyle a_\tau c_\tau\int_k\kappa(a)J_\pi^d(at, au)\pi^{-1}\operatorname{sgn}_\tau(a)d^\times a, & \text{for } \pi\in\Pi_d\cup\{\pi_0\}.\end{cases}\tag{9.20}$$

For a given $\pi\in\Pi_{pr}\cup\{\pi_{sp}\}$ (resp. $\pi\in\Pi_d\cup Q_d$), there exists a neighborhood of a fixed point $(u_0, t_0, a_0)\in k^\times\times k^\times\times k^\times$ on which the function $J_\pi(at, au)$ (resp. $J_\pi^d(at, au)$) takes a non-zero constant value. This makes clear to be possible to choose $\xi$, $\eta$ and $\kappa$ such that

$$\Phi(t;\pi,1)=\int_{(k^\times)^2}A(\pi,s)(u)\pi^{-1}(u)K_\pi(\check{f}\mid t,u)du\neq 0\,,$$

where $A(\pi,s)$ is as in (8.1) and (8.2). Thus we get $\mathfrak{M}_1=\mathfrak{H}_1$.

Step 2. To prove $\mathfrak{M}=\mathfrak{H}$, it is enough to show that, for arbitrary $\Lambda\in\mathfrak{H}$ and $\varepsilon>0$, we have $\varphi\in\mathcal{H}$ such that $\|\Lambda-\Phi\|<4\varepsilon$ where $\Phi=W\varphi$. According to (9.13), $\Lambda$ is decomposed as $\Lambda=\Lambda_1+\Lambda_\varepsilon+\Lambda_p+\Lambda_{\varepsilon p}$, $\Lambda_s\in\mathfrak{H}_s$. If we have $\varphi_s\in\mathcal{H}$ for every $s\in E$ such that $\Phi_s=W\varphi_s\in\mathfrak{H}_s$ and $\|\Lambda_s-\Phi_s\|<\varepsilon$, then for $\Phi=\sum_{s\in E}\Phi_s=\sum_{s\in E}W\varphi_s$ it holds $\|\Lambda-\Phi\|<4\varepsilon$ and so $\varphi=\sum_{r\in E}\varphi_s$ is a required function.

From Step (1) there exists $\psi\in\mathcal{H}$ such that $\|P_sW\psi-\Lambda_s\|<\varepsilon/2$. On the other hand, from the next lemma, there exists $\varphi_s\in\mathcal{H}$ such that $W\varphi_s\in\mathfrak{H}_s$ and $\|W\varphi_s-P_sW\psi\|<\varepsilon/2$. Hence $\|W\varphi_s-\Lambda_s\|<\varepsilon$, and this completes the poof of Proposition 9.1. Q. E. D.

Now the following lemma is left to be proved.

**Lemma 9.4.** *Let $\psi\in\mathcal{H}$, $\varepsilon>0$, and $s\in E$. Then there exists a function $\varphi_s\in\mathcal{H}$ such that $W\varphi_s\in\mathfrak{H}_s$ and $\|W\varphi_s-P_sW\psi\|<\varepsilon$.*

*Proof*. Let $\psi=Uh$, $h\in\mathcal{S}(G)$. We can assume $h$ is as in (9.19). Then $K_\pi(\check{h}\mid t,u)=\hat{\eta}(t)\hat{\xi}(u)M_\pi(t,u)$ where $M_\pi(t,u)$ is as in (9.20). For given $\delta>0$, let $k$ be a natural number such that, for $\pi\in\Pi_{pr}\cup\{\pi_{sp}\}$, it holds

$$\left|\int_{P^k}(\pi_1\pi_2^{-1}\pi^{-1}\rho)^{1/2}\rho^{-1}(u)\hat{\xi}(u)du\right|<\delta\,,$$

and if $u\in P^k$, then $|a^2ut|<1$ for all $t\in\mathrm{Supp}[\hat{\eta}]$, $a\in\mathrm{Supp}[\kappa]$. Let $\zeta(u)$ be the function equal to $\hat{\xi}(u)$ if $u\in P^k$ and zero otheerwise, $\hat{\xi}_s$ be the function such that $\hat{\xi}_s=\hat{\xi}-\zeta$ on $s(k^\times)^2$ and zero outside. Since $\hat{\xi}-\zeta\in\mathcal{S}^\times$, $\hat{\xi}_s\in\mathcal{S}^\times$ and whence $\xi_s\in\mathcal{S}$. We set $f_s=\xi_s(-x)\eta(-y)\kappa(a^{-1})\in\mathcal{S}(G)$. We prove for $s=1$ that $\varphi_s=Uf_s$ is a required function, namely, prove that $W\varphi_s=Vf_s\in\mathfrak{H}_s$ and $\|Vf_s-P_sVh\|<\varepsilon$ for $\delta$ small enough. For another $s$, the proof is similar.

Put $\Phi_1=Vf_1$. Its component $\Phi_1(t;\pi,r)$ for $\pi\in\Pi$ and $r\in E$ is given by

$$\Phi_1(t;\pi,r)=\int_k A(\pi,r)(u)\pi^{-1}(u)\hat{\eta}(t)\hat{\xi}_1(u)M_\pi(t,u)du\,,$$

where $M_\pi(t,u)$ is in (9.20). Since $\mathrm{Supp}[\hat{\xi}_1]\subset(k^\times)^2$, the above integral is actually taken over $r(k^\times)^2\cap(k^\times)^2$. Hence $\Phi_1(t;\pi,r)=0$ if $r\neq 1$. Thus $Vf_1\in\mathfrak{H}_1$. Put $\Psi_1=P_1Vh$ and let $\Psi_1(t;\pi,1)$ be its component. Then

$$\Psi_1(t;\pi,1)-\Phi_1(t;\pi,1)=\int_{(k^\times)^2}A(\pi,1)(u)\pi^{-1}(u)\zeta(u)\hat{\eta}(t)M_\pi(t,u)du\,.$$

Since the support of $\zeta(u)\hat{\eta}(t)\kappa(a)$ is contained in $\{(u,t,a);\,|a^2ut|<1\}$, it follows from the discussions in § 5 that if $\zeta(u)\hat{\eta}(t)\neq 0$, then

$$M_\pi(t, u)=\begin{cases} \pi(u)\Gamma(\pi^{-1})\tilde{\kappa}_1(\pi)+\pi^{-1}(t)\Gamma(\pi)\tilde{\kappa}_1(\pi^{-1}), & \text{for } \pi\in\Pi_{pr} \\ \pi_{sp}^{-1}(t)\Gamma(\pi_{sp})\tilde{\kappa}_1(\pi_{sp}^{-1}), & \text{for } \pi=\pi_{sp}, \\ 0, & \text{for } \pi\in\Pi_d\cup Q_d, \end{cases}$$

where $\tilde{\kappa}_1(\pi)=\int_k \kappa(a)|a|\pi(a)d^\times a$. Thus

$$|\Psi_1(t;\pi, s)-\Phi_1(t;\pi, s)|=\left|\int_k A(\pi, 1)(u)\pi^{-1}(u)\zeta(u)\hat{\eta}(t)M_\pi(t, u)du\right|$$

is less than $\delta|\hat{\eta}(t)||\Gamma(\pi^{-1})\tilde{\kappa}_1(\pi)+\Gamma(\pi)\tilde{\kappa}_1(\pi^{-1})|$ if $\pi\in\Pi_{pr}$, and less than $\delta|\pi_{sp}^{-1}(t)\hat{\eta}(t)\Gamma(\pi_{sp})\tilde{\kappa}_1(\pi_{sp}^{-1})|$ if $\pi=\pi_{sp}$ and equal to 0 if $\pi\in\Pi_d\cup Q_d$. Thus we have

$$\|\Psi_1-\Phi_1\|^2<\delta^2\Big\{\|\hat{\eta}\|^2\int_{\Pi_{pr}}|\Gamma(\pi^{-1})\tilde{\kappa}_1(\pi)+\Gamma(\pi)\tilde{\kappa}_1(\pi^{-1})|^2 m(\pi)d\pi$$
$$+\|\pi_{sp}^{-1}\hat{\eta}\|^2|\Gamma(\pi_{sp})\tilde{\kappa}_1(\pi_{sp}^{-1})|^2\Big\}.$$

Since $\Gamma(\pi^{-1})\tilde{\kappa}_1((\pi)+\Gamma(\pi)\tilde{\kappa}_1(\pi^{-1})$ can be extended as a continuous function, even at $\pi\equiv 1$, and is compactly supported, then the integral converges. Taking $\Phi_1$ for a sufficiently small $\delta$, we have the lemma for $s=1$. Q. E. D.

**9.4.** Now we arrive at one of our main results.

**Theorem 9.5.** *Let $\pi_1$, $\pi_2$ be fixed unitary characters in $\tilde{k}^\times$, $\Pi=\Pi(\pm 1)$ be in* (9.9) *or* (9.14). *Let $\mathfrak{H}=\mathfrak{H}^{(\pm)}$ be the Hilbert spoce of the functions on $k\times\Pi\times E$ satisfying the conditions* ($\mathfrak{H}$.1), ($\mathfrak{H}$.2) *and* ($\mathfrak{H}^{(\pm)}$.3) *in* § 9.2. *Then there exists a unitary mapping $W:\varphi\to\Lambda$ of $L^2\otimes L^2$ onto $\mathfrak{H}$, $\mathfrak{H}=\mathfrak{H}^{(+)}$ or $\mathfrak{H}^{(-)}$ according as $\pi_1\pi_2(-1)=1$ or $\pi_1\pi_2(-1)=-1$. $W$ is given on $\mathcal{K}$ $(\subset\mathcal{S}_{\pi_1}\otimes\mathcal{S}_{\pi_2})$ by $WU=V$, where $U$ and $V$ are defined in* (6.2) *and* (9.5) *respectively. Moreover $W$ is a G-morphism, $\Pi_g W=WT_g$ $(g\in G)$, where $T_g$ is an operator of the tensor product $\mathfrak{R}_{\pi_1}\bar{\otimes}\mathfrak{R}_{\pi_2}$: for $\varphi\in L^2\otimes L^2$,*

$$T_g\varphi(x_1, x_2)=\pi_1\rho^{-1}(\beta x_1+\delta)\pi_2\rho^{-1}(\beta x_2+\delta)\varphi\Big(\frac{\alpha x_1+\gamma}{\beta x_1+\delta}, \frac{\alpha x_2+\gamma}{\beta x_2+\delta}\Big),$$

*and $\boldsymbol{T}_g$ is given as follows: for $\Lambda\in\mathfrak{H}$,*

$$\boldsymbol{T}_g\Lambda=[\mathfrak{T}_g^\pi\Lambda(t;\pi, s), \pi\in\Pi, s\in E],$$

*where*

$$\begin{aligned} \mathfrak{T}_g^\pi\Lambda(t;\pi, s)&=\pi\rho(a)\Lambda(a^2t;\pi, s), & g&=d(a), \\ &=\chi(-tx)\Lambda(t;\pi, s), & g&=n(x), \\ &=H_\pi\Lambda(t;\pi, s), & g&=w \text{ and } \pi\in\Pi_{pr}\cup Q_{sp}, \\ &=H_\pi^d\Lambda(t;\pi, s), & g&=w \text{ and } \pi\in\Pi_d\cup Q_d, \end{aligned}$$

*Here $H_\pi$ and $H_\pi^d$ are defined in* (3.2) *and* (4.22) *respectively.*

In other words,

**Theorem 9.6.** *The unitary transformation $W$ realizes the decomposition of the tensor product $\mathfrak{R}_{\pi_1}\bar{\otimes}\mathfrak{R}_{\pi_2}$ into irreducibles as follows: In case $-1\in(k^\times)^2$, if $\pi_1\pi_2(-1)=1$*

$$\mathfrak{R}_{\pi_1}\bar{\otimes}\mathfrak{R}_{\pi_2}\simeq[4]\int_{\Pi_{pr(+1)'}}\mathfrak{R}_\pi m(\pi)d\pi\oplus[4]\mathfrak{R}_{sp}\oplus[2]\int_{\Pi_{d(+)'}}(\mathfrak{R}_\pi^+\oplus\mathfrak{R}_\pi^-), \tag{9.21}$$

*and if* $\pi_1\pi_2(-1)=-1$

$$\mathfrak{R}_{\pi_1}\bar{\otimes}\mathfrak{R}_{\pi_2}\simeq[4]\int_{\Pi_{pr(-1)'}}\mathfrak{R}_\pi m(\pi)d\pi\oplus[2]\sum_{\pi\in\Pi_{d(-1)'}}(\mathfrak{R}_\pi^+\oplus\mathfrak{R}_\pi^-) \tag{9.22}$$

$$\oplus(\mathfrak{R}_0^1\oplus\mathfrak{R}_0^\varepsilon\oplus\mathfrak{R}_0^p\oplus\mathfrak{R}_0^{\varepsilon p}).$$

*In case* $-1\notin(k^\times)^2$, *if* $\pi_1\pi_2(-1)=1$

$$\mathfrak{R}_{\pi_1}\bar{\otimes}\mathfrak{R}_{\pi_2}\simeq[4]\int_{\Pi_{pr(+1)'}}\mathfrak{R}_\pi m(\pi)d\pi\oplus[4]\mathfrak{R}_{sp} \tag{9.23}$$

$$\oplus[2]\sum_{\pi\in\Pi_{d(+1)'}}(\mathfrak{R}_\pi^+\oplus\mathfrak{R}_\pi^-)\oplus(\mathfrak{R}_0^1\oplus\mathfrak{R}_0^\varepsilon\oplus\mathfrak{R}_0^p\oplus\mathfrak{R}_0^{\varepsilon p}),$$

*and if* $\pi_1\pi_2(-1)=-1$,

$$\mathfrak{R}_{\pi_1}\bar{\otimes}\mathfrak{R}_{\pi_2}\simeq[4]\int_{\Pi_{pr(-1)'}}\mathfrak{R}_\pi m(\pi)d\pi\oplus[2]\sum_{\pi\in\Pi_{d(-1)'}}(\mathfrak{R}_\pi^+\oplus\mathfrak{R}_\pi^-). \tag{9.24}$$

**9.5.** We give the direct form of the intertwining projection for $\pi\in\Pi_{pr}\cup Q_{sp}$. First let for $r\in E$,

$$\Phi_r(t;\pi)=\sqrt{2}^{-1}\sum_{s\in E}(\mathrm{sgn}_r s)\Phi(t;\pi,s)$$

$$=\sqrt{2}^{-1}\int_k(\pi_1\pi_2^{-1}\pi^{-1}\rho)^{1/2}\rho^{-1}\,\mathrm{sgn}_r(u)K_\pi(\check{f}\,|\,t,u)du.$$

Let $\hat{\Phi}_r(x;\pi)$ be the (principal value integral) Fourier transform of $\Phi_r(t;\pi)$ with respect to $t$. Then we have the following direct formula of the intertwining projection: $\varphi\to\hat{\Phi}_r(x;\pi)$. This is quite analogous to that given in [9, p. 124] for the decomposition of the tensor product for $SL_2(\boldsymbol{C})$.

**Proposition 9.7.** *For* $\pi\in\Pi_{pr}\cup Q_{sp}$,

$$\hat{\Phi}_r(x;\pi)=\sqrt{2}^{-1}\Gamma((\pi_1\pi_2^{-1}\pi^{-1}\rho)^{1/2}\,\mathrm{sgn}_r)$$

$$\times\int_k\int_k(\pi_1^{-1}\pi_2\pi\rho)^{1/2}\rho^{-1}\,\mathrm{sgn}_r(z_1)(\pi_1\pi_2^{-1}\pi\rho)^{1/2}\rho^{-1}\,\mathrm{sgn}_r(z_2)$$

$$\times(\pi_1^{-1}\pi_2^{-1}\pi^{-1}\rho)^{1/2}\rho^{-1}\,\mathrm{sgn}_r(z_2-z_1)\varphi(z_1+x,z_2+x)dz_1dz_2.$$

*Proof.* We set, for $f\in\mathcal{S}(G)$,

$$F(x,x_1,\pi)=\int_k\int_k\check{f}(n(-x)d(a)n^+(y)n(x_1))\pi\rho^{-1}(a)d^\times a\,dy$$

$$=\int_k\int_k f(n(-x_1)d(a)n^+(y)n(x))\pi^{-1}\rho^{-1}(a)d^\times a\,dy.$$

For a fixed $\pi\in\Pi_{pr}\cup Q_{sp}$, $F(x, x_1, \pi)\in\mathcal{S}_\pi\otimes\mathcal{S}_{\pi^{-1}}$ and $K_\pi(\check{f}|t, u)$ is given by

$$K_\pi(\check{f}|t, u)=P\text{-}\int_k\int_k F(x, x_1, \pi)\chi(tx)\chi(-ux_1)dxdx_1 .$$

Then

$$\hat{\Phi}_r(x\,;\pi)=\mathcal{F}_t\sqrt{2}^{-1}\Big\{\int_k \widehat{[(\pi_1\pi_2^{-1}\pi^{-1}\rho)^{1/2}\rho^{-1}\operatorname{sgn}_r]}(u)K_\pi(\check{f}|t, u)du\Big\}$$

$$=\sqrt{2}^{-1}\Gamma((\pi_1\pi_2^{-1}\pi^{-1}\rho)^{1/2}\operatorname{sgn}_r)\int_k(\pi_1^{-1}\pi_2\pi\rho)^{1/2}\rho^{-1}\operatorname{sgn}_r(x_1)F(x, x_1, \pi)dx_1$$

$$=\sqrt{2}^{-1}\Gamma((\pi_1\pi_2^{-1}\pi^{-1}\rho)^{1/2}\operatorname{sgn}_r)\int_k\int_k\int_k(\pi_1^{-1}\pi_2\pi\rho)^{1/2}\rho^{-1}\operatorname{sgn}_r(x_1)$$

$$f(n(-x_1)d(a)n^+(y)n(x))\pi^{-1}\rho^{-1}(a)d^\times a\,dy\,dx_1 .$$

According to the decomposition (4.3), we have

$$n(-x_1)n^+(y)=d(-x_1y+1)n^+((-x_1y+1)y)n(-x_1(-x_1y+1)^{-1}),$$

then

$$A=\int_k\int_k\int_k(\pi_1^{-1}\pi_2\pi\rho)^{1/2}\rho^{-1}\operatorname{sgn}_r(x_1)\pi^{-1}\rho^{-1}(a)f(n(-x_1)d(a)n^+(y)n(x))d^\times a\,dy\,dx_1$$

$$=\int_k\int_k\int_k(\pi_1^{-1}\pi_2\pi\rho)^{1/2}\rho^{-1}\operatorname{sgn}_r(x_1)\pi_1^{-1}\pi_2(a)f(d(a)n(-x_1)n^+(y)n(x))d^\times a\,dy\,dx_1$$

$$=\int_k\int_k\int_k(\pi_1^{-1}\pi_2\pi\rho)^{1/2}\rho^{-1}\operatorname{sgn}_r(x_1)\pi_1^{-1}\pi_2(a)$$

$$\times f(d(a)d(-x_1y+1)n^+((-x_1y+1)y)n(-x_1(-x_1y+1)^{-1}+x))d^\times a\,dy\,dx_1 ,$$

Take, for given $\varphi\in\mathcal{H}$, $f(g)=\pi_1\pi_2^{-1}(a)\kappa(a)\pi_2\rho^{-1}(y)\varphi(x, x+y^{-1})$ in $\mathcal{S}(G)$, where $\kappa\in\mathcal{S}^\times$ is such that $\int_k\kappa d^\times a=1$. In the last side of $A$, replace a by $a(-x_1y+1)^{-1}$, then we obtain

$$A=\int_k\int_k(\pi_1^{-1}\pi_2\pi\rho)^{1/2}\rho^{-1}\operatorname{sgn}_r(x_1)\pi_1\pi_2^{-1}(-x_1y+1)\pi_2\rho^{-1}((-x_1y+1)y)$$

$$\varphi(-x_1(-x_1y+1)^{-1}+x,\ y^{-1}+x)dx_1dy ,$$

because $(-x_1y+1)^{-1}y^{-1}-x_1(-x_1y+1)^{-1}+x=y^{-1}+x$. Now we change the variable; $z_1=-x_1(-x_1y+1)^{-1}$ and $z_2=y^{-1}$. Then $(-x_1y+1)y=(z_1-z_2)^{-1}$, $-x_1y+1=z_2(z_1-z_2)^{-1}$, $x_1=-z_1z_2(z_1-z_2)^{-1}$, and $dx_1dy=\rho^{-2}(z_2-z_1)dz_1dz_2$. Then we come to the desired formula for $\hat{\Phi}_r(x\,;\pi)$. Q. E. D.

## §10. The decomposition formula in Case (II).

In this section, we give the decomposition formula of the tensor product of a supplementary series representation with a principal series one. Let $\pi_1$ be a character of the form $\pi_1(x)=|x|^{\alpha_1}$, $-1<\alpha_1<0$, and $\pi_2\in\tilde{k}^\times$. Note that in this case the equalities $\pi_1\pi_2(-1)=1$ or $=-1$ turn out to $\pi_2(-1)=1$ or $=-1$. So,

$\Pi_{pr}$ and other sets in (7.6) depend only on $\pi_2(-1)$. We have the following theorem quite similar to Theorem 8.1.

**Theorem 10.1.** *Let $H$ in* (H. II), $H(g)=\Gamma(\pi_1^{-1})^{-1}\pi_1^{-1}\pi_2(a)\pi_1^{-1}\rho^{-1}(x)\Delta(y)$ *for* $g=d(a)n^+(y)n(x)$. *Then the Plancherel transform $\hat{H}$ is given as follows:*

$$\hat{H}(u, v, \pi)=2\sum_{s\in E} A(\pi, s)(u)\bar{A}(\pi, s)(v), \qquad for\ \pi\in\Pi_{pr}\cup\Pi_d\cup Q_d,$$

$$\hat{H}(u, v, \pi)=2\sum_{s\in E} A(\pi, s)(u)\bar{A}(\pi, s)(v)\pi_{sp}^{-1}(v), \qquad for\ \pi\in Q_{sp},$$

*where $A(\pi, s)$ are similar to* (8.1) *and* (8.2).

This theorem is reduced to the following.

**Proposition 10.2.** *The Plancherel transform $\hat{H}$ of $H$ in* (H. II) *is given as follows: for $\pi\in\Pi_{pr}\cup Q_{sp}$,*

$$\hat{H}(u, v, \pi)=\frac{1}{2}\sum_{\tau\in E}(\pi_1\pi_2^{-1}\pi\rho)^{1/2}\rho^{-1}\,\mathrm{sgn}_\tau(u)(\pi_1\pi_2\pi^{-1}\rho)^{1/2}\rho^{-1}\,\mathrm{sgn}_\tau(v),$$

*and for $\pi\in(\Pi_d\cap\tilde{C}_\tau)\cup Q_d$,*

$$\hat{H}(u, v, \pi)=\frac{1}{2}\sum_{\tau\in E}(\pi_1\pi_2^{-1}\pi\rho\,\mathrm{sgn}_\tau)^{1/2}\rho^{-1}\,\mathrm{sgn}_\tau(u)(\pi_1\pi_2\pi^{-1}\rho\,\mathrm{sgn}_\tau)^{1/2}\rho^{-1}\,\mathrm{sgn}_\tau(v).$$

*Proof.* Let $H(g)=\Gamma(\pi_1^{-1})^{-1}\pi_1^{-1}\pi_2(a)\pi_1^{-1}\rho^{-1}(x)\Delta(y)$, where $\pi_1(x)=|x|^{\alpha_1}$, $1<\alpha_1<0$, and $\pi_2\in\tilde{k}^\times$. Remark $d(a)n(x)=n(a^2x)d(a)$, then replace $a^2x$ by $x$, and put $f_1=L_{n(-x)}f$. So, we get

$$\Gamma(\pi_1^{-1})\int_G H(g)f(g)dg=\int_k\pi_1^{-1}\rho^{-1}(x)\left\{\int_k\pi_1\pi_2(a)f_1(d(a))d^\times a\right\}dx$$

$$=\int_k\pi_1^{-1}\rho^{-1}(x)\{(\mathrm{i})+(\mathrm{ii})+(\mathrm{iii})+(\mathrm{iv})\}\,dx,$$

where as in the proof of Proposition 8.2,

$$(\mathrm{i})\quad =\frac{1}{2}\sum_{s\in E}\int_{\Pi_{pr}}\int_k\int_k(\pi_1^{-1}\pi_2^{-1}\pi\rho)^{1/2}\rho^{-1}\,\mathrm{sgn}_s(u)$$
$$(\pi_1\pi_2\pi^{-1}\rho)^{1/2}\rho^{-1}\,\mathrm{sgn}_s(v)K_\pi(f_1|u, v)dudvm(\pi)d\pi,$$

$$(\mathrm{ii})\quad =\frac{1}{2}[Q_{sp}]m(\pi_{sp})\sum_s\int_k\int_k(\pi_1^{-1}\pi_2^{-1}\pi_{sp}\rho)^{1/2}\rho^{-1}\,\mathrm{sgn}_s(u)$$
$$(\pi_1\pi_2\pi_{sp}^{-1}\rho)^{1/2}\rho^{-1}\,\mathrm{sgn}_s(v)K_{\pi_{sp}}(f_1|u, v)dudv,$$

$$(\mathrm{iii})\quad =\frac{1}{2}\sum_{\pi\in\Pi_d}m(\pi)\sum_s\int_k\int_k(\pi_1^{-1}\pi_2^{-1}\pi\rho\,\mathrm{sgn}_\tau)^{1/2}\rho^{-1}\,\mathrm{sgn}_s(u)$$
$$(\pi_1\pi_2\pi^{-1}\rho\,\mathrm{sgn}_\tau)^{1/2}\rho^{-1}\,\mathrm{sgn}_s(v)K_\pi(f_1|u, v)dudv,$$

(iv) $=\frac{1}{2}[Q_d]m(\pi_0)\sum_s\int_k\int_k(\pi_1^{-1}\pi_2^{-1}\pi_0\rho\,\mathrm{sgn}_\varepsilon)^{1/2}\rho^{-1}\,\mathrm{sgn}_s(u)$

$$(\pi_1\pi_2\pi_0^{-1}\rho\,\mathrm{sgn}_\varepsilon)^{1/2}\rho^{-1}\,\mathrm{sgn}_s(v)K_{\pi_0}(f_1|u,v)du\,dv\,.$$

First consider the integral $A=\int_k\pi_1^{-1}\rho^{-1}(x)(\mathrm{i})dx$. Put $\lambda=\pi_1\pi_2^{-1}$, then

$$A=\frac{1}{2}\sum_{s\in E}\int_k\pi_1^{-1}\rho^{-1}(x)\left\{\int_{\Pi_{pr}}\int_k\int_k\pi_1^{-1}(\lambda\pi\rho)^{1/2}\rho^{-1}\,\mathrm{sgn}_s(u)\right.$$

$$\left.\times\pi_2(\lambda\pi^{-1}\rho)^{1/2}\rho^{-1}\,\mathrm{sgn}_s(v)K_\pi(f_1|u,v)du\,dv\,m(\pi)d\pi\right\}dx\,.$$

Next we change the order of integration with respect to $dx$ and $dvm(\pi)d\pi$, then

$$A=\frac{1}{2}\sum_{r\in E}\int_{\Pi_{pr}}\int_k\pi_2(\lambda\pi^{-1}\rho)^{1/2}\rho\,\mathrm{sgn}_s(v)\left\{\int_k\int_k\pi_1^{-1}(\lambda\pi\rho)^{1/2}\rho^{-1}\,\mathrm{sgn}_s(u)\right.$$

$$\left.\times\pi_1^{-1}\rho^{-1}(x)\chi(xu)K_\pi(f|u,v)du\,dx\right\}dv\,m(\pi)d\pi\,.$$

By Corollary 5.3, as a function of $u$, $K_\pi(f|u,v)$ is in $\mathcal{S}_\pi$ for a fixed $v$ and $\pi\in\tilde{k}^\times$. Then it is easy to see that $F(u)=\pi_1^{-1}(\lambda\pi\rho)^{1/2}\rho^{-1}\,\mathrm{sgn}_s(u)K_\pi(f|u,v)$ is a linear combination of functions in $\hat{S}_\mu$ and $\hat{S}_{\mu\pi^{-1}}$, where $\mu=\pi_1^{-1}(\lambda\pi\rho)^{1/2}\rho^{-1}\,\mathrm{sgn}_s=|\cdot|^\beta\theta$. Since $\mathrm{Re}(\beta)=\mathrm{Re}((-\alpha_1-\alpha_2+1)/2-1)<1$ and $0<-\alpha_1<1$, we apply Corollary 2.5, and obtain

$$\int_k\int_k\pi_1^{-1}\rho^{-1}(x)\chi(xu)F(u)du\,dx=\Gamma(\pi_1^{-1})\int_k\pi_1(u)F(u)du$$

$$=\Gamma(\pi_1^{-1})\int_k(\lambda\pi\rho)^{1/2}\rho^{-1}\,\mathrm{sgn}_s(u)K_\pi(f|u,v)du$$

Thus

(10.1) $$A=\frac{1}{2}\sum_{r\in E}\Gamma(\pi_1^{-1})\int_{\Pi_{pr}}\int_k\int_k(\pi_1\pi_2^{-1}\pi\rho)^{1/2}\rho^{-1}\,\mathrm{sgn}_s(u)$$

$$\times(\pi_1\pi_2\pi^{-1}\rho)^{1/2}\rho^{-1}\,\mathrm{sgn}_s(v)K_\pi(f|u,v)du\,dv\,m(\pi)d\pi\,.$$

The integral in the last side in $A$ is absolutely convergent and so the above change of order of integrations is justified.

The calculation for (ii), (iii) and (iv) are similar. Q. E. D.

We put for $\pi\in\Pi=\Pi_{pr}\cup Q_{sp}\cup\Pi_d\cup Q_d$,

(10.2) $$\Phi(t;\pi,s)=\sqrt{2}\int_k A(\pi,s)(u)\pi^{-1}(u)K_\pi(\check{f}|t,u)du$$

Then, as a function in $t$, $\Phi(t;\pi,s)$ is in one of the spaces of representations $\mathfrak{R}_\pi$, $\mathfrak{R}_{sp}$, $\mathfrak{R}_\pi^\pm$ and $\mathfrak{R}_0^s$ corresponding to $\pi$ or $(\pi,s)$. This is similar as in § 9.1. For $\pi\in\Pi_{pr}\cup\Pi_d$, $\Phi$ satisfies the condition;

(10.3) $$\Phi(t\,;\,\pi^{-1},\,s)=\Phi(t\,;\,\pi,\,s)\pi(t)\,.$$

We define a linear mapping

(10.4) $$V\,;\,f\longrightarrow\Phi=\Phi(t\,;\,\pi,\,s)\,.$$

We have the same diagram as (9.7). From Theorem 10.1 and (7.8), we obtain

(10.5)

$$\|\varphi\|_{\rm II}^2=\sum_{s\in E}\int_{\Pi_{pr}}\int_k|\Phi(t\,;\,\pi,\,s)|^2dtm(\pi)d\pi+[Q_s]m(\pi_{sp})\sum_s\int_k|(\Phi t\,;\,\pi_{sp},\,s)|^2\pi_{sp}(t)dt$$

$$+\sum_{\pi\in\Pi_d}m(\pi)\sum_s\int_k|\Phi(t\,;\,\pi,\,s)|^2dt+[Q_d]m(\pi_0)\sum_s\int_k|\Phi(t\,;\,\pi_0,\,s)|^2dt=\|\Phi\|^2\ (\text{put})\,,$$

where $\|\ \|_{\rm II}$ is in § 6.1 (II). The right hand side above has the same form as in (9.10). Therefore $\Phi\in\mathfrak{H}$, where $\mathfrak{H}=\mathfrak{H}^{(+)}$ or $\mathfrak{H}^{(-)}$ is the separable Hilbert space in § 9.2. Thus we get a linear isometric $G$-morphism $W:\varphi\rightarrow\Phi$ of $\mathcal{H}$ into $\mathfrak{H}$ such that $WU=V$.

Let $L^2_{\pi_1}\otimes L^2$ be the Hilbert space of all measurable functions $\varphi$ on $k\times k$ such that $\|\varphi\|_{\rm II}<\infty$. Again by Proposition 9.2, $W$ is extended to a unitary $G$-morphism of $\bar{\mathcal{H}}=L^2_{\pi_1}\otimes L^2$ onto $\mathfrak{H}$. Thus we obtain another one of our main results.

**Theorem 10.3.** *Let $\pi_1$ and $\pi_2$ be characters of $k^\times$ as at the beginning of this section. Then there exists a unitary mapping $W$ of $L^2_{\pi_1}\otimes L^2$ onto $\mathfrak{H}$, which is given on $\mathcal{H}$ by $WU=V$, where $U$ and $V$ are defined in* (6.2) *and* (10.4) *respectively. Moreover $W$ is a $G$-morphism, that is, $WT_g=\boldsymbol{T}_gW$, where representations $T_g$ and $\boldsymbol{T}_g$ are as in Theorem* 9.5. *Thus $W$ realizes the decomposition of the tensor product $\mathfrak{R}_{\pi_1}\bar{\otimes}\mathfrak{R}_{\pi_2}$ into irreducibles for this case.*

*In case $-1\in(k^\times)^2$ and $\pi_2(-1)=1$, it is given by the formula* (9.21).

*In case $-1\in(k^\times)^2$ and $\pi_2(-1)=-1$, by* (9.22). *In case $-1\notin(k^\times)^2$ and $\pi_2(-1)=1$, by* (9.23). *In case $-1\notin(k^\times)^2$ and $\pi_2(-1)=1$, by* (9.24).

## § 11. The decomposition formula in Case (III. A).

The decomposition of the tensor product of two supplementary series representations is studied according to the following two cases: for $\pi_1(x)=|x|^{\alpha_1}$, $\pi_2(x)=|x|^{\alpha_2}$ such that $-1<\alpha_1,\ \alpha_2<0$, we say

Case (III. A) if $0<1+\alpha_1+\alpha_2$, and Case (III. B) if $-1<1+\alpha_1+\alpha_2<0$.

In this section, we give the formula for (III. A), calculating the Plancherel transform $\hat{H}$ of $H$. In the next section we give the formula for (III. B) by an analytic continuation of $\hat{H}$. Note that, for these cases, in (7.6) and (7.7) it is only $\pi_1\pi_2(-1)=1$, therefore $\Pi=\Pi(+1)$ etc.

**11.1.** Let $\pi_1$ and $\pi_2$ be as in (III. A). We consider the following products of gamma functions: for $\pi\in\Pi_{pr}\cup\{\pi_{sp}\}$

$$\boldsymbol{\Gamma}_s(\pi_1, \pi_2, \pi)=\Gamma((\pi_1\pi_2\pi\rho)^{1/2}\,\mathrm{sgn}_s)\Gamma((\pi_1\pi_2\pi^{-1}\rho)^{1/2}\,\mathrm{sgn}_s)$$
$$\Gamma((\pi_1^{-1}\pi_2\pi\rho)^{1/2}\,\mathrm{sgn}_s)\Gamma((\pi_1^{-1}\pi_2\pi^{-1}\rho)^{1/2}\,\mathrm{sgn}_s)\,. \tag{11.1}$$

From the property $\Gamma(\bar{\lambda})=\bar{\lambda}(-1)\overline{\Gamma(\lambda)}$, $\lambda$ a non-unitary character of $k^\times$, we see that $\boldsymbol{\Gamma}_s(\pi_1, \pi_2, \pi)$ is positive. For $\pi=\pi_{sp}$ and $\pi_1=\pi_2$ we should understand $\Gamma((\pi_1^{-1}\pi_2\pi_{sp}\rho)^{1/2}(\Gamma((\pi_1^{-1}\pi_2\pi_{sp}^{-1}\rho)^{1/2})=1$. It is also that $\boldsymbol{\Gamma}_s(\pi_1, \pi_2, \pi_{sp})$ is positive.

For a character $\nu$ of $k^\times$ and $\pi\in\Pi_d\cup\tilde{C}_\tau$, we define a gamma function on $L_\tau=k(\sqrt{\tau})$ by

$$\Gamma_\tau(\nu\pi^{-1}, \nu)=\int_{L_\tau}\nu(z\bar{z})\pi^{-1}(z)\chi(S_\tau(z))d^\times z\,. \tag{11.2}$$

Put

$$\boldsymbol{g}_s(\pi_1, \pi_2, \pi)=c_\tau^2\Gamma_\tau((\pi_1\pi_2\pi^{-1}\rho\,\mathrm{sgn}_\tau)^{1/2}\,\mathrm{sgn}_s,\ (\pi_1\pi_2\pi\rho\,\mathrm{sgn}_\tau)^{1/2}\,\mathrm{sgn}_s)$$
$$\Gamma_\tau((\pi_1^{-1}\pi_2\pi^{-1}\rho\,\mathrm{sgn}_\tau)^{1/2}\,\mathrm{sgn}_s,\ (\pi_1^{-1}\pi_2\pi\rho\,\mathrm{sgn}_\tau)^{1/2}\,\mathrm{sgn}_s)\,, \tag{11.3}$$

where $c_\tau$ is in (4.15). We assert that $\boldsymbol{g}_s>0$. Since

$$z\longrightarrow\nu(z\bar{z})\pi^{-1}(z)=(\pi_1\pi_2\pi\rho\,\mathrm{sgn}_\tau)^{1/2}\,\mathrm{sgn}_s(z\bar{z})\pi^{-1}(z)$$

is a ramified character of $L_\tau^\times$, $\boldsymbol{g}_s=c_\tau^2\pi(-1)a$ with $a>0$, and $c_\tau^2=\mathrm{sgn}_\tau(-1)\ b$ with $b>0$. So we have $\boldsymbol{g}_s=ab\pi\ \mathrm{sgn}_\tau(-1)=ab>0$.

We need the following lemma, which is analogous to Proposition 3.7.

**Lemma 11.1.** *Let $\nu$ be a character (not necessarly unitary) of $k^\times$ such that $\nu(x)=|x|^\alpha\theta(x)$, $0<\mathrm{Re}(\alpha)<1$. Then for $\pi\in\tilde{C}_\tau$ and $\varphi\in\mathcal{S}^\times(k_\tau^\times)$,*

$$\int_k\nu\rho^{-1}(x)H_\pi^d\varphi(x)dx=c_\tau\Gamma_\tau(\nu\pi^{-1}, \nu)\int_k\nu^{-1}\pi(x)\varphi(x)dx\,,$$

*where $H_\pi^d$ is as in* (4.22) *and $\Gamma_\tau(\nu\pi^{-1}, \nu)$ is as in* (11.2).

*Proof.* Take $z\in L_\tau^\times$ such that $z\bar{z}=x$. As we studied in §4.4, there exist $\Phi(z)\in\mathcal{S}(L_\tau)$ such that $\varphi(x)=\Phi_\pi(z)\pi^{-1}(z)$ with $\Phi_\pi(z)=\int_{C_\tau}\Phi(tz)\pi^{-1}(t)d^\times t$. Then

$$H_\pi^d\varphi(x)=c_\tau\int_K J_\pi^d(x, y)\varphi(y)dy$$
$$=c_\tau\int_{L_\tau}\chi(S_\tau(z\bar{z}'))\pi^{-1}(z)\Phi_\pi(z')dz'=c_\tau\hat{\Phi}_\pi(z)\pi^{-1}(z)\,.$$

From (4.14), we have

$$\int_k\nu\rho^{-1}(x)H_\pi^d\varphi(x)dx=c_\tau\int_k\nu\rho^{-1}(z\bar{z})\pi^{-1}(z)\hat{\Phi}_\pi(z)dx$$
$$=a_\tau^{-1}c_\tau\int_{L_\tau}\nu\rho^{-1}(z\bar{z})\pi^{-1}(z)\hat{\Phi}_\pi(z)dz$$
$$=a_\tau^{-1}c_\tau\boldsymbol{\Gamma}_\tau(\nu\pi^{-1}, \nu)\int_{L_\tau}\nu^{-1}(z\bar{z})\pi(\bar{z})\Phi_\pi(z)dz\quad(\because\ \Phi_\pi\in\mathcal{S}(L_\tau))$$

$$=c_\tau\Gamma_\tau(\nu\pi^{-1}, \nu)\int_k \nu^{-1}\pi(x)\varphi(x)dx . \qquad \text{Q. E. D,}$$

**11.2.** We have the following proposition analogous to Proposition 8.2.

**Theorem 11.2.** *Let $\pi_1$, $\pi_2$ be in Case* (III. A) *and $H$ be in* (H. III), *that is, $H(g)=\Gamma(\pi_1^{-1})^{-1}\Gamma(\pi_2^{-1})^{-1}\pi_1^{-1}\pi_2(a)\pi_1^{-1}\rho^{-1}(x)\pi_2^{-1}\rho^{-1}(y)$ for $g=d(a)n^+(y)n(x)$. Then the Plancherel transform $\hat{H}$ of $H$ is given as follows.*

*For $\pi\in\Pi_{pr}\cup\{\pi_{sp}\}$,*

$$\hat{H}(u, v, \pi)=\frac{1}{2}\sum_{s\in E}\boldsymbol{\Gamma}_s(\pi_1, \pi_2, \pi)(\pi_1\pi_2^{-1}\pi\rho)^{1/2}\rho^{-1}\,\mathrm{sgn}_s(u)$$
$$(\pi_1\pi_2^{-1}\pi^{-1}\rho)^{1/2}\rho^{-1}\,\mathrm{sgn}_s(v) .$$

*For $\pi\in(\Pi_d\cap\tilde{C}_\tau)\cup Q_d$,*

$$\hat{H}(u, v, \pi)=\frac{1}{2}\sum_{s\in E}\boldsymbol{g}_s(\pi_1, \pi_2, \pi)(\pi_1\pi_2^{-1}\pi\rho\,\mathrm{sgn}_\tau)^{1/2}\rho^{-1}\,\mathrm{sgn}_s(u)$$
$$(\pi_1\pi_2^{-1}\pi^{-1}\rho\,\mathrm{sgn}_\tau)^{1/2}\rho^{-1}\mathrm{sgn}_s(v) .$$

*Proof.* For $f\in\mathcal{S}(G)$,

$$\Gamma(\pi_1^{-1})\Gamma(\pi_2^{-1})\int_G H(g)f(g)dg$$
$$=\int_k\int_k\int_k \pi_1^{-1}\pi_2(a)\pi_1^{-1}\rho^{-1}(x)\pi_2^{-1}\rho^{-1}(y)f(d(a)u^+(y)n(x))d^\times a\,dx\,dy .$$

Remark that $d(a)n^+(y)n(x)=n^+(a^{-2}y)n(a^2y)d(a)$. Replace $x$ by $a^{-2}x$, $y$ by $a^2y$, and put $f_1=L_{n^+(-y)}f$. So, we have

$$=\int_k \pi_2^{-1}\rho^{-1}(y)\left\{\int_k\int_k \pi_1\pi_2^{-1}(a)\pi_1^{-1}\rho^{-1}(x)f_1(n(x)d(a))d^\times a\,dx\right\}dy .$$
$$=\int_k \pi_2^{-1}\rho^{-1}(y)\{\text{(i)}+\text{(ii)}+\text{(iii)}+\text{(iv)}\}\,dy ,$$

where as in the proof of Proposition 10.2,

$$\text{(i)}\quad =\frac{1}{2}\Gamma(\pi_1^{-1})\sum_{s\in E}\int_{\Pi_{pr}}\int_k\int_k (\pi_1\pi_2\pi\rho)^{1/2}\rho^{-1}\,\mathrm{sgn}_s(u)$$
$$\times(\pi_1\pi_2^{-1}\pi^{-1}\rho)^{1/2}\rho^{-1}\,\mathrm{sgn}_s(v)K_\pi(f_1|\,u, v)du\,dv\,m(\pi)d\pi ,$$

and (ii), (iii) and (iv) are similarly calculated.

I. First we consider the integral $A=\int_k \pi_2^{-1}\rho^{-1}(y)$ (i) $dy$. Put $\lambda=\pi_1\pi_2^{-1}$, then

$$A=\frac{1}{2}\Gamma(\pi_1^{-1})\sum_{s\in E}\int_{\Pi_{pr}}\int_k (\lambda\pi^{-1}\rho)^{1/2}\rho^{-1}\,\mathrm{sgn}_s(v)S(v, \pi)dv\,m(\pi)d\pi ,$$

where

(11.4) $$S(v,\pi)=\int_k\int_k \pi_2^{-1}\rho^{-1}(y)(\pi_2(\lambda\pi\rho)^{1/2}\rho^{-1}\,\mathrm{sgn}_s)(u)K_\pi(f_1\,|\,u,v)du\,dy\,.$$

Since $n^+(-y)=w^{-1}n(y)w$, $K_\pi(f_1|u,v)=H_\pi^{-1}K_\pi(L_{n(y)w}f|u,v)$, where $H_\pi$ acts on $u$. $H_\pi^{-1}=H_\pi$ by Corollary 3.2. Moreover since $0<1+\alpha_1+\alpha_2<1$, we can apply Proposition 3.7 and obtain

(11.5) $$S(v,\pi)=aN\int_k\int_k \pi_2^{-1}\rho^{-1}(y)(\pi_2^{-1}(\lambda\pi\rho)^{-1/2}\pi\,\mathrm{sgn}_s)(u)K_\pi(L_{n(y)w}f\,|\,u,v)du\,dy\,,$$

where $N=\Gamma(\pi_2(\lambda\pi\rho)^{1/2}\,\mathrm{sgn}_s)\Gamma(\pi_2(\lambda\pi\rho)^{1/2}\pi^{-1}\,\mathrm{sgn}_s)$ and $a=\pi_2(\lambda\pi\rho)^{1/2}\pi\,\mathrm{sgn}_s(-1)$. Note that

$$K_\pi(L_{n(y)w}f\,|\,u,v)=\chi-(yu)K_\pi(L_wf\,|\,u,v)=\chi(-yu)H_\pi K_\pi(f\,|\,u,v)\,.$$

Apply Corollary 2.6 to (11.5) as in the proof of Proposition 8.2. Then apply Proposition 3.7. So we see that $S(v,\pi)$ equals

$$aN\int_k\int_k \pi_2^{-1}\rho^{-1}(y)\chi(-yu)(\pi_2^{-1}(\lambda\pi\rho)^{-1/2}\pi\,\mathrm{sgn}_s)(u)K_\pi(L_wf\,|\,u,v)du\,dy$$

$$=aN\Gamma(\pi_2^{-1})\int_k(\lambda\pi\rho)^{-1/2}\pi\,\mathrm{sgn}_s(u)H_\pi K_\pi(f\,|\,u,v)du$$

$$=aa'NN'\Gamma(\pi_2^{-1})\int_k(\lambda\pi\rho)^{1/2}\rho^{-1}\,\mathrm{sgn}_s(u)K_\pi(f\,|\,u,v)du\,,$$

where $N'=\Gamma((\lambda\pi\rho)^{-1/2}\pi\,\mathrm{sgn}_s)\Gamma((\lambda\pi\rho)^{-1/2}\,\mathrm{sgn}_s)$ and $a'=(\lambda\pi\rho)^{1/2}\,\mathrm{sgn}_s(-1)$. It is easy seen that $NN'=\boldsymbol{\Gamma}_s(\pi_1,\pi_2,\pi)$ and $aa'=1$. Substituting the last side above in $A$, we obtain the desired formula for $\pi\in\Pi_{pr}$.

II. Next we consider the integral for (ii). This case can be treated similarly as I.

III. We discuss the integral $A=\int_k\pi_2^{-1}\rho^{-1}(y)(\mathrm{iii})dy$. By changing the integration order,

$$A=\frac{1}{2}\Gamma(\pi_1^{-1})\sum_{s\in E}\int_k\pi_2^{-1}\rho^{-1}(y)\Big\{\int_k\int_k\pi_2(\lambda\pi\rho\,\mathrm{sgn}_\tau)^{1/2}\rho^{-1}\,\mathrm{sgn}_s(u)$$

$$\times(\lambda\pi^{-1}\rho\,\mathrm{sgn}_\tau)^{1/2}\rho^{-1}\,\mathrm{sgn}_s(v)K_\pi(f_1\,|\,u,v)du\,dv\Big\}dy$$

$$=\frac{1}{2}\Gamma(\pi_1^{-1})\sum_{s\in E}\int_k(\lambda\pi\rho\,\mathrm{sgn}_\tau)^{1/2}\pi^{-1}\rho^{-1}\,\mathrm{sgn}_s(v)S(v,\pi)dv\,,$$

where

(11.6) $$S(v,\pi)=\int_k\pi_2^{-1}\rho^{-1}(y)\Big\{\int_k\pi_2(\lambda\pi\rho\,\mathrm{sgn}_\tau)^{1/2}\rho^{-1}\,\mathrm{sgn}_s(u)K_\pi(f_1\,|\,u,v)du\Big\}dy\,.$$

Note that

$$K_\pi(f_1\,|\,u,v)=K_\pi(L_{w^{-1}n(y)w}f\,|\,u,v)$$
$$=\pi\,\mathrm{sgn}_\tau(-1)H_\pi^d\{\chi(-yu)H_\pi^dK_\pi(f\,|\,u,v)\}=H_\pi^d\{\chi(-yu)H_\pi^dK_\pi(f\,|\,u,v)\}\,.$$

Apply Lemma 11.1 and Corollary 2.6 repeatedly to (11.6), then we see that the $S(v, \pi)$ equals

$$\Gamma(\pi_2^{-1})\boldsymbol{g}_s(\pi_1, \pi_2, \pi)\int_k (\pi_1\pi_2^{-1}\pi\rho\,\mathrm{sgn}_\tau)^{1/2}\rho^{-1}\,\mathrm{sgn}_s(u)K_\pi(f\,|\,u, v)du\,,$$

Substituting this equality to $A$, we have the formula for $\pi\in\Pi_d$.

IV. The integral $A=\int_k \pi_2^{-1}\rho^{-1}(y)(\mathrm{iv})dy$ is treated similarly as (III).

Summing up these four terms we get the desired formula. Q. E. D.

**11.3.** We study the formula which gives the decomposition. Put for $\pi\in\Pi=\Pi(+1)$,

(11.7) $\quad \boldsymbol{\Phi}_s(t\,;\pi)$

$$=\begin{cases} \sqrt{2}^{-1}\int_k (\pi\pi_2^{-1}\pi^{-1}\rho)^{1/2}\rho^{-1}\,\mathrm{sgn}_s(u)K_\pi(\check{f}\,|\,t, u)du & \text{for } \pi\in\Pi_{pr}\cup\{\pi_{sp}\} \\ \sqrt{2}^{-1}\int_k (\pi_1\pi_2^{-1}\pi^{-1}\rho\,\mathrm{sgn}_\tau)^{1/2}\rho^{-1}\,\mathrm{sgn}_s(u)K_\pi(\check{f}\,|\,t, u)du & \text{for } \pi\in\Pi_d\cup Q_d\,. \end{cases}$$

For $\varphi\in\mathcal{H}$ and $f\in\mathcal{S}(G)$ such that $Uf=\varphi$, we apply Theorem 11.2 to (7.8). Then we get

$$\begin{aligned}(11.8)\qquad \|\varphi\|_{\mathrm{III}}^2=&\sum_{s\in E}\int_{\Pi_{pr}}\boldsymbol{\Gamma}_s(\pi_1, \pi_2, \pi)\int_k|\boldsymbol{\Phi}_s(t\,;\pi)|^2dtm(\pi)d\pi\\ &+m(\pi_{sp})\sum_s\boldsymbol{\Gamma}_s(\pi_1, \pi_2, \pi_{sp})\int_k|\boldsymbol{\Phi}_s(t\,;\pi_{sp})|^2\pi_{sp}(t)dt\\ &+\sum_{\pi\in\Pi_d}m(\pi)\sum_s\boldsymbol{g}_s(\pi_1, \pi_2, \pi)\int_k|\boldsymbol{\Phi}_s(t\,;\pi)|^2dt\\ &+[Q_d]m(\pi_0)\sum_s\boldsymbol{g}_s(\pi_1, \pi_2, \pi_0)\int_k|\boldsymbol{\Phi}_s(t\,;\pi_0)|^2dt\,.\end{aligned}$$

where $\|\ \|_{\mathrm{III}}$ is as in § 6.1 (III).

To make the decomposition formula, we normalize the formula (11.8). We define $\Theta(t\,;\pi, s)$ for $\boldsymbol{\Phi}_s(t\,;\pi)$ or $\boldsymbol{\Phi}(t\,;\pi, s)$ in (9.3): for $\pi\in\Pi_{pr}\cup\{\pi_{sp}\}$,

$$(11.9)\qquad \Theta(t\,;\pi, s)=\boldsymbol{\Gamma}_s(\pi_1, \pi_2, \pi)^{1/2}\boldsymbol{\Phi}_s(t\,;\pi)\,.$$

Let $\pi$ fix in $\Pi_d\cap\tilde{C}_\tau$. Let $\tau', \tau''\in E$ such that $\{1, \tau, \tau', \tau''\}=E$, and $\tau_1\in E$ such that $k_\tau^\times=(k^\times)^2\cup\tau_1(k^\times)^2$, and $\tau_2, \tau_3\in E$ such that $\{1, \tau_1, \tau_2, \tau_3\}=E$. Then we put

$$(11.10)\qquad \begin{cases}\Theta(t\,;\pi, 1)=\dfrac{1}{2}(\boldsymbol{g}_1+\boldsymbol{g}_\tau)^{1/2}\{\boldsymbol{\Phi}(t\,;\pi, 1)+\boldsymbol{\Phi}(t\,;\pi, \tau_1)\}\,,\\ \Theta(t\,;\pi, \tau_1)=\dfrac{1}{2}(\boldsymbol{g}_{\tau'}+\boldsymbol{g}_{\tau''})^{1/2}\{\boldsymbol{\Phi}(t\,;\pi, 1)-\boldsymbol{\Phi}(t\,;\pi, \tau_1)\}\,,\\ \Theta(t\,;\pi, \tau_2)=\dfrac{1}{2}(\boldsymbol{g}_1+\boldsymbol{g}_\tau)^{1/2}\{\boldsymbol{\Phi}(t\,;\pi, \tau_2)+\boldsymbol{\Phi}(t\,;\pi, \tau_3)\}\,,\\ \Theta(t\,;\pi, \tau_3)=\dfrac{1}{2}(\boldsymbol{g}_{\tau'}+\boldsymbol{g}_{\tau''})^{1/2}\{\boldsymbol{\Phi}(t\,;\pi, \tau_2)-\boldsymbol{\Phi}(t\,;\pi, \tau_3)\}\,.\end{cases}$$

For $\pi\in Q_d$,

(11.11) $$\Theta(t;\pi_0, s)=2^{-1}(\sum_{r}g_r)^{1/2}\Phi(t;\pi_0, s).$$

We set for $\pi\in\Pi_d\cap\tilde{C}_\tau$

(11.12) $$K=\sum_{s\in E}g_s(\pi_1, \pi_2, \pi)\int_k|\Phi_s(t;\pi)|^2dt$$

$$=\frac{1}{4}\sum_{s}g_s\int_k|\sum_{\mathrm{sgn}_\tau r=1}(\mathrm{sgn}_s r)\Phi(t;\pi, r)+\sum_{\mathrm{sgn}_\tau r=-1}(\mathrm{sgn}_s r)\Phi(t;\pi, r)|^2dt.$$

For $r=1$ or $\tau_1$, $\mathrm{sgn}_\tau r=1$ and then $\Phi(t;\pi, r)\in\mathcal{S}^\times(k_\tau^\times)$. For $r=\tau_2$ or $\tau_3$, $\mathrm{sgn}_\tau r=-1$ and $\Phi(t;\pi, r)\in\mathcal{S}^\times((k_\tau^\times)^c)$. Then $\Theta(t;\pi, r)\in\mathcal{S}^\times(k_\tau^\times)$ if $\mathrm{sgn}_\tau r=-1$ and $\Theta(t;\pi, r)\in\mathcal{S}^\times((k_\tau^\times)^c)$ if $\mathrm{sgn}_\tau r=-1$. Therefore

$$K=\frac{1}{4}\sum_{s\in E}g_s\Big\{\int_k|\Phi(t;\pi, 1)+(\mathrm{sgn}_s\tau_1)\Phi(t;\pi, \tau_1)|^2dt$$

$$+\int_k|(\mathrm{sgn}_s\tau_2)\Phi(t;\pi, \tau_2)+(\mathrm{sgn}_s\tau_3)\Phi(t;\pi, \tau_3)|^2dt\Big\}$$

$$=\int_k|\Theta(t;\pi, 1)|^2dt+\int_k|\Theta(t;\pi, \tau_1)|^2dt$$

$$+\int_k|\Theta(t;\pi, \tau_2)|^2dt+\int_k|\Theta(t;\pi, \tau_3)|^2dt.$$

For $\pi\in Q_d$, it holds

(11.13) $$K'=\sum_{s\in E}g_s(\pi_1, \pi_2, \pi_0)\int_k|\Phi_s(t;\pi_0)|^2dt=\sum_{s\in E}\int_k|\Theta(t;\pi_0, s)|^2dt.$$

Substituting (11.9), $K$ and $K'$ to (11.8), (11.8) is rewritten as

(11.14)

$$\|\varphi\|_{\mathrm{III}}^2=\sum_{s\in E}\int_{\Pi_{pr}}\int_k|\Theta(t;\pi, s)|^2dtm(\pi)d\pi+m(\pi_{sp})\sum_s\int_k|\Theta(t;\pi_{sp}, s)|^2\pi_{sp}(t)dt$$

$$+\sum_{\pi\in\Pi_d}m(\pi)\sum_s\int_k|\Theta(t;\pi, s)|^2dt+[Q_d]m(\pi_0)\sum_s\int_k|\Theta(t;\pi_0, s)|^2dt.$$

**11.4.** We note that for $\pi\in\Pi_{pr}\cup\Pi_d$, $\Theta(t;\pi, s)$ satisfies the condition

(11.15) $$\Theta(t;\pi^{-1}, s)=\Theta(t;\pi, s)\pi(t).$$

We define the mapping of $\mathcal{S}(G)$ by

$$V':f\longrightarrow\Theta=\Theta(t;\pi, s)\in\mathfrak{H}=\mathfrak{H}^{(+)}.$$

$\varphi\to\Phi(t;\pi, s)$ is a $G$-morphism as in (9.7) and it is easily seen that $\Phi\to\Theta$ is also a $G$-morphism. So, by (11.14), $W:\varphi\to\Theta$ is an isometric $G$-morphism of $\mathcal{H}$ into $\mathfrak{H}$, and it is given by

(11.16) $$WU=V'.$$

It is extended to that of $\mathscr{K}=L^2_{\pi_1}\otimes L^2_{\pi_2}$ into $\mathfrak{H}$, where $L^2_{\pi_1}\otimes L^2_{\pi_2}$ is the Hilbert space of all measurable functions $\varphi$ on $k\times k$ such that $\|\varphi\|_{\mathrm{III}}<\infty$.

**Proposition 11.3.** *The image of $L^2_{\pi_1}\otimes L^2_{\pi_2}$ under $W$ is the whole space $\mathfrak{H}$.*

*Proof.* The proposition is proved by modifying that of Proposition 9.1, in particular, of Lemma 9.4. Q. E. D.

Thus we obtain the result for this case.

**Theorem 11.4.** *Let $W$ be linear mapping of $L^2_{\pi_1}\otimes L^2_{\pi_2}$ onto $\mathfrak{H}$ given in* (11.16). *Then $W$ is a unitary $G$-morphism and it realizes the decomposition of the tensor product $\mathfrak{R}_{\pi_1}\overline{\otimes}\mathfrak{R}_{\pi_2}$ into irreducibles as follows. In case $-1\in(k^\times)^2$, it is given by the formula* (9.21). *In case $-1\notin(k^\times)^2$, by* (9.23).

## § 12. The decomposition formula for Case (III. B).

In this section, we give the decomposition formula for Case (III. B): $\pi_i(x)=|x|^{\alpha_i}$ $(i=1, 2)$ such that $-1<\alpha_1, \alpha_2<0$, $-1<1+\alpha_1+\alpha_2<0$. For this case the formula (11.8) does not holds, because we can not apply Proposition 3.7 to compute (11.5). To modify (11.8), we apply the method of analytic continuation, so that we extend $\alpha_1$ and $\alpha_2$ to complex numbers. We set

(12.1) $$D=\{(\alpha_1, \alpha_2)\in C^2;\ -1<\mathrm{Re}\,(\alpha_1),\ \mathrm{Re}\,(\alpha_2)<0\}.$$

**12.1.** Suppose $\varphi\in\mathcal{S}\otimes\mathcal{S}\cap\mathscr{H}$, that is, $\varphi$ has the compact support on $k\times k$ and vanishes on a neighborhood of the diagonal "$x_1=x_2$". Put

(12.2) $$f(d(a)n^+(y)n(x))=\pi_1\pi_2^{-1}(a)\kappa(a)\pi_2\rho^{-1}(y)\varphi(x,\ x+1/y),$$

where $\kappa\in\mathcal{S}^\times$ such that $\int_k\kappa(a)d^\times a=1$. Let $f'$ correspond to $\bar{\varphi}$ similarly. Then $f$ and $f'$ are in $\mathcal{S}(G)$. We consider them as functions on $(\alpha_1, \alpha_2)$. The mapping $U=U(\alpha_1, \alpha_2)$: $\mathcal{S}(G)\to\mathscr{H}$ defined in (6.2) also depend on $(\alpha_1, \alpha_2)$, and $Uf=\varphi$ and $Uf'=\bar{\varphi}$.

We put $\Phi_s(t;\pi)=\Phi_s(t;\pi, \pi_1, \pi_2)$ for $f$ and $\Phi'_s(t;\pi)$ for $f'$ as in (11.7). Since $\Phi'_s(t;\pi^{-1})=\bar{\Phi}_s(t;\pi)$ for $\alpha_1$, $\alpha_2$ real and $0<1+\alpha_1+\alpha_2$, we get the following formula from (11.8).

$$\Gamma(\pi_1^{-1})^{-1}\Gamma(\pi_2^{-1})^{-1}\int_k\int_k\int_k\int_k\pi_1^{-1}\rho^{-1}(x_1-x'_1)\pi_2^{-1}\rho^{-1}(x_2-x'_2)$$
$$\times\varphi(x_1,\ x_2)\overline{\varphi(x'_1,\ x'_2)}dx_1dx'_1dx_2dx'_2$$

(12.4) $$=\sum_{s\in E}\int_{\Pi_{pr}(+1)}\int_k\boldsymbol{\Gamma}_s(\pi_1,\ \pi_2,\ \pi)\Phi_s(t;\pi)\Phi'_s(t;\pi^{-1})dtm(\pi)d\pi$$

(12.5) $$+m(\pi_{sp})\sum_s\boldsymbol{\Gamma}_s(\pi_1,\ \pi_2,\ \pi_{sp})\int_k\Phi_s(t;\pi_{sp})\Phi'_s(t;\pi_{sp})\pi_{sp}(t)dt$$

$$(12.6)\qquad +\sum_{\pi\in\Pi_{d(+1)}} m(\pi)\sum_s \boldsymbol{g}_s(\pi_1, \pi_2, \pi)\int_k \Phi_s(t;\pi)\Phi'_s(t;\pi^{-1})dt$$

$$(12.7)\qquad +[Q_d]m(\pi_0)\sum_s \boldsymbol{g}_s(\pi_1, \pi_2, \pi_0)\int_k \Phi_s(t;\pi_0)\Phi'_s(t;\pi_0^{-1})dt\,,$$

where $\boldsymbol{\Gamma}_s(\pi_1, \pi_2, \pi)$ and $\boldsymbol{g}_s(\pi_1, \pi_2, \pi)$ are defined in (11.1) and (11.3) respectively. This formula holds even for the case $(\alpha_1, \alpha_2)\in D$ and $0<\mathrm{Re}\,(1+\alpha_1+\alpha_2)$. The left hand side is an analytic function on the whole $D$. Therefore the right hand side have an analytic continuation to any $(\alpha_1, \alpha_2)\in D$, in particular, to $(\alpha_1, \alpha_2)$ such that $\alpha_1, \alpha_2$ real and $-1<1+\alpha_1+\alpha_2\leqq 0$. We shall observe each integration term in the right hand side.

First we note the following. The function $f\in\mathcal{S}(G)$ in (12.2) is expressed as $f(g)=\sum\mu_i(\alpha_1, \alpha_2)f^i(g)$ (finite sum), where $\mu_i(\alpha_1, \alpha_2)$ is an analytic function on $D$ and $f^i\in\mathcal{S}(G)$ is independent of $(\alpha_1, \alpha_2)$. In fact, put $f^{lm}(g)=\kappa(a)\varphi(x, x+y^{-1})$ for $|a|=q^l$ and $|y|=q^m$, and zero otherwise. Then $f(g)=\sum_{l,m} q^{l(\alpha_1-\alpha_2)}q^{m(\alpha_2-1)}f^{lm}(g)$ is of a desired form. Since $K_\pi(\check{f}|t, u)=\sum_i \mu_i(\alpha_1, \alpha_2)K_\pi(\check{f}^i|t, u)$, we may consider that the kernel $K_\pi(\check{f}|u, v)$ in the formula of $\Phi_s(t;\pi)$ is independent of $(\alpha_1, \alpha_2)$.

**12.2.** Now, take an integration term in (12.6) or (12.7). For a fixed $\pi\in\Pi_d$ or $\pi\in Q_d$, $K_\pi(\check{f}|t, u)$ is a linear combination of functions of the form $\xi(t)\eta(u)$, $\xi, \eta\in\mathcal{S}^\times$. Therefore $\Phi_s(t;\pi)$ is that of functions of the form $c(\alpha_1, \alpha_2)\xi(t)$, where $c(\alpha_1, \alpha_2)=\int_k(\pi_1\pi_2^{-1}\pi^{-1}\rho\ \mathrm{sgn}_\tau)^{1/2}\rho^{-1}\mathrm{sgn}_s(u)\eta(u)du$ is analytic on $D$. As a function of $(\alpha_1, \alpha_2)$, $\boldsymbol{g}_s$ is analytic on $D$, because each character in gamma function factors of $\boldsymbol{g}_s$ is a ramified character of $L_\tau^\times$. Hence we conclude that each term in (12.6) and (12.7) is analytic on the whole $D$. As to the terms in (12.5), $\Phi_s(t;\pi_{sp})$ is similarly a linear combination of functions $c(\alpha_1, \alpha_2)\xi(t)$, $\xi\in\mathcal{S}_{sp}$. $\boldsymbol{\Gamma}_s(\pi_1, \pi_2, \pi_{sp})$ is also analytic on $D$. Thus each term in (12.5) is analytic.

We discuss the terms in (12.4). Let $\tilde{O}_{pr}^\times=\{\theta\in\tilde{O}^\times;\ \theta(-1)=1\}$, $\pi(x)=|x|^{\gamma i}\theta(x)$, $\theta\in\tilde{O}_{pr}^\times$ and $\gamma$ in the torus $T=[-\pi/\log q, \pi/\log q)$. Then (12.4) equals

$$\sum_{\theta\in\tilde{O}_{pr}}\sum_{s\in E}\int_T\int_k \frac{\boldsymbol{\Gamma}_s(\pi_1, \pi_2, \pi)}{2\Gamma(\pi)\Gamma(\pi^{-1})}\Phi_s(t;\pi)\Phi'_s(t;\pi^{-1})dtd\gamma$$

$$(12.8)\qquad =\sum_{\substack{\theta\neq 1\\ \theta\in\tilde{O}_{pr}}}\sum_{s\in E}\int_T\int_k\cdots dtd\gamma+\sum_{\substack{s=p,\varepsilon p\\(\theta=1)}}\int_T\int_k\cdots dtd\gamma$$

$$(12.9)\qquad +\int_T\int_k\frac{\boldsymbol{\Gamma}_\varepsilon(\pi_1, \pi_2, \pi)}{2\Gamma(\pi)\Gamma(\pi^{-1})}\Phi_\varepsilon(t;\pi)\Phi'_\varepsilon(t;\pi^{-1})dtd\gamma\qquad(\theta=1)$$

$$(12.10)\qquad +\int_T\int_k\frac{\boldsymbol{\Gamma}_1(\pi_1, \pi_2, \pi)}{2\Gamma(\pi)\Gamma(\pi^{-1})}\Phi_1(t;\pi)\Phi'_1(t;\pi^{-1})dtd\gamma\qquad(\theta=1)\,,$$

where the summation over $\tilde{O}_{pr}^\times$ is actually taken over only a finite number of $\theta$.

From Theorem 5.2 on the form of $K_\pi(\check{f}|t, u)$, it is easy to see that the integral $\int_k (2\Gamma(\pi)\Gamma(\pi^{-1}))^{-1}\Phi_s(t\,;\,\pi)\Phi'_s(t\,;\,\pi^{-1})dt$ is analytic in $(\alpha_1, \alpha_2)\in D$ and continuous in $\gamma\in T$. So the singularity of integrals in (12.8), (12.9) and (12.10) come from only the gamma factors $\boldsymbol{\Gamma}_s(\pi_1, \pi_2, \pi)$. On the other hand, since the characters in gamma function in $\boldsymbol{\Gamma}_s(\pi_1, \pi_2, \pi)$ in (12.8) are all ramified, $\boldsymbol{\Gamma}_s(\pi_1, \pi_2, \pi)$ are analytic on $D$ and continuous in $\gamma\in T$. The integrals in (12.8) are analytic functions of $(\alpha_1, \alpha_2)$ in $D$. As for the integral in (12.9),

$$\begin{aligned}\boldsymbol{\Gamma}_\varepsilon(\pi_1, \pi_2, \pi)=&(1+q^{(b+1+i\gamma)/2-1})(1+q^{-(b+1+i\gamma)/2})^{-1}(1+q^{(b+1-i\gamma)/2-1})\\ &\times(1+q^{-(b+1-i\gamma)/2})^{-1}(1+q^{(a+1+i\gamma)/2-1})(1+q^{-(a+1+i\gamma)/2})^{-1}\\ &\times(1+q^{(a+1-i\gamma)/2-1})(1+q^{-(a+1-i\gamma)/2})^{-1},\end{aligned}$$

where $a=\alpha_1-\alpha_2$ and $b=\alpha_1+\alpha_2$. Since complex numbers $a$ and $b$ are just given by the conditions $-1<\mathrm{Re}(a)<1$ and $-2\mathrm{Re}(b)<0$ respectively, $\boldsymbol{\Gamma}_\varepsilon(\pi_1, \pi_2, \pi)$ is analytic in $(\alpha_1, \alpha_2)\in D$ and continuous in $\gamma\in T$. Hence the integral (12.9) is also analytic in $(\alpha_1, \alpha_2)\in D$.

**12.3.** Now, we discuss the term (12.10). In this case we use the variable $(a, b)=(\alpha_1-\alpha_2, \alpha_1+\alpha_2)$. The integral (12.10) is analytic on $\{(a, b)\,;\,-1<\mathrm{Re}(a)<1 \text{ and } -1<\mathrm{Re}(b)<0\}$. So, our problem is reduced to study the analytic continuation with respect to $b$ to the domain $-2<\mathrm{Re}(b)\leqq-1$ for a fixed $a$.

When $\gamma$, $\pi(x)=|x|^{\gamma i}$, is extended to a complex variable, the integral $\boldsymbol{\Gamma}_1(\pi_1, \pi_2, \pi)(2\Gamma(\pi)\Gamma(\pi^{-1}))^{-1}\int_k \Phi_1(t\,;\,\pi)\Phi'_1(t\,;\,\pi^{-1})dt$ is analytic on $\gamma$ on the domain $\{\mathrm{Re}(a)-1<\mathrm{Im}(\gamma)<\mathrm{Re}(a)+1\}$. Put

$$B(b, \gamma)=\Gamma((\pi_1\pi_2\pi\rho)^{1/2})\Gamma((\pi_1\pi_2\pi^{-1}\rho)^{1/2})(b+1+i\gamma)(b+1-i\gamma)\,.$$

For $b+1+i\gamma=0$, the valus of $B(b, \gamma)$ should be

(12.11) $$B(b, i(b+1))=\lim_{b+1+i\gamma\to 0} B(b, \gamma)=4(b+1)(1-q^{-1})(\log q)^{-1}\Gamma(\pi_1\pi_2\rho)\,.$$

Put

(12.12) $$A(b, \gamma)=\frac{B(b, \gamma)\Gamma((\pi_1^{-1}\pi_2\pi\rho)^{1/2})\Gamma(\pi_1^{-1}\pi_2\pi^{-1}\rho)^{1/2})}{2\Gamma(\pi)\Gamma(\pi^{-1})}\int_k \Phi_1(t\,;\,\pi)\Phi'_1(t\,;\,\pi^{-1})dt\,.$$

Then the integral (12.10) equals $\int_T A(b, \gamma)\{(b+1)^2+\gamma^2\}^{-1}d\gamma$. For a fixed $a$, $A(b, \gamma)$ is analytic on $K=\{(b, \gamma)\,;\,-1<\mathrm{Re}(b)<0,\ \mathrm{Re}(a)-1<\mathrm{Im}\,\gamma<\mathrm{Re}(a)+1 \text{ and } \mathrm{Re}\,\gamma\in T\}$. If $\gamma=-i(b+1)$, then $(b, \gamma)$ is in $K$, and so $A(b, i(b+1))$ is analytic in $b$. On the other hand, it is easy to see that

$$\int_k \Phi_1(t\,;\,\pi^{-1})\Phi'_1(t\,;\,\pi)dt=\int_k \Phi_1(t\,;\,\pi)\Phi'_1(t\,;\,\pi^{-1})dt\,.$$

Therefore, it holds that $A(b, \gamma)=A(b, -\gamma)$. Then $A(b, \gamma)-A(b, i(b+1))$ is factored by $(b+1)^2+\gamma^2$ and $A_1(b, \gamma)=\{A(b, \gamma)-A(b, i(b+1))\}\{(b+1)^2+\gamma^2\}^{-1}$ is analytic in $(b, \gamma)\in K$. The integral (12.10) equals

$$\int_T \frac{A(b, \gamma)d\gamma}{(b+1)^2+\gamma^2} = \int_T A_1(b, \gamma)d\gamma + A(b, i(b+1))\int_T \frac{d\gamma}{(b+1)^2+\gamma^2}.$$

The first term in the left hand side is analytic on $b$, $-2<\mathrm{Re}(b)<0$. But in the second term

$$\int_T \frac{d\gamma}{(b+1)^2+\gamma^2} = \begin{cases} \dfrac{2}{b+1}\tan^{-1}\dfrac{\pi}{2(b+1)\log q}, & \mathrm{Re}(b+1)>0, \\ \dfrac{-2}{b+1}\tan^{-1}\dfrac{\pi}{2(b+1)\log q}, & \mathrm{Re}(b+1)\leqq 0. \end{cases}$$

The analytic continuation of (12.10) to the domain $\{b; -2<\mathrm{Re}(b)<-1\}$ is given by

$$\text{(12.13)} \quad \int_T \frac{A(b, \gamma)d\gamma}{(b+1)^2+\gamma^2} + 4\frac{A(b, i(b+1))}{b+1}\tan^{-1}\frac{\pi}{2(b+1)\log q}$$

$$= \int_T\int_k \Gamma_1(\pi_1, \pi_2, \pi)\{2\Gamma(\pi)\Gamma(\pi^{-1})\}^{-1}\Phi_1(t; \pi)\Phi_1'(t; \pi^{-1})dtd\gamma$$

$$+2\Gamma(\pi_2^{-1})^{-2}r(\pi_1, \pi_2)\int_k \Phi_1(t; \pi_1\pi_2\rho)\Phi_1'(t; (\pi_1\pi_2\rho)^{-1})dt,$$

where,

$$\text{(12.14)} \quad r(\pi_1, \pi_2) = 4(1-q^{-1})\frac{\Gamma(\pi_1^{-1})\Gamma(\pi_2^{-1})}{(\log q)\Gamma((\pi_1\pi_2\rho)^{-1})}\tan^{-1}\frac{\pi}{2(b+1)\log q},$$

and,

$$\Phi_1(t; \pi_1\pi_2\rho) = \sqrt{2}^{-1}\int_k \pi_2^{-1}\rho^{-1}(u)K_{\pi_1\pi_2\rho}(\check{f}|t, u)du,$$

$$\Phi_1'(t; (\pi_1\pi_2\rho)^{-1}) = \sqrt{2}^{-1}\int_k \pi_1(u)K_{(\pi_1\pi_2\rho)^{-1}}(\check{f}'|t, u)du$$

$$= \sqrt{2}^{-1}\int_k \pi_2^{-1}\rho^{-1}(u)K_{\pi_1\pi_2\rho}(\check{f}'|t, u)(\pi_1\pi_2\rho)(t)du.$$

Thus the analytic continuation had been completely done.

**12.4.** By Proposition 9.7 valid for $\pi_1$, $\pi_2$ in (III.B), we have

$$\text{(12.15)} \quad \begin{cases} \hat{\Phi}_1(x; \pi_1\pi_2\rho) = \sqrt{2}^{-1}\Gamma(\pi_2^{-1})\phi(x) \in \mathcal{S}_{\pi_1\pi_2\rho}, \\ \hat{\Phi}_1'(x; (\pi_1\pi_2\rho)^{-1}) = \dfrac{\Gamma(\pi_2^{-1})}{\sqrt{2}\,\Gamma((\pi_1\pi_2\rho)^{-1})}\displaystyle\int_k \overline{\phi(x')}(\pi_1\pi_2\rho)^{-1}\rho^{-1}(x-x')dx'. \end{cases}$$

where,

$$\phi(x) = \int_k\int_k \pi_2(z_1)\pi_1(z_2)(\pi_1\pi_2\rho)^{-1}(z_1-z_2)\varphi(z_1+x, z_2+x)dz_1dz_2.$$

Thus the second term in (12.13) is rewritten as $r(\pi_1, \pi_2)\|\phi\|^2_{\alpha_1+\alpha_2+1}$, where

$$\text{(12.16)} \quad \|\phi\|^2_{\alpha_1+\alpha_2+1} = \frac{1}{\Gamma((\pi_1\pi_2\rho)^{-1})}\int_k\int_k (\pi_1\pi_2\rho)^{-1}\rho^{-1}(x-x')\phi(x)\overline{\phi(x')}dxdx'.$$

When $T_g=T_g^{\pi_1}\otimes T_g^{\pi_2}$ acts on $\varphi$, it occurs the supplementary series representation $T_g^{\pi_1\pi_2\rho}$ on $\phi(x)$.

Now, we obtain the following proposition:

**Proposition 12.1.** *For $\pi_1$, $\pi_2$ in case* (III.B), *the following formula holds: for compactly supported function $\varphi\in\mathcal{H}$*

(12.17)
$$\|\varphi\|_{\mathrm{III}}^2=\sum_{s\in E}\int_{\Pi_{pr}(+1)}\int_k \boldsymbol{\Gamma}_s(\pi_1, \pi_2, \pi)|\Phi_s(t;\pi)|^2 dt m(\pi)\,d\pi$$
$$+m(\pi_{sp})\sum_{s\in E}\boldsymbol{\Gamma}_s(\pi_1, \pi_2, \pi_{sp})\int_k|\Phi_s(t;\pi_{sp})|^2\pi_{sp}(t)dt$$
$$+\sum_{\pi\in\Pi_d(+1)}m(\pi)\sum_{s\in E}\boldsymbol{g}_s(\pi_1, \pi_2, \pi)\int_k|\Phi_s(t;\pi)|^2dt$$
$$+[Q_d]m(\pi_0)\sum_{s\in E}\boldsymbol{g}_s(\pi_1, \pi_2, \pi_0)\int_k|\Phi_s(t;\pi_0)|^2dt$$
$$+r(\pi_1, \pi_2)\|\phi\|_{\alpha_1+\alpha_2+1}^2.$$

*The last term vanishes when $\pi_1\pi_2\rho=1$ $(\alpha_1+\alpha_2+1=0)$.*

**Remark.** Except the last term in the formula, the right hand side can be rewritten by means of (11.14).

**12.5.** Let $\mathfrak{H}_{\pi_1\pi_2\rho}$ be the Hilbert space of all measurable functions $\phi$ on $k$ such that $\|\phi\|_{\alpha_1+\alpha_2+1}<\infty$. Let $\mathfrak{H}'=\mathfrak{H}\oplus\mathfrak{H}_{\pi_1\pi_2\rho}$, $\mathfrak{H}=\mathfrak{H}^{(+)}$, be the Hilbert space with the inner product given for $A'=A\oplus\phi$ by

(12.18)
$$\|A'\|^2=\|A\|^2+r(\pi_1, \pi_2)\|\phi\|_{\alpha_1+\alpha_2+1}^2.$$

On $\mathfrak{H}'$, we consider the representation $\boldsymbol{T}'_g=\boldsymbol{T}_g\oplus T_g^{\pi_1\pi_2\rho}$, where $\boldsymbol{T}$ be as in Theorem 9.5, and $T^{\pi_1\pi_2\rho}$ is of supplementary series. We define a mapping of $\mathcal{S}(G)$ into $\mathfrak{H}'$ by

(12.19)
$$V': f\longrightarrow \Theta'=\Theta(t;\pi, s)\oplus\phi(x),$$

where $\Theta$ are in (11.9), (11.10) and (11.11), and $\phi$ in (12.15). Then $V'$ induces an isometric mapping $W$ of $\mathcal{H}$ into $\mathfrak{H}'$ by

(12.20)
$$WU=V'.$$

**Proposition 12.2.** *Let $\mathfrak{K}=L_{\pi_1}^2\otimes L_{\pi_2}^2$ be as in* § 11.3. *Then $W$ is extended to an isometric mapping of $\mathfrak{K}$ onto $\mathfrak{H}'$.*

*Proof.* As is already seen, the space $\mathfrak{H}$ is decomposed as $\mathfrak{H}=\mathfrak{H}_1\oplus\mathfrak{H}_\varepsilon\oplus\mathfrak{H}_p\oplus\mathfrak{H}_{\varepsilon p}$. The space $\mathfrak{H}'$ is decomposed as $\mathfrak{H}'=\mathfrak{H}'_1\oplus\mathfrak{H}_\varepsilon\oplus\mathfrak{H}_p\oplus\mathfrak{H}_{\varepsilon p}$, where $\mathfrak{H}'_1=\mathfrak{H}_1\oplus\mathfrak{H}_{\pi_1\pi_2\rho}$. Put $P'_1$ the projection of $\mathfrak{H}'$ onto $\mathfrak{H}'_1$. We show $P'_1W$ is extended to the mapping of $\mathfrak{K}$ onto $\mathfrak{H}'_1$ by applying Lemma 9.3. For this, it is enough to see that

(1) for $\pi\in\Pi$, there is an $f\in\mathcal{S}(G)$ such that $\Phi(t;\pi, 1)\neq0$,

(2) there is an $f\in\mathcal{S}(G)$ such that $\psi(x)\neq 0$.
The assertions are proved analogously to Step (1) in the proof of Proposition 9.1. Thus $P_1'W\mathfrak{H}=\mathfrak{H}_1'$. It is also proved by modifying Step (2) in the same proof that the image of $\mathfrak{H}$ under $W$ is the whole space $\mathfrak{H}'$. Q. E. D.

**Theorem 12.3.** *The mapping $W$ in (12.20) is a unitary $G$-morphism of $L_{\pi_1}^2\otimes L_{\pi_2}^2$ onto $\mathfrak{H}'$ and realizes the decomposition of the tensor product $\mathfrak{R}_{\pi_1}\overline{\otimes}\mathfrak{R}_{\pi_2}$ into irreducibles. There appears a representation $\mathfrak{R}_{\pi_1\pi_2\rho}$ of supplementary series as a new component.*

*In case* $-1\in(k^\times)^2$,

$$\mathfrak{R}_{\pi_1}\overline{\otimes}\mathfrak{R}_{\pi_2}\simeq[4]\int_{\Pi_{pr(+1)'}}\mathfrak{R}_\pi m(\pi)d\pi\oplus[4]\mathfrak{R}_{sp}\oplus[2]\sum_{\pi\in\Pi_{d(+1)'}}(\mathfrak{R}_\pi^+\oplus\mathfrak{R}_\pi^-)\oplus\mathfrak{R}_{\pi_1\pi_2\rho}\,.$$

*In case* $-1\notin(k^\times)^2$,

$$\mathfrak{R}_{\pi_1}\overline{\otimes}\mathfrak{R}_{\pi_2}\simeq[4]\int_{\Pi_{pr(+1)'}}\mathfrak{R}_\pi m(\pi)d\pi\oplus[4]\mathfrak{R}_{sp}\oplus[2]\sum_{\pi\in\Pi_{d(+1)'}}(\mathfrak{R}_\pi^+\oplus\mathfrak{R}_\pi^-)$$
$$\oplus(\mathfrak{R}_0^1\oplus\mathfrak{R}_0^\varepsilon\oplus\mathfrak{R}_0^p\oplus\mathfrak{R}_0^{\varepsilon p})\oplus\mathfrak{R}_{\pi_1\pi_2\rho}\,.$$

## § 13. Decomposition formulas for limiting cases.

As the limiting cases, we obtain the decomposition formulas for tensor products of the special representation with one of representations of principal series, supplementary series and the special representation itself. These tensor products are realized explicity in (IV), (V) and (VI) in § 6.1.

**Case (IV).** The tensor product of the special representation with a principal series one, the limiting case of (II). Taking the limits as $\alpha_1\to -1$ of the both sides of the formula (10.5), we get the decomposition formula for this case.

Let $\varphi(x_1, x_2)$ be the following function: (*) $\varphi$ is locally constant, compactly supported and zero on a neighborhood of the diagonal and satisfies the condition $\int_k\varphi(x_1, x_2)dx_1=0$. Let $\pi_1(x)=|x|^{\alpha_1}$, $-1<\alpha_1<0$, and fix $\pi_2\in\tilde{k}^\times$. Put

$$f(d(a)n^+(y)n(x))=\pi_2^{-1}\rho^{-1}(a)\kappa(ay^{-1})\pi_2\rho^{-1}(y)\varphi(x, x+y^{-1})\in\mathcal{S}(G)\,, \tag{13.1}$$

where $\kappa\in\mathcal{S}^\times$ such that $\int_k\kappa(a)d^\times a=1$. Then it is proved by changing variables that

$$(Uf)(x_1, x_2)=(U(\alpha_1, \alpha_2)f)(x_1, x_2)=\tilde{\kappa}(\pi_1^{-1}\rho^{-1})\varphi_{\alpha_1}(x_1, x_2)\,, \tag{13.2}$$

where $\varphi_{\alpha_1}(x_1, x_2)=\pi_1\rho(x_2-x_1)\varphi(x_1, x_2)$ and $\tilde{\kappa}(\pi_1^{-1}\rho^{-1})=\int_k\pi_1^{-1}\rho^{-1}(a)\kappa(a)d^\times a$. As $\alpha_1\to -1$ we have $\varphi_{\alpha_1}\to\varphi$ and $Uf\to\varphi$. It is also proved that

$$\int_k f(n(x')g)dx'=0 \qquad \text{for all } g\in G\,. \tag{13.3}$$

We get from (10.5) that

$$(13.4)\qquad |\kappa(\pi_1^{-1}\rho^{-1})|^2\|\varphi_{\alpha_1}\|_{\mathrm{II}}^2=\sum_{s\in E}\int_{\Pi_{pr}}\int_k|\Phi(t\,;\pi,\pi_1,s)|^2dtm(\pi)d\pi$$

$$+[Q_{sp}]m(\pi_{sp})\sum_s\int_k|\Phi(t\,;\pi_{sp},\pi_1,s)|^2\pi_{sp}(t)dt$$

$$+\sum_{\pi\in\Pi_d}m(\pi)\sum_s\int_k|\Phi(t\,;\pi,\pi_1,s)|^2dt$$

$$+[Q_d]m(\pi_0)\sum_s\int_k|\Phi(t\,;\pi_0,\pi_1,s)|^2dt\,,$$

where $\|\ \|_{\mathrm{II}}$ is as in § 6.1 (II), and $\Phi(t\,;\pi,\pi_1,s)=\Phi(t\,;\pi,s)$ in (10.2). Let $\|\ \|_{\mathrm{IV}}$ be as in § 6.1 (V).

**Proposition 13.1.** *For a function $\varphi$ of* (*), $\|\varphi\|_{\mathrm{IV}}^2$ *equals the sum obtained the right hand side of* (13.4) *by replacing $\pi_1$ by $\pi_{sp}$.*

*Proof.* (1) First we prove that the left hand side of (13.4) tends to $\|\varphi\|_{\mathrm{IV}}^2$. Since $\bar{\kappa}(\pi_1^{-1}\rho^{-1})\to 1$, it is enough to prove $\|\varphi_{\alpha_1}\|_{\mathrm{II}}\to\|\varphi\|_{\mathrm{IV}}$. We show that $\|\varphi_{\alpha_1}-\varphi\|_{\mathrm{II}}\to 0$. The function $\varphi_{\alpha_1}-\varphi=\{\pi_1\rho(x_2-x_1)-1\}\varphi$ is expressed as $(\alpha_1+1)a(x_1,x_2,\pi_1)\varphi$, where the function $a(x_1,x_2,\pi_1)$ is, for every $\alpha_1$, locally constant on Supp[$\varphi$] and it is uniformly bounded as $\alpha_1\to-1$. Since $(\alpha_1+1)^2\Gamma(\pi_1^{-1})^{-1}\to 0$, the assertion follows from

$$\|\varphi_{\alpha_1}-\varphi\|_{\mathrm{II}}^2=(\alpha_1+1)^2\Gamma(\pi_1^{-1})^{-1}\int_k\int_k\int_k\pi_1\rho(x_1-x_1')$$

$$\times a(x_1,x_2,\pi_1)a(x_1',x_2,\pi_1)\varphi(x_1,x_2)\overline{\varphi(x_1',x_2)}dx_1dx_1'dx_2\,.$$

Thus

$$\|\varphi_{\alpha_1}\|_{\mathrm{II}}^2=\|\varphi_{\alpha_1}-\varphi\|_{\mathrm{II}}^2+\langle\varphi,\varphi_{\alpha_1}-\varphi\rangle_{\mathrm{II}}+\langle\varphi_{\alpha_1}-\varphi,\varphi\rangle_{\mathrm{II}}+\|\varphi\|_{\mathrm{II}}^2\longrightarrow\|\varphi\|_{\mathrm{IV}}^2\,.$$

(2) Note that $f$ supported in $wG^0$ is a linear combination of the form $\xi(-x)\eta(-y)\kappa(a^{-1})$ for $g=n(x)d(a)wn(y)$ where $\xi,\eta\in\mathcal{S}$ and $\kappa\in\mathcal{S}^\times$. In our discussion we may assume $f$ in (13.1) is of this form. Then $\check{f}(g)=f(g^{-1})=\xi(y)\eta(x)\kappa(a)$, and $K_\pi(\check{f}|t,u)=\hat{\eta}(t)\hat{\xi}(u)M_\pi(t,u)$, where $M_\pi(t,u)$ as in (9.20). The condition (13.3) is equivalent to "$\hat{\xi}\in\mathcal{S}^\times$". So, on $k\times k\times\Pi_{pr}K_\pi(\check{f}|t,u)$ is a linear combination of functions of the type $a(t)b(u)\tilde{c}(\pi)$ where $a$, $b$ and $c\in\mathcal{S}^\times$. We make $\alpha_1$ tend to $-1$ in the right side of (13.4). Let us discuss the first terms. $\Phi(t\,;\pi,\pi_1,s)$ $(s\in E)$ are linear combinations of functions of the type $a(t)\tilde{b}(\pi,\pi_1)\tilde{c}(\pi)$, where $\tilde{b}(\pi,\pi_1)=\int_{s(k^\times)^2}(\pi_1\pi_2^{-1}\pi^{-1}\rho)^{1/2}\rho^{-1}(u)b(u)du$. Since $b\in\mathcal{S}^\times$, the integral $\tilde{b}(\pi,\pi_{sp})$ converges, and the continous functions $\tilde{b}(\pi,\pi_1)$ in $\pi$ tend uniformly to $\tilde{b}(\pi,\pi_{sp})$ as $\alpha_1\to-1$. Thus we have the limit of $\Phi(t\,;\pi,\pi_1,s)$ and

$$\lim_{\alpha_1\to-1}\int_{\Pi_{pr}}\int_k|\Phi(t\,;\pi,\pi_1,s)|^2dtm(\pi)d\pi=\int_{\Pi_{pr}}\int_k|\Phi(t\,;\pi,\pi_{sp},s)|^2dtm(\pi)d\pi\,.$$

By the similar discussion, we get for $\pi\in Q_{sp}$,

$$\lim_{\alpha_1\to-1}\int_k |\Phi(t\,;\,\pi_{sp},\,\pi_1,\,s)|^2\pi_{sp}(t)dt=\int_k |\Phi(t\,;\,\pi_{sp},\,\pi_{sp},\,s)|^2\pi_{sp}(t)dt\,,$$

and for $\pi\in\Pi_d\cup Q_d$,

$$\lim_{\alpha_1\to-1}\int_k |\Phi(t\,;\,\pi,\,\pi_1,\,s)|^2dt=\int_k |\Phi(t\,;\,\pi_{sp},\,\pi_{sp},\,s)|^2dt\,.$$

Thus each term in the right hand side of (13.4) tends to the analogous one obtained by replacing $\pi_1$ by $\pi_{sp}$. This completes the proof. Q. E. D.

Let $\mathcal{S}_{-1}(G)$ be the space of functions $f$ in $\mathcal{S}(G)$ satisfying (13.3), and $\mathcal{H}_{-1}$ the space of functions $\varphi\in\mathcal{H}$ such that $\int_k\varphi(x_1,\,x_2)dx_1=0$. Then the mapping $U=U(\pi_{sp},\,\pi_2)$: $Uf=\varphi$, is of $\mathcal{S}_{-1}(G)$ onto $\mathcal{H}_{-1}$. Indeed, for $g=d(a_1)n^+(y_1)n(x_1)$ it holds from (B) in Lemma 6.4 that

$$0=\int_k\int_k \pi_{sp}\pi_2^{-1}(a)f(n(x)d(a)g)d^\times a\,dx=\int_k\int_k \pi_{sp}^{-1}\pi_2^{-1}(a)f(d(a)n(x)g)d^\times a\,dx$$

$$=\pi_{sp}^{-1}(a_1)\pi_2\rho^{-1}(y_1)\int_k\varphi(x+x_1,\,x_1+y_1^{-1})dx$$

Thus we have $\varphi\in\mathcal{H}_{-1}$. By (13.1) and (13.3), the mapping $U$ is surjective.

By the mapping

$$V: f\in\mathcal{S}_{-1}(G)\longrightarrow\Phi=\Phi(t\,;\,\pi,\,\pi_{sp},\,s)\in\mathfrak{H}\,, \tag{13.6}$$

and the formula in Proposition 13.1, we can define an isometric $G$-morphism $W$ of $\mathcal{H}_{-1}$ into $\mathfrak{H}$ by $WU=V$. Here $\mathfrak{H}=\mathfrak{H}^{(+)}$ or $\mathfrak{H}^{(-)}$ according as $\pi_2(-1)=1$ or $\pi_2(-1)=-1$. We can extend $W$ to an isometry of the Hilbert space $L_{sp}^2\otimes L^2$ into $\mathfrak{H}$, where $L_{sp}^2\otimes L^2$ is a space functions $\varphi(x_1,\,x_2)$ on $k\times k$ such that $\|\varphi\|_{\mathrm{IV}}<\infty$. We can see from the proof of Proposition 9.1 that the surjectivity of $W$ is also valid for this case.

**Theorem 13.2.** *$W$ is a unitary $G$-morphism of $L_{sp}^2\otimes L^2$ onto $\mathfrak{H}$, $\mathfrak{H}=\mathfrak{H}^{(+)}$ or $\mathfrak{H}^{(-)}$ according as $\pi_2(-1)=1$ or $=-1$, and realizes the decomposition of the tensor produdt $\mathfrak{R}_{sp}\bar{\otimes}\mathfrak{R}_{\pi_2}$ into irreducibles as follows.*

*In case $-1\in(k^\times)^2$ and $\pi_2(-1)=1$, by* (9.21). *In case $-1\in(k^\times)^2$ and $\pi_2(-1)=-1$, by* (9.22). *In case $-1\notin(k^\times)^2$ and $\pi_2(-1)=1$, by* (9.23). *In case $-1\notin(k^\times)^2$ and $\pi_2(-1)=-1$, by* (9.24).

**Remark.** The result for this case is of the same form as that in Theorem 10.3.

**Case (V).** The tensor product of the special representation with a supplementary series one, the limiting case of (III. B). Let $\pi_i(x)=|x|^{\alpha_i}$ $(i=1,\,2)$, $-1<\alpha_i<0$. We fix $\pi_2$ and make $\alpha_1$ tend to $-1$. So, $\pi_1$, $\pi_2$ are in Case (III. B). Let $\varphi$ be of (*) and $f$ as in (13.1). Then $Uf=\tilde{\kappa}(\pi_1^{-1}\rho^{-1})\varphi_{\alpha_1}$ as in (13.2). By formula in Proposition 12.1,

$$(13.7)\qquad |\tilde{\kappa}(\pi_1^{-1}\rho^{-1})|^2\|\varphi_{\alpha_1}\|_{\mathrm{III}}^2$$

$$=\sum_{\tau\in E}\int_{\Pi_{pr}}\int_k \boldsymbol{\Gamma}_\tau(\pi_1,\ \pi_2,\ \pi)|\Phi_\tau(t\,;\ \pi,\ \pi_1,\ \pi_2)|^2 dt m(\pi)d\pi$$

$$+m(\pi_{sp})\sum_\tau \boldsymbol{\Gamma}_\tau(\pi_1,\ \pi_2,\ \pi_{sp})\int_k |\Phi_\tau(t\,;\ \pi_{sp},\ \pi_1,\ \pi_2)|^2\pi_{sp}(t)dt$$

$$+\sum_{\pi\in\Pi_d} m(\pi)\sum_\tau \boldsymbol{g}_\tau(\pi_1,\ \pi_2,\ \pi)\int_k |\Phi_\tau(t\,;\ \pi,\ \pi_1,\ \pi_2)|^2 dt$$

$$+[Q_d]m(\pi_0)\sum_\tau g_\tau(\pi_1,\ \pi_2,\ \pi_0)\int_k |\Phi_\tau(t\,;\ \pi_0,\ \pi_1,\ \pi_2)|^2 dt$$

$$+r(\pi_1,\ \pi_2)\|\psi\|_{\alpha_1+\alpha_2+1}^2,$$

where $\|\ \|_{\mathrm{III}}$ is as in § 6.1 (III), $\Phi_\tau$ in (11.7), $r(\pi_1,\ \pi_2)$ in (12.14) and

$$(13.8)\qquad \psi(x)=\int_k\int_k \pi_2(z_1)\pi_1(z_2)(\pi_1\pi_2\rho)^{-1}(z_1-z_2)\varphi(z_1+x,\ z_2+x)dz_1dz_2\,.$$

We prove the last term in (13.7) tends to zero as $\alpha_1\to-1$. The last term is rewritten as

$$(13.9)\qquad \frac{r(\pi_1,\ \pi_2)}{\Gamma(\pi_1^{-1})\Gamma(\pi_2^{-1})}\int_k \psi(x)\overline{\psi'(x)}dx\,,$$

where

$$(13.10)\qquad \psi'(x)=\int_k\int_k \pi_1^{-1}\rho^{-1}(z_1)\pi_2^{-1}\rho^{-1}(z_2)\varphi(z_1+x,\ z_2+x)dz_1dz_2\,.$$

Since $\int_k \varphi(x_1,\ x_2)dx_1=0$, we have that $\Gamma(\pi_1^{-1})^{-1}\psi'$ is in $\mathcal{S}$ and, as $\alpha_1\to-1$, it uniformly converges to

$$c\int_k\int_k \log|z_1|\pi_2^{-1}\rho^{-1}(z_2)\varphi(z_1+x,\ z_2+x)dz_1dz_2\in\mathcal{S}\,.$$

Let $P^n\times P^n$ be a neighborhood of $(0,0)$ in $k\times k$ such that $\varphi(x_1,\ x_2)=0$ on $P^n\times P^n$. We divide the integration domain in (13.8) as $k\times k=(k\times P^n)\cup(k\times(P^n)^c)=I_1\cup I_2$. Then (13.8) equals

$$\iint_{I_1}\cdots\ dz_1dz_2+\iint_{I_2}\cdots\ dz_1dz_2=J_1+J_2\,.$$

The integrand of the first term is equal to 0 if $z_1\in P^n$, and to $\pi_1^{-1}\rho^{-1}(z_1)\pi_1(z_2)\varphi(z_1+x,\ z_2+x)$ if $z_1\in(P^n)^c$. Thus

$$J_1=\frac{1}{\Gamma(\pi_1^{-1})\Gamma(\pi_1\rho)}\int_{(P^n)^c}\int_{P^n}\pi_1^{-1}\rho^{-1}(z_1)\pi_1(z_2)\varphi(z_1+x,\ z_2+x)dz_1dz_2$$

$$\longrightarrow c\int_k \log|z_1|\varphi(z_1+x,\ x)dz_1\in\mathcal{S}\,.$$

The integral $J_2$ also converges to a function in $\mathcal{S}$. So, we get that the integral (13.9)$\sim r(\pi_1, \pi_2)\sim(\alpha_1+1)\to 0$ as $\alpha_1\to -1$.

Similarly as in Case (IV), $|\tilde{\kappa}(\pi_1^{-1}\rho^{-1})|\,\|\varphi_{\alpha_1}\|_{\mathrm{III}\alpha_1}\to\|\varphi\|_{\mathrm{V}}$. For other terms in the right hand side of (13.7), we can change the order of $\lim_{\alpha_1\to -1}$ and integrations. Thus, we get

**Proposition 13.3.** *For $\varphi(x_1, x_2)$ be of* $(*)$, $\|\varphi\|_{\mathrm{V}}^2$ *equals the sum obtained from right side of* (13.7) *by replacing $\pi_1$ by $\pi_{sp}$. Here, the last term vonishes.*

Similarly as in § 11, we get the decomposition formula for this case.

**Theorem 13.4.** *There exists a unitary G-morphism W of $L^2_{sp}\otimes L^2_{\pi_2}$ onto $\mathfrak{H}^{(+)}$, which realized the decomposition of the tensor product $\mathfrak{R}_{sp}\bar{\otimes}\mathfrak{R}_{\pi_2}$ into irreducibles as follows. In case $-1\in(k^\times)^2$, it is given by* (9.21). *In case $-1\notin(k^\times)^2$, by* (9.23).

**Remark.** The supplementary series representation appeared in Case (III. B) vanishes here.

**Case (IV).** The tensor product of two special representations. Let again $\pi_1(x)=|x|^{\alpha_i}$ $(i=1, 2)$ as in (III. B). Let $\varphi$ be of $(*)$ and satisfy $\int_k\varphi(x_1, x_2)dx_2=0$. Put $f(g)=f(d(a)n^+n(y)n(x))=\kappa(ay^{-1})\rho^2(y)\varphi(x, x+y^{-1})\in\mathcal{S}(G)$, where $\kappa\in\mathcal{S}^\times$ such that $\int_k\kappa(a)d^\times a=1$. It holds that $(Uf)(x_1, x_2)=\tilde{\kappa}(\pi_1^{-1}\pi_2)\pi_1\rho(x_2-x_1)\varphi(x_1, x_2)=\tilde{\kappa}(\pi_1^{-1}\pi_2)\varphi_{\alpha_1}$. Then we have

$$|\tilde{\kappa}(\pi_1^{-1}\pi_2)|^2\|\varphi_{\alpha_1}\|^2_{\mathrm{II}}=\text{the right hand side of (13.7)}. \tag{13.11}$$

We make $\alpha_1$ and $\alpha_2$ tend to $-1$.

First we show that the last term in the right hand side of (13.7) vanishes as $\alpha_1, \alpha_2\to -1$. As $\alpha_1, \alpha_2\to -1$, $\Gamma(\pi_1^{-1})\Gamma(\pi_2^{-1})\psi'$ with $\psi'$ in (13.10) converges uniformly to

$$c^2\int_k\int_k \log|z_1|\log|z_2|\varphi(z_1+x, z_2+x)dz_1dz_2\in\mathcal{S}.$$

We divide the integration domain in (13.8) into three parts. Let $\varepsilon>0$ such that, if $|x_1|, |x_2|<\varepsilon$, then $\varphi(x_1, x_2)=0$. We set $I_1=\{(z_1, z_2)\in\operatorname{Supp}\varphi;\ |z_2|<\varepsilon\}$, $I_2=\{(z_1, z_2)\in\operatorname{Supp}\varphi;\ |z_1|<\varepsilon\}$ and $I_3$ the other part in the support of $\varphi$. Since $|z_1-z_2|=|z_1|$ for $(z_1, z_2)\in I_1$ and $|z_1-z_2|=|z_2|$ for $(z_1, z_2)\in I_2$, therefore

$$\begin{aligned}\psi(x)=&\iint_{I_1}\pi_1^{-1}\rho^{-1}(z_1)\pi_1(z_2)\varphi(z_1+x, z_2+x)dz_1dz_2\\&+\iint_{I_2}\pi_2(z_1)\pi_2^{-1}\rho^{-1}(z_2)\varphi(z_1+x, z_2+x)dz_1dz_2\\&+\iint_{I_3}\pi_2(z_1)\pi_1(z_2)(\pi_1\pi_2\rho)^{-1}(z_1-z_2)\varphi(z_1+x, z_2+x)dz_1dz_2.\end{aligned}$$

As in Case (V), when $\alpha_1, \alpha_2 \to -1$, these three terms converge each to functions in $\mathcal{S}$. So, we have the integral (13.9)$\sim r(\pi_1, \pi_2) \sim \frac{(\alpha_1+1)(\alpha_2+1)}{(\alpha_1+\alpha_2+1)} \to 0$.

Next we should discuss the integral

$$S=\boldsymbol{\Gamma}_1(\pi_1, \pi_2, \pi)\int_k |\Phi_1(t\,;\pi, \pi_1, \pi_2)|^2\pi(t)dt$$

$$=\boldsymbol{\Gamma}_1(t\,;\pi_1, \pi_2, \pi)\int_k \Phi_1(x\,;\pi, \pi_1, \pi_2)\Phi_1(x'\,;\pi, \pi_1, \pi_2)\pi(x-x')dxdx',$$

where $\pi(x)=|x|^\alpha$, $-1<\alpha<0$.

Since, for $\mu(x)=|x|^\beta$, $\beta\to 0$, it holds $\Gamma(\mu)\sim 1/\beta$ and $\Gamma(\mu\rho)\sim\beta$, we have

$$\boldsymbol{\Gamma}_1(\pi_1, \pi_2, \pi)\sim\frac{-\alpha_1+\alpha_2-\alpha-1}{(\alpha_1+\alpha_2-\alpha+1)(-\alpha_1+\alpha_2+\alpha+1)},$$

as $\alpha_1$, $\alpha_2$ and $\alpha\to -1$.

On the other hand, for $\Phi_1$ we use the formula in Proposition 9.6 which is applicable for this case. That is,

$$\Phi_1(x\,;\pi, \pi_1, \pi_2)=\sqrt{2}^{-1}\Gamma((\pi_1\pi_2^{-1}\pi^{-1}\rho)^{1/2})\int_k\int_k A(z_1, z_2)\varphi(z_1+x, z_2+x)dz_1dz_2,$$

where $A(z_1, z_2)=(\pi_1^{-1}\pi_2\pi\rho)^{1/2}\rho^{-1}(z_1)(\pi_1\pi_2^{-1}\pi\rho)^{1/2}\rho^{-1}(z_2)\ (\pi_1^{-1}\pi_2^{-1}\pi^{-1}\rho)^{1/2}\rho^{-1}(z_1-z_2)$. We divide the integration domain.

$$\Phi_1(x\,;\pi, \pi_1, \pi_2)=\iint_{I_1} A(z_1, z_2)\varphi(z_1+x, z_2+x)dz_1dz_2$$

$$+\iint_{I_2}\cdots dz_1dz_2+\iint_{I_3}\cdots dz_1dz_2.$$

By similar method as above, we can prove that $\Phi_1\sim(\alpha_1-\alpha_2-\alpha-1)l(x)$, $l(x)\in\mathcal{S}_{sp}$.

Thus we get $S\sim\frac{(-\alpha_1+\alpha_2-\alpha-1)(\alpha_1-\alpha_2-\alpha-1)}{\alpha_1+\alpha_2-\alpha-1}\to 0$ with $\alpha_1$, $\alpha_2$ and $\alpha\to -1$.

So, we should understand that the term $\Gamma_1(\pi_1, \pi_2, \pi_{sp})\iint_k |\Phi_1(t\,;\pi_{sp}, \pi_1\pi_2)|^2\pi_{sp}(t)dt$ vanishes.

**Proposition 13.5.** *Let $\varphi$ be of* (*) *and satisfy* $\int_k\varphi(x_1, x_2)dx_2=0$. *Then* $\|\varphi\|_{\mathrm{IV}}^2$ *equals the sum obtained from the right hand side of* (13.7) *by replacing* $\pi_1$, and $\pi_2$ *by* $\pi_{sp}$. *Here, the term* $\Gamma_1\int_k|\Phi_1|^2\pi_{sp}(t)dt$ *and the last term vanish.*

Through the analogous discussion to Case (IV) and (V), we get the decomposition formula.

**Theorem 13.6.** *There exists a unitary G-morphism $W$ of* $L_{sp}^2\otimes L_{sp}^2$ *onto* $\mathfrak{H}^{(+)}\ominus L_{sp}^2$. *It realizes the decomposition of the tensor product* $\mathfrak{R}_{sp}\overline{\otimes}\mathfrak{R}_{sp}$ *into* *irreducibles as follows.*

*In case* $-1\in(k^{\times})^2$, $\mathfrak{R}_{sp}\overline{\otimes}\mathfrak{R}_{sp}\simeq$ *the right hand side of* (9.21) $\ominus\mathfrak{R}_{sp}$.
*In case* $-1\notin(k^{\times})^2$, $\mathfrak{R}_{sp}\overline{\otimes}\mathfrak{R}_{ap}\simeq$ *the right hand side of* (9.23) $\ominus\mathfrak{R}_{sp}$.

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**Added in proof.** After this paper had been accepted for publication, the author was informed that the decomposition formulas in Theorems 11.4, 12.3, 13.4 and 13.6 was obtained in the following note. The formulas was proved by the method adopted in [13].

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