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THE PLATEAU PROBLEM FOR SURFACES OF PRESCRIBED MEAN CURVATURE IN A RIEMANNIAN MANIFOLD

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1. Introduction

In this work we treat the problem of finding a surface of prescribed mean curvature in a three-dimensional riemannian manifold M, with a given closed curve as boundary. That is, given a real-valued function $H(z)$ defined on M, we wish to find a mapping $z: B \to M$, B denoting the two-dimensional unit disk, which satisfies the following conditions:

(i) $z \in C^2(B) \cap C^0(\bar{B})$,
(ii) z maps ∂B homeomorphically onto Γ,
(iii) z satisfies in B the systems

$$\nabla_{z_u} z_u + \nabla_{z_v} z_v = 2H(z)^*(z_u \wedge z_v) , \tag{1.1}$$

$$\langle z_u, z_u \rangle - \langle z_v, z_v \rangle = \langle z_u, z_v \rangle = 0 . \tag{1.2}$$

Here $\langle \, , \, \rangle$ denotes the inner product on the tangent bundle of M, ∇ the associated Levi-Civita connection, $^*P$ the tangent vector associated with a two-vector P using $\langle \, , \, \rangle$. Let g_{ij} be the coefficients of $\langle \, , \, \rangle$ in some coordinate system. We may write explicity

$$^*(z_u \wedge z_v)^k = \sqrt{g} \begin{vmatrix} g^{1k} & g^{2k} & g^{3k} \\ z_u^1 & z_u^2 & z_u^3 \\ z_v^1 & z_v^2 & z_v^3 \end{vmatrix} ,$$

where $g_{ij}g^{jk} = \delta_i^k$ and $g = \det(g_{ij})$. (1.2) states that z is a conformal mapping on its image (possibly with degenerate points); under that condition, (1.1) become the equations for mean curvature $H(z)$ at regular points.

The basic result of the present paper for smooth complete M may be stated as follows. Let K_0 denote an upper bound on sectional curvatures of M, and $\Phi(r)$ the mean curvature with respect to an inward normal of the geodesic sphere of radius r in the space of constant curvature K_0. Explicitly, $\Phi(r) = \sqrt{K_0} \cot(\sqrt{K_0}\, r)$. In the case $K_0 > 0$, replace Φ by any smaller function ϕ

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which is monotone decreasing. Then for a rectifiable Jordan curve Γ contained in the geodesic ball $B_r(m)$, where $\exp_m$ is injective on $B_r(0) \subset M_m$, and for a Hölder-continuous function $H(z)$ satisfying $|H(z)| \leq \phi(r)$ in that ball, the problem has a solution (Theorem 2). The injectivity of $\exp_m$ on $B_r(0)$ is not essential (Theorem 3).

For the minimal surface case, i.e., $H \equiv 0$, this problem was considered by Morrey [14] after pioneering work in euclidean space by Radó [16] and Douglas [2]. Heinz [7] considered the case of constant mean curvature H in euclidean space, showing existence under the condition that Γ be contained in a ball of radius $(\sqrt{17} - 1)/(8|H|)$. This radius was sharpened by Werner [19] to $\frac{1}{2}|H|^{-1}$. Hildebrandt [11] improved this to the best possible, requiring radius $|H|^{-1}$. This was accomplished, using regularity results of Morrey, via the introduction of a restricted variational problem in combination with a new maximum principle valid for solutions which are only continuous. Hildebrandt has generalized this result to prescribed mean curvature using a more elegant proof which involves a modified free variational problem [10]. This method appears to run into difficulty in the riemannian context if positive sectional curvatures are allowed. However a variant of the method of [11] is applied to the problem successfully in the present work: the result stated above is a direct generalization of the result of [10]. In the case of nonpositive sectional curvatures our method provides a generalization of Morrey's result in [14]. As in [11], the core of this work is a maximum principle for continuous solutions to the variational problem, which we present in § 4. A similar maximum principle, requiring the mapping to be smooth, has been recently obtained by Kaul [12].

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2. The functional and its first variation

In order to define the variational problem we shall use, we assume for some point $m \in M$ the map $\exp = \exp_m: M_m \to M$ is a diffeomorphism of the ball $B_R(0)$ onto its image, which is the geodesic ball B_R of radius R and center m. If S is a set in M_m we define $C(S)$, the cone on S, to be the set $\{tx: 0 \leq t \leq 1, x \in S\}$. If $S_1 = \exp(S) \subset B_R$ we define the geodesic cone on $S_1, C(S_1) = \exp(C(S))$. If S_1 is an oriented 2-chain, $C(S_1)$ is an oriented 3-chain. Let a mapping $z: B \to B_R$ and a measurable real-valued function $H(x)$ defined on B_R be given. Then we may define the functional

$$W[z] = 4 \int_{C(z(B))} H(x) dV(x) ,$$

where dV refers to oriented riemannian volume. For vectors V_1, V_2 tangent to M_m we write the euclidean inner product as $V_1 \cdot V_2$, reserving the symbol $\langle V_1, V_2 \rangle$ for the riemannian inner product on tangent vectors to M. For

$z: B \to M$ and $y: B \to M_m$ we use the notation $|\nabla y|^2 = y_u \cdot y_u + y_v \cdot y_v$ and $|\nabla z|_M^2 = \langle z_u, z_u \rangle + \langle z_v, z_v \rangle$. We may then write the euclidean and riemannian Dirichlet integrals as

$$\bar{D}[y] = \iint_B |\nabla y|^2 dudv , \qquad D[z] = \iint_B |\nabla z|_M^2 dudv .$$

Finally define the functional for our variational problem:

$$E[z] = D[z] + W[z] .$$

For a mapping z into B_R we introduce the notation $\tilde{z} = \exp^{-1} \circ z$. The manifold M is said to be of class C^k if it has a C^k differentiable structure and the inner product $\langle \, , \, \rangle$ is of class C^{k-1}.

Proposition 1. *If M is of class C^2 and $H \in C^1$, then the Euler equations for E are the system* (1.1).

Proof. We may write

$$\begin{aligned} W[z] &= 4 \iint_B \int_0^1 H(tz) < \gamma_*(t|\tilde{z}|), {}^*(z_u(t|\tilde{z}|) \wedge z_v(t|\tilde{z}|)) > |\tilde{z}| \, dtdudv \\ &= 4 \iint_B \omega(z, \nabla z) dudv . \end{aligned}$$

Here we define $tz = \exp(t\tilde{z})$; γ is the arc-length geodesic from m to z; for a vector $V \in M_z$, $V(t|\tilde{z}|)$ denotes the element at $tz = \gamma(t|\tilde{z}|)$ of the Jacobi field along γ determined by $V(|\tilde{z}|) = V$ and $V(0) = 0$. Let g_{ij} be the coefficients of the inner product of M with respect to normal coordinates at m, $g = \det(g_{ij})$. Then

$$\begin{aligned} \omega(z, \nabla z) &= \int_0^1 H(tz) \sqrt{g}(tz) \frac{\tilde{z}}{|\tilde{z}|} \cdot (t\tilde{z}_u) \wedge (t\tilde{z}_v) |\tilde{z}| \, dt \\ &= \int_0^1 t^2 H(tz) \sqrt{g}(tz) dt \, \tilde{z} \cdot \tilde{z}_u \wedge \tilde{z}_v = Q(\tilde{z}) \cdot \tilde{z}_u \wedge \tilde{z}_v . \end{aligned} \tag{2.1}$$

The hypotheses imply $Q \in C^1$. By Lemma 8 of [5] the first variation (in the euclidean context)

$$[\omega]_z = \operatorname{div} Q \tilde{z}_u \wedge \tilde{z}_v ,$$

where the divergence operator is euclidean. But $\operatorname{div} Q = H(z) \sqrt{g}(z)$, by way of integration by parts. The first variation (again in the euclidean context) of D in the lth component z^l is readily calculated as

$$-2g_{kl}(z^k_{uu} + z^k_{vv}) - 2\Gamma_{ij|l}(z^i_u z^j_u + z^i_v z^j_v) .$$

Here the Γ's are the coefficients of ∇. Thus the Euler equations of E are

$$g_{kl}[z^k_{uu} + z^k_{vv} + \Gamma^k_{ij}(z^i_u z^j_u + z^i_v z^j_v)] = 2H(z)\sqrt{g}\,(\tilde{z}_u \wedge \tilde{z}_v)^l ,$$

or

$$z^k_{uu} + z^k_{vv} + \Gamma^k_{ij}(z^i_u z^j_u + z^i_v z^j_v) = 2H(z)^*(z_u \wedge z_v)^k , \tag{2.2}$$

which is the same as

$$\nabla_{z_u} z_u + \nabla_{z_v} z_v = 2H(z)^*(z_u \wedge z_v) .$$

3. Variation with fixed boundary mapping

Let M be a three-dimensional riemannian manifold of class C^3, with sectional curvatures $K \le b^2$, b either real or imaginary. Choose $r_0 > 0$; if $b^2 > 0$ we require $|b|r_0 \le \frac{1}{2}\pi$. We assume for the present that exp is defined on $B_{r_0}(0) \subset M_m$ and is a diffeomorphism of $B_{r_0}(0)$ onto B_{r_0} (this requirement will be dropped in § 6). The radius r_0 will be fixed in this section and the one following.

Define $\mathscr{D}_R = \{x\colon B \to M_m\colon x \in H_1(B), |x(w)| \le R$ for almost all $w \in B\}$. Here H_1 denotes the space of L^2 functions with L^2 first derivatives and norm $\|x\|_1$ given by $\|x\|_1^2 = \|x\|_{L^2}^2 + \bar{D}[x]$. For $f \in \mathscr{D}_R$, we define $\mathscr{D}_R(f) = \{x \in \mathscr{D}_R\colon x - f \in \mathring{H}_1(B)\}$, where $\mathring{H}_1$ denotes the closure in H_1 of smooth functions with compact support. For a given function $H \in L^\infty(M)$, we denote by $P(f, R, H)$ the variational problem: $E[y] \to \min$ among mappings y such that $\tilde{y} \in \mathscr{D}_R(f)$ and by $e(f, R, H)$ this minimum. Denote $h = \operatorname{ess\,sup}_{x \in B_{r_0}} |H(x)|$.

Consider a mapping $z\colon B \to B_{r_0}$ at a point $w_0 \in B$. We need to bound $\omega(z, \nabla z)$ in terms of the riemannian Dirichlet integrand $|\nabla z|_M^2$.

Define a function on C:

$$G(\zeta) = \csc^2 \zeta \cot \zeta(\zeta - \cos \zeta \sin \zeta) \qquad \zeta \ne 0 ,$$
$$G(0) = 2/3 .$$

Observe G is continuous, with $0 \le G(\zeta) < 1$ for $-\frac{1}{2}\pi \le \zeta \le \frac{1}{2}\pi$ and for all imaginary ζ.

Lemma 1. *If $r = |z(w_0)| \le r_0$ and $h \le b \cot(br_0)$, then*

$$|\omega(z(w_0), \nabla z(w_0))| \le \tfrac{1}{4}|\nabla z|_M^2 G(br_0) .$$

Proof. Let $F(\rho) = \sqrt{g}\,(\rho z_0/r)$ be the Jacobian of exp at $\rho\tilde{z}_0/r \in M_m$. Here $z_0 = z(w_0)$. Then from (2.1) we have

$$\omega(z_0, \nabla z(w_0)) = \int_0^1 t^2 H(tz_0)F(tr)dt\, \tilde{z}_0 \cdot \tilde{z}_u(w_0) \wedge \tilde{z}_v(w_0) .$$

Writing $A = \tilde{z}_0 \cdot \tilde{z}_u(w_0) \wedge \tilde{z}_v(w_0)$, note that

$$|AF(r)| = r| < \gamma_*, {}^*(z_u(w_0) \wedge z_v(w_0)) > | \leq \tfrac{1}{2} r |\nabla z(w_0)|_M^2 ,$$

where γ is the arc-length geodesic from m to z_0. Now F satisfies the growth condition

$$\frac{\rho^2 F(\rho)}{r^2 F(r)} \leq \frac{\sin^2(b\rho)}{\sin^2(br)}$$

for $\rho \leq r$ [6]. So we have

$$\omega(z_0, \nabla z(w_0)) = A \int_0^1 t^2 H(tz_0)F(tr)dt \leq AF(r)h \int_0^1 \frac{t^2 F(tr)}{F(r)} dt ,$$

so that

$$|\omega(z_0, \nabla z(w_0))| \leq \tfrac{1}{2} r |\nabla z(w_0)|_M^2 h \int_0^1 \frac{\sin^2(btr)}{\sin^2(br)} dt$$

$$= \tfrac{1}{2} |\nabla z(w_0)|_M^2 h \int_0^r \frac{\sin^2(b\rho)}{\sin^2(br)} d\rho .$$

The integral is increasing as a function of r since

$$\frac{d}{dr} \int_0^r \frac{\sin^2(b\rho)}{\sin^2(br)} d\rho = 1 - G(br) > 0 .$$

Thus

$$|\omega(z_0, \nabla z(w_0))| \leq \tfrac{1}{2} |\nabla z(w_0)|_M^2 b \cot(br_0) \int_0^{r_0} \frac{\sin^2(b\rho)}{\sin^2(br_0)} d\rho$$

$$= \tfrac{1}{4} |\nabla z(w_0)|_M^2 G(br_0) . \quad \text{q.e.d.}$$

In the case $b = 0$, read r for $\sin(br)/b$ and 1 for $\cos(br)$. Thus $\Phi(r) = b \cot(br)$ becomes the familiar $1/r$ for $b = 0$.

For a vector V tangent to B_{r_0} denote $\tilde{V} = (\exp^{-1})_*(V)$.

Lemma 2. *Assume M is of class C^1. There exists N such that for any tangent vectors $V \in M_z, |\tilde{z}| \leq r_0$, we have*

$$\frac{1}{N} \tilde{V} \cdot \tilde{V} \leq \langle V, V \rangle \leq N \tilde{V} \cdot \tilde{V} .$$

Proof. At each point z of $\bar{B}_{r_0}$, let $N(z) = \sup\{\langle V, V\rangle, 1/\langle V, V\rangle : V \in M_z, \tilde{V}\cdot\tilde{V} = 1\}$. Since $\langle\ ,\ \rangle$ is continuous and positive definite, $N(z)$ is continuous and finite, and hence bounded on $\bar{B}_{r_0}$.

Corollary 1. *If $h \leq b \cot(br_0)$ and $\tilde{z} \in \mathscr{D}_{r_0}$, then for any measurable $B' \subset B$ we have*

$$\frac{1}{N}[1 - G(br_0)]\bar{D}_{B'}[\tilde{z}] \leq E_{B'}[z] \leq N[1 + G(br_0)]\bar{D}_{B'}[\tilde{z}] .$$

Here the subscripted B' denotes integration over that set.

Lemma 3. *Assume $h \leq b \cot(br_0)$. Let $\{\tilde{y}_n\}$ be a sequence from $\mathscr{D}_R$, $R \leq r_0$, such that $\tilde{y}_n$ converges weakly to $\tilde{y}$ in $H_1(B)$. Then $\tilde{y} \in \mathscr{D}_R$ and $E[y] \leq \liminf E[y_n]$.*

Proof. Using Lemma 1, as in [11, Lemma 1].

Lemma 4. *Suppose $h \leq b \cot(br_0)$ and $f \in \mathscr{D}_R$ for $R \leq r_0$. Then there exists a solution z to the variational problem $P(f, R, H)$.*

Proof. Choose a sequence $\tilde{z}_n \in \mathscr{D}_R(f)$ such that, with $z_n = \exp\circ\tilde{z}_n$, $\lim E[z_n] = e(f, R, H)$. Then the numbers $E[z_n]$ are uniformly bounded, hence by Corollary 1, $\bar{D}[\tilde{z}_n] <$ uniform bound. So $\|\tilde{z}_n\|_1^2 \leq R^2 \iint_B du dv + \bar{D}[\tilde{z}_n] <$ uniform bound, and some subsequence converges weakly to a function $\tilde{z} \in H_1(B)$, with $\tilde{z} - f \in \mathring{H}_1(B)$. Using Lemma 3, $\tilde{z} \in \mathscr{D}_R(f)$ and

$$e(f, R, H) \leq E[z] \leq \lim E[z_n] = e(f, R, H) .$$

Thus z solves $P(f, R, H)$. q.e.d.

As a consequence of its minimizing property and Corollary 1, this z satisfies a uniform Hölder condition in B. If, moreover, $f \in C^0(\partial B)$ then $z \in C^0(\bar{B})$ and $z = f$ on B. These properties follow from results of Morrey [13, Theorem 2.2] using a glueing technique (cf. [11, Lemma 4]). For any subdomain $B' \subset B$ such that $\sup_{w\in B'} |\tilde{z}(w)| < R$, the first variation of E will vanish with respect to any smooth test function with compact support; that is, for $H \in C^1$, z is a weak solution to the Euler equations (1.1) or the equivalent form (2.2). It then follows from a result of Heinz and Tomi (cf. [18]) that $z \in C^{1+\beta}$ for all $\beta < 1$ and has the representation

$$z(w) = y(w) + \iint_{B'} G(w, \zeta)\{2H(z(\zeta))^*(z_\xi \wedge z_\eta) - \Gamma^k_{ij}(z^i_\xi z^j_\xi + z^i_\eta z^j_\eta)V_k\}d\xi d\eta , \tag{3.1}$$

where y is the harmonic function with $z = y$ on $\partial B'$, $G(w, \zeta)$ the Green's function for B', and V_k the kth coordinate vector. Assuming only that H is C^α, it follows by methods of potential theory that $z \in C^{2+\alpha}$ and satisfies (1.1). It is the purpose of the next section to show that under appropriate hypotheses these considerations may be applied with $B' = B$ itself.

4. The maximum principle; smoothness

Let $r_1 \in (0, r_0)$ be chosen. We construct a C^1 mapping $\tilde{T}: M_m \to M_m$ by defining $\tilde{T}(y) = \sigma(|y|)y/|y|$ for a C^1 function σ with the properties: $\sigma(r) \leq r$ for all $r, \sigma(r) = r$ for $r \in [0, r_1]$, and $\sigma''(r_1+) < 0$. Now define $T: B_{r_0} \to B_{r_0}$ by $T(x) = \exp(\tilde{T}(\tilde{x}))$. Observe that if $y \in \mathscr{D}_R$ then $\tilde{T} \circ y \in \mathscr{D}_{\sigma(R)}$.

Lemma 5. *Suppose $h < b \cot(br_1)$. Then there exists R_1, $r_1 < R_1 \leq r_0$, such that for $\tilde{z} \in \mathscr{D}_{R_1} \cap C^\circ(\bar{B})$ with $\inf_{w \in B} |\tilde{z}(w)| \leq r_1 < \sup_{w \in B} |\tilde{z}(w)|$ there holds $E[T \circ z] < E[z]$. Thus, if $z \in C^\circ(\bar{B})$ solves $P(f, R_1, H)$ where $f \in C^\circ(\partial B)$ and $\sup_{w \in \partial B} |f(w)| \leq r_1$, then $\tilde{z} \in \mathscr{D}_{r_1}$.*

Proof. We first estimate the effect of T_* on the length of vectors. For $V \in M_p$ we define an orthogonal decomposition $V = V^r + V^s$ where $V^r = \langle V, \gamma_* \rangle \gamma_*, \gamma =$ arc-length geodesic from m to p. We have an analogous decomposition for $\tilde{V} \in (M_m)_{\tilde{p}}$, with $V^r = \exp_*(\tilde{V}^r)$ and $V^s = \exp_*(\tilde{V}^s)$. Writing $R = |\tilde{p}|$ we see that $(\tilde{T}_* \tilde{V})^s = \tilde{V}^s \sigma(R)/R$ and $(\tilde{T}_* \tilde{V})^r = \sigma'(R)\tilde{V}^r$, modulo the identification of tangents to M_m at different points by parallel translation. Write $\tilde{V}(\rho)$ for the Jacobi field along the ray through the origin and $\tilde{p}$, determined by $\tilde{V}(R) = \tilde{V}$ and $\tilde{V}(0) = 0$. Namely $\tilde{V}(\rho) = (\rho/R)\tilde{V}$ translated to $(\rho/R)\tilde{p}$. Let $f(\rho)$ be the Jacobian of exp restricted to the subspace generated by $\tilde{V}(\rho)^s$. For $\rho_1 \leq \rho_2$ we have the inequality

$$\frac{\rho_1 f(\rho_1)}{\rho_2 f(\rho_2)} \leq \frac{\sin(b\rho_1)}{\sin(b\rho_2)},$$

(cf. [6] or [17, proof of Theorem 3]). This now yields:

$$\begin{aligned}
|T_* V|^2 &= |(T_* V)^r|^2 + |(T_* V)^s|^2 = |(\tilde{T}_* \tilde{V})^r|^2 + f(\sigma(R))^2 |(\tilde{T}_* \tilde{V})^s|^2 \\
&= (\sigma'(R))^2 |\tilde{V}^r|^2 + (\sigma(R)/R)^2 f(\sigma(R))^2 |\tilde{V}^s|^2 \\
&= (\sigma'(R))^2 |V^r|^2 + \left(\frac{\sigma(R) f(\sigma(R))}{R f(R)}\right)^2 |V^s|^2 \\
&\leq (\sigma'(R))^2 |V^r|^2 + \frac{\sin^2(b\sigma(R))}{\sin^2(bR)} |V^s|^2 .
\end{aligned}$$

Now the function $\phi(R) = \sin(b\sigma(R))/\sin(bR)$ has $\phi(r_1) = 1$ and $\phi'(r_1) = 0$. Since $\sigma'(r_1) = 1$ and $\sigma''(r_1+) < 0$, we see that $0 < \sigma' \leq \phi$ on some interval $[0, R_0]$ where $R_0 > r_1$, equality holding on $[0, r_1]$. Then for $R \leq R_0$

$$|T_* V|^2 \leq (\phi(R))^2 (|V^r|^2 + |V^s|^2) = \frac{\sin^2(b\sigma(R))}{\sin^2(bR)} |V|^2 . \tag{4.1}$$

We need next to estimate the effect of T on the volume integrand $\omega(z, \nabla z)$.

Denote $y = T \circ z$, and let $z_u^s(\rho), z_v^s(\rho)$ be the Jacobi fields generated by z_u^s, z_v^s. Thus $z_u^s(\rho) \in M_{\gamma(\rho)}$. Observe that $y_u^s(\rho) = z_u^s(\rho), y_v^s(\rho) = z_v^s(\rho)$ for $\rho \leq \sigma(|\tilde{z}|)$. First assume that z_u and z_v are independent, and let $F(\rho)$ be the Jacobian of exp at $\rho\tilde{z}/R$. Then for $\rho_1 \leq \rho_2$

$$\frac{\rho_1{}^2 F(\rho_1)}{\rho_2{}^2 F(\rho_2)} \leq \frac{\sin^2(b\rho_1)}{\sin^2(b\rho^2)}$$

[6]. Now

$$\begin{aligned}\langle \gamma_*(\rho), *(y_u(\rho) \wedge y_v(\rho))\rangle &= \langle \gamma_*(\rho), *(z_u(\rho) \wedge z_v(\rho))\rangle \\ &= F(\rho)\frac{\tilde{z}}{R}\cdot\left(\frac{\rho}{R}\tilde{z}_u\right) \wedge \left(\frac{\rho}{R}\tilde{z}_v\right) C\rho^2 F(\rho) ,\end{aligned}$$

where C is independent of ρ. Thus using (2.1),

$$\begin{aligned}|\omega(z, \nabla z) - \omega(z, \nabla y)| &= \left|\int_{(\sigma R)}^{R} H\left(\frac{\rho z}{R}\right) C\rho^2 F(\rho) d\rho\right| \\ &\leq h\,|CR^2 F(R)| \int_{\sigma(R)}^{R} \frac{\rho^2 F(\rho)}{R^2 F(R)} d\rho \\ &\leq h\,|\langle \gamma_*(R), *(z_u \wedge z_v)\rangle| \int_{\sigma(R)}^{R} \frac{\sin^2(b\rho)}{\sin^2(bR)} d\rho \\ &\leq \tfrac{1}{2} h\,|\nabla z|_M^2 \int_{\sigma(R)}^{R} \frac{\sin^2(b\rho)}{\sin^2(bR)} d\rho .\end{aligned}$$

If z_u and z_v are not independent, this relation holds trivially.

Finally, with $i(z, \nabla z) = |\nabla z|_M^2 + 4\omega(z, \nabla z)$, and supposing $R = |\tilde{z}| \leq R_0$ we have from (4.1) and (4.2) that

$$\begin{aligned}i(z, \nabla z) - i(y, \nabla y) &= |\nabla z|_M^2 - |\nabla y|_M^2 + 4(\omega(z, \nabla z) - \omega(y, \nabla y)) \\ &\geq |\nabla z|_M^2 \left\{1 - \frac{\sin^2(b\sigma(R))}{\sin^2(bR)} - 2h\int_{\sigma(R)}^{R} \frac{\sin^2(br)}{\sin^2(bR)} d\rho\right\} \\ &= |\nabla z|_M^2 g(R) .\end{aligned}$$

In straightforward fashion we compute $g(R) = 0$ for $R \leq r_1$, $g'(r_1) = 0$, and

$$g''(r_1+) = 2\sigma''(r_1+)[h - b\cot(br_1)] > 0 .$$

Thus there exists $R_1 \in (r_1, R]$ such that $g > 0$ on $(r_1, R_1]$. Now assume $z \in \mathscr{D}_{R_1}$. We have $i(z, \nabla z) - i(y, \nabla y) \geq 0$ everywhere, i.e., $E_{B''}[z] - E_{B''}[y] \geq 0$ for all measurable $B'' \subset B$. Assume further $z \in C^\circ(\bar{B})$ with

$$\inf_{w\in B}|\tilde{z}(w)| \le r_1 < R_2 = \sup_{w\in B}|\tilde{z}(w)| \le R_1 .$$

Then $|\tilde{z}|$ takes every value in $[r_1, R_2]$. In particular, z is not constant on the open set

$$B' = \{w \in B: \tfrac{1}{2}(r_1 + R_2) < |\tilde{z}(w)| < R_2\} ,$$

and thus $D_{B'}[z] > 0$. But there exists $\delta > 0$ such that $g \ge \delta$ on $[\frac{1}{2}(r_1 + R_2), R_2]$. Let $B'' = B\backslash B'$. Hence

$$E[z] - E[y] = E_{B''}[z] - E_{B''}[y] + E_{B'}[z] - E_{B'}[y] \ge \delta D_{B'}[z] > 0 ,$$

as claimed.

Now, if $\sup_{w\in\partial B}|f(w)| \le r_1$ and $z \in \mathscr{D}_{R_1}(f)$ then $y \in \mathscr{D}_{R_1}(f)$; thus if z solves $P(f, R_1, H)$, $\sup_{w\in B}|\tilde{z}(w)| > r_1$ is impossible.

Theorem 1. *Suppose M^3 is a riemannian manifold of class C^3 with sectional curvatures $\le b^2$. For some $m \in M$ and $r_0 > 0$, with $4b^2r_0^2 < \pi^2$, suppose that the restriction of* $\exp = \exp_m$ *to $B_{r_0}(0) \subset M_m$ is a diffeomorphism onto its image. Let a C^1 function $H: B_{r_0} \to \boldsymbol{R}$ be given with $h = \sup_{x\in B_{r_0}}|H(x)| < b \cot(br_1)$, where $r_1 < r_0$. Then given $f \in \mathscr{D}_{r_1} \cap C^\circ(\partial B)$ there exists a solution $z \in C^2(B) \cap C^\circ(\bar{B})$ to $P(f, r_1, H)$, which satisfies (1.1) in B and agrees with* $\exp\circ f$ *on ∂B.*

Proof. Let R_1 be as given by Lemma 5. Let z be a solution to $P(f, R_1, H)$ as given by Lemma 4. From results of Morrey we have $z \in C^\circ(\bar{B})$, as remarked at the end of § 3, and $z \in C^2(B')$ for any $B' \subset B$ with $\sup_{w\in B'}|\tilde{z}(w)| < R_1$. But Lemma 5 shows that $\sup_{w\in B}|\tilde{z}(w)| \le r_1 < R_1$, so that $z \in C^2(B)$ and satisfies the Euler equations (1.1). q.e.d.

We now drop the condition that $H \in C^1$. Given any function $H \in C^\alpha(\bar{B}_{r_1})$ with $\sup_{x\in B_{r_1}}|H(x)| \le b \cot(br_1)$ we approximate H uniformly in B_{r_1} by a sequence of functions $H_n \in C^1(B_{r_0})$ with $\sup_{x\in B_{r_0}}|H_n(x)| < b \cot(br_1)$. Then for each H_n Theorem 1 gives a solution z_n to $P(f, r_1, H_n)$. As in [5, § 5] we find z such that some subsequence of the z_n converges in $H_1(B)$ to z. Since each z_n has a representation (3.1) with respect to H_n, we obtain that representation for z with respect to H. It may then be shown, using a standard argument of potential theory, that $z \in C^{2+\alpha}$ and satisfies (1.1). This shows

Corollary 2. *Theorem 1 continues to hold if the function H satisfies only $H \in C^\alpha(\bar{B}_{r_1})$ and $\sup_{x\in B_{r_1}}|H(x)| \le \cot(br_1)$.*

5. The Plateau problem

Let Γ be an oriented closed Jordan curve in B_{r_0}. Denote by $\mathscr{D}(\Gamma, R)$ the set of mappings $\tilde{x} \in \mathscr{D}_R$ such that $x|_{\partial B}$ is equal almost everywhere to a continuous, monotone mapping of degree 1 over the integers onto Γ. Define a variational problem $P_H(\Gamma, R)$ by $E[x] \to \min$ among mappings x such that $\tilde{x} \in \mathscr{D}(\Gamma, R)$.

Theorem 2. *Let M be a riemannian manifold of dimension 3 and of class C^3 with sectional curvatures $\leq b^2$. For $m \in M$ and $r_1 > 0$ with $4r_1^2b^2 < \pi^2$, assume $\exp_m$ is defined on $\bar{B}_{r_1}(0) \subset M_m$ and maps $\bar{B}_{r_1}(0)$ diffeomorphically onto $\bar{B}_{r_1} = \bar{B}_{r_1}(m)$. Let Γ be an oriented closed Jordan curve in $\bar{B}_{r_1}$ such that $\mathscr{D}(\Gamma, \infty)$ is nonempty. Let H be a uniformly Hölder-continuous function: $\bar{B}_{r_1} \to \boldsymbol{R}$ with $h = \sup_{z \in B_{r_1}} |H(z)| \leq b \cot (br_1)$. Then there exists a solution $z \in C^2(B) \cap C^\circ(\bar{B})$ to the variational problem $P_H(\Gamma, r_1)$, mapping ∂B homeomorphically onto Γ in an orientation-preserving fashion and satisfying* (1.1) *and* (1.2) *in B.*

Proof. Let $r_0 > r_1$ be chosen so that $4b^2r_0^2 < \pi^2$ and so that $\exp_m$ is a diffeomorphism of $B_{r_0}(0)$ onto $B_{r_0}(m)$. The theorem now follows from Corollary 2 in essentially the same fashion as in [11, Theorem 2]. In the process of the proof, we modify a minimizing sequence $\{x_n\}$ by requiring each x_n to satisfy a three-point condition. This can be done without changing $E[x_n]$ since E is conformally invariant. We require the choice of the three points to be such that every monotone map: $\partial B \to \Gamma$ satisfying the three-point condition will be of degree 1. In particular, the limiting mapping $z|_{\partial B}$ will be of degree 1. Observe that for any C^1-diffeomorphism $\phi\colon \bar{B} \to \bar{B}$ there holds $W[\phi \circ z] = W[z]$; therefore since $\phi \circ z \in \mathscr{D}(\Gamma, r_1)$ we have $D[z] \leq D[\phi \circ z]$. From this it follows that z satisfies (1.2) by a straightforward adaptation of the method of [1, pp. 107–112].

To show that $z|_{\partial B}$ is a homeomorphism, it suffices to show that for any $w_0 \in \partial B$, a neighborhood of which in ∂B is mapped into a C^2 curve, there holds an asymptotic representation

$$z_u - iz_v = a(w - w_0)^l + 0(|w - w_0|^l)$$

for some integer $l \geq 1$ and $a \in \mathbb{C}^3 \backslash \{0\}$. This may be obtained by suitable modification of an argument of Heinz [9, relations (14) and (30)] to allow isothermal parameters in the sense of (1.2).

6. Globalization

We now drop the requirement that exp be injective on $\bar{B}_{r_1}(0)$. We shall need the following fact, which may be expressed as the statement that exp behaves like a covering projection with respect to curves which are not too long.

Lemma 6. *Let M be a complete riemannian manifold of class C^3 with sectional curvatures $\leq b^2$. Suppose a C^1 curve $\gamma\colon [0, 1] \to M$ is given with $\gamma(0) = m$ and $r = \text{length}\,(\gamma) < r_1$, where $r_1^2b^2 < \pi^2$. Then there is a unique mapping*

$\tilde{\gamma}: [0, 1] \to M_m$ with $\tilde{\gamma}(0) = 0$ and $\exp \circ \tilde{\gamma} = \gamma$. Moreover, suppose $\{\gamma_s\}$ is a family of such curves such that $g(s, t) = \gamma_s(t)$ defines a continuous mapping $g: (-\varepsilon, \varepsilon) \times [0, 1] \to M$. Then the family $\{\tilde{\gamma}_s\}$ of liftings yields a continuous mapping $\tilde{g}: (-\varepsilon, \varepsilon) \times [0, 1] \to M_m$ by defining $\tilde{g}(s, t) = \tilde{\gamma}_s(t)$.

Proof. Every point $q \in B_{r_1}(0) \subset M_m$ has a neighborhood $U(q)$ such that exp is a diffeomorphism of $U(q)$ onto its image. This follows from the condition $r_1^2 b^2 < \pi^2$ using a comparison technique (cf. [3, pp. 176–179]). Let S be the set of $t \in [0, 1]$ such that there exists a unique continuous lifting $\tilde{\gamma}: [0, t] \to M_m$ with $\exp \circ \tilde{\gamma} = \gamma|_{[0,t]}$ and $\tilde{\gamma}(0) = 0$. Thus $0 \in S$. Suppose $t \in S$. Then for $t_1 \leq t$,

$$|\tilde{\gamma}(t_1)| = \int_0^{t_1} \tilde{\gamma}_*(t) \cdot \tilde{W} dt = \int_0^{t_1} \langle \gamma_*(t), W \rangle dt \leq \text{length}(\gamma) = r\,,$$

where $\tilde{W}$ is the radial unit vector field in M_m, and $W = \exp_*(\tilde{W})$. Thus $\tilde{\gamma}([0, t]) \subset B_r(0) \subset\subset B_{r_1}(0)$.

In particular, for sufficiently small $\varepsilon > 0$, $\gamma([t, t + \varepsilon]) \subset \exp(U(\tilde{\gamma}(t)))$ so that for $t_1 \in [t, t + \varepsilon]$ defining $\tilde{\gamma}(t_1) = (\exp|_{U(\tilde{\gamma}(t))})^{-1} \circ \gamma(t_1)$ extends $\tilde{\gamma}$ over $[0, t + \varepsilon]$. The extended curve $\tilde{\gamma}$ must be unique, since otherwise this process would provide a contradiction to the uniqueness of $\tilde{\gamma}|_{[0,t]}$. Thus $t + \varepsilon \in S$. This shows $S_0 = \{t_0: [0, t_0] \subset S\}$ is open.

Now suppose $\{t_n\}$ is an increasing sequence from S with $t_n \to t_0$. Uniqueness implies that the curves $\tilde{\gamma}$ associated with different values t_n are merely restrictions of one another. This defines $\tilde{\gamma}: [0, t_0) \to B_r(0)$. Among the various points of the finite set $\{q \in \bar{B}_r(0): \exp(q) = \gamma(t_0)\}$, at most one can be a cluster point of $\tilde{\gamma}(t)$ as $t \to t_0$; otherwise there would be a continuum of cluster points, each of which must be mapped to $\gamma(t_0)$ by exp. It follows that $\tilde{\gamma}(t)$ approaches a limit as $t \to t_0$, and we define $\tilde{\gamma}(t_0)$ to be this limit. The uniqueness of this extended $\tilde{\gamma}$ is clear. Thus $t_0 \in S$. This shows S_0 is closed. Therefore $S_0 = [0, 1]$, i.e., $S = [0, 1]$.

For the second part of the conclusion, it suffices to show $\tilde{g}$ is continuous at $s = 0$. The compact set $\tilde{\gamma}([0, 1])$ is covered by a finite number of neighborhoods U_i, $1 \leq i \leq n$, where each U_i is $U(\tilde{\gamma}(t))$ for some $t \in [0, 1]$. Choose $\delta \in (0, \varepsilon)$ small enough that for each $t \in [0, 1]$, $g([-\delta, \delta] \times \{t\}) \subset \exp(U_i)$ for some i. Define $T = \{t \in [0, 1]: \tilde{g}$ is continuous on $[-\delta, \delta] \times [0, t]\}$. Thus $0 \in T$. For some $t \geq 0$, suppose $[0, t) \subset T$. There exist i and $\eta > 0$ such that $g([-\delta, \delta] \times [t - \eta, t + \eta]) \subset \exp(U_i)$. For $s \in [-\delta, \delta]$ and $t_1 \in [t - \eta, t + \eta]$ define $\tilde{\gamma}_s'(t_1) = (\exp|_{U_i})^{-1} \circ \gamma_s(t_1)$; this defines a continuous lifting $\tilde{\gamma}_s'$ of $\gamma_s|_{[0,t+\eta]}$. By the uniqueness of $\tilde{\gamma}_s$, we have $\tilde{\gamma}_s' = \tilde{\gamma}_s$, i.e.,

$$\tilde{g}|_{[-\delta,\delta]\times[t-\eta,t+\eta]} = (\exp|_{U_i})^{-1} \circ g|_{[-\delta,\delta]\times[t-\eta,t+\eta]}\,.$$

Thus $\tilde{g}$ is continuous on $[-\delta, \delta] \times [t - \eta, t + \eta]$ and hence on $[-\delta, \delta] \times [0, t + \eta]$ via a glueing lemma. So $t + \eta \in T$. This shows $T = [0, 1]$, i.e., $\tilde{g}$ is continuous. q.e.d.

We shall need a new way of limiting the extent of a closed contractible curve $\Gamma: [0,1] \to M$. Let a contraction of Γ be given by $g: [0,1] \times [0,1] \to M$ with $g(s,0) = m, g(s,1) = \Gamma(s)$ and $g(1,t) = g(0,t)$ for all $s, t \in [0,1]$. We may assume the transverse curves $g_s(t) = g(s,t)$ are uniformly smooth: $g_s \in C^1([0,1])$ and sup length $(g_s) < \infty$. We make the following definition: if g is a contraction of Γ such that each g_s is rectifiable and length $(g_s) \leq r$, we call g an *r-contraction* of Γ; if Γ has an r-contraction, it is called *r-contractible*. Thus any contractible curve is r-contractible for sufficiently large r.

Lemma 7. *Let N be a complete riemannian manifold of class C^3 with sectional curvatures $\leq b^2$. If a continuous closed curve $\Gamma: [0,1] \to N$ is r_1-contractible, where $b^2r_1^2 < \pi^2$, then there exist $n \in N$ and a continuous closed curve $\tilde{\Gamma}: [0,1] \to \bar{B}_{r_1}(0) \subset N_n$ such that $\Gamma = \exp_n \circ \tilde{\Gamma}$.*

Proof. Let $g: [0,1] \times [0,1] \to N$ be an r_1-contraction of Γ, and n the common point $g(s,0)$. Write $g_s(t) = g(s,t)$; we have length $(g_s) \leq r_1$. Applying Lemma 6 to the family of curves $\{g_s\}$, there is a family of liftings $\{\tilde{g}_s\}$ such that $\tilde{g}(s,t) = \tilde{g}_s(t)$ defines a continuous mapping $\tilde{g}: [0,1] \times [0,1] \to N^n$. Since $g_0 = g_1$, it follows from the uniqueness of liftings that $\tilde{g}_1 = \tilde{g}_0$. Let $\tilde{\Gamma}(s) = \tilde{g}_s(1)$. Then $\tilde{\Gamma}$ is a continuous closed curve with $\Gamma = \exp_n \circ \tilde{\Gamma}$.

Theorem 3. *Let Γ be an r_1-contractible Jordan curve in a complete riemannian manifold N^3 of class C^3 and with sectional curvatures $\leq b^2$. Assume there is a mapping $x_0: \bar{B} \to N$ such that x_0 maps ∂B continuously and monotonically onto Γ, and $D[x_0] < \infty$. Suppose that $H \in C^\alpha(N)$ satisfies $\sup_{x \in N} |H(x)| \leq b \cot(br_1)$ and that $4b^2r_1^2 < \pi^2$. Then there is a mapping $z: \bar{B} \to N, z \in C^2(B) \cap C^\circ(\bar{B})$, taking ∂B homeomorphically onto Γ and satisfying* (1.1) *and* (1.2) *in B.*

Proof. By Lemma 7, there exist $n \in N$ and a continuous closed curve $\tilde{\Gamma}: [0,1] \to \bar{B}_{r_1}(0) \subset N_n$ such that $\Gamma = \exp_n \circ \tilde{\Gamma}$. Thus $\tilde{\Gamma}$ is a Jordan curve. We shall define a new manifold M as follows. Let $r_0 > r_1$ be chosen with $b^2r_0^2 < \pi^2$. Then $\exp_n$ has full rank on $B_{r_0}(0) \subset N_n$. Let M be $B_{r_0}(0)$ with the riemannian structure which makes $\exp_n$ a local isometry, and denote $m = 0 \in N_n$. Then clearly $\exp_m$ is a diffeomorphism of $B_{r_0}(0) \subset M_m$ onto $M = B_{r_0}(m)$. Define $\tilde{H}: M \to R$ by $\tilde{H}(x) = H \circ \exp_n(x)$. We need to find $y_0 \in \mathscr{D}(\tilde{\Gamma}, \infty)$. We may assume x_0 is smooth in B. It is then possible to modify x_0 on some compact subdomain of B to a mapping x_1 which is smooth in B and describes an r_0-contraction of Γ to n such that x_1 is homotopic through r_0-contractions of Γ to $\exp_n(C(\tilde{\Gamma}))$. Lemma 6 may then be applied to find a lifting $y_0: \bar{B} \to N_n$ with $x_1 = \exp_n \circ y_0$. Thus $y_0 \in \mathscr{D}(\tilde{\Gamma}, \infty)$. Now apply Theorem 2 to the curve $\tilde{\Gamma}$ in the manifold M with the prescribed function $\tilde{H}$: this gives a mapping $y: \bar{B} \to M$. Define $z = \exp_n \circ y$. Then z has the required properties.

Remarks. 1) It is clear from the proof that weaker hypotheses will suffice: if Γ is r_1-contractible to a point $n \in N$, then we may replace the requirement that N be complete by the requirement that $\exp_n$ be defined on $\bar{B}_{r_1}(0) \subset N_n$, and require H to be defined only on $\bar{B}_{r_1}(n)$, of class $C^\alpha(\bar{B}_{r_1}(n))$, with

$\sup_{x \in \bar{B}_{r_1}(n)} |H(x)| \leq b \cot (br_1)$.

2) Observe that the solution mapping is homotopic to the particular r_1-contraction g employed in the proof, up to sign; that is, the two mappings represent the same element or inverse elements in $\pi_2(N, \Gamma)$. In fact either orientation could be specified for Γ so that $z \in [g]$ or $z \in [g]^{-1}$ could be obtained at will. Thus, we may obtain a solution z in any homotopy class in which Γ is r_1-contractible.

3) The author [4] has recently demonstrated that the solution mapping z is an immersion, that is, $\langle z_u, z_u \rangle = \langle z_v, z_v \rangle \neq 0$ in B.

4) By a result of Heinz [8], the restriction on h is the best possible for the case $b = 0$; it is reasonable to suppose that it continues to be sharp for other values of b.

5) The requirement that Γ be r_1-contractible may not be replaced by the condition of contractibility in conjunction with a general restriction on diameter. This may be seen by considering a flat three-torus T^3 of arbitrarily small diameter, letting Γ be the image of a plane circle of radius $> h^{-1}$ under the locally isometric covering map $E^3 \to T^3$. Using the result of [8], this problem has no solution with $H(x) \equiv h$.

7. Minimal surfaces

In the case $H \equiv 0$, we may ignore the volume term $W[z]$ entirely, and the restriction on the dimension of M is no longer necessary. The same considerations, with inessential modifications, now yield:

Theorem 4. *Let Γ be an r_1-contractible Jordan curve in a complete riemannian manifold N, of class C^3 and with sectional curvatures $K \leq K_0$. Assume there is a mapping $x_0\colon \bar{B} \to N$ which takes ∂B continuously and monotonically onto Γ, with $D[x_0] < \infty$. Suppose $4K_0 r_1^2 < \pi^2$. Then there is a minimal surface in N with a conformal representation $z \in C^2(B) \cap C^\circ(\bar{B})$ mapping ∂B homeomorphically onto Γ.*

This is a partial generalization of the theorem of Morrey [14].

Remarks. 1) If $\dim N > 3$, we make no claim that the solution mapping will be an immersion.

2) In [14], Morrey constructs an example to shed light on his hypothesis of homogeneous regularity. The example occurs in a manifold of negative sectional curvature; but in such a manifold Theorem 4 gives a minimal surface spanning every contractible rectifiable Jordan curve. Thus no example has yet come to light of a contractible rectifiable Jordan curve in a complete manifold which cannot be spanned by a minimal surface.

Bibliography

[1] R. Courant, *Dirichlet's principle, conformal mapping, and minimal surfaces,* Interscience, New York, 1950.

[2] J. Douglas, *Solution of the problem of Plateau,* Trans. Amer. Math. Soc. **33** (1931) 263–321.

[3] D. Gromoll, W. Klingenberg & W. Meyer, *Riemannsche Geometrie im Grossen,* Lecture Notes in Math. Vol. 55, Springer, Berlin, 1968.

[4] R. Gulliver, *Regularity of minimizing surfaces of prescribed mean curvature,* Ann. of Math. **97** (1973) 275–305.

[5] R. Gulliver & J. Spruck, *The Plateau problem for surfaces of prescribed mean curvature in a cylinder,* Invent. Math. **13** (1971) 169–178.

[6] P. Günther, *Einige Sätze über das Volumenelement eines Riemannschen Raumes,* Publ. Math. Debrecen **7** (1960) 78–93.

[7] E. Heinz, *Über die Existenz einer Fläche konstanter mittlerer Krümmung bei vorgegebener Berandung,* Math. Ann. **127** (1954) 258–287.

[8] ——, *On the non-existence of a surface of constant mean curvature with finite area and prescribed rectifiable boundary,* Arch. Rational Mech. Anal. **35** (1969) 249–252.

[9] ——, *Über das Randverhalten quasilinearer elliptischer Systeme mit isothermen Parametern,* Math. Z. **113** (1970) 99–105.

[10] S. Hildebrandt, *Randwertprobleme für Flächen mit vorgeschriebener mittlerer Krümmung und Andwendungen auf die Kapillaritätstheorie.* I, Math. Z. **112** (1969) 205–213.

[11] ——, *On the Plateau problem for surfaces of constant mean curvature,* Comm. Pure Appl. Math. **23** (1970) 97–114.

[12] H. Kaul, *Ein Einschliessungssatz für* H-*Flächen in Riemannschen Mannigfaltigkeiten,* Manuscripta Math. **5** (1971) 103–112.

[13] C. B. Morrey, Jr., *Multiple integral problems in the calculus of variations and related topics,* Univ. California Publ. Math., New series, Vol. 1, No. 1, 1943, 1-130.

[14] ——, *The problem of Plateau on a Riemannian manifold,* Ann. of Math. **49** (1948) 807–851.

[15] ——, *Multiple integrals in the calculus of variations,* Springer, New York, 1966.

[16] T. Radó, *The problem of the least area and the problem of Plateau,* Math. Z. **32** (1930) 763–796.

[17] H. E. Rauch, *A contribution to Riemannian geometry in the large,* Ann. of Math. **54** (1951) 38–55.

[18] F. Tomi, *Ein einfacher Beweis eines Regularitätssatzes für schwache Lösungen gewisser elliptischer Systeme,* Math. Z. **112** (1969) 214–218.

[19] H. Werner, *Das Problem von Douglas für Flächen konstanter mittlerer Krümmung,* Math. Ann. **133** (1957) 303–319.

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