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THE PLURI-GENERA OF SURFACE SINGULARITIES

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Abstract. We give a criterion, in terms of pluri-genera, for a normal surface singularity over the complex number field to be a simple elliptic or cusp singularity (resp. quotient singularity, log-canonical singularity).

Introduction. Let (X, x) be a normal n-dimensional isolated singularity over the complex number field $\boldsymbol{C}$ and $f: (M, A) \to (X, x)$ a resolution of the singularity (X, x) with the exceptional locus $A = f^{-1}(x)$. We say a resolution f to be good if A is a divisor with normal crossings. The geometric genus of the singularity (X, x) is defined by $p_g(X, x) = \dim_{\boldsymbol{C}}(R^{n-1} f_* \mathcal{O}_M)_x$. Watanabe [15] introduced pluri-genera $\{\delta_m(X, x)\}_{m \in \boldsymbol{N}}$ which carry more precise information of the singularity, where $\boldsymbol{N}$ is the set of positive integers. The pluri-genera $\{\delta_m(X, x)\}_{m \in \boldsymbol{N}}$ can be computed on a good resolution, and $\delta_1(X, x) = p_g(X, x)$.

In this paper, we work only on surface singularities, so "a singularity" always means a normal surface singularity over $\boldsymbol{C}$.

A singularity (X, x) is said to be rational (resp. elliptic) if $p_g(X, x) = 0$ (resp. 1). Watanabe [15] proved that a singularity (X, x) is a quotient singularity if and only if $\delta_m(X, x) = 0$ for all $m \in \boldsymbol{N}$. A singularity (X, x) is said to be purely elliptic if $\delta_m(X, x) = 1$ for all $m \in \boldsymbol{N}$. Ishii [6] proved that a singularity (X, x) is a purely elliptic singularity if and only if (X, x) is a cusp or a simple elliptic singularity, while (X, x) is a log-canonical singularity if and only if $\delta_m(X, x) \leq 1$ for all $m \in \boldsymbol{N}$.

We will show that a singularity (X, x) is a quotient singularity if and only if $\delta_m(X, x) = 0$ for $m = 4, 6$, while (X, x) is a purely elliptic singularity if and only if $\delta_m(X, x) = 1$ for $m = 1, 4, 6$. We also prove similar assertions for log-canonical singularities.

Our result is a partial answer to the following question: Can $\{\delta_m(X, x)\}_{m \in \boldsymbol{N}}$ be determined by $\{\delta_m(X, x)\}_{m \in N}$ for some finite subset N of $\boldsymbol{N}$?

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1. Preliminaries.

(1.1) Let (X, x) be a surface singularity and $f: (M, A) \to (X, x)$ a resolution of the singularity (X, x). Let $A = \bigcup_{i=1}^{k} A_i$ be the decomposition of the exceptional set A into

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irreducible components. A cycle D is an integral combination of the A_i, i.e., $D=\sum_{i=1}^{k} d_i A_i$ with $d_i \in \boldsymbol{Z}$, where $\boldsymbol{Z}$ is the set of rational integers. There exists a natural partial ordering for cycles by comparison of the coefficients. A cycle D is said to be positive if $D \geq 0$ and $D \neq 0$. For any two positive cycles V and W, there exists an exact sequence

$$0 \to \mathcal{O}_W \otimes_{\mathcal{O}_M} \mathcal{O}_M(-V) \to \mathcal{O}_{V+W} \to \mathcal{O}_V \to 0 . \tag{1.1.1}$$

A resolution $f: (M, A) \to (X, x)$ is called a minimal good resolution, if f is the smallest resolution for which A consists of non-singular curves interesecting among themselves transversally, with no three through one point. It is well known that there is a unique minimal good resolution. Let us assume that $f: (M, A) \to (X, x)$ is the minimal good resolution of the singularity (X, x). The weighted dual graph of (X, x) is the graph such that each vertex represents a component of A weighted by the self-intersection number, while each edge connecting the vertices corresponding to A_i and A_j, $i \neq j$, corresponds to the point $A_i \cap A_j$. Giving the weighted dual graph is equivalent to giving the information on the genera of the A_i's and the intersection matrix $(A_i \cdot A_j)$. A string S in A is a chain of smooth rational curves $A_1, \dots, A_n$ so that $A_i \cdot A_{i+1} = 1$ for $i=1, \dots, n-1$, and these account for all intersections in A among the A_i's, except that A_1 intersects exactly one other curve. The weighted dual graph of the singularity (X, x) is said to be star-shaped, if the divisor A is written as $A = A_0 + \sum S_j$, where A_0 is a curve and S_j are maximal strings. Then A_0 is called the central curve, and S_j are called branches.

(1.2) Let $f: (M, A) \to (X, x)$ be a resolution of a singularity (X, x), $\mathcal{F}$ a sheaf of $\mathcal{O}_M$-modules and D a divisor on M. We will use the following notation: $\mathcal{F}(D) = \mathcal{F} \otimes_{\mathcal{O}_M} \mathcal{O}_M(D)$, $H^i(\mathcal{F}) = H^i(M, \mathcal{F})$, $H^i_A(\mathcal{F}) = H^i_A(M, \mathcal{F})$, $h^i(\mathcal{F}) = \dim_{\boldsymbol{C}} H^i(\mathcal{F})$ and $h^i_A(\mathcal{F}) = \dim_{\boldsymbol{C}} H^i_A(\mathcal{F})$.

We denote by K the canonical divisor on M. The Riemann-Roch theorem implies, for any positive cycle V and any invertible sheaf $\mathcal{L}$ on M, that

$$\chi(\mathcal{O}_V) = h^0(\mathcal{O}_V) - h^1(\mathcal{O}_V) = -V \cdot (V+K)/2 ,$$

and

$$\chi(\mathcal{O}_V \otimes \mathcal{L}) = h^0(\mathcal{O}_V \otimes \mathcal{L}) - h^1(\mathcal{O}_V \otimes \mathcal{L}) = \mathcal{L} \cdot V + \chi(\mathcal{O}_V) .$$

DEFINITION 1.3. A positive cycle E is minimally elliptic if $\chi(\mathcal{O}_E) = 0$ and $\chi(\mathcal{O}_D) > 0$ for all cycles D such that $0 < D < E$.

(1.4) There is a unique fundamental cycle Z (cf. [2]) such that $Z > 0$, $A_i \cdot Z \leq 0$ for all i, and that Z is minimal with respect to these two properties. Note that $h^0(\mathcal{O}_Z) = 1$ (cf. [9]).

PROPOSITION 1.5 (Laufer [9, Theorem 3.4]). *Let $f: (M, A) \to (X, x)$ be the minimal resolution of the singularity (X, x), Z the fundamental cycle and K the canonical divisor on M. Then the following are equivalent.*

(1) *Z is a minimally elliptic cycle.*
(2) *$A_i \cdot Z = -A_i \cdot K$ for all A_i.*

DEFINITION 1.6. A singularity (X, x) is minimally elliptic if the minimal resolution $f: (M, A) \to (X, x)$ satisfies the conditions of Proposition 1.5.

THEOREM 1.7 (cf. [9, Theorem 3.10]). *A singularity (X, x) is minimally elliptic if and only if (X, x) is an elliptic Gorenstein singularity.*

(1.8) Let $f: (M, A) \to (X, x)$ be the minimal resolution of the singularity (X, x) and Z the fundamental cycle. By the natural surjective map $H^1(\mathcal{O}_M) \to H^1(\mathcal{O}_Z)$, we have $p_g(X, x) \geq h^1(\mathcal{O}_Z)$. Artin [2] proved that $p_g(X, x) = 0$ if and only if $h^1(\mathcal{O}_Z) = 0$. If $p_g(X, x) = 1$, then $h^1(\mathcal{O}_Z) = 1$, and there exists a unique minimally elliptic cycle E by [9, Proposition 3.1]. The support of E is the exceptional set of a minimally elliptic singularity by [9, Lemma 3.3].

(1.9) We take the following characterization of du Bois singularities as its definition.

PROPOSITION 1.10 (Steenbrink [13, (3.6)]). *A normal surface singularity (X, x) is a du Bois singularity if and only if the natural map $H^1(\mathcal{O}_M) \to H^1(\mathcal{O}_A)$ is an isomorphism, where $f: (M, A) \to (X, x)$ is a good resolution.*

THEOREM 1.11 (Ishii [3, Theorem 2.3]). *Every resolution of a du Bois singularity is a good resolution.*

2. The pluri-genera.

(2.1) Let (X, x) be a singularity and $f: (M, A) \to (X, x)$ a resolution. We denote by K the canonical divisor on M, and set $U = X - \{x\} \cong M - A$.

DEFINITION 2.2 (Watanabe [15]). We define the pluri-genera $\{\delta_m(X, x)\}_{m \in \boldsymbol{N}}$ by

$$\delta_m(X, x) = \dim_{\boldsymbol{C}} H^0(\mathcal{O}_U(mK_X))/L^{2/m}(U),$$

where $L^{2/m}(U)$ denotes the set of all $L^{2/m}$-integrable m-ple holomorphic 2-forms on U.

PROPOSITION 2.3 (cf. [15, p. 67]). *If $f: (M, A) \to (X, x)$ is a good resolution, then $\delta_m(X, x)$ is expressed as*

$$\delta_m(X, x) = \dim_{\boldsymbol{C}} H^0(\mathcal{O}_U(mK))/H^0(\mathcal{O}_M(mK + (m-1)A)).$$

THEOREM 2.4 (cf. [15, Theorem 2.8]). *Let A' be a connected proper subvariety of A, and (X', x') the singularity obtained by contracting A' in M. Then $\delta_m(X, x) \geq \delta_m(X', x')$ for all $m \in \boldsymbol{N}$.*

THEOREM 2.5 (Ishii [5]). *Let $\pi: \bar{X} \to (\boldsymbol{C}, 0)$ be a small deformation of a singularity $(X, x) = \pi^{-1}(0)$. Let $Y = \pi^{-1}(c)$, with $c \in \boldsymbol{C}$ near 0, and $\{y_i\}$ the set of singular points of Y. Then*

$$\delta_m(X, x) \geq \sum \delta_m(Y, y_i) .$$

THEOREM 2.6 (Kato [8, p. 246]). *Let $\mathscr{L}$ be an invertible sheaf on M. If $\mathscr{L} \cdot A_i \geq K \cdot A_i$ for all i, then $H^1(\mathscr{L})=0$.*

LEMMA 2.7. *If $f: (M, A) \to (X, x)$ is minimal, i.e., $K \cdot A_i \geq 0$ for all i, and if (X, x) is not a rational double point, then $H^1(\mathcal{O}_M(nK+A))=0$ for $n \geq 2$.*

PROOF. There exists an exact sequence

$$0 \to \mathcal{O}_M(nK) \to \mathcal{O}_M(nK+A) \to \mathcal{O}_A(nK+A) \to 0 .$$

By Theorem 2.6, $H^1(\mathcal{O}_M(nK))=0$, and hence $H^1(\mathcal{O}_M(nK+A)) \cong H^1(\mathcal{O}_A(nK+A))$. By the Serre duality, $h^1(\mathcal{O}_A(nK+A))=h^0(\mathcal{O}_A((1-n)K))$. We will show that $H^0(\mathcal{O}_A(-nK))=0$ for $n \geq 1$. Since (X, x) is not a rational double point, we may assume that $K \cdot A_1 > 0$. Let $\{Z_j\}_{j=0,1,\dots,k}$ be a computation sequence for A: $Z_0=0$, $Z_1=A_1=A_{i_1}, \dots, Z_j= Z_{j-1}+A_{i_j}, \dots, Z_k=Z_{k-1}+A_{i_k}=A$, where $Z_{j-1} \cdot A_{i_j} > 0$ for $j=2, \dots, k$. For $j=1, \dots, k$, $H^0(\mathcal{O}_{A_{i_j}}(-nK-Z_{j-1}))=0$, since $(-nK-Z_{j-1}) \cdot A_{i_j} < 0$. From the exact sequences (cf. (1.1.1))

$$0 \to \mathcal{O}_{A_{i_j}}(-nK-Z_{j-1}) \to \mathcal{O}_{Z_j}(-nK) \to \mathcal{O}_{Z_{j-1}}(-nK) \to 0 ,$$

we inductively see that $H^0(\mathcal{O}_{Z_j}(-nK))=0$ for $j=1, \dots, k$. In particular, $H^0(\mathcal{O}_A(-nK))=0$. □

THEOREM 2.8. *Let (X, x) be a du Bois singularity, and $f: (M, A) \to (X, x)$ the minimal resolution of the singularity (X, x). Then*

$$\delta_2(X, x)=h^1_A(\mathcal{O}_M(2K+A))=h^1(\mathcal{O}_M(-K-A)) .$$

PROOF. By the Serre duality, $h^1_A(\mathcal{O}_M(2K+A))=h^1(\mathcal{O}_M(-K-A))$. We assume that (X, x) is not a rational double point. By Lemma 2.7, there exists an exact sequence

$$0 \to H^0(\mathcal{O}_M(2K+A)) \to H^0(\mathcal{O}_U(2K)) \to H^1_A(\mathcal{O}_M(2K+A)) \to 0 .$$

From Theorem 1.11 and Proposition 2.3, $\delta_2(X, x)=h^1_A(\mathcal{O}_M(2K+A))$.

Let (X, x) be a rational double point. Then $K=0$ and $H^1(\mathcal{O}_M(-A))=0$. Hence $H^1(\mathcal{O}_M(-K-A))=0$. Since (X, x) is a quotient singularity (see Theorem 2.11), $\delta_2(X, x)= 0$. □

COROLLARY 2.9. *In the situation above, let V be a positive cycle. Then*

$$\delta_2(X, x) \geq V \cdot (K+A) - \chi(\mathcal{O}_V) .$$

PROOF. Theorem 2.8 implies that

$$\delta_2(X, x) \geq h^1(\mathcal{O}_V(-K-A)) \geq -\chi(\mathcal{O}_V(-K-A)) = V \cdot (K+A) - \chi(\mathcal{O}_V) .$$

□

DEFINITION 2.10. A singularity (X, x) is called a $\boldsymbol{Q}$-Gorenstein singularity if there exists a positive integer r such that $\mathcal{O}_X(rK_X)$ is invertible at x. It is well known that any rational singularity is a $\boldsymbol{Q}$-Gorenstein singularity. For a $\boldsymbol{Q}$-Gorenstein singularity (X, x), the minimal positive integer r which satisfies the condition above is called the index of (X, x), and denoted by $I(X, x)$.

For any singularity (X, x), the minimal positive integer m such that $\delta_m(X, x) \neq 0$ is called the δ-index of (X, x), and denoted by $I_\delta(X, x)$. If $\delta_m(X, x)=0$ for all $m \in \boldsymbol{N}$, we set $I_\delta(X, x)=\infty$.

THEOREM 2.11 (cf. [15, Theorem 3.9]). *A singularity (X, x) is a quotient singularity if and only if $I_\delta(X, x)=\infty$.*

THEOREM 2.12 (cf. [6]). *Let (X, x) be a singularity such that $\{\delta_m(X, x)\}_{m \in \boldsymbol{N}}$ is bounded, i.e., there exists an integer B such that $\delta_m(X, x) \leq B$ for all $m \in \boldsymbol{N}$. Assume that (X, x) is not a quotient singularity. Then (X, x) is a $\boldsymbol{Q}$-Gorenstein singularity with $I(X, x)=I_\delta(X, x)$, and $\delta_m(X, x) \leq 1$ for all $m \in \boldsymbol{N}$. Let $I=I(X, x)$. Then we have the following*:

(1) *$\delta_m(X, x)=1$ for $m \equiv 0 \pmod{I}$ and $\delta_m(X, x)=0$ for $m \not\equiv 0 \pmod{I}$.*

(2) *$I=1$ if and only if (X, x) is a simple elliptic or a cusp singularity.*

(3) *If $I>1$, then (X, x) is the quotient with respect to a cyclic group of a simple elliptic or a cusp singularity.*

(2.13) A $\boldsymbol{Q}$-Gorenstein singularity (X, x) is said to be log-canonical if the following condition is satisfied: For a good resolution $f\colon (M, A) \to (X, x)$, we have, as $\boldsymbol{Q}$-divisor,

$$K_M = f^*K_X + \sum a_i A_i \quad \text{with} \quad a_i \geq -1 \quad \text{for all} \quad i\,.$$

The singularities in Theorem 2.12 are log-canonical by [4, Theorem 2.1].

(2.14) A singularity with $\boldsymbol{C}^*$-action is called a $\boldsymbol{C}^*$-singularity.

Let (X, x) be a $\boldsymbol{C}^*$-singularity and $f\colon (M, A) \to (X, x)$ the minimal good resolution. It is well known that the weighted dual graph of (X, x) is a star-shaped graph. The weighted dual graph of a cyclic quotient singularity is regarded as a start-shaped graph without central curve (note that it is a chain of rational curves).

We set $A = A_0 + \sum_{i=1}^{\beta} S_i$, where A_0 is the central curve, and S_i the branches. The curves of S_i are denoted by $A_{i,j}$, $1 \leq j \leq r_i$, where $A_0 \cdot A_{i,1} = A_{i,j} \cdot A_{i,j+1} = 1$. Let $b_{i,j} = -A_{i,j} \cdot A_{i,j}$. For each branch S_i, positive integers e_i and d_i are defined by

$$\frac{d_i}{e_i} = b_{i,1} - \cfrac{1}{b_{i,2} - \cfrac{1}{\cdots - \cfrac{1}{b_{i,r_i}}}}$$

where $e_i < d_i$, and e_i and d_i are relatively prime.

For any integers $m \geq 1$ and $k \geq 0$, we define the divisors on A_0 by

$$D_m^{(k)}=kD-\sum_{i=1}^{\beta}[(ke_i+m(d_i-1))/d_i]P_i\,,$$

where D is a divisor such that $\mathcal{O}_{A_0}(D)$ is the conormal sheaf of A_0, $P_i=A_0\cap A_{i,1}$, and for any $a\in\boldsymbol{R}$, $[a]$ is the greatest integer not more than a.

The following is an extended version of Pinkham's formula (cf. [12, Theorem 5.7]).

THEOREM 2.15 (Watanabe [16, Corollary 2.22]). *In the situation above,*

$$\delta_m(X,x)=\sum_{k\geq 0}h^0(\mathcal{O}_{A_0}(mK_{A_0}-D_m^{(k)}))\,.$$

THEOREM 2.16 (Tomaru [14]). *In the situation above, let g be the genus of the central curve A_0.*

(1) *(X,x) is a log-canonical singularity with $I(X,x)>1$ if and only if $g=0$ and $\sum_{i=1}^{\beta}(d_i-1)/d_i=2$. In this case, $I(X,x)=\mathrm{lcm}(d_1,\dots,d_\beta)$.*

(2) *(X,x) is a quotient singularity if and only if $g=0$ and $\sum_{i=1}^{\beta}(d_i-1)/d_i<2$.*

3. Rational singularities.

(3.1) Let (X,x) be a rational singularity and $f\colon(M,A)\to(X,x)$ the minimal resolution of the singularity (X,x). Since $H^1(\mathcal{O}_M)=H^1(\mathcal{O}_A)=0$, f is a minimal good resolution by Proposition 1.10 and Theorem 1.11. Note that the weighted dual graph of a rational singularity is a tree. For any component A_i of A, we set $t_i=(A-A_i)\cdot A_i$, the cardinality of the intersection points on A_i.

In this section, except in Corollary 3.6, (X,x) denotes a rational singularity and $f\colon(M,A)\to(X,x)$ the minimal resolution.

LEMMA 3.2. *If the weighted dual graph of (X,x) is a star-shaped graph, then*

$$\delta_m(X,x)=\sum_{k\geq 0}h^0(\mathcal{O}_{A_0}(mK_{A_0}-D_m^{(k)}))\,,$$

where A_0 and $D_m^{(k)}$ are as in (2.14).

PROOF. By the Riemann-Roch theorem of [10, p. 196], $\delta_m(X,x)+h^1(\mathcal{O}_M(mK+(m-1)A))$ is determined by the weighted dual graph. Let $L_m=mK+(m-1)A$. From the exact sequence

$$0\to\mathcal{O}_M(mK)\to\mathcal{O}_M(L_m)\to\mathcal{O}_{(m-1)A}(L_m)\to 0\,,$$

using Theorem 2.6, we have $h^1(\mathcal{O}_M(L_m))=h^1(\mathcal{O}_{(m-1)A}(L_m))$. Since $H^1(\mathcal{O}_M)=0$, we have $H^1(\mathcal{O}_{(m-1)A})=0$. By [1, (1.7)], invertible sheaves on $(m-1)A$ are classified by their degree. Thus $h^1(\mathcal{O}_{(m-1)A}(L_m))$ is determined by the weighted dual graph and the variety A, hence so is $\delta_m(X,x)$.

Let A_0, D, $D_m^{(k)}$, P_i, e_i and d_i be as in (2.14) (note that they are defined for star-shaped graphs). For any $k\geq 0$, let $D^{(k)}$ be the divisor

$$D^{(k)}=kD-\sum_{i=1}^{\beta}\{ke_i/d_i\}P_i$$

on A_0, where for any $a\in\boldsymbol{R}$, $\{a\}$ denotes the least integer not less than a. Let $R=\bigoplus_{k\geq 0}H^0(\mathcal{O}_{A_0}(D^{(k)}))$. By [12], Spec$(R)$ is a singularity of which the exceptional set of the minimal good resolution and the weighted dual graph are the same as those of (X, x). Then $\delta_m(X, x)=\delta_m(\mathrm{Spec}(R))$. Since Spec$(R)$ is a $\boldsymbol{C}^*$-singularity, $\delta_m(\mathrm{Spec}(R))$ is computed by the formula in Theorem 2.15. □

(3.3) Let (X, x) be a rational singularity with a star-shaped graph. Then the central curve is a non-singular rational curve. Using the notation of (2.14), we set

$$F_m^{(k)}=-2m-kb+\sum_{i=1}^{\beta}[(ke_i+m(d_i-1))/d_i]\,,$$

where $b=-A_0\cdot A_0$. By Lemma 3.2,

$$\delta_m(X, x)=\sum_{k\geq 0}h^0(\mathcal{O}_{A_0}(F_m^{(k)}))\,.$$

We always assume that $d_1\leq\cdots\leq d_\beta$.

LEMMA 3.4. *If $\delta_2(X, x)=0$, then the weighted dual graph of (X, x) is either a chain (if (X, x) is a cyclic quotient singularity), or a star-shaped graph with three branches.*

PROOF. For any component A_i of A, we have $t_i\leq 3$ by Corollary 2.9. If $t_i\leq 2$ for all i, then A is a chain of curves.

We assume that $t_1=3$. Let A_n be any component of A. Let $\sum_{i=1}^{n}A_i$ be the minimal connected cycle containing A_1 and A_n. Then $t_i\geq 2$ for $i\leq n-1$. Applying Corollary 2.9 to the positive cycle $\sum_{i=1}^{n-1}A_i$, we have $0\geq\sum_{i=2}^{n-1}(t_i-2)$. Hence $t_i=2$ for $i=2,\ldots,n-1$. □

THEOREM 3.5 (Okuma [11]). *If $\delta_m(X, x)=0$ for $m=4, 6$, then (X, x) is a quotient singularity.*

PROOF. Note that the assumption implies $\delta_m(X, x)=0$ for $m=1, 2$ (cf. Proposition 2.3). We assume that (X, x) is not a cyclic quotient singularity. By Lemma 3.4, the weighted dual graph of (X, x) is a star-shaped graph with three branches. Then

$$F_4^{(0)}=-8+\sum_{i=1}^{3}[4-4/d_i]\quad\text{and}\quad F_6^{(0)}=-12+\sum_{i=1}^{3}[6-6/d_i]\,.$$

Note that $[m-m/a_1]\leq[m-m/a_2]$ if $a_1\leq a_2$.

Since $\delta_6(X, x)=0$, we have $F_6^{(0)}\leq -1$. If $d_1\geq 3$, then $F_6^{(0)}\geq 0$. Hence $d_1=2$. Since $\delta_4(X, x)=0$, we have $F_4^{(0)}=-6+[4-4/d_2]+[4-4/d_3]\leq -1$. Thus $d_2\leq 3$.

If $d_1=d_2=2$, then $\sum_{i=1}^{3}(d_i-1)/d_i<2$, and hence (X, x) is a quotient singularity by Theorem 2.16.

Assume $d_2=3$. Since $F_6^{(0)}=-5+[6-6/d_3]\leq -1$, we have $d_3\leq 5$. Again, we get $\sum_{i=1}^3 (d_i-1)/d_i<2$, and hence (X,x) is a quotient singularity by Theorem 2.16. □

COROLLARY 3.6. *Let (X,x) be any singularity. If (X,x) is not a quotient singularity, then $I_\delta(X,x)\leq 6$.*

PROOF. The result is an immediate consequence of Theorems 2.11 and 3.5. □

PROPOSITION 3.7. *Let (X,x) be a singularity with $I_\delta(X,x)=6$ and $\delta_{14}(X,x)=0$. Then (X,x) is a log-canonical singularity with $I(X,x)=6$.*

PROOF. By assumption, $\delta_m(X,x)=0$ for $m=1,2,3,4,5$. By Lemma 3.4, (X,x) has a star-shaped graph with three branches. Since $\delta_3(X,x)=0$, we have $F_3^{(0)}=-6+\sum_{i=1}^3[3-3/d_i]\leq -1$. Thus $d_1=2$. Similarly, we have $d_2\leq 3$ by $d_1=2$ and $F_4^{(0)}\leq -1$. If $d_2=2$ or $d_3\leq 5$, then $I_\delta(X,x)=\infty$ by the proof of Theorem 3.5. Hence we get $d_1=2$, $d_2=3$ and $d_3\geq 6$. Since $\delta_{14}(X,x)=0$, we have $F_{14}^{(0)}=-12+[14-14/d_3]\leq -1$. Thus $d_3=6$. By Theorem 2.16, (X,x) is a log-canonical singularity with $I(X,x)=6$. □

(3.8) We note that if $I_\delta(X,x)=5$, then (X,x) is not a log-canonical singularity by Theorems 2.12 and 2.16 (cf. Theorem 3.11).

PROPOSITION 3.9. *Let (X,x) be a singularity with $I_\delta(X,x)=4$ and $\delta_{14}(X,x)=0$. Then (X,x) is a log-canonical singularity with $I(X,x)=4$.*

PROOF. As in the proof of the proposition above, we have $d_1=2$ and $d_2\geq 3$. However, $d_2=3$ implies the same result as in the proposition above. Hence $d_2\geq 4$. Then $d_2=d_3=4$ by $F_{14}^{(0)}\leq -1$. By Theorem 2.16, (X,x) is a log-canonical singularity with $I(X,x)=4$. □

PROPOSITION 3.10. *Let (X,x) be a singularity with $I_\delta(X,x)=3$ and $\delta_{14}(X,x)=0$. Then (X,x) is a log-canonical singularity with $I(X,x)=3$.*

PROOF. If $d_1=2$, we have the same result as in the proposition above. Hence $d_1\geq 3$. Then $d_1=d_2=d_3=3$ by $F_{14}^{(0)}\leq -1$. Again by Theorem 2.16, (X,x) is a log-canonical singularity with $I(X,x)=3$. □

THEOREM 3.11. *Let (X,x) be a singularity with $\delta_{14}(X,x)=0$. Then (X,x) is a log-canonical singularity.*

PROOF. Since $\delta_{14}(X,x)=0$, we have $\delta_1(X,x)=\delta_2(X,x)=0$, and hence $I_\delta(X,x)\geq 3$.

If $I_\delta(X,x)=\infty$, then (X,x) is a quotient singularity, and it is log-canonical (more precisely, log-terminal). Assume that $I_\delta(X,x)\leq 6$ (cf. Corollary 3.6). If $I_\delta(X,x)\neq 5$, then we are done. By the proof of the propositions above, there exists no singularity (X,x) with $I_\delta(X,x)=5$ and $\delta_{14}(X,x)=0$. □

LEMMA 3.12. *Let (X,x) be a singularity with $\delta_2(X,x)=1$. Then we have one of the following:*

(1) *(X, x) has a star-shaped graph with three branches.*

(2) *(X, x) has a star-shaped graph with four branches.*

(3) *The exceptional divisor A is written as $\sum_{i=0}^{4} S_i$, where S_i, $i \geq 1$, are the maximal strings, and S_0 is a chain of curves.*

PROOF. By Corollary 2.9, we have $t_i \leq 4$ for all A_i. Since (X, x) is not a cyclic quotient singularity, there exists a component A_j such that $t_j \geq 3$. Assume that (X, x) is not in the case (1). If $t_1 = 4$, then as in the proof of Lemma 3.4, we have a star-shaped graph with four branches. If $t_i \leq 3$ for all A_i, then we may assume that $t_1 = t_2 = 3$. Then, as in the proof of Lemma 3.4, we have $t_i \leq 2$ for $i \geq 3$. Thus $A - A_1 - A_2$ is a disjoint union of chains of curves. Since the weighted dual graph is a tree, there exists a unique minimal connected cycle S_0 containing A_1 and A_2. Since $t_1 = t_2 = 3$, a cycle $A - S_0$ is a disjoint union of four maximal strings in A. □

LEMMA 3.13. *Let (X, x) be a singularity with $\delta_{14}(X, x) = 1$. If (X, x) has a star-shaped graph with three branches, then $\delta_2(X, x) = 0$.*

PROOF. Assume that (X, x) has a star-shaped graph with three branches. Using the notation of (3.3), we have

$$F_m^{(k)} = m - kb + \sum_{i=1}^{3} [(ke_i - m)/d_i] .$$

If $b \geq 3$, then $F_2^{(k)} \leq F_2^{(k-1)} \leq \cdots \leq F_2^{(0)} < 0$, and hence $\delta_2(X, x) = 0$. If $\sum 1/d_i \geq 1$, then $\delta_2(X, x) = 0$ by Theorem 2.16. Assume that $b = 2$ and $\sum 1/d_i < 1$. We define a subset Δ^* of N^6 as follows: $(e, d) = (e_1, e_2, e_3, d_1, d_2, d_3) \in N^6$ is an element of Δ^* if and only if $d_1 \leq d_2 \leq d_3$, $\sum 1/d_i < 1$, $\sum e_i/d_i < 2$ (cf. [12, p. 185]), $e_i < d_i$, and e_i and d_i are relatively prime for $i = 1, 2, 3$. We regard $F_m^{(k)}$ as a function of k, m and $(e, d) \in \Delta^*$, and write $F_m^{(k)}(e, d)$. Let

$$G^{(k)}(e, d) = k\left(\sum e_i/d_i - 2\right) + 2\left(1 - \sum 1/d_i\right) .$$

Then

$$F_2^{(k)}(e, d) \leq 2 - 2k + \sum (ke_i - 2)/d_i = G^{(k)}(e, d) .$$

Since $\sum e_i/d_i - 2 < 0$, we have $F_2^{(k)}(e, d) < 0$ for $k \geq 2$ (resp. $k \geq 3$) if $G^{(2)}(e, d) < 0$ (resp. $= 0$).

Let

$$\Delta = \{d \in N^3 \mid (e, d) \in \Delta^* \text{ for some } e \in N^3, \text{ and } F_{14}^{(0)} \leq 0\} .$$

Let $\Delta_1 = \{(2, 3, d_3) \mid 7 \leq d_3 \leq 13\}$ and $\Delta_2 = \{(2, 4, 5), (2, 4, 6)\}$. As in the proof of the propositions above, we have $\Delta = \Delta_1 \cup \Delta_2 \cup \{(3, 4, 4)\}$.

Assume that $d \in \Delta_1$. Since $\delta_{14}(X, x) = 1$ and $F_{14}^{(0)} = 0$, we have

$$F_{14}^{(3)} = -3 + e_2 + [(3e_3 - 14)/d_3] \leq -1 .$$

Let $\Delta_1' = \{(e, d) \in \Delta^* \mid d \in \Delta_1, F_{14}^{(3)} \leq -1\}$. We can easily get $F_2^{(k)}(e, d) < 0$ for $(e, d) \in \Delta_1'$ and $k = 0, 1, 2$. We will show $G^{(2)}(e, d) = 2(\sum (e_i - 1)/d_i - 1) \leq 0$ for $(e, d) \in \Delta_1'$. For $(e, d) \in \Delta_1'$ with $e_2 = 1$, we have $G^{(2)}(e, d) = 2((e_3 - 1)/d_3 - 1) < 0$. Let $e_2 = 2$. Then $3e_3 - 14 < d_3$, and $e_3/d_3 < 5/6$. The maximum of $\{(e_3 - 1)/d_3\}$ is $(7-1)/9 = 2/3$. Hence $G^{(2)}(e, d) = 2((e_3 - 1)/d_3 - 2/3) \leq 0$. Then we have $F_2^{(k)} < 0$, for $k \geq 0$ and $(e, d) \in \Delta_1'$.

Assume that $d \in \Delta_2$. If $e_2 = 1$, then $G^{(2)}(e, d) = 2((e_3 - 1)/d_3 - 1) < 0$. Let $e_2 = 3$. As above, we have $e_3 + d_3 < 7$ from $F_{14}^{(2)} \leq -1$. Hence $e_3 = 1$. Then $G^{(2)}(e, d) = 2(1/2 - 1) < 0$. Clearly, $F_2^{(0)}$ and $F_2^{(1)}$ are negative. Hence $F_2^{(k)} < 0$ for $k \geq 0$.

If $d = (3, 3, 4)$, then $e = (e_1, e_2, e_3)$ $(e_1 \leq e_2)$ such that $(e, d) \in \Delta^*$ is one of $(1, 1, 1)$, $(1, 1, 3)$, $(1, 2, 1)$, $(1, 2, 3)$ and $(2, 2, 1)$. Again, we have $F_2^{(k)} < 0$ for $k \geq 0$.

Thus in any of the cases, we get $\delta_2(X, x) = 0$. □

PROPOSITION 3.14. *Let (X, x) be a singularity with $I_\delta(X, x) = 2$ and $\delta_{14}(X, x) = 1$. Then (X, x) is a log-canonical singularity with $I(X, x) = 2$.*

PROOF. Since $\delta_{14}(X, x) = 1$ and $\delta_2(X, x) \neq 0$, we have $\delta_2(X, x) = 1$ (cf. Proposition 2.3). By the lemmas above, we have the weighted dual graph in (2) or (3) of Lemma 3.12.

Suppose (X, x) has a star-shaped graph. Then $d_1 = \cdots = d_4 = 2$ by $F_{14}^{(0)} \leq 0$, and hence (X, x) is a log-canonical singularity with $I(X, x) = 2$ by Theorem 2.16.

Assume that $A = \sum_{i=0}^{4} S_i$ as in (3) of Lemma 3.12. By [7, Theorem 3.7], there exists a deformation $\pi: \bar{M} \to (\boldsymbol{C}, 0)$ of $M = \pi^{-1}(0)$ which induces a trivial deformation of S_i for $i = 1, 2, 3, 4$, and for $c \neq 0$ near 0, $\pi^{-1}(c)$ has a connected component of the exceptional set $A_0 + \sum_{i=1}^{4} S_i$, where A_0 is a rational curve. Note that π blows down to a deformation of (X, x). Let (Y, y) be a singularity obtained by contracting the exceptional divisor $A_0 + \sum_{i=1}^{4} S_i$ above. By Theorem 2.5, we have $p_g(Y, y) = 0$, $\delta_2(Y, y) \leq 1$ and $\delta_{14}(Y, y) \leq 1$. Thus (Y, y) is a rational singularity which has a star-shaped graph with four branches. By Lemma 3.4, we have $\delta_2(Y, y) = \delta_{14}(Y, y) = 1$. Applying the argument above to (Y, y), we have $d_1 = \cdots = d_4 = 2$. By the definition of d_i, we see that S_i is a curve with $S_i \cdot S_i = -2$, for $i \geq 1$. Recall that π induces a trivial deformation of S_i for $i \geq 1$. Let B be a cycle on M defined by $B = A + S_0$. Then $-B$ is numerically equivalent to $2K$. Since any rational singularity is a $\boldsymbol{Q}$-Gorenstein singularity, (X, x) is a log-canonical singularity with $I(X, x) = 2$ (cf. Theorem 2.12 and (2.13)). □

4. Elliptics singularities.

(4.1) Let (X, x) be an elliptic singularity, $f: (M, A) \to (X, x)$ a resolution of the singularity (X, x) and K the canonical divisor on M.

LEMMA 4.2. *Let (X, x) be a Gorenstein singularity. Then $\delta_{m_1}(X, x) \leq \delta_{m_2}(X, x)$ if $m_1 \leq m_2$.*

PROOF. Let $f: (M, A) \to (X, x)$ be the minimal good resolution of the singularity (X, x). It is well known that there exists a positive cycle $D \geq A$ such that $\mathcal{O}_M(K) \cong \mathcal{O}_M(-D)$.

Then $H^0(\mathcal{O}_{M-A}(mK)) \cong H^0(\mathcal{O}_M)$ and $O_M(mK+(m-1)A) \cong \mathcal{O}_M((m-1)(A-D)+K)$. Since $A-D \leq 0$, we have

$$\mathcal{O}_M((m_1-1)(A-D)+K) \supset \mathcal{O}_M((m_2-1)(A-D)+K)$$

for $m_1 \leq m_2$. Thus Proposition 2.3 implies the assertion. □

LEMMA 4.3. *Let (X, x) be a minimally elliptic singularity which is not a du Bois singularity. Then $\delta_6(X, x) \geq 2$.*

PROOF. First, we assume that the minimal resolution of the singularity (X, x) is a good resolution. Let f: $(M, A) \to (X, x)$ be the minimal resolution. By Lemma 2.7, we have $H^1(\mathcal{O}_M(2K+A))=0$. By Proposition 2.3 and [8, Corollary 1], we have

$$\delta_2(X, x) = -(K+A) \cdot (2K+A)/2 + 1 .$$

Since (X, x) is not a du Bois singularity, we have $H^1(\mathcal{O}_A)=0$, and hence $-A \cdot (A+K)/2 = \chi(\mathcal{O}_A) = 1$. Then we have $\delta_2(X, x) = -(K+A) \cdot K + 2$. Since f is minimal and $-(K+A) \geq 0$, we get $\delta_2(X, x) \geq 2$. By Lemma 4.2, we have $\delta_6(X, x) \geq 2$.

Now we assume that the minimal resolution of (X, x) is not good. Let $f: (M, A) \to (X, x)$ be the minimal good resolution of the singularity (X, x). By [9, Proposition 3.5], (X, x) has a star-shaped graph with three branches, and the divisor A can be written as $A = \sum_{i=1}^4 A_i$, where A_1 is the central curve with $A_1 \cdot A_1 = -1$, and $A_2 \cdot A_2 \geq A_3 \cdot A_3 \geq A_4 \cdot A_4$. Then $-K = 2A_1 + \sum_{i=2}^4 A_i$. Let $Z = \sum_{i=1}^4 n_i A_i$ be the fundamental cycle on M. Then $(n_1, \ldots, n_4)$ is one of $(6, 3, 2, 1)$, $(4, 2, 1, 1)$ or $(3, 1, 1, 1)$. Let $\mathcal{M}$ be the maximal ideal in $\mathcal{O}_X$ which defines the singular point x. By [9, Theorem 3.13], there exists a function $g \in H^0(\mathcal{M})$ (under the assumption that X is sufficiently small) such that $f^*(g)$ has a zero of order n_1 on A_1. Since (X, x) is minimally elliptic, we have $f_*\mathcal{O}_M(K) \cong \mathcal{M}$. On the other hand, we have

$$\mathcal{O}_M(6K+5A) \cong \mathcal{O}_M(K-5A) \cong \mathcal{O}_M\left(-7A_1 - \sum_{i=2}^4 A_i\right).$$

Hence

$$f^*(g) \in H^0(\mathcal{O}_M(K)) \setminus H^0(\mathcal{O}_M(6K+5A)) .$$

Since $H^0(\mathcal{O}_M) \supsetneqq H^0(\mathcal{O}_M(K)) \supsetneqq H^0(\mathcal{O}_M(6K+5A))$, we have $\delta_6(X, x) \geq 2$ by Proposition 2.3. □

PROPOSITION 4.4. *Let (X, x) be an elliptic singularity which is not a du Bois singularity. Then $\delta_6(X, x) \geq 2$.*

PROOF. (1.8), Theorem 2.4 and Lemma 4.3 imply the assertion. □

EXAMPLE 4.5. There exists a singularity (X, x) with $\delta_m(X, x) = 1$ for $m = 1, \ldots, 5$ which is not a du Bois singularity, but a minimally elliptic singularity.

Let (X, x) be a minimally elliptic singularity such that the minimal resolution of (X, x) is not good. Using the notation in the proof of Lemma 4.3, we assume that $A_2 \cdot A_2 = -2$, $A_3 \cdot A_3 = -3$ and $A_4 \cdot A_4 \leq -7$. Then $Z = 6A_1 + 3A_2 + 2A_3 + A_4 = -K + 4A_1 + 2A_2 + A_3$. Note that there exists such a minimally elliptic singularity. Since $Z > A$, we have $H^1(\mathcal{O}_A) = 0$ (cf. Definition 1.3). Thus (X, x) is not a du Bois singularity by Proposition 1.10. As in the proof of Lemma 4.3, we have

$$\begin{aligned}\delta_5(X, x) &= \dim_{\mathbf{C}} H^0(\mathcal{O}_M)/H^0(\mathcal{O}_M(K)) + \dim_{\mathbf{C}} H^0(\mathcal{O}_M(K))/H^0(\mathcal{O}_M(5K+4A)) \\ &= 1 + \dim_{\mathbf{C}} H^0(\mathcal{O}_M(K))/H^0(\mathcal{O}_M(K-4A_1)) .\end{aligned}$$

From the exact sequence

$$0 \to \mathcal{O}_M(K-4A_1) \to \mathcal{O}_M(K) \to \mathcal{O}_{4A_1}(K) \to 0 ,$$

we have

$$\dim_{\mathbf{C}} H^0(\mathcal{O}_M(K))/H^0(\mathcal{O}_M(K-4A_1)) = 6 - h^1(\mathcal{O}_M(K-4A_1)) .$$

We will show that $h^1(\mathcal{O}_M(K-4A_1)) = 6$. Since $H^1(\mathcal{O}_M) \cong H^1(\mathcal{O}_Z)$, we have $H^1(\mathcal{O}_M(-Z)) = 0$. From the exact sequence

$$0 \to \mathcal{O}_M(-Z) \to \mathcal{O}_M(K-4A_1) \to \mathcal{O}_{2A_2+A_3}(K-4A_1) \to 0 ,$$

we have $H^1(\mathcal{O}_M(K-4A_1)) \cong H^1(\mathcal{O}_{2A_2+A_3}(K-4A_1))$. Let $L = K - 4A_1$. Consider the exact sequences

$$0 \to \mathcal{O}_{2A_2}(L-A_3) \to \mathcal{O}_{2A_2+A_3}(L) \to \mathcal{O}_{A_3}(L) \to 0 ,$$

$$0 \to \mathcal{O}_{A_2}(L-A_3-A_2) \to \mathcal{O}_{2A_2}(L-A_3) \to \mathcal{O}_{A_2}(L-A_3) \to 0 .$$

Then we get

$$\begin{aligned}h^1(\mathcal{O}_{2A_2+A_3}(K-4A_1)) &= h^1(\mathcal{O}_{A_3}(L)) + h^1(\mathcal{O}_{A_2}(L-A_3)) + h^1(\mathcal{O}_{A_2}(L-A_3-A_2)) \\ &= 2+3+1 = 6 .\end{aligned}$$

Hence $\delta_5(X, x) = 1$. By Lemma 4.2, $\delta_m(X, x) = 1$ for $m = 1, \ldots, 5$.

(4.6) Let (X, x) be an elliptic du Bois singularity and $f\colon (M, A) \to (X, x)$ the minimal resolution. Since $H^1(\mathcal{O}_A) = 1$, the divisor A is decomposed as $A = E_1 + E_2$, where E_1 is either a non-singular elliptic curve or a cycle of r rational curves with $r \geq 1$ (a cycle of one rational curve means a rational curve with an ordinary double point), and E_2 is void or a disjoint union of trees of non-singular rational curves. If $E_2 = 0$, then (X, x) is a simple elliptic or a cusp singularity.

We will use this notation in Lemma 4.7, Lemma 4.8 and Proposition 4.9 below.

LEMMA 4.7. *If E_2 is a rational curve with $E_2 \cdot E_2 \leq -3$, then $\delta_3(X, x) \geq 2$.*

PROOF. For any component A_i of A, we have $(2K + 2A - E_2) \cdot A_i \geq 0$. By Theorem 2.6, $H^1(\mathcal{O}_M(3K+2A)) \cong H^1(\mathcal{O}_{E_2}(3K+2A))$. Since $(3K+2A) \cdot E_2 = K \cdot E_2 - 2 \geq -1$, we have $H^1(\mathcal{O}_M(3K+2A))=0$. Let $L=3K+2A$. Then we get

$$0 \to H^0(\mathcal{O}_M(L)) \to H^0(\mathcal{O}_M(L+E_1)) \to H^0(\mathcal{O}_{E_1}(L+E_1)) \to 0 ,$$

and

$$\dim_{\mathbf{C}} H^0(\mathcal{O}_M(L+E_1))/H^0(\mathcal{O}_M(L)) = h^0(\mathcal{O}_{E_1}(L+E_1)) \geq \chi(\mathcal{O}_{E_1}(L+E_1)) = 2 .$$

Since

$$\delta_3(X, x) = \dim_{\mathbf{C}} H^0(\mathcal{O}_{M-A}(3K))/H^0(\mathcal{O}_M(L))$$

and

$$H^0(\mathcal{O}_{M-A}(3K)) \supset H^0(\mathcal{O}_M(L+E_1)) \supset H^0(\mathcal{O}_M(L)) ,$$

we have $\delta_3(X, x) \geq 2$. □

Lemma 4.8. *If E_2 is a rational curve with $E_2 \cdot E_2 = -2$, then $\delta_4(X, x) \geq 2$.*

Proof. As above, we have $H^1(\mathcal{O}_M(4K+3A)) \cong H^1(\mathcal{O}_{2E_2}(4K+3A))$. Let $L=4K+3A$. From the exact sequence

$$0 \to \mathcal{O}_{E_2}(L-E_2) \to \mathcal{O}_{2E_2}(L) \to \mathcal{O}_{E_2}(L) \to 0 ,$$

we have $h^1(\mathcal{O}_{2E_2}(L))=2$. Consider the exact sequence

$$0 \to \mathcal{O}_M(L) \to \mathcal{O}_M(L+E_1) \to \mathcal{O}_{E_1}(L+E_1) \to 0 .$$

As in the proof of Lemma 4.7,

$$\delta_4(X, x) \geq \dim_{\mathbf{C}} H^0(\mathcal{O}_M(L+E_1))/H^0(\mathcal{O}_M(L)) = 1 + h^1(\mathcal{O}_M(L+E_1)) .$$

Since $h^1(\mathcal{O}_M(L+E_1)) \geq h^1(\mathcal{O}_{E_2}(L+E_1)) = 1$, we have $\delta_4(X, x) \geq 2$. □

Proposition 4.9. *Let (X, x) be an elliptic du Bois singularity such that $E_2 \neq 0$. Then $\delta_3(X, x) \geq 2$ or $\delta_4(X, x) \geq 2$.*

Proof. Let A_1 be a curve in E_2 intersecting E_1. Then $h^1(\mathcal{O}_{E_1+A_1})=1$. Let (X', x') be the singularity obtained by contracting E_1+A_1 in M. By Theorem 2.4, we have $p_g(X', x') \leq 1$. Hence $p_g(X', x') = h^1(\mathcal{O}_{E_1+A_1}) = 1$. By Proposition 1.10, the singularity (X', x') is an elliptic du Bois singularity. Thus the result is an immediate consequence of Theorem 2.4 and Lemmas 4.7 and 4.8. □

Theorem 4.10. *Let (X, x) be a singularity with $\delta_m(X, x)=1$ for $m=1, 4, 6$. Then (X, x) is a simple elliptic or cusp singularity.*

Proof. Note that $\delta_1(X, x) = \delta_6(X, x) = 1$ implies $\delta_3(X, x)=1$. By Proposition 4.4, (X, x) is an elliptic du Bois singularity. Then Proposition 4.9 implies the assertion (cf. (4.6)). □

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