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THE POINCARÉ–CARTAN FORM IN SUPERFIELD THEORY

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An intrinsic description of the Hamilton–Cartan formalism for first-order Berezinian variational problems determined by a submersion of supermanifolds is given. This is achieved by studying the associated higher-order graded variational problem through the Poincaré–Cartan form. Noether theorem and examples from superfield theory and supermechanics are also discussed.

Keywords: Berezinian sheaf; graded manifold; Hamilton–Cartan formalism in graded manifolds; Hamilton equations; infinitesimal contact transformations; infinitesimal supersymmetries; Lagrangian superdensity; Poincaré–Cartan form; Berezinian and graded variational problems.

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Contents

1. Introduction

In this paper, we generalize some of the results already presented in [33, 35], where supermechanics (that is, variational problems defined for supercurves $\sigma\colon \mathbb{R}^{1|1} \to \mathbb{R}^{1|1} \times (M, \mathcal{A})$ with $(M, \mathcal{A})$ a supermanifold and $\mathbb{R}^{1|1}$ the parameter superspace), is considered from the viewpoint of Poincaré–Cartan theory. Now, we intend to deal with superfield theory; that is, with first order variational problems defined for superfields $\sigma : (M, \mathcal{A}) \to (N, \mathcal{B})$ (here $(M, \mathcal{A}), (N, \mathcal{B})$ are supermanifolds).

The basic object in our study is the Poincaré–Cartan form, for which we present an intrinsic construction in the context of Berezinian variational problems (intrinsic up to a volume form on the base manifold, as we will see).

Let us recall that there are two kind of integration theories defined on supermanifolds: the one associated to the Berezin integral and the other associated to what is called the graded integral. The first one is more suitable to state physical problems in the supermanifold setting, but it lacks from an associated theory of Berezinian superdifferential forms. So, it is not possible to work directly with a Poincaré–Cartan form and to develop a Hamilton–Cartan formalism from it.

The second theory of integration does not have a good physical interpretation but, conversely, a consistent theory of differential forms is available and therefore, it is possible to define a Poincaré–Cartan form and to develop the corresponding Hamilton–Cartan formalism.

According to these two possibilities, variational problems can be stated using either the Berezin integral or the graded integral; we call them Berezinian or graded variational problems, respectively. However, there is a deep connection between both problems. In brief, the relationship is based on the fact that to each first-order Berezinian variational problem over a graded submersion $p : (N, \mathcal{B}) \to (M, \mathcal{A})$ we can associate a graded variational problem of order $n + 1$ over p, where $(m|n)$ is the dimension of $(M, \mathcal{A})$ (see Sec. 4.4 below); we refer the reader to Theorem 4.1 for formal definitions and statement of this result, known as the Comparison Theorem.

With the help of the Comparison Theorem, the way to build a Poincaré–Cartan form and to develop a Hamilton–Cartan formalism for a *first-order* Berezinian variational problem is clear: Firstly, we define the graded Poincaré–Cartan form for the associated graded variational problem, *now of order* $n + 1$, and secondly we translate, with the hint offered by the Comparison Theorem, this form to an object which will play the role of Berezinian Poincaré–Cartan form for the Berezinian variational problem. From this object, it is possible to obtain the Euler–Lagrange superequations and a Noether Theorem.

A question arises at this point. In the classical case, it is well known that a canonical Poincaré–Cartan form of higher order does *not* exist. Of course, objects which can be called higher-order Poincaré–Cartan forms can be defined, but the problem is that they depend on some additional parameters (such as a connection, see [12, 15]). Nevertheless, here we give a *canonical* formulation of the graded Poincaré–Cartan form for higher-order graded variational problems; the key to understand how this is achieved is to note that we deal with a special subclass of these problems: those coming from first-order Berezinian variational ones through the Comparison Theorem. Actually, our purpose is to solve these first order Berezinian problems, so we could consider this feature as a byproduct.

Another very important consequence of this formalism in the classical case, is the existence of a Noether Theorem, which is a basic tool in the study of the symmetries of a variational problem. We present here a generalization to the graded setting.

In order to make the paper relatively self-contained, the first Sections contains a review of previous results on jet bundles and calculus of variations on supermanifolds.

Finally, there are some worked out examples (the $(m|2)$ field theory) and we analyze a particular case of interest in Physics (supermechanics) showing the coincidence with the results obtained by other methods [33, 35].

2. Basics of Supermanifold Theory

2.1. *General definitions*

For general references, we refer the reader to [43], [10, Chaps. 2 and 3], [27], [28], [3] [29] and [45]. The basic idea underlying the definition of a graded manifold is the substitution of the commutative sheaf of algebras of differentiable functions on a smooth manifold by another sheaf in which we can accommodate some objects with a $\mathbb{Z}_2$-grading (in what follows, all the gradings considered are assumed to be $\mathbb{Z}_2$-gradings, unless otherwise explicitly stated.)

A graded manifold (or a supermanifold) of dimension $(m|n)$ on a C^∞-manifold M of dimension m, is a sheaf $\mathcal{A}$ on M of graded $\mathbb{R}$-commutative algebras — the structure sheaf — such that,

1. There exists an exact sequence of sheaves,
$$0 \to \mathcal{N} \to \mathcal{A} \overset{\sim}{\to} C^\infty(M) \to 0, \tag{2.1}$$
where $\mathcal{N}$ is the sheaf of nilpotents in $\mathcal{A}$ and $\sim$ is a surjective morphism of graded $\mathbb{R}$-commutative algebras.
2. $\mathcal{N}/\mathcal{N}^2$ is a locally free module of rank n over $C^\infty(M) = \mathcal{A}/\mathcal{N}$, and $\mathcal{A}$ is locally isomorphic, as a sheaf of graded $\mathbb{R}$-commutative algebras, to the exterior bundle $\bigwedge_{C^\infty(M)}(\mathcal{N}/\mathcal{N}^2)$.

For any open subset $U \subset M$, from the exact sequence (2.1) we obtain the exact sequence of graded algebras,
$$0 \to \mathcal{N}(U) \to \mathcal{A}(U) \overset{\sim}{\to} C^\infty(U) \to 0.$$
A section f of $\mathcal{A}$ is called a graded function (or a superfunction). The image of such a graded function $f \in \mathcal{A}(U)$ by the structure morphism $\sim$ is denoted by $\tilde{f}$.

The fact that $\mathcal{A}$ is a sheaf of graded $\mathbb{R}$-commutative algebras induces a grading on its sections, and we denote the degree of such an f by $|f|$.

From the very definition of a supermanifold the structure sheaf of $(M, \mathcal{A})$ is locally isomorphic to $\bigwedge_{C^\infty(M)}(\mathcal{N}/\mathcal{N}^2)$. An important theorem (known as Batchelor Theorem [4, 5], but also see [16]), guarantees that in the C^∞ category this holds not only locally, but also globally, although this is no longer true in the complex analytic category. Thus, for any smooth supermanifold $(M, \mathcal{A})$ there exists a vector bundle $E \to M$ which is isomorphic to $\mathcal{N}/\mathcal{N}^2$ and such that $\mathcal{A} \cong \bigwedge_{C^\infty(M)}(E)$, but this isomorphism is not canonical.

A splitting neighborhood of a supermanifold $(M, \mathcal{A})$ is an open subset U in M such that the bundle $E = \mathcal{N}/\mathcal{N}^2$ is trivial over U and

$$\mathcal{A}|_U \cong \bigwedge_{C^\infty(U)} (E|_U).$$

If U is a splitting neighborhood, there exists a basis of sections for $E|_U$, denoted by $(x^{-1}, \ldots, x^{-n})$, along with an isomorphism

$$\mathcal{A}(U) \cong C^\infty(U) \otimes_{\mathbb{R}} \bigwedge E_n, \tag{2.2}$$

where E_n denotes the vector $\mathbb{R}$-space generated by $(x^{-1}, \ldots, x^{-n})$. Therefore, the natural projection $\mathcal{A}(U) \to C^\infty(U)$, $f \mapsto \tilde{f}$, admits a global section of $\mathbb{R}$-algebras, $\sigma : C^\infty(U) \hookrightarrow \mathcal{A}(U)$. If U is a splitting neighborhood, a family of superfunctions (x^i, x^{-j}), $1 \leq i \leq m$, $1 \leq j \leq n$, $|x^i| = 0$, $|x^{-j}| = 1$, is called a graded coordinate system (or a supercoordinate system) if,

1. $x^i = \sigma(\tilde{x}^i)$, $1 \leq i \leq m$, where $(\tilde{x}^1, \ldots, \tilde{x}^m)$ is an ordinary coordinate system on U,
2. $\{x^{-1}, \ldots, x^{-n}\}$ is a basis of sections of $E|_U$; i.e., $x^{-1}, \ldots, x^{-n} \in \bigwedge E_n$ and $\prod_{j=1}^n x^{-j} \neq 0$.

A morphism of graded manifolds $\phi : (M, \mathcal{A}) \to (N, \mathcal{B})$ is a pair of mappings $(\tilde{\phi}, \phi^*)$ where $\tilde{\phi} : M \to N$ is a differentiable mapping of smooth manifolds and for every open subset $U \subset N$, $\phi^* : \mathcal{B}(U) \to (\tilde{\phi}_* \mathcal{A})(U) = \mathcal{A}(\tilde{\phi}^{-1}(U))$ is an even morphism of graded algebras compatible with the restrictions, and all such that the diagram

$$\begin{array}{ccc} \mathcal{B}(U) & \xrightarrow{\phi^*} & \mathcal{A}(\tilde{\phi}^{-1}(U)) \\ \downarrow & & \downarrow \\ C^\infty(U) & \xrightarrow[\tilde{\phi}^*]{} & C^\infty(\tilde{\phi}^{-1}(U)) \end{array}$$

commutes.

Throughout this paper, we assume that M is connected and oriented by a volume form η. We confine ourselves to consider coordinate systems adapted to this volume form; i.e.,

$$\eta = d\tilde{x}^1 \wedge \cdots \wedge d\tilde{x}^m.$$

We refer all our constructions to this volume, but we simply call "intrinsic constructions" those results which are independent of η, in order to avoid continuous mention to η. Note that, by Batchelor's Theorem (see [4]), the natural projection $\mathcal{A}(M) \to C^\infty(M)$ admits a global section $\sigma : C^\infty(M) \to \mathcal{A}(M)$. Thus, once a section σ has been fixed, every ordinary volume form η on M induces a graded volume η^G on $(M, \mathcal{A})$.

Let $\mathcal{F}, \mathcal{G}$ be sheaves on a topological space X. For any open subset U *subset* M, $\mathrm{Hom}(\mathcal{F}|_U, \mathcal{G}|_U)$ denotes the space of morphisms between the sheaves $\mathcal{F}|_U$ and $\mathcal{G}|_U$; this is an abelian group in a natural way. The sheaf of homomorphisms is the sheaf $\mathrm{Hom}(\mathcal{F}, \mathcal{G})$ given by $\mathrm{Hom}(\mathcal{F}, \mathcal{G})(U) = \mathrm{Hom}(\mathcal{F}|_U, \mathcal{G}|_U)$ with the natural restriction morphisms.

The sheaf of left $\mathcal{A}$-modules of derivations of a graded manifold $(M, \mathcal{A})$ is the subsheaf of $\mathrm{End}_{\mathbb{R}}(\mathcal{A})$ whose sections on an open subset $U \subseteq M$ are $\mathbb{R}$-linear graded derivations $D : \mathcal{A}|_U \to \mathcal{A}|_U$. This sheaf is denoted by $\mathrm{Der}_{\mathbb{R}}(\mathcal{A})$ or simply $\mathrm{Der}(\mathcal{A})$, and its elements are called graded vector fields (or supervector fields) on the graded manifold $(M, \mathcal{A})$. The notation $\mathcal{X}_G(M)$ is also often used.

Let U be a coordinate neighborhood for a graded manifold $(M, \mathcal{A})$ with graded coordinates (x^i, x^{-j}), $1 \le i \le m$, $1 \le j \le n$. There exist even derivations $\partial/\partial x^1, \dots, \partial/\partial x^m$ and odd derivations $\partial/\partial x^{-1}, \dots, \partial/\partial x^{-m}$ of $\mathcal{A}(U)$ uniquely characterized by the conditions

$$\frac{\partial x^j}{\partial x^i} = \delta_i^j, \quad \frac{\partial x^{-j}}{\partial x^i} = 0, \quad \frac{\partial x^j}{\partial x^{-i}} = 0, \quad \frac{\partial x^{-j}}{\partial x^{-i}} = \delta_i^j$$

(negative indices running from $-n$ to -1, positive ones from 1 to m) and such that every derivation $D \in \mathrm{Der}\mathcal{A}(U)$ can be written as

$$D = \sum_{i=1}^{m} D(x^i) \frac{\partial}{\partial x^i} + \sum_{j=1}^{m} D(x^{-j}) \frac{\partial}{\partial x^{-j}}.$$

In particular, $\mathrm{Der}(\mathcal{A}(U))$ is a free right $\mathcal{A}(U)$-module with basis

$$\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^m}; \frac{\partial}{\partial x^{-1}}, \dots, \frac{\partial}{\partial x^{-m}}.$$

If $U \subseteq M$ is an open subset, the algebraic dual of the graded $\mathcal{A}$-module $\mathrm{Der}(\mathcal{A}(U))$ is $(\mathrm{Der}\mathcal{A}(U))^* = \mathrm{Hom}_{\mathcal{A}}(\mathrm{Der}(\mathcal{A}(U)), \mathcal{A}(U))$, which has itself a natural structure of graded $\mathcal{A}$-module and it defines a sheaf $U \mapsto (\mathrm{Der}\, \mathcal{A}(U))^*$.

The sheaves of right $\mathcal{A}$-modules of graded differential forms on $(M, \mathcal{A})$ are the sheaves

$$\Omega_G^p(M) = \bigwedge^p (\mathrm{Der}\, \mathcal{A})^*.$$

We also set $\Omega_G(M) = \sum_{p \in \mathbb{N}} \Omega_G^p(M)$, with $\Omega_G^0(M) = \mathcal{A}$.

The graded differential forms on $(M, \mathcal{A})$ are simply called graded forms. The three usual operators: insertion of a graded vector field, graded Lie derivative with respect to a graded vector field and the graded exterior differential, are defined in a similar way to the classical case (e.g., see [27]), and denoted by ι_X, $\mathcal{L}_X^G$, and d^G, respectively.

2.2. *Supervector bundles*

Let $\mathrm{GL}(V)$ be the general linear supergroup of a supervector space $V = V_0 \oplus V_1$. We set $\mathrm{GL}(p|q) = \mathrm{GL}(\mathbb{R}^{p|q})$. For the definition of the graded structure of $\mathrm{GL}(p|q)$

as a super Lie group, we refer the reader to [3, I, Sec. 3], [7, Chap. 2, Sec. 1], [8, Sec. 1.5], [10, Sec. 2.11], [11], [29, Chap. 4, Sec. 10], [38, Sec. 2.14], [39, Sec. 2], [42, Sec. 4.19], and [44, Sec. 2.2.1].

Let $(M, \mathcal{A})$ be an $(m|n)$-dimensional supermanifold. As is well known (e.g., see [10, Sec. 3.2], [42, 7.10]), a supervector bundle of rank $(p|q)$ over $(M, \mathcal{A})$ can be described either (i) as a fiber bundle V over M with typical fiber $\mathbb{R}^{p|q}$ and structure group $\mathrm{GL}(p|q)$, or (ii) as a locally free sheaf of $\mathcal{A}$-modules $\mathcal{V}$ of rank $(p|q)$. The description in (ii) means that every point $x \in M$ admits an open neigborhood $U \subseteq M$ such that $\mathcal{V}|_U$ is isomorphic — as a sheaf of $\mathcal{A}|_U$-modules — to $\mathcal{A}^{p|q}|_U = \mathcal{A}^p|_U \oplus \Pi\mathcal{A}^q|_U$ (direct sum of p copies of $\mathcal{A}$ and q copies of $\Pi\mathcal{A}$), where Π denotes the functor of change of parity; precisely, for every open subset $O \subseteq U$ we have $\mathcal{V}(O) \cong \mathcal{A}^p(O) \oplus \Pi\mathcal{A}^q(O)$.

More formally, we can state (see [37, 2.11 Theorem]): There is a one-to-one (functorial) correspondence between the set of isomorphism classes of locally free sheaves of (left) graded $\mathcal{A}$-modules of rank $(p|q)$ over M and the set of isomorphisms classes of supervector bundles of rank $(p|q)$ over the graded manifold $(M, \mathcal{A})$. Also see [42, Theorem 7.10.] for a slightly different approach.

We remark that the tangent and cotangent "supervector bundles" introduced in [27] are not supervector bundles in the previous sense, as they are not locally trivial. Because of this, we prefer to work with the supertangent bundle $\mathcal{ST}(M, \mathcal{A})$ of $(M, \mathcal{A})$ introduced by Sánchez-Valenzuela, which corresponds to the locally free sheaf of $\mathcal{A}$-modules of derivations, $\mathrm{Der}\mathcal{A}$. For our purposes, another important reason to do this, is that the graded manifold of 1-jets of graded curves from $\mathbb{R}^{1|1}$ to a graded manifold $(M, \mathcal{A})$ is isomorphic to $\mathcal{ST}(M, \mathcal{A})$; i.e., $J^1_G(p) \simeq \mathcal{ST}(M, \mathcal{A})$, where $J^1_G(p)$ is the graded manifold of graded 1-jets of sections of the natural projection onto the first factor, $p : \mathbb{R}^{1|1} \times (M, \mathcal{A}) \to \mathbb{R}^{1|1}$.

Let $\pi : (E, \mathcal{E}) \to (M, \mathcal{A})$ be a supervector bundle. For any $x \in M$, we denote by $\pi^{-1}(x)$ the superfiber over x, i.e., the supermanifold whose underlying topological space is $\tilde{\pi}^{-1}(x)$ and whose structure sheaf is

$$\mathcal{A}_x = (\mathcal{E} / \mathcal{K}_x)|_{\tilde{\pi}^{-1}(x)},$$

where $\mathcal{K}_x$ is the subsheaf of $\mathcal{E}$ whose sections vanish when restricted to $\tilde{\pi}^{-1}(x)$.

For any $x \in M$, $\pi^{-1}(x)$ is isomorphic with the standard fiber of π.

A supervector bundle morphism from the vector bundle $\pi_E : (E, \mathcal{E}) \to (M, \mathcal{A})$ to the vector bundle $\pi_F : (F, \mathcal{F}) \to (M, \mathcal{A})$ is a supermanifold morphism

$$H : (E, \mathcal{E}) \to (F, \mathcal{F})$$

such that $\pi_F \circ H = \pi_E$ the restriction of which to each superfiber $\pi_E^{-1}(x)$ is superlinear. The following consequence can be proved:

Proposition 2.1 [Proposition 3.3 in [37]]. *Let $(M, \mathcal{A})$ be a graded manifold, let $\mathcal{K}, \mathcal{L}$ be two locally free sheaves of graded $\mathcal{A}$-modules of ranks $(p|q)$ and $(r|s)$, respectively, and let $\pi_E : (E, \mathcal{E}) \to (M, \mathcal{A})$, $\pi_F : (F, \mathcal{F}) \to (M, \mathcal{A})$ be the supervector* *bundles that $\mathcal{K}$ and $\mathcal{L}$ give rise to, respectively. Each morphism $\psi : \mathcal{K} \to \mathcal{L}$ of sheaves of graded $\mathcal{A}$-modules over M defines a morphism*

$$H_\psi : (E, \mathcal{E}) \to (F, \mathcal{F})$$

such that $\pi_F \circ H_\psi = \pi_E$ and it restricts to a superlinear morphism over each fiber.

Another construction which we will use is the pull-back (or inverse image) of a supervector bundle along a graded submersion, which is a particular case of the pull-back of modules over ringed spaces. For our purposes, it suffices the following description.

Let $p : (N, \mathcal{B}) \to (M, \mathcal{A})$ be a graded submersion, and let $\mathcal{K}$ be a sheaf of graded $\mathcal{A}$-modules over M with projection π. The pull-back $p^*\mathcal{K}$ is the sheaf of $p^*\mathcal{A}$-modules over N where to each open $V \subset N$, it corresponds

$$p^*\mathcal{K}(V) = \{(k, y) \in \mathcal{K}(\tilde{p}(V)) \times V : \pi(k) = \tilde{p}(y)\}.$$

It is customary to write $p^*\mathcal{K} = \mathcal{K} \times_{(M,\mathcal{A})} (N, \mathcal{B})$. Note that if we consider the supervector bundle on $(M, \mathcal{A})$ given by $\mathcal{K}$, then $p^*\mathcal{K}$ gives a supervector bundle on $(N, \mathcal{B})$.

3. Graded Jet Bundles

3.1. *Notations and definitions*

For the details of the construction of graded jet bundles associated to a graded submersion $p : (N, \mathcal{B}) \to (M, \mathcal{A})$, we refer the reader to [21, 22, 26, 30, 31]. We also note that other approaches to superjet bundles of interest in Physics are possible, see [19].

We denote by

$$p_k : \left(J^k_G(p), \mathcal{A}_{J^k_G(p)}\right) \to (M, \mathcal{A})$$

the graded k-jet bundle of local sections of p, with natural projections

$$p_{kl} : \left(J^k_G(p), \mathcal{A}_{J^k_G(p)}\right) \to \left(J^l_G(p), \mathcal{A}_{J^l_G(p)}\right), \quad k \geq l.$$

Remark 3.1. Sometimes we will write $p_{k,l}$ in order to avoid confusions, as in the case of the projection $p_{k,k-1}$ (of $J^k_G(p)$ onto $J^{k-1}_G(p)$) and even we will employ p^k_l indistinctly.

Each section $\sigma : (M, \mathcal{A}) \to (N, \mathcal{B})$ of the graded submersion p induces a closed embedding of graded manifolds

$$j^k\sigma : (M, \mathcal{A}) \to \left(J^k_G(p), \mathcal{A}_{J^k_G(p)}\right),$$

which is called the graded k-jet extension of σ.

We set $(m|n) = \dim(M, \mathcal{A})$, $(m + r|n + s) = \dim(N, \mathcal{B})$, and let

$$\left.\begin{array}{ll} (x^\alpha), & \alpha = -n, \ldots, -1, 1, \ldots, m, \\ (y^\mu), & \mu = -s, \ldots, -1, 1, \ldots, r. \end{array}\right\} \tag{3.1}$$

be a fibered coordinate system for the submersion $p:(N,\mathcal{B})\to(M,\mathcal{A})$, defined over an open domain $V\subseteq N$. This means that the graded functions (x^α), $i=-n,\dots,-1,1,\dots,m$, belong to $p^*\mathcal{A}(U)$, where $U=\tilde{p}(V)$.

System (3.1) induces a coordinate system for $J^k_G(p)$ on $(\tilde{p}_{k0})^{-1}(V)$, denoted by y^μ_{IA}, where $\mu=-s,\dots,-1,1,\dots,r$, $I=(i_1,\dots,i_m)\in\mathbb{N}^m$, and $A=(-\alpha_1,\dots,-\alpha_l)\in(\mathbb{Z}^-)^l$, for $l=0,\dots,n$, is a strictly decreasing multi-index, such that $|I|+|A|\le k$, with the assumption $y^\mu_{0\emptyset}=y^\mu$. This system of coordinates is determined by the following equations:

$$(j^k\sigma)^*y^\mu_{IA}=\frac{\partial^{i_1}}{(\partial x^1)^{i_1}}\circ\cdots\circ\frac{\partial^{i_m}}{(\partial x^m)^{i_m}}\circ\frac{\partial}{\partial x^{-\alpha_l}}\circ\cdots\circ\frac{\partial}{\partial x^{-\alpha_1}}(\sigma^*y^\mu),$$

for every smooth section $\sigma:(U,\mathcal{A}|_U)\to(V,\mathcal{B}|_V)$ of the given graded submersion.

Sometimes we will write expressions such as $y^\mu_{I,A}$ instead of y^μ_{IA}. This will be done in order to avoid confusions involving positive and negative multiindices.

The parity of y^μ_{IA} is the sum modulo 2 of the parity of y^μ and $|A|$. In particular, the parity of the coordinate system induced by (3.1) on $J^1_G(p)$ is explicitly given by

$$\left.\begin{array}{lll}|y^\mu_i|=0, & i=-n,\dots,-1, & \mu=-s,\dots,-1\\ |y^\mu_i|=1, & i=1,\dots,m, & \mu=-s,\dots,-1\\ |y^\mu_i|=1, & i=-n,\dots,-1, & \mu=1,\dots,r\\ |y^\mu_i|=0, & i=1,\dots,m, & \mu=1,\dots,r\end{array}\right\}$$

and we accordingly have,

$$\dim\bigl(J^1_G(p),\mathcal{A}_{J^1_G(p)}\bigr)=(m+r+mr+ns|n+s+ms+nr).$$

We also work with the inverse limit

$$\bigl(J^\infty_G(p)=\lim_{\leftarrow}J^k_G(p),\mathcal{A}_{J^\infty_G(p)}=\lim_{\rightarrow}\mathcal{A}_{J^k_G(p)}\bigr)$$

of the system $\bigl(J^k_G(p),\mathcal{A}_{J^k_G(p)};p_{kl},k\ge l\bigr)$, with natural projections

$$\begin{aligned}p_\infty&:\bigl(J^\infty_G(p),\mathcal{A}_{J^\infty_G(p)}\bigr)\to\bigl(M,\mathcal{A}\bigr),\\ p_{\infty k}&:\bigl(J^\infty_G(p),\mathcal{A}_{J^\infty_G(p)}\bigr)\to\bigl(J^k_G(p),\mathcal{A}_{J^k_G(p)}\bigr).\end{aligned}$$

Given the submersion $p:(N,\mathcal{B})\to(M,\mathcal{A})$, we denote by $\mathcal{V}(p)$ the vertical subspace of $S\mathcal{T}(N,\mathcal{B})$. In particular, this applies to the various p_k and p_{kl} submersions derived from p, so we will write $\mathcal{V}(p_k),\mathcal{V}(p_{kl})$, etc.

In the following, we will work with differential operators acting on the spaces $J^k_G(p)$, and in order to deal with the multi-index notation (especially for negative multi-indices) it will be useful to establish the following conventions.

1. We will denote positive multi-indices by the capital letters $I,J,K,\dots$ and the negative ones by $A,B,C,\dots$. An arbitrary multi-index (containing both positive and negative indices) will be denoted $P,Q,R,\dots$. By $\mathbb{I}_n$ we will understand the set $\mathbb{I}_n=\{1,2,\dots,n\}$.

2. The multi-index $\emptyset$ amounts to take 0 within any expression in which it appears, that is:

$$\frac{\partial^{\emptyset}}{\partial x^{\emptyset}} F_{IA} = 0, \quad G_{J\emptyset} = 0.$$

The multi-index (0) amounts to take the identity:

$$\frac{\partial^{0}}{\partial x^{0}} F_{IA} = F_{IA}, \quad G_{J0} = G_{J}.$$

3. A negative multi-index A with length l in $J^k_G(p)$ has the structure

$$A = (-\alpha_1, \ldots, -\alpha_l)$$

with $l \leq k$, where $\alpha_i \in \mathbb{I}_n$, $\dim(M, \mathcal{A}) = (m|n)$, $1 \leq i \leq l$. Each $-\alpha_i$ gives the odd coordinate of $(M, \mathcal{A})$ with respect to which we are computing the derivative; that is, the place occupied by $-\alpha_i$ in the multi-index only expresses the order in which the corresponding derivative appears from left to right. Thus, if $\dim(M, \mathcal{A}) = (3|6)$, we could consider $J^4_G(p)$ and $A = (-3, -5, -2)$, then $\frac{\partial^{|A|}}{\partial x^A}$ would represent

$$\frac{\partial^{|A|}}{\partial x^A} = \frac{\partial}{\partial x^{-3}} \circ \frac{\partial}{\partial x^{-5}} \circ \frac{\partial}{\partial x^{-2}}.$$

4. If we are dealing with $J^k_G(p)$, a negative multi-index A always has length $l \leq k$. By convention, if the length of A is $l > k$, then $A = \emptyset$. Note that if $l > n$, automatically $A = \emptyset$. Generally, if a negative multi-index A contains two repeated indices, $A = \emptyset$.
5. In principle, a negative multi-index does not need to be ordered, but nothing prevents from having such ordered indices as the length 5 multi-index

$$B = (-9, -7, -4, -2, -1)$$

in $J^8_G(p)$, with $\dim(M, \mathcal{A}) = (2|9)$.

6. For negative multi-indices, we define the operation of (non-ordered) *juxtaposition.* If A has length l and B has length q,

$$\begin{aligned} A &= (-\alpha_1, \ldots, -\alpha_l) \quad \text{with } \alpha_i \in \mathbb{I}_n \\ B &= (-\beta_1, \ldots, -\beta_q) \quad \text{with } \beta_j \in \mathbb{I}_n, \end{aligned}$$

then their juxtaposition is given by:

$$A \star B = \begin{cases} (-\alpha_1, \ldots, -\alpha_l, -\beta_1, \ldots, -\beta_q) & \text{if } l + q \leq k \quad \text{and} \quad -\xi_i \neq -\xi_j \\ & \quad \text{being } \xi_i \in \{\alpha_1, \ldots, \alpha_l, \beta_1, \ldots, \beta_q\} \\ \emptyset & \text{other case.} \end{cases}$$

Note that $A \star B \neq B \star A$. In particular, if $A = (-j)$ and $B = (-\beta_1, \ldots, -\beta_q)$, then

$$A \star B = (-j, -\beta_1, \ldots, -\beta_q)$$

and that means

$$\frac{\partial^{q+1}}{\partial x^{A\star B}} = \frac{\partial}{\partial x^{-j}} \circ \frac{\partial}{\partial x^{-\beta_1}} \circ \cdots \circ \frac{\partial}{\partial x^{-\beta_q}},$$

provided $1 + q \leq k$ *and there are no repeated indices.*

7. If we take a positive multiindex I and a negative one A (or a pair of positive multiindices) their juxtaposition is analogously defined, but in this case it is a *commutative* operation. To stress this fact we then write $I + A, I + J$, etc.

3.2. *Graded contact forms*

Let $p : (M, \mathcal{A}) \to (N, \mathcal{B})$ be a graded submersion with $(m|n) = \dim(M, \mathcal{A})$, $(m + r|n + s) = \dim(N, \mathcal{B})$. The graded manifold $(J^k_G(p), \mathcal{A}_{J^k_G(p)})$ is endowed with a differential system, which characterizes the holonomy of the sections of $p_k : (J^k_G(p), \mathcal{A}_{J^k_G(p)}) \to (M, \mathcal{A})$. Precisely, a graded 1-form ω on $(J^k_G(p), \mathcal{A}_{J^k_G(p)})$ is said to be a contact form if $(j^k\sigma)^*\omega = 0$, for every local section σ of p. With the same assumptions and notations as in Subsec. 3.1, the set of contact forms is a sheaf of $\mathcal{A}_{J^k_G(p)}$-modules locally generated by the forms

$$\theta^\mu_{IA} = d^G y^\mu_{IA} - \sum_{h=1}^{m} d^G x^h \cdot y^\mu_{\{h\}\star I,A} - \sum_{j=1}^{n} \varepsilon(j, A) d^G x^{-j} \cdot y^\mu_{I,\{-j\}\star A}, \tag{3.2}$$

where $\alpha = -s, \ldots, -1, 1, \ldots, r$, $|I| + |A| \leq k - 1$.

These forms fit together in order to define a global $(p_{k,k-1})^*\mathcal{V}(p_k)$-valued 1-form on $(J^k_G(p), \mathcal{A}_{J^k_G(p)})$, called the structure form on the graded k-jet bundle, given by

$$\theta^k = \theta^\mu_{IA} \otimes \frac{\partial}{\partial y^\mu_{IA}}, \tag{3.3}$$

which characterizes graded k-jet extensions of sections of p, as follows: a section $\bar{\sigma} : (M, \mathcal{A}) \to \left(J^k_G(p), \mathcal{A}_{J^k_G(p)}\right)$ of p_k coincides with the k-jet extension of a certain section of p if and only if, $\bar{\sigma}^*\theta^k = 0$.

3.3. *Graded lifts of vector fields*

Consider a graded submersion $p : (N, \mathcal{B}) \to (M, \mathcal{A})$. We will define liftings of graded vector fields to superjet bundles $J^k_G(p)$, $1 \leq k \leq \infty$.

3.3.1. *Horizontal lifts*

Let X be a vector field on $(M, \mathcal{A})$. The horizontal or total graded lift X^H of X is the vector field on $\left(J^\infty_G(p), \mathcal{A}_{J^\infty_G(p)}\right)$ uniquely determined by the following equations:

$$j^k(\sigma)^*(X^H(f)) = X(j^k(\sigma)^*(f)), \quad \forall\, k \in \mathbb{N},$$

for all open subsets $V \subseteq N$, $W \subseteq p^{-1}_{k0}(V)$, every $f \in \mathcal{A}_{J^k_G(p)}(W)$, and every smooth section $\sigma : (U, \mathcal{A}(U)) \to (V, \mathcal{B}(V))$ of p, with $U = \tilde{p}(V)$. A vector field X on

$J_G^\infty(p)$ is said to be horizontal if vector fields $X_1, \dots, X_r$ on $(M, \mathcal{A})$ and functions $f^1, \dots, f^r \in \mathcal{A}_{J_G^\infty(p)}$ exist, such that $X = f^i(X_i)^H$.

If (x^α, y^μ) is a fibered coordinate system for the submersion p, then the expression for the horizontal lift of the basic vector field $\partial/\partial x^\alpha$ in the induced coordinate system, is

$$\begin{aligned} \frac{d}{dx^\alpha} &= \left(\frac{\partial}{\partial x^\alpha}\right)^H \\ &= \frac{\partial}{\partial x^\alpha} + y^\mu_{\{\alpha\}\star Q}\frac{\partial}{\partial y^\mu_Q}. \end{aligned} \tag{3.4}$$

The map $X \mapsto X^H$ is an $\mathcal{A}$-linear injection of Lie algebras (cf. [30, 31]). Note that X^H is p_∞-projectable onto X. Moreover, we can consider $\mathcal{A}_{J_G^{k+1}(p)}$ as a sheaf of $\mathcal{A}_{J_G^k(p)}$-algebras via the natural injection

$$p^*_{k+1,k} : \mathcal{A}_{J_G^k(p)} \to \mathcal{A}_{J_G^{k+1}(p)},$$

and, for every $k \in \mathbb{N}$, X^H induces a derivation of $\mathcal{A}_{J_G^k(p)}$-modules,

$$X^H : \mathcal{A}_{J_G^k(p)} \to \mathcal{A}_{J_G^{k+1}(p)}.$$

Let $\Omega_G^k(J_G^\infty(p))$ be the space of graded differential k-forms on $J_G^\infty(p)$. We denote by $H_r^s(J_G^\infty(p))$ the module of $(r+s)$-forms on $J_G^\infty(p)$ that are r-times horizontal and s-times vertical; that is, such that they vanish when acting on more than s p_∞-vertical vector fields or more than r p_∞-horizontal vector fields.

Let d^G be the exterior differential, and let

$$\begin{aligned} D &: H_r^s(J_G^\infty(p)) \to H_{r+1}^s(J_G^\infty(p)) \\ \partial &: H_r^s(J_G^\infty(p)) \to H_r^{s+1}(J_G^\infty(p)) \end{aligned}$$

be the horizontal and vertical differentials, respectively. We have

$$\begin{aligned} d^G &= D + \partial, \\ D^2 &= 0, \\ \partial^2 &= 0, \\ D \circ \partial + \partial \circ D &= 0. \end{aligned}$$

We can make a local refinement of the bigrading above, which depends on the chart chosen, but we will make use of it only when computing in local coordinates. Let $(W, \mathcal{A}_{J_G^\infty(p)}(W))$ be an open coordinate domain in $J_G^\infty(p)$. Since

$$\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^{-j}}\right)_{1\le i\le m, 1\le j\le n}$$

is a basis of vector fields for $(M, \mathcal{A})$, we can define $H^s_{r_1,r_2}(W)$ to be the submodule of differential forms in $H^s_{r_1+r_2}(W)$ such that they vanish when acting on more than

r_1 vector fields among the $\partial/\partial x^i$, or when acting on more than r_2 vector fields among the $\partial/\partial x^{-j}$. Therefore,

$$H_r^s(W) = \bigoplus_{r_1+r_2=r} H_{r_1,r_2}^s(W),$$

with projections $\pi_{r_1,r_2} : H_r^s(W) \to H_{r_1,r_2}^s(W)$. Considering the action of D on a fixed $H_{r_1,r_2}^s(W)$, we define

$$\begin{aligned} D_0 &= \pi_{r_1+1,r_2} \circ D, \\ D_1 &= D - D_0. \end{aligned}$$

3.3.2. *Infinitesimal contact transformations*

Let $p : (M, \mathcal{A}) \to (N, \mathcal{B})$ be a graded submersion. A homogeneous vector field Y on $(J_G^k(p), \mathcal{A}_{J_G^k(p)})$ is said to be a k-order graded infinitesimal contact transformation if an endomorphism h of $\mathcal{A}_{J_G^k(p)} \otimes_{\mathcal{B}} \mathrm{Der}_{\mathcal{A}}(\mathcal{B})$ — considered as a left $\mathcal{A}_{J_G^k(p)}$-module — exists such that,

$$\mathcal{L}_Y^G \theta^k = h \circ \theta^k,$$

where θ^k is the structure form (recall (3.3)).

Theorem 3.2 [26]. *Let $p : (N, \mathcal{B}) \to (M, \mathcal{A})$ be a graded submersion. For every graded vector field X on $(N, \mathcal{B})$, there exists a unique k-order graded infinitesimal contact transformation $X_{(k)}$ on $(J_G^k(p), \mathcal{A}_{J_G^k(p)})$ projecting onto X.*

Moreover, for every $k > l$, the vector field $X_{(k)}$ projects onto $X_{(l)}$ via the natural map $p_{kl} : (J_G^k(p), \mathcal{A}_{J_G^k(p)}) \to (J_G^l(p), \mathcal{A}_{J_G^l}(p))$.

4. Berezinian Sheaves

4.1. ***The Berezinian sheaf of a supermanifold***

The Berezinian sheaf is a geometrical object designed to make possible an integration theory in supermanifolds, tailored to the needs coming from Physics. A global description of it can be given as follows (see [25, 30], cf. [29]).

Let $(M, \mathcal{A})$ be a graded manifold, of dimension $(m|n)$, and let $\mathrm{P}^k(\mathcal{A})$ be the sheaf of graded differential operators of $\mathcal{A}$ of order k. This is the submodule of $\mathrm{End}(\mathcal{A})$ whose elements P satisfy the following conditions:

$$[\ldots[[P, a_0], a_1], \ldots, a_k] = 0, \quad \forall\, a_0, \ldots, a_k \in \mathcal{A}.$$

Here the element $a \in \mathcal{A}$ is identified with the endomorphism $b \mapsto ab$. The sheaf $P^k(\mathcal{A})$ has two essentially different structures of $\mathcal{A}$-module: For every $P \in P^k(\mathcal{A})$ and every $a, b \in \mathcal{A}$,

1. the left structure is given by $(a \cdot P)(b) = a \cdot P(b)$; and
2. the right structure is given by $(P \cdot a)(b) = P(a \cdot b)$.

This is important as the Berezinian sheaf is considered with its structure of *right* $\mathcal{A}$-module.

One has that if (x^i, x^{-j}), $1 \leq j \leq n$, $1 \leq i \leq m$, are supercoordinates for a splitting neighborhood $U \subset M$, $\mathrm{P}^k(\mathcal{A}(U))$ is a free module (for both structures, left and right) with basis

$$\left.\begin{aligned}\left(\frac{\partial}{\partial x^1}\right)^{\alpha_1} \circ \cdots \circ \left(\frac{\partial}{\partial x^m}\right)^{\alpha_m} \circ \left(\frac{\partial}{\partial x^{-1}}\right)^{\beta_1} \circ \cdots \circ \left(\frac{\partial}{\partial x^{-n}}\right)^{\beta_n}, \\ \alpha_1 + \cdots + \alpha_m + \beta_1 + \cdots + \beta_n \leq k.\end{aligned}\right\}$$

Let us consider the sheaf $P^k(\mathcal{A}, \Omega_G^m) = \Omega_G^m(M) \otimes_{\mathcal{A}} P^k(\mathcal{A})$, of m-form valued differential operators on $\mathcal{A}$ of order k, and for every open subset $U \subset M$, let $K^n(U)$ be the set of operators $P \in P^n(\mathcal{A}(U), \Omega_G^m(U))$ such that, for every $a \in \mathcal{A}(U)$ with compact support, there exists an ordinary $(m-1)$-form of compact support, ω, satisfying

$$\widetilde{P(a)} = d\omega.$$

The idea is to take the quotient of $P^n(\mathcal{A}, \Omega_G^m)$ by K^n; in this way, when we later define the integral operator, two sections differing in a total differential will be regarded as equivalent (Stokes Theorem). Having this in mind, we observe that K^n is a submodule of $P^n(\mathcal{A}, \Omega_G^m)$ for its *right structure*, so we can take quotients and obtain the following description of the Berezinian sheaf, $\mathrm{Ber}(\mathcal{A})$:

$$\mathrm{Ber}(\mathcal{A}) \simeq P^n(\mathcal{A}, \Omega_G^m) / K^n.$$

We write this as an equivalence because there are other definitions of the Berezinian sheaf. For us, however, this is *the* definition.

According to this description, a local basis of $\mathrm{Ber}(\mathcal{A})$ can be given explicitly: If (x^i, x^{-j}), $1 \leq j \leq n$, $1 \leq i \leq m$, are supercoordinates for a splitting neighborhood $U \subset M$, the local sections of the Berezinian sheaf are written in the form

$$\Gamma(U, \mathrm{Ber}(\mathcal{A})) = \left[d^G x^1 \wedge \cdots \wedge d^G x^m \otimes \frac{\partial}{\partial x^{-1}} \circ \cdots \circ \frac{\partial}{\partial x^{-n}}\right] \cdot \mathcal{A}(U), \tag{4.1}$$

where $[\cdot]$ stands for the equivalence class modulo K^n.

4.2. *Higher order Berezinian sheaf*

Let $p : (N, \mathcal{B}) \to (M, \mathcal{A})$ be a graded submersion. Given $P \in \mathrm{P}^l(\mathcal{A}, \Omega_G^m)$, let

$$P^H : \mathcal{A}_{J_G^k(p)} \to H_m^0(J_G^k(p))$$

be the first-order operator defined by the condition,

$$j^k(\sigma)^* P^H f = P j^k(\sigma)^* f,$$

for every $f \in \mathcal{A}_{J_G^k(p)}$ and every local section σ of p. We call P^H the total or horizontal lift of P. Let us denote by $PH^l(\mathcal{A}_k, H_m^0)$ (resp. $KH_l(\mathcal{A}_k)$) the sheaf of those operators in

$$P^l\big(\mathcal{A}_{J_G^k(p)}, H_m^0\big(J_G^k(p)\big)\big)$$

that are horizontal lifts of operators of $P^l(\mathcal{A},\Omega^m_G)$ (resp. $K_l(\mathcal{A})$). Then, the k-order Berezinian sheaf is defined as

$$\mathrm{Ber}^k(\mathcal{A}_k) = \frac{PH^n(\mathcal{A}_k, H^0_m)}{KH_n(\mathcal{A}_k)} \otimes \mathcal{A}_{J^k_G(p)}.$$

According to this description, a local basis for $\mathrm{Ber}^k(\mathcal{A}_k)$ can be given explicitly: If (x^i, x^{-j}), $1 \le i \le m$, $1 \le j \le n$, are the graded $\mathcal{A}$-coordinates for the coordinate open domain $(U, \mathcal{A}(U))$ and $(V, \mathcal{B}(V))$ is a $\mathcal{B}$-coordinate open domain with a suitable $V \subseteq \tilde{p}^{-1}(U)$, then, if W is an open subset in $J^k_G(p)$ such that $W \subseteq \tilde{p}_k^{-1}(U)$ we have

$$\Gamma(W, \mathrm{Ber}^k(\mathcal{A}_k)) = \left[d^G x^1 \wedge \cdots \wedge d^G x^m \otimes \frac{d}{dx^{-1}} \circ \cdots \circ \frac{d}{dx^{-n}} \right] \cdot \mathcal{A}_{J^k_G(p)}(W).$$

4.3. *The Berezin integral*

Given a supermanifold $(M, \mathcal{A})$, the Berezin integral can be defined over the sections of the Berezinian sheaf with compact support, by means of the formula

$$\begin{aligned} \int_{\mathrm{Ber}} : \Gamma^c_U(\mathrm{Ber}(\mathcal{A})) &\to \mathbb{R} \\ [P] &\mapsto \int_U \widetilde{P(1)}. \end{aligned} \tag{4.2}$$

In this expression, M is assumed to be oriented, and the right integral is taken with respect to that orientation. In this sense, having a fixed volume form on M is not a loss of generality.

4.3.1. *An example*

Let $(M, \mathcal{A}) = (\mathbb{R}^m, C^\infty(\mathbb{R}^m) \otimes \Omega(\mathbb{R}^n))$ be the standard graded manifold. A section of $\mathcal{A}$ is just a differential form $\rho = f_I(x^1, \ldots, x^m)x^{-I}$, $0 \le |I| \le n$, where (x^i), $1 \le i \le m$, are the coordinates of $\mathbb{R}^m$ and we write $x^{-j} = dx^j$ for the odd coordinates; thus

$$\rho = f_0 + f_j x^{-j} + \cdots + f_{1\ldots n} x^{-1} \cdots x^{-n},$$

and we recover the formula for Berezin's expression common in Physics textbooks (except for a global sign); i.e., "to integrate the component of highest odd degree" (see [6]):

$$\begin{aligned} &\int_{\mathrm{Ber}} \left[d^G x^1 \wedge \cdots \wedge d^G x^m \otimes \frac{\partial}{\partial x^{-1}} \circ \cdots \circ \frac{\partial}{\partial x^{-n}} \right] \cdot \rho \\ &\quad = (-1)^{\binom{n}{2}} \int_{\mathbb{R}^m} f_{1\ldots n} dx^1 \cdots dx^m. \end{aligned}$$

4.3.2. *Lie derivative on the Berezinian sheaf*

If X is a graded vector field, it is possible to define the notion of graded Lie derivative of sections of the Berezinian sheaf with respect to X. This is the mapping

$$\mathcal{L}^G_X : \Gamma(\mathrm{Ber}(\mathcal{A})) \to \mathrm{P}^{n+1}(\mathcal{A}, \Omega^m_G)/K^{n+1} = \Gamma(\mathrm{Ber}(\mathcal{A})),$$

given by

$$\mathcal{L}_X^G[\eta^G \otimes P] = (-1)^{|X||\eta^G \otimes P|+1}[\eta^G \otimes P \circ X], \tag{4.3}$$

for homogeneous X and $\eta^G \otimes P$.

This Lie derivative has the properties that one could expect:

1. For homogeneous $X \in \mathrm{Der}(\mathcal{A})$, $\xi \in \Gamma(\mathrm{Ber}(\mathcal{A}))$ and $a \in \mathcal{A}$,
$$\mathcal{L}_X^G(\xi \cdot a) = \mathcal{L}_X^G(\xi) \cdot a + (-1)^{|X||\xi|}\xi \cdot X(a).$$
2. For homogeneous $X \in \mathrm{Der}(\mathcal{A})$, $\xi \in \Gamma(\mathrm{Ber}(\mathcal{A}))$ and $a \in \mathcal{A}$,
$$\mathcal{L}_{a \cdot X}^G(\xi) = (-1)^{|a|(|X|+|\xi|)}\mathcal{L}_X^G(\xi \cdot a).$$
3. Given a system of supercoordinates (x^i, x^{-j}), $1 \leq j \leq n$, $1 \leq i \leq m$. If
$$\xi_{x^i,x^{-j}} = \left[d^G x^1 \wedge \cdots \wedge d^G x^m \otimes \frac{\partial}{\partial x^{-1}} \circ \cdots \circ \frac{\partial}{\partial x^{-n}}\right]$$
is the local generator of the Berezinian sheaf, then
$$\mathcal{L}^G_{\frac{\partial}{\partial x^i}}(\xi_{x^i,x^{-j}}) = 0,$$
$$\mathcal{L}^G_{\frac{\partial}{\partial x^{-j}}}(\xi_{x^i,x^{-j}}) = 0.$$

4.3.3. *Berezinian divergence*

We can now introduce the notion of Berezinian divergence. Let $(M, \mathcal{A})$ be a graded manifold whose Berezinian sheaf is generated by a section ξ. The graded function $\mathrm{div}_B^\xi(X)$ given — for homogeneous X — by the formula

$$\mathcal{L}_X^G(\xi) = (-1)^{|X||\xi|}\xi \cdot \mathrm{div}_B^\xi(X)$$

(and extended by $\mathcal{A}$-linearity) is called the Berezinian divergence of X with respect to ξ. When there is no risk of confusion, we simply write $\mathrm{div}_B(X)$.

For example, if we consider the standard graded manifold of Example 4.3.1; i.e., $(M, \mathcal{A}) = (\mathbb{R}^m, C^\infty(\mathbb{R}^m) \otimes \Omega(\mathbb{R}^n))$, then $\mathrm{Ber}(\mathcal{A})$ is trivial and generated by

$$\xi = \left[d^G x^1 \wedge \cdots \wedge d^G x^m \otimes \frac{\partial}{\partial x^{-1}} \circ \cdots \circ \frac{\partial}{\partial x^{-n}}\right],$$

and the Berezinian divergence of a graded vector field $X = f_i \partial/\partial x^i + g_j \partial/\partial x^{-j}$ with respect to ξ is given by

$$\mathrm{div}_B(X) = \sum_{i=1}^m \frac{\partial f_i}{\partial x^i} + \sum_{j=1}^n (-1)^{|g_j|}\frac{\partial g_j}{\partial x^{-j}}. \tag{4.4}$$

Having in mind the previous section, these notions can be carried over to higher orders with the appropriate modifications.

4.4. *Graded and Berezinian Lagrangian densities*

Let us introduce the notion of variational problems in terms of the Berezinian sheaf.

A *Berezinian Lagrangian density* of order k for a graded submersion $p:(N,\mathcal{B}) \to (M,\mathcal{A})$, is a section

$$[P^H]\cdot L \in \Gamma(\mathrm{Ber}^k(\mathcal{A}_k)).$$

In particular, a first-order Berezinian Lagrangian density can locally be written as $\xi \cdot L$, where

$$\xi = \left[d^G x^1 \wedge \cdots \wedge d^G x^m \otimes \frac{d}{dx^{-1}} \circ \cdots \circ \frac{d}{dx^{-n}} \right]$$

and $L \in \mathcal{A}_{J^k_G(p)}$ is an element of the ring of graded functions on the graded bundle of 1-jets $(J^1_G(p), \mathcal{A}_{J^1_G(p)})$. In this paper, we only consider first-order Berezinian Lagrangian densities and assume that M is oriented by an ordinary volume form η.

The variation of the Berezinian functional associated to $\xi \cdot L$, along a section s of $p:(N,\mathcal{B}) \to (M,\mathcal{A})$, is the mapping

$$\begin{aligned} \delta_s I_{\mathrm{Ber}}(L)\colon \quad & \mathcal{V}^c(N) \to \mathbb{R} \\ & Y \mapsto \int_{\mathrm{Ber}} (j^1 s)^* (\mathcal{L}^G_{Y_{(1)}}(\xi \cdot L)), \end{aligned}$$

where $\mathcal{V}^c(N)$ denotes the space of graded vector fields on $(N,\mathcal{B})$, which are vertical over $(M,\mathcal{A})$ and whose support has compact image on M; $Y_{(1)}$ is the graded infinitesimal contact transformation associated to Y, and $\mathcal{L}^G_{Y_{(1)}}(\xi \cdot L)$ is defined by means of (4.3), which makes sense as Y is p-projectable. A section s is called a Berezinian extremal if $\delta_s \mathrm{I}_{\mathrm{Ber}} = 0$.

Finally, we turn our attention to the relation between Berezinian and graded variational problems. As we will see shortly, even restricting ourselves to first-order Berezinian Lagrangian densities we must consider higher-order graded Lagrangian ones.

A *graded Lagrangian density* of order k for a graded submersion $p:(N,\mathcal{B}) \to (M,\mathcal{A})$ is a section

$$\eta^G \cdot L \in \Omega^m_G(M) \otimes_{\mathcal{A}} \mathcal{A}_{J^k_G(p)},$$

where $(m|n) = \dim(M,\mathcal{A})$, η^G is a graded m-form on $(M,\mathcal{A})$, and L is an element of the graded ring $\mathcal{A}_{J^k_G(p)}$, of graded functions on the graded k-jet bundle $J^k_G(p)$.

The variation of the functional associated to a graded k-order Lagrangian density $\eta^G \cdot L$ along a section s of $p:(N,\mathcal{B}) \to (M,\mathcal{A})$, is the mapping

$$\begin{aligned} \delta_s \mathrm{I}^k_{\mathrm{grad}}(L): \quad & \mathcal{V}^c_G(N) \to \mathbb{R} \\ & Y \mapsto \int_M \widetilde{(j^k s)^* (\mathcal{L}^G_{Y_{(k)}}(\eta^G \cdot L))}, \end{aligned}$$

where $\mathcal{V}^c(N)$ is as before and $Y_{(k)}$ is the k-graded infinitesimal contact transformation prolongation of Y.

Berezinian and graded variational problems are related through the following result (usually known as the Comparison Theorem):

Theorem 4.1 [26, 30]. *Let $p : (N, \mathcal{B}) \to (M, \mathcal{A})$ be a graded submersion, with $(m|n) = \dim(M, \mathcal{A})$. Every first-order Berezinian Lagrangian density $\xi \cdot L$ for p is equivalent to a graded Lagrangian density of order $n+1$ in the following sense: There exists an element L' in the graded ring $\mathcal{A}_{J_G^{n+1}(p)}$ of the graded $(n+1)$-jet bundle $J_G^{n+1}(p)$ such that the Berezinian variation of the functional associated to $\xi \cdot L$ equals the graded variation of the functional associated to $\eta^G \cdot L' = d^G x^1 \wedge \cdots \wedge d^G x^m \cdot L'$; that is,*

$$(\delta_s I_{\mathrm{Ber}}(L))(Y) = \left(\delta_s I_{\mathrm{grad}}^{n+1}(L')\right)(Y),$$

for every section s of p, and every graded p-vertical $Y \in \mathcal{V}^c(N, \mathcal{B})$.

5. The $\mathcal{J}_k$ Operators

As stated in the Introduction, our intention is to study the Cartan formalism for variational problems and in this formalism a central object is the so-called Cartan form for field theory, denoted Θ_0^L and locally given by

$$\begin{aligned}\Theta_0^L = \sum_{i=1}^{m} \sum_{\mu=-s}^{r} &(-1)^{m+i} d^G x^1 \wedge \cdots \wedge \widehat{d^G x^i} \wedge \cdots \wedge d^G x^m \\ &\wedge \left(d^G y^\mu - \sum_{\alpha=-n}^{m} d^G x^\alpha \cdot y_\alpha^\mu \right) \frac{\partial L}{\partial y_i^\mu} + \eta^G \cdot L. \end{aligned} \tag{5.1}$$

We will provide an intrinsic construction of Θ_0^L and we will develop from it a consistent theory of the first-order calculus of variations on supermanifolds. The idea is the same as those used in the formulation of mechanics (see [20, 41]), but with some new details that arise because this time we deal with fields (for an interesting discussion of the classical formalism in this case see also [17] and [18]); let us describe it very briefly.

The graded generalization of the vertical endomorphism of the tangent bundle used in classical mechanics would be (unlike the case of mechanics, note that $\tilde{\mathcal{J}}$ is *not* an endomorphism here):

$$\tilde{\mathcal{J}} \doteq \sum_{i=1}^{m} \sum_{\mu=-s}^{r} (-1)^{m+i} d^G x^1 \wedge \cdots \wedge \widehat{d^G x^i} \wedge \cdots \wedge d^G x^m \wedge d^G y^\mu \otimes \frac{\partial}{\partial y_i^\mu}.$$

Also, for each $\alpha \in \{-n, \ldots, -1, 1, \ldots, m\}$, $i \in \{1, \ldots, m\}$, the graded analogue of the Liouville vector field would be

$$\Delta_{\alpha i} = \sum_{\mu=-s}^{r} (-1)^{m+i} y_\alpha^\mu \frac{\partial}{\partial y_i^\mu},$$

and finally (by using Einstein's convention, from now on we omit the summation symbols),

$$\mathcal{J} \doteq \tilde{\mathcal{J}} - d^G x^1 \wedge \cdots \wedge \widehat{d^G x^i} \wedge \cdots \wedge d^G x^m \wedge d^G x^\alpha \otimes \Delta_{\alpha i}.$$

Let us evaluate $\mathcal{L}^G_{\mathcal{J}}(L)$. It will be useful to bear in mind the developed expression for $\mathcal{J}$:

$$\mathcal{J} = (-1)^{m+i} d^G x^1 \wedge \cdots \wedge \widehat{d^G x^i} \wedge \cdots \wedge d^G x^m \wedge \left(d^G y^\mu - d^G x^\alpha \cdot y^\mu_\alpha\right) \otimes \frac{\partial}{\partial y^\mu_i}. \tag{5.2}$$

Note that

$$\begin{aligned}\iota_{\frac{\partial}{\partial y^\mu_i}}(d^G L) &= \iota_{\frac{\partial}{\partial y^\mu_i}}\left(d^G x^\alpha \cdot \frac{dL}{dx^\alpha} + d^G y^\nu \cdot \frac{\partial L}{\partial y^\nu} + d^G y^\nu_\alpha \cdot \frac{\partial L}{\partial y^\nu_\alpha}\right) \\ &= \frac{\partial L}{\partial y^\nu_i}.\end{aligned}$$

We then have

$$\begin{aligned}\mathcal{L}^G_{\mathcal{J}}(L) &= \iota_{\mathcal{J}}(d^G L) \\ &= (-1)^{m+i} d^G x^1 \wedge \cdots \wedge \widehat{d^G x^i} \wedge \cdots \wedge d^G x^m \\ &\quad \wedge \left(d^G y^\mu - d^G x^\alpha \cdot y^\mu_\alpha\right) \cdot \iota_{\frac{\partial}{\partial y^\mu_i}}(d^G L) \\ &= (-1)^{m+i} d^G x^1 \wedge \cdots \wedge \widehat{d^G x^i} \wedge \cdots \wedge d^G x^m \\ &\quad \wedge \left(d^G y^\mu - d^G x^\alpha \cdot y^\mu_\alpha\right) \cdot \frac{\partial L}{\partial y^\mu_i} \\ &= \Theta^L_0 - d^G x^1 \wedge \cdots \wedge d^G x^m \cdot L,\end{aligned}$$

so that

$$\Theta^L_0 = \mathcal{L}^G_{\mathcal{J}}(L) + \eta^G \cdot L.$$

Thus, to have Θ^L_0 intrinsically defined, there remains to prove that this is the case for $\mathcal{J}$. Notice that $\mathcal{J}$ is the graded analogue of the $(1, m)$-tensor field S_η that appears in [41] (for arbitrary m, see pp. 156–158). We will now study the intrinsic construction of these objects in the graded context, but the generalization is not straightforward, as the classical point constructions are not applicable now.

5.1. *Algebraic preliminaries*

Let $p : (N, \mathcal{B}) \to (M, \mathcal{A})$ be a graded submersion. Consider the cotangent supervector bundle $\mathcal{ST}^*(M, \mathcal{A}) \to (M, \mathcal{A})$, and its pull-back $p^*\mathcal{ST}^*(M, \mathcal{A})$ to $(N, \mathcal{B})$. Furthermore, let $\mathcal{V}(p) \subset \mathcal{ST}(N, \mathcal{B})$ be the vertical sub-bundle of p. This is the supervector bundle on $(N, \mathcal{B})$ defined by the short exact sequence

$$0 \to \mathcal{V}(p) \to \mathcal{ST}(N, \mathcal{B}) \xrightarrow{p_*} p^*\mathcal{ST}(M, \mathcal{A}) \to 0. \tag{5.3}$$

We can thus construct the tensor product supervector bundle

$$\pi : p^*\mathcal{ST}^*(M, \mathcal{A}) \otimes \mathcal{V}(p) \to J^0_G(p) \simeq (N, \mathcal{B}).$$

From a homological point of view, we have a natural identification

$$p^*\mathcal{ST}^*(M,\mathcal{A})\otimes\mathcal{V}(p)\simeq\operatorname{Hom}\left(p^*\mathcal{ST}(M,\mathcal{A}),\mathcal{V}(p)\right),$$

and within this algebraic setting, we can obtain a representation for $J^1_G(p)$ by considering the short exact sequence (5.3) and thinking of $J^1_G(p)$ as being the space of its splittings.

Proposition 5.1. *Let $p:(N,\mathcal{B})\to(M,\mathcal{A})$ be a graded submersion with dimensions $(m|n)=\dim(M,\mathcal{A})$, $(m+r|n+s)=\dim(N,\mathcal{B})$. A unique isomorphism*

$$p^*_{10}(p^*\mathcal{ST}^*(M,\mathcal{A})\otimes\mathcal{V}(p))\simeq\mathcal{V}(p_{10}),\tag{5.4}$$

exists, which is given by

$$d^Gx^i\otimes\frac{\partial}{\partial y^\mu}\mapsto\frac{\partial}{\partial y^\mu_i},\tag{5.5}$$

on every fibered coordinate system (3.1).

Proof. As the formula (5.5) completely determines isomorphism (5.4), we need only to prove that the isomorphism is independent of the fibered coordinate system chosen. This reduces to compute how the tensor fields in formula (5.5) transform under a change of fibered coordinates, from

$$\left.\begin{array}{ll}(x^\alpha), & \alpha=-n,\dots,-1,1,\dots,m,\\ (y^\mu), & \mu=-s,\dots,-1,1,\dots,r,\end{array}\right\}\tag{5.6}$$

to

$$\left.\begin{array}{ll}(\bar{x}^\beta), & \alpha=-n,\dots,-1,1,\dots,m,\\ (\bar{y}^\nu), & \nu=-s,\dots,-1,1,\dots,r,\end{array}\right\}\tag{5.7}$$

and the corresponding change in $J^1_G(p)$. From the very definition of y^ρ_γ as a coordinate in $J^1_G(p)$ (e.g. see [24, Section 1]), we can compute

$$\begin{aligned}d^G\bar{y}^\nu\otimes\frac{\partial}{\partial\bar{x}^\beta}&=\left(d^Gx^\alpha\frac{\partial\bar{y}^\nu}{\partial x^\alpha}+d^Gy^\mu\frac{\partial\bar{y}^\nu}{\partial y^\mu}\right)\otimes\frac{\partial x^\sigma}{\partial\bar{x}^\beta}\frac{\partial}{\partial x^\sigma}\\&=(-1)^{\alpha(\alpha+\nu+\sigma+\beta)}\frac{\partial\bar{y}^\nu}{\partial x^\alpha}\frac{\partial x^\sigma}{\partial\bar{x}^\beta}d^Gx^\alpha\otimes\frac{\partial}{\partial x^\sigma}\\&\quad+(-1)^{\mu(\mu+\nu+\sigma+\beta)}\frac{\partial\bar{y}^\nu}{\partial y^\mu}\frac{\partial x^\sigma}{\partial\bar{x}^\beta}d^Gy^\mu\otimes\frac{\partial}{\partial x^\sigma}.\end{aligned}$$

By passing to coordinates in $J^1_G(p)$ this tells us the following:

$$\bar{y}^\nu_\beta=(-1)^{\alpha(\nu+\beta)}\frac{\partial\bar{y}^\nu}{\partial x^\alpha}\frac{\partial x^\alpha}{\partial\bar{x}^\beta}+(-1)^{\mu(\mu+\nu+\sigma+\beta)}\frac{\partial\bar{y}^\nu}{\partial y^\mu}\frac{\partial x^\sigma}{\partial\bar{x}^\beta}y^\mu_\sigma.\tag{5.8}$$

With this expression in mind, we are going to compute the graded 1-form $d^G\bar{y}^\nu_\beta$. Initially, we should have

$$d^G\bar{y}^\nu_\beta = d^G x^\gamma \frac{\partial \bar{y}^\nu_\beta}{\partial x^\gamma} + d^G y^\mu \frac{\partial \bar{y}^\nu_\beta}{\partial y^\mu} + d^G y^\mu_\alpha \frac{\partial \bar{y}^\nu_\beta}{\partial y^\mu_\alpha},$$

so that we should consider each term separately, but, in fact, as we will compute $\partial/\partial\bar{y}^\nu_\beta$ by applying duality, we need only to compute the coefficient of $d^G y^\mu_\alpha$, which is given by (5.8):

$$\begin{aligned}\frac{\partial \bar{y}^\nu_\beta}{\partial y^\mu_\alpha} &= \frac{\partial}{\partial y^\mu_\alpha}\left\{(-1)^{\rho(\rho+\nu+\sigma+\beta)}\frac{\partial \bar{y}^\nu}{\partial y^\rho}\frac{\partial x^\sigma}{\partial \bar{x}^\beta} y^\rho_\sigma\right\}\\ &= (-1)^{\rho(\rho+\nu+\sigma+\beta)+(\mu+\alpha)(\rho+\nu+\sigma+\beta)}\frac{\partial \bar{y}^\nu}{\partial y^\rho}\frac{\partial x^\sigma}{\partial \bar{x}^\beta}\delta^\rho_\mu\delta^\alpha_\sigma\\ &= (-1)^{\alpha(\mu+\nu+\alpha+\beta)}\frac{\partial \bar{y}^\nu}{\partial y^\mu}\frac{\partial x^\alpha}{\partial \bar{x}^\beta}.\end{aligned}$$

We can then write

$$d^G\bar{y}^\nu_\beta = d^G x^\gamma A^\nu_{\beta\gamma} + d^G y^\mu B^\nu_{\beta\mu} + (-1)^{\alpha(\mu+\nu+\alpha+\beta)} d^G y^\mu_\alpha \frac{\partial \bar{y}^\nu}{\partial y^\mu}\frac{\partial x^\alpha}{\partial \bar{x}^\beta}, \tag{5.9}$$

where $A^\nu_{\beta\gamma}$, $B^\nu_{\beta\mu}$ are coefficients whose explicit expression is not needed.

We also remark

$$\begin{cases} d^G\bar{y}^\nu = d^G x^\alpha \dfrac{\partial \bar{y}^\nu}{\partial x^\alpha} + d^G y^\mu \dfrac{\partial \bar{y}^\nu}{\partial y^\mu},\\ d^G\bar{x}^\beta = d^G x^\alpha \dfrac{\partial \bar{x}^\beta}{\partial x^\alpha}.\end{cases} \tag{5.10}$$

Next, we consider the expression for $\partial/\partial\bar{y}^\mu_\gamma$ as a tangent vector on $J^1_G(p)$. Initially, we should have

$$\frac{\partial}{\partial \bar{y}^\mu_\gamma} = K^\sigma \frac{\partial}{\partial x^\sigma} + L^\eta \frac{\partial}{\partial y^\eta} + P^{\rho\tau}_{\sigma\gamma}\frac{\partial}{\partial y^\tau_\gamma},$$

and we can compute the action of the basic differentials (5.9) and (5.10) on it. This gives

$$\begin{aligned} 0 &= \left\langle \frac{\partial}{\partial \bar{y}^\mu_\gamma}; d^G\bar{x}^\beta\right\rangle\\ &= K^\sigma \frac{\partial \bar{x}^\beta}{\partial x^\sigma},\\ 0 &= \left\langle \frac{\partial}{\partial \bar{y}^\mu_\gamma}; d^G\bar{y}^\nu\right\rangle\\ &= L^\eta \frac{\partial \bar{y}^\nu}{\partial y^\eta},\\ \delta^\rho_\nu\delta^\beta_\sigma &= \left\langle \frac{\partial}{\partial \bar{y}^\mu_\gamma}; d^G\bar{y}^\nu_\beta\right\rangle\\ &= (-1)^{\alpha(\mu+\nu+\alpha+\beta)} P^{\rho\mu}_{\sigma\alpha}\frac{\partial \bar{y}^\nu}{\partial y^\mu}\frac{\partial x^\alpha}{\partial \bar{x}^\beta},\end{aligned}$$

and from these equations, we obtain

$$\begin{cases} K^\gamma = 0, \\ L^\mu = 0, \\ P^{\rho\mu}_{\sigma\alpha} = (-1)^{\alpha(\mu+\nu+\alpha+\beta)} \dfrac{\partial \bar{x}^\sigma}{\partial x^\alpha} \dfrac{\partial y^\mu}{\partial \bar{y}^\rho}. \end{cases}$$

Hence, the law of transformation for $\partial/\partial\bar{y}^\rho_\sigma$ is

$$\frac{\partial}{\partial \bar{y}^\rho_\sigma} = (-1)^{\alpha(\mu+\rho+\alpha+\sigma)} \frac{\partial \bar{x}^\sigma}{\partial x^\alpha} \frac{\partial y^\mu}{\partial \bar{y}^\rho} \frac{\partial}{\partial y^\mu_\alpha}.$$

This coincides with the law of transformation for $d^G\bar{x}^\sigma \otimes \partial/\partial\bar{y}^\rho$. Indeed,

$$\begin{aligned} d^G\bar{x}^\sigma \otimes \frac{\partial}{\partial \bar{y}^\rho} &= d^G x^\alpha \frac{\partial \bar{x}^\sigma}{\partial x^\alpha} \otimes \frac{\partial y^\mu}{\partial \bar{y}^\rho} \frac{\partial}{\partial y^\mu} \\ &= (-1)^{\alpha(\alpha+\sigma+\rho+\mu)} \frac{\partial \bar{x}^\sigma}{\partial x^\alpha} \frac{\partial y^\mu}{\partial \bar{y}^\rho} d^G x^\alpha \otimes \frac{\partial}{\partial y^\mu}. \end{aligned}$$

Thus, the isomorphism in the statement of the proposition is well defined. □

5.2. *Intrinsic construction of $\mathcal{J}$*

Let $p : (N, \mathcal{B}) \to (M, \mathcal{A})$ be a graded submersion. On the module of the graded vector fields on the graded 1-jet bundle $(J^1_G(p), \mathcal{A}_{J^1_G(p)})$, a $\mathcal{V}(p_{10})$-valued mapping acting upon m arguments, is defined as follows:

$$\mathcal{J}(D_1, \dots, D_m) = (-1)^{j+m} \iota_{D_{(\hat{\jmath})}} \left(p_1^*(d^G x^1 \wedge \cdots \wedge d^G x^m)\right) \otimes \theta(D_j), \tag{5.11}$$

where $\iota_{D_{(\hat{\jmath})}} = \iota_{D_m} \circ \cdots \circ \widehat{\iota_{D_j}} \circ \cdots \circ \iota_{D_1}$ and

$$\theta = (d^G y^\mu - d^G x^\alpha \cdot y^\mu_\alpha) \otimes \frac{\partial}{\partial y^\mu}, \tag{5.12}$$

is intrinsically defined in [22, Theorem 1.7], and $d^G x^1 \wedge \cdots \wedge d^G x^m$ comes from a volume form η on M, $\eta = dx^1 \wedge \cdots \wedge dx^m$, so that $\mathcal{J}$ is an intrinsic object.

Proposition 5.2. *The operator $\mathcal{J}$ defined by* (5.11) *is a graded m-form.*

Proof. First, multilinearity is a consequence of that of θ and the properties of the insertion operator,

$$\begin{aligned} \iota_{\alpha\cdot D}\Lambda &= \alpha \wedge \iota_D \Lambda, \\ \iota_{D_1+D_2}\Lambda &= \iota_{D_1}\Lambda + \iota_{D_2}\Lambda, \end{aligned}$$

for all $D \in \mathcal{X}_G(J^1_G(p))$, $\alpha \in \mathcal{A}_{J^1_G(p)}$, $\Lambda \in \Omega^1_G(J^1_G(p))$. Second, we have skew symmetry, which is rather obvious in view that $\iota_{D_1}\iota_{D_2} = -\iota_{D_2}\iota_{D_1}$. □

Now, we must check if the local expression for J obtained from (5.11) gives the expression we want (5.2). Because of the term $p_1^*(d^G x^1 \wedge \cdots \wedge d^G x^m)$, we just need to evaluate

$$\mathcal{J}\left(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^m}\right), \quad \mathcal{J}\left(\frac{\partial}{\partial x^1}, \dots, \widehat{\frac{\partial}{\partial x^i}}, \dots, \frac{\partial}{\partial x^m}, \frac{\partial}{\partial x^{-j}}\right),$$

and

$$\mathcal{J}\left(\frac{\partial}{\partial x^1},\dots,\widehat{\frac{\partial}{\partial x^i}},\dots,\frac{\partial}{\partial x^m},\frac{\partial}{\partial y^\nu}\right),$$

where $i \in \{1,\dots,m\}$, $\nu \in \{-s,\dots,-1,1,\dots,r\}$. Now, we have

$$\begin{aligned}
&(-1)^{m-1}\mathcal{J}\left(\frac{\partial}{\partial x^1},\dots,\frac{\partial}{\partial x^m}\right)\\
&\quad= (-1)^{j-1}\iota_{\frac{\partial}{\partial x^m}}\circ\cdots\circ\widehat{\iota_{\frac{\partial}{\partial x^j}}}\circ\cdots\circ\iota_{\frac{\partial}{\partial x^1}}p_1^*(d^Gx^1\wedge\cdots\wedge d^Gx^m)\otimes\theta\left(\frac{\partial}{\partial x^j}\right)\\
&\quad= (-1)^{j-1}\iota_{\frac{\partial}{\partial x^m}}\circ\cdots\circ\widehat{\iota_{\frac{\partial}{\partial x^j}}}\circ\cdots\circ\iota_{\frac{\partial}{\partial x^1}}p_1^*(d^Gx^1\wedge\cdots\wedge d^Gx^m)\otimes\left(-y_j^\mu\frac{\partial}{\partial y^\mu}\right)\\
&\quad= (-1)^m d^Gx^j\otimes y_j^\mu\frac{\partial}{\partial y^\mu}\\
&\quad= (-1)^m d^Gx^j\cdot y_j^\mu\otimes\frac{\partial}{\partial y^\mu}\\
&\quad\simeq (-1)^m y_j^\mu\frac{\partial}{\partial y_j^\mu},
\end{aligned}$$

where the last identification comes from (5.5). Also,

$$\begin{aligned}
&(-1)^{m-1}\mathcal{J}\left(\frac{\partial}{\partial x^1},\dots,\widehat{\frac{\partial}{\partial x^i}},\dots,\frac{\partial}{\partial x^m},\frac{\partial}{\partial x^{-j}}\right)\\
&\quad= (-1)^{m-1}\iota_{\frac{\partial}{\partial x^m}}\circ\cdots\circ\widehat{\iota_{\frac{\partial}{\partial x^i}}}\circ\cdots\circ\iota_{\frac{\partial}{\partial x^1}}p_1^*(d^Gx^1\wedge\cdots\wedge d^Gx^m)\otimes\theta\left(\frac{\partial}{\partial x^{-j}}\right)\\
&\quad= (-1)^{m-1}(-1)^{m-i}d^Gx^i\otimes\theta\left(\frac{\partial}{\partial x^{-j}}\right)\\
&\quad= -(-1)^i d^Gx^i\otimes\left(-y_{-j}^\mu\frac{\partial}{\partial y^\mu}\right)\\
&\quad= (-1)^i d^Gx^i\cdot y_{-j}^\mu\otimes\frac{\partial}{\partial y^\mu}\\
&\quad= (-1)^i y_{-j}^\mu d^Gx^i\otimes\frac{\partial}{\partial y^\mu}\\
&\quad\simeq (-1)^i y_{-j}^\mu\frac{\partial}{\partial y_i^\mu}.
\end{aligned}$$

Moreover, by noting that each term $\iota_{\frac{\partial}{\partial y^\gamma}}p_1^*(d^Gx^1\wedge\cdots\wedge d^Gx^m)$ vanishes, we obtain

$$\begin{aligned}
&(-1)^{m-1}\mathcal{J}\left(\frac{\partial}{\partial x^1},\dots,\widehat{\frac{\partial}{\partial x^i}},\dots,\frac{\partial}{\partial x^m},\frac{\partial}{\partial y^\nu}\right)\\
&\quad= (-1)^{m-1}\iota_{\frac{\partial}{\partial x^m}}\circ\cdots\circ\widehat{\iota_{\frac{\partial}{\partial x^i}}}\circ\cdots\circ\iota_{\frac{\partial}{\partial x^1}}p_1^*(d^Gx^1\wedge\cdots\wedge d^Gx^m)\otimes\theta\left(\frac{\partial}{\partial y^\nu}\right)
\end{aligned}$$

$$= (-1)^{i-1} d^G x^i \otimes \frac{\partial}{\partial y^\nu}$$
$$\simeq (-1)^{i-1} \frac{\partial}{\partial y_i^\nu}.$$

Thus, we conclude that the local expression for $\mathcal{J}$ is

$$\begin{aligned}
(-1)^{m-1}\mathcal{J} &= (-1)^m d^G x^1 \wedge \cdots \wedge d^G x^m \otimes y_i^\mu \frac{\partial}{\partial y_i^\mu} \\
&\quad + (-1)^i d^G x^1 \wedge \cdots \wedge \widehat{d^G x^i} \wedge \cdots \wedge d^G x^m \wedge d^G x^{-j} \otimes y_{-j}^\mu \frac{\partial}{\partial y_i^\mu} \\
&\quad + (-1)^{i-1} d^G x^1 \wedge \cdots \wedge \widehat{d^G x^i} \wedge \cdots \wedge d^G x^m \wedge d^G y^\nu \otimes \frac{\partial}{\partial y_i^\nu} \\
&= (-1)^i d^G x^1 \wedge \cdots \wedge \widehat{d^G x^i} \wedge \cdots \wedge d^G x^m \wedge d^G x^i \cdot y_i^\mu \otimes \frac{\partial}{\partial y_i^\mu} \\
&\quad + (-1)^i d^G x^1 \wedge \cdots \wedge \widehat{d^G x^i} \wedge \cdots \wedge d^G x^m \wedge d^G x^{-j} \cdot y_{-j}^\mu \otimes \frac{\partial}{\partial y_i^\mu} \\
&\quad + (-1)^{i-1} d^G x^1 \wedge \cdots \wedge \widehat{d^G x^i} \wedge \cdots \wedge d^G x^m \wedge d^G y^\mu \otimes \frac{\partial}{\partial y_i^\mu} \\
&= (-1)^i d^G x^1 \wedge \cdots \wedge \widehat{d^G x^i} \wedge \cdots \wedge d^G x^m \wedge \left(d^G x^\alpha \cdot y_\alpha^\mu - d^G y^\mu \right) \otimes \frac{\partial}{\partial y_i^\mu}.
\end{aligned}$$

Hence,

$$\mathcal{J} = (-1)^{m-1} \iota_{\frac{\partial}{\partial x^i}} \eta^G \wedge \theta^\mu \otimes \frac{\partial}{\partial y_i^\mu},$$

where $\theta^\mu = d^G y^\mu - d^G x^\alpha \cdot y_\alpha^\mu$ is the horizontal differential of y^μ, and this is precisely (5.2) written in a more compact form.

In this way, we have constructed a canonical $\mathcal{V}(p_{10})$-valued m-form $\mathcal{J}$ for any graded submersion $p : (N, \mathcal{B}) \to (M, \mathcal{A})$. This is appropriate for the case of graded mechanics, but if we want to study graded fields, we must go on to higher-order jet bundles; let us see how to extend the previous construction to $J_G^k((N, \mathcal{B}), (M, \mathcal{A})) \equiv (J_G^k(p), \mathcal{A}_{J_G^k(p)})$ for any k.

5.3. *Intrinsic construction of $\mathcal{J}_k$*

Consider the following submersion playing the rôle of p in previous sections: $p_{k-1} : J_G^{k-1}(p) \to (M, \mathcal{A})$. Then, the preceding construction tells us that we have defined a $\mathcal{J}$ on the graded bundle $J_G^1((J_G^{k-1}(p), \mathcal{A}_{J_G^{k-1}(p)}), (M, \mathcal{A})) \doteq J_G^1(p_{k-1})$, which is a graded m-form with values on $\mathcal{V}((p_1^{k-1})_{10})$ that will be denoted $\mathcal{J}_k$ (here, p_1^{k-1} is defined by $p_1^{k-1} : J_G^1(p_{k-1}) \to (M, \mathcal{A})$, and $(p_1^{k-1})_{10} : J_G^1(p_1^{k-1}) \to J_G^1(p_{k-1})$ the target projection).

If $(x^\alpha, y^\mu, z_Q^\mu)$, $1 \le |Q| \le k-1$, is a system of coordinates for $J_G^{k-1}(p)$, then $(x^\alpha, y^\mu, z_Q^\mu, w_R^\mu)$ (with $1 \le |R| \le k$, $1 \le |Q| \le k-1$, recall that Q, R denote arbitrary multi-indices) is a system for $J^1_G(p_{k-1})$, and we have the local expression (with $1 \leq i \leq m$ a *positive* index)

$$\mathcal{J}_k = (-1)^{m-1} \iota_{\frac{\partial}{\partial x^i}} \eta^G \wedge \left(\theta^{y^\mu} \otimes \frac{\partial}{\partial y^\mu_i} + \theta^{z^\mu_Q} \otimes \frac{\partial}{\partial z^\mu_{i+Q}} + \theta^{w^\mu_R} \otimes \frac{\partial}{\partial w^\mu_{i+R}} \right) \tag{5.13}$$

(it must be noted that we are using the canonical identification (5.5) in writing $\frac{\partial}{\partial w^\mu_{i+R}}$, also, note that the sum $\iota_{\frac{\partial}{\partial x^i}} \eta^G \wedge \theta^{w^\mu_R} \otimes \frac{\partial}{\partial w^\mu_{i+R}}$ only runs up to $|R| = k-1$), where $\theta^{z^\mu_Q} = d^G z^\mu_Q - d^G x^\alpha \cdot z^\mu_{\alpha\star Q}$ and so on. Now, we observe that there exists a canonical graded immersion $J^k_G(p) \overset{\Psi}{\hookrightarrow} J^1_G(p_{k-1})$, which expressed through its action on coordinates, reads (here, $(x^\alpha, y^\mu, v^\mu_Q)$, $1 \leq |Q| \leq k$, is a system of coordinates for $J^k_G(p)$):

$$\left.\begin{aligned} \Psi^*(x^\alpha) &= x^\alpha \\ \Psi^*(y^\mu) &= y^\mu \\ \Psi^*(z^\mu_Q) &= v^\mu_Q \\ \Psi^*(w^\mu_R) &= v^\mu_R \end{aligned}\right\}. \tag{5.14}$$

Of course, when acting upon jet extensions of sections σ, this action reads

$$\Psi^*(j^1(j^{k-1}(\sigma))) = j^k(\sigma).$$

Now, it is clear that (as Ψ^* commutes with d^G),

$$\begin{aligned} \Psi^*\left(\iota_{\frac{\partial}{\partial x^i}} \eta^G\right) &= (-1)^{i-1} \Psi^*(d^G x^1 \wedge \cdots \wedge \widehat{d^G x^i} \wedge \cdots \wedge d^G x^m) \\ &= \iota_{\frac{\partial}{\partial x^i}} \eta^G, \\ \Psi^*(\theta^{y^\mu}) &= \Psi^*(d^G y^\mu - d^G x^\alpha \cdot y^\mu_\alpha) \\ &= d^G y^\mu - d^G x^\alpha \cdot y^\mu_\alpha \\ &= \theta^\mu, \\ \Psi^*(\theta^{z^\mu_Q}) &= \Psi^*(d^G z^\mu_Q - d^G x^\alpha \cdot z^\mu_{\alpha\star Q}) \\ &= d^G y^\mu_Q - d^G x^\alpha \cdot y^\mu_{\alpha\star Q} \\ &= \theta^\mu_Q, \end{aligned}$$

and so on. We can apply Ψ^* to (5.13) to obtain a graded m-form on $J^k_G(p)$; according to the preceding observations, the only terms that represent some problem are those duplicated in $\partial/\partial z^\mu_{i+Q}$ and $\partial/\partial w^\mu_{i+R}$ (see (5.14)). But these terms are precisely the ones coming from a single supervector on $J^k_G(p)$ through (5.14); to be more precise, let us study $\Psi_*(\partial/\partial v^\mu_{i+Q})$. We would like to see that

$$\Psi_*\left(\frac{\partial}{\partial v^\mu_{i+Q}}\right) = \frac{\partial}{\partial z^\mu_{i+Q}} + \frac{\partial}{\partial w^\mu_{i+Q}},$$

as an extreme case we have $|Q| = k$, but then this reduces to

$$\Psi_*\left(\frac{\partial}{\partial v^\mu_{i+Q}}\right) = \frac{\partial}{\partial w^\mu_{i+Q}},$$

and as we have the canonical identification (5.5), what we really want is to prove

$$\Psi_*\left(\frac{\partial}{\partial v^\mu_Q}\right) = \frac{\partial}{\partial z^\mu_Q} + \frac{\partial}{\partial w^\mu_Q},$$

for an arbitrary multiindex Q.

Thus, consider the action of

$$\Psi_*\left(\frac{\partial}{\partial v^\mu_Q}\right).$$

We have

$$\begin{aligned}
\Psi_*\left(\frac{\partial}{\partial v^\mu_Q}\right)(z^\nu_R) &= \frac{\partial}{\partial v^\mu_Q}(\Psi^*(z^\nu_R)) \\
&= \frac{\partial}{\partial v^\mu_Q} v^\nu_R \\
&= \delta^\nu_\mu \delta^R_Q, \\
\Psi_*\left(\frac{\partial}{\partial v^\mu_Q}\right)(w^\nu_R) &= \frac{\partial}{\partial v^\mu_Q}(\Psi^*(w^\nu_R)) \\
&= \frac{\partial}{\partial v^\mu_Q} v^\nu_R \\
&= \delta^\nu_\mu \delta^R_Q,
\end{aligned}$$

and this is precisely the action of $\partial/\partial z^\mu_Q + \partial/\partial w^\mu_Q$, as wanted.

As a consequence, we have the following result (see [1, 2] for a classical version):

Theorem 5.3. *On $J^k_G(p)$ (for any k) there is defined a canonical graded m-form with values on $\mathcal{V}((p_k)_{10}) \subset \mathcal{V}((p_1^{k-1})_{10})$, which we denote by $\mathcal{J}_k$, and whose local expression is*

$$\mathcal{J}_k = (-1)^{m-1} \iota_{\frac{\partial}{\partial x^i}} \eta^G \wedge \theta^\mu_Q \otimes \frac{\partial}{\partial y^\mu_{i+Q}} \quad (1 \leq i \leq m = \dim M),$$

being $0 \leq |Q| \leq k-1$, with the usual convention $\theta^\mu_Q = \theta^\mu$ when $|Q| = 0$.

Remark 5.4. In the statement of the theorem, we are writing collectively θ^μ_Q instead of $\theta^{z^\mu_Q}$ and $\theta^{w^\mu_R}$ (it is a shorthand for (5.13)).

Generalizing the classical expression (see, for instance [41, Theorem 5.5.2]), for any $L \in \mathcal{A}_{J^k_G(p)}$, we define the graded m-form (the so called *Poincaré–Cartan form of order k*)

$$\tilde{\Theta}^L = \mathcal{L}^G_{\mathcal{J}_k}(L) + \eta^G \cdot L.$$

Let us make a remark. Let $\mathcal{A} \xrightarrow{\sim} C^\infty(M)$ be the structural morphism and $C^\infty(M) \xrightarrow{\sigma} \mathcal{A}$ a global section of it. Then, to every volume form η on M we can associate a graded volume form $\eta^G = \sigma(\eta)$ on $(M, \mathcal{A})$. On the other hand, note that a graded Lagrangian density is an m-form of the type

$$\eta^G \cdot L, L \in \mathcal{A}_{J^k_G(p)}.$$

Thus, if we change the volume form η on M to a form $\mu = \eta \cdot f$ where f is a differentiable function on M, $f \in C^\infty(M)$, we will have a induced change in the Lagrangian:

$$\eta^G \cdot f \cdot L.$$

Moreover, recall that from the local expression of the $\mathcal{J}_k$ morphism (5.13) it is clear that replacing η for μ amounts to passing from $\mathcal{J}_k$ to $f \cdot \mathcal{J}_k$. Putting these observations together we get (introducing temporarily an obvious notation to distinguish which graded volume form is in use):

$$\begin{aligned}\tilde{\Theta}^L_\mu &= \mathcal{L}^G_{\mathcal{J}^\mu_k}(L) + \mu^G \cdot L \\ &= \mathcal{L}^G_{f \cdot \mathcal{J}^\eta_k}(L) + \eta^G \cdot f \cdot L \\ &= \mathcal{L}^G_{\mathcal{J}^\eta_k}(f \cdot L) + \eta^G (f \cdot L),\end{aligned}$$

where in the last step use has been made of the fact that $f \in C^\infty(M)$ is not affected by the derivative on the fiber coordinates, carried on by $\mathcal{L}^G_{\mathcal{J}^\mu_k}$. If we denote $f \cdot L \in \mathcal{A}_{J^k_G(p)}$ by L_f, what we have obtained is

$$\tilde{\Theta}^L_\mu = \tilde{\Theta}^{L_f}_\eta,$$

so the graded Poincaré–Cartan form $\tilde{\Theta}^L$ is well behaved under the decomposition "graded Lie derivative plus graded Lagrangian density".

6. Equivalence between Graded and Berezinian Variational Problems

Let us make some remarks about the correspondence between Berezinian and graded variational problems. It is well known how to obtain the equations of the solutions to a graded variational problem (see [22]); on the contrary, for Berezinian problems an intrinsic formulation in Cartan's spirit has been not available up until now. What does exist, is a way (based on the Comparison Theorem) to associate to each graded problem a Berezinian one and to establish a correspondence between their solutions. The basic idea is as follows: given a Lagrangian $L \in \mathcal{A}_{J^1_G(p)}$, let ξ_L be the first-order Berezinian density that it determines, which is given by

$$\left[d^G x^1 \wedge \cdots \wedge d^G x^m \otimes \frac{d}{dx^{-1}} \circ \cdots \circ \frac{d}{dx^{-n}} \right] \cdot L,$$

and let

$$\lambda_{\xi_L} = d^G x^1 \wedge \cdots \wedge d^G x^m \cdot \frac{d^n L}{dx^{-1} \cdots dx^{-n}}$$

be the corresponding graded Lagrangian density. In [22], to each first order graded Lagrangian density λ a canonical graded m-form is associated, the graded Poincaré–Cartan form for the Lagrangian density λ_{ξ_L}. Here, we denote by Θ_0^L the graded Poincaré–Cartan form corresponding to $-\lambda_L$; in local coordinates, it is given by expression (5.1) and, as we have proved, it can be constructed as an intrinsic object. Now, as

$$\lambda_{\xi_L} = \mathcal{L}^G_{\frac{d}{dx^{-1}}} \circ \cdots \circ \mathcal{L}^G_{\frac{d}{dx^{-n}}}(d^G x^1 \wedge \cdots \wedge d^G x^m \cdot L),$$

it is natural to consider the graded m-form

$$\Theta^L = \mathcal{L}^G_{\frac{d}{dx^{-1}}} \circ \cdots \circ \mathcal{L}^G_{\frac{d}{dx^{-n}}} \Theta_0^L \tag{6.1}$$

as the graded Poincaré–Cartan form for the Berezinian density ξ_L. But we could as well follow other way to define Θ^L: Instead of taking the first-order Lagrangian density $d^G x^1 \wedge \cdots \wedge d^G x^m \cdot L$, construct $\Theta_0^L = \mathcal{L}^G_{\mathcal{J}}(L) + \eta^G \cdot L$ and then apply

$$\mathcal{L}^G_{\frac{d}{dx^{-1}}} \circ \cdots \circ \mathcal{L}^G_{\frac{d}{dx^{-n}}},$$

we could have considered the Lagrangian density, of order $(n+1)$, $\frac{d^n L}{dx^{-1} \cdots dx^{-n}}$ and apply $\mathcal{L}^G_{\mathcal{J}_{n+1}}$ to obtain

$$\tilde{\Theta}^L = \mathcal{L}^G_{\mathcal{J}_{n+1}}\left(\frac{d^n L}{dx^{-1} \cdots dx^{-n}}\right) + \eta^G \cdot \frac{d^n L}{dx^{-1} \cdots dx^{-n}}.$$

The first procedure is designed to take benefit of the graded variational calculus, but has the handicap of presenting an expression like (6.1), with the factors

$$\mathcal{L}^G_{\frac{d}{dx^{-1}}} \circ \cdots \circ \mathcal{L}^G_{\frac{d}{dx^{-n}}}$$

destroying, at a first glance, covariance. On the other hand, once a volume form $\eta = dx^1 \wedge \cdots \wedge dx^m$ has been fixed on the base, the second one proceeds intrinsically to obtain

$$\frac{d^n L}{dx^{-1} \cdots dx^{-n}}$$

from the Berezinian density ξ_L and then $\tilde{\Theta}^L$, thus developing a Cartan formalism in an analogous manner to the classical tangent bundle formulation. Nevertheless, it would be desirable the convergence of the two ways, in the sense that $\Theta^L = \tilde{\Theta}^L$ for any $L \in \mathcal{A}_{J^1_G(p)}$; indeed, this is the case as we will see in Theorem 6.5.

We will need some notations and technical lemmas that also will be useful later.

6.1. *Preliminaries*

Let $B \in (\mathbb{Z}^-)^k$ be a strictly decreasing multi-index. For every $b \in B$, we define $p(b)$, $q(b)$ as follows:

$$p(b) = ((\text{position of } b \text{ in } B) - 1) \bmod 2,$$
$$q(b) = (\text{position of } b \text{ in } B) \bmod 2.$$

For example, if $B = (-1, -5, -7, -8)$, then $p(-7) = 0$, $q(-7) = 1$. The symbol $B-\{b\}$ denotes the $(|B|-1)$-multi-index obtained by removing b from B; e.g., in the previous example we have $B - \{-5\} = (-1, -7, -8)$. We also set $|Z|_2 = |Z| \bmod 2$, for every multi-index Z. For any pair of multi-indices

$$Q = (i_1, \dots, i_{|Q|}) \in \mathbb{Z}^{|Q|}, \quad B = (b_1, \dots, b_{|B|}) \in (\mathbb{Z}^-)^{|B|},$$

such that $|B| \geq |Q|$, we define $\varphi(Q, B)$ as follows:

$$\varphi(Q, B) = \sum_{k=1}^{|I|} i_k \varphi_k(b_k),$$

where

$$\varphi_k(b) = \begin{cases} p(b), & \text{if } k \equiv 1 \bmod 2 \\ q(b), & \text{if } k \equiv 0 \bmod 2 \end{cases}$$

and $\varphi(Q, B) = 0$, if $|Q| = 0$.

Finally, as usual, the symbol $\star$, applied to a pair of multi-indices, means juxtaposition.

In what follows, we denote by $\frac{d}{dx^\alpha}$ the graded horizontal lift of $\partial/\partial x^\alpha$ to $J^\infty_G(p)$, whose local expression is given in formula (3.4).

Lemma 6.1. *For any strictly decreasing multi-index $B \in (\mathbb{Z}^-)^k$, we have*

$$\left[\frac{\partial}{\partial y^\mu}, \frac{d^{|B|}}{dx^B}\right] = 0,$$

when acting on $\mathcal{A}_{J^r_G(p)}$.

Lemma 6.2. *Let k be a positive integer. Given $i_0 \in \mathbb{Z}$ and $j \in \{1, \dots, n\}$, for every $Q \in \mathbb{Z}^k$, we have*

$$\left[\frac{\partial}{\partial y^\mu_{\{i_0\}\star Q}}, \frac{d}{dx^{-j}}\right] = \delta^{-j}_{i_0} \frac{\partial}{\partial y^\mu_Q},$$

both sides acting on $\mathcal{A}_{J^r_G}(p)$.

Note $\partial/\partial y^\mu_{\{i_0\}\star Q}$ vanishes on $\mathcal{A}_{J^r_G}(p)$ whenever $|Q| > r$. In particular, for every $L \in \mathcal{A}_{J^1_G(p)}$ we have

$$\frac{\partial}{\partial y^\mu_\alpha}\left(\frac{dL}{dx^{-j}}\right) = \delta^{-j}_\alpha \frac{\partial L}{\partial y^\mu}, \quad \frac{\partial}{\partial y^\mu_{\alpha\beta}}\left(\frac{dL}{dx^{-j}}\right) = \delta^{-j}_\alpha \frac{\partial L}{\partial y^\mu_\beta},$$

but

$$\frac{\partial}{\partial y^\mu_Q}\left(\frac{dL}{dx^{-j}}\right) = 0, \quad \text{for } |Q| > 2.$$

Lemma 6.3. *For any strictly decreasing multi-index $B \in (\mathbb{Z}^-)^k$, we have*

$$\left[\frac{\partial}{\partial y^\mu_\alpha}, \frac{d^{|B|}}{dx^B}\right] = \sum_{b \in B} (-1)^{\mu(|B|_2+1)+\alpha \cdot p(b)} \delta^\alpha_b \frac{d^{|B|-1}}{dx^{B-\{b\}}} \frac{\partial}{\partial y^\mu},$$

when acting on $\mathcal{A}_{J^r_G(p)}$, *where it is assumed*

$$\frac{d^0 F}{dx^{\emptyset}} = F, \quad \forall F \in \mathcal{A}_{J^r_G(p)}.$$

In particular, we have

$$\frac{\partial}{\partial y^\mu_i}\frac{d^{|B|}F}{dx^B} = (-1)^{\mu|B|_2}\frac{d^{|B|}}{dx^B}\frac{\partial}{\partial y^\mu_i}, \quad \forall i > 0.$$

Proposition 6.4. *For every* $L \in \mathcal{A}_{J^1_G(p)}$, *every strictly decreasing* $B \in (\mathbb{Z}^-)^k$, *and every* $Q \in \mathbb{Z}^r$ *such that* $k \geq 2$, $1 \leq r \leq k$, *we have*

$$\frac{\partial}{\partial y^\mu_{\{i\}\star Q}}\frac{d^{|B|}L}{dx^B} = \sum_{\substack{-b_{c_1} > \cdots > -b_{c_{|Q|}} \\ b_{c_1},\ldots,b_{c_{|Q|}} \in B}} (-1)^{\mu(|B|_2+|Q|_2)+\varphi(Q,B)}\delta^{i_1}_{b_{c_1}}\cdots\delta^{i_{|Q|}}_{b_{c_{|Q|}}}\frac{d^{|B|-|Q|}}{dx^{B-\{b_{c_1},\ldots,b_{c_{|Q|}}\}}}\frac{\partial L}{\partial y^\mu_i}.$$

The proof of these results is a lengthy induction, but only involving standard computations.

6.2. *The main theorem*

In this subsection, as announced in the Introduction, we study the equivalence between first-order Berezinian variational problems and higher-order graded variational problems. As the computations are rather cumbersome, we will illustrate the general situation by considering the case $n = 2$ (that is, a base manifold of graded dimension $(m|2)$).

Theorem 6.5. *Let* ξ_L *be a first-order Berezinian density,*

$$\xi_L = \left[d^G x^1 \wedge \cdots \wedge d^G x^m \otimes \frac{d}{dx^{-1}} \circ \cdots \circ \frac{d}{dx^{-n}}\right] \cdot L, \quad L \in \mathcal{A}_{J^1_G(p)}, \tag{6.2}$$

and let

$$\lambda_{\xi_L} = d^G x^1 \wedge \cdots \wedge d^G x^m \frac{d^n L}{dx^{-1}\cdots dx^{-n}}.$$

Let Θ^L_0 *be the graded Poincaré–Cartan form corresponding to* $-\lambda_{\xi_L}$, *and let us set*

$$\Theta^L = \mathcal{L}^G_{\frac{d}{dx^{-1}}} \circ \cdots \circ \mathcal{L}^G_{\frac{d}{dx^{-n}}}\Theta^L_0$$

and

$$\tilde{\Theta}^L = \mathcal{L}^G_{\mathcal{J}_{n+1}}\left(\frac{d^n L}{dx^{-1}\cdots dx^{-n}}\right) + \eta^G \cdot \frac{d^n L}{dx^{-1}\cdots dx^{-n}}.$$

Then, we have

$$\Theta^L = \tilde{\Theta}^L.$$

Proof. Let T be the totally odd multi-index $T = (-1,\ldots,-n)$, so that $|T| = n$. We also write $\varepsilon = (-1)^{\mu(|B|_2+|Q|_2)+\varphi(Q,B)}$. By applying the preceding proposition, we obtain

$$\begin{aligned}
\tilde{\Theta}^L - \eta^G \cdot \frac{d^n L}{dx^T} &= \mathcal{L}^G_{\mathcal{J}_{n+1}} \left(\frac{d^n L}{dx^T} \right) \\
&= \sum_{0 \le |Q| \le n} (-1)^{m-1} \iota_{\frac{\partial}{\partial x^i}} \eta^G \wedge \theta^\mu_Q \cdot \frac{\partial}{\partial y^\mu_{i+Q}} \frac{d^n L}{dx^T} \\
&= \sum_{0 \le |Q| \le n} (-1)^{m-1} \iota_{\frac{\partial}{\partial x^i}} \eta^G \wedge \theta^\mu_Q \\
&\quad \cdot \left(\sum_{\substack{-b_{c_1} > \cdots > -b_{c_{|Q|}} \\ b_{c_1}, \ldots, b_{c_{|Q|}} \in B}} \varepsilon \delta^{i_1}_{b_{c_1}} \cdots \delta^{i_{|Q|}}_{b_{c_{|Q|}}} \frac{d^{n-|Q|}}{dx^{T - \{b_{c_1}, \ldots, b_{c_{|Q|}}\}}} \frac{\partial L}{\partial y^\mu_i} \right) \\
&= \sum_{0 \le |Q| \le n} (-1)^{m-1} \varepsilon \cdot \iota_{\frac{\partial}{\partial x^i}} \eta^G \wedge \theta^\mu_{i_1 \cdots i_{|Q|}} \frac{d^{n-|Q|}}{dx^{T - \{i_1, \ldots, i_{|Q|}\}}} \frac{\partial L}{\partial y^\mu_i} \\
&= \mathcal{L}^G_{\frac{d}{dx^{-1}}} \circ \cdots \circ \mathcal{L}^G_{\frac{d}{dx^{-n}}} \left((-1)^{m-1} \iota_{\frac{\partial}{\partial x^i}} \eta^G \wedge \theta^\mu \otimes \frac{\partial L}{\partial y^\mu_i} \right) \\
&= \Theta^L - \eta^G \cdot \frac{d^n L}{dx^T}. \qquad \square
\end{aligned}$$

6.3. *(m|2)-superfield theory*

As the use and notations for multi-indices are rather cumbersome, let us analyze a specific case in detail, that of supermanifold with m even and 2 odd coordinates. We start with a Berezinian density

$$\xi_L = \left[d^G x^1 \wedge \cdots \wedge d^G x^m \otimes \frac{d}{dx^{-1}} \circ \frac{d}{dx^{-2}} \right] \cdot L,$$

where $L \in \mathcal{A}_{J^1_G(p)}$; i.e., $L = L(x^\alpha, y^\mu, y^\mu_\alpha)$. The associated graded Lagrangian density is

$$d^G x^1 \wedge \cdots \wedge d^G x^m \cdot \frac{d^2 L}{dx^{-1} dx^{-2}}.$$

Next, from L we can obtain, by applying

$$\mathcal{J}_1 = (-1)^{m-1} \iota_{\frac{\partial}{\partial x^i}} \eta^G \wedge \theta^\mu \otimes \frac{\partial}{\partial y^\mu_i} \quad (1 \le i \le m = \dim M),$$

the graded form

$$\begin{aligned}
\Theta^L - \eta^G \cdot \frac{d^2 L}{dx^{-1} dx^{-2}} &= \mathcal{L}^G_{\frac{d}{dx^{-1}}} \circ \mathcal{L}^G_{\frac{d}{dx^{-2}}} \mathcal{L}^G_{\mathcal{J}_1} L \\
&= \mathcal{L}^G_{\frac{d}{dx^{-1}}} \circ \mathcal{L}^G_{\frac{d}{dx^{-2}}} \left((-1)^{m-1} \iota_{\frac{\partial}{\partial x^i}} \eta^G \wedge \theta^\mu \cdot \frac{\partial L}{\partial y^\mu_i} \right) \\
&= (-1)^{m-1} \mathcal{L}^G_{\frac{d}{dx^{-1}}} \left(\iota_{\frac{\partial}{\partial x^i}} \eta^G \wedge \left(\theta^\mu_{-2} \cdot \frac{\partial L}{\partial y^\mu_i} + (-1)^\mu \theta^\mu \cdot \frac{d}{dx^{-2}} \frac{\partial L}{\partial y^\mu_i} \right) \right)
\end{aligned}$$

$$
\begin{aligned}
&= (-1)^{m-1}\iota_{\frac{\partial}{\partial x^i}}\eta^G \wedge \left(\theta^\mu_{-1,-2}\cdot\frac{\partial L}{\partial y^\mu_i} + (-1)^{\mu+1}\theta^\mu_{-2}\cdot\frac{d}{dx^{-1}}\frac{\partial L}{\partial y^\mu_i}\right.\\
&\quad\left. + (-1)^\mu\theta^\mu_{-1}\cdot\frac{d}{dx^{-2}}\frac{\partial L}{\partial y^\mu_i} + \theta^\mu\cdot\frac{d}{dx^{-1}}\frac{d}{dx^{-2}}\frac{\partial L}{\partial y^\mu_i}\right).
\end{aligned}
\tag{6.3}
$$

Moreover, we can apply the $\mathcal{V}((p_3)_{10})$-valued m-form $\mathcal{J}_3$ on $J^3_G(p)$ to the superfunction

$$
\frac{d^2L}{dx^{-1}dx^{-2}} \in \mathcal{A}_{J^3_G(p)},
$$

the result being

$$
\begin{aligned}
\tilde{\Theta}^L - \eta^G\cdot\frac{d^2L}{dx^{-1}dx^{-2}} &= \mathcal{L}^G_{\mathcal{J}_3}\frac{d^2L}{dx^{-1}dx^{-2}}\\
&= (-1)^{m-1}\iota_{\frac{\partial}{\partial x^i}}\eta^G \wedge \left(\theta^\mu\cdot\frac{\partial}{\partial y^\mu_i}\frac{d^2L}{dx^{-1}dx^{-2}}\right.\\
&\quad\left. + \theta^\mu_\alpha\cdot\frac{\partial}{\partial y^\mu_{\alpha i}}\frac{d^2L}{dx^{-1}dx^{-2}} + \theta^\mu_{\alpha\beta}\cdot\frac{\partial}{\partial y^\mu_{\alpha\beta i}}\frac{d^2L}{dx^{-1}dx^{-2}}\right).
\end{aligned}
\tag{6.4}
$$

The factor

$$
\frac{d^2L}{dx^{-1}dx^{-2}}
$$

can be evaluated in two different ways:

$$
\begin{aligned}
\frac{d^2L}{dx^{-1}dx^{-2}} &= \frac{d}{dx^{-1}}\left(\frac{dL}{dx^{-2}}\right)\\
&= \frac{d}{dx^{-1}}\left(\frac{\partial L}{\partial x^{-2}} + y^\nu_{-2}\frac{\partial L}{\partial y^\nu} + y^\nu_{-2,\alpha}\frac{\partial L}{\partial y^\nu_\alpha}\right)\\
&= \frac{d}{dx^{-1}}\frac{\partial L}{\partial x^{-2}} + y^\nu_{-1,-2}\frac{\partial L}{\partial y^\nu} - (-1)^\nu y^\nu_{-2}\frac{d}{dx^{-1}}\frac{\partial L}{\partial y^\nu}\\
&\quad + y^\nu_{-1,-2,\alpha}\frac{\partial L}{\partial y^\nu_\alpha} + (-1)^{\nu+\alpha+1}y^\nu_{-2,\alpha}\frac{d}{dx^{-1}}\frac{\partial L}{\partial y^\nu_\alpha},
\end{aligned}
\tag{6.5}
$$

or else,

$$
\begin{aligned}
\frac{d^2L}{dx^{-1}dx^{-2}} &= -\frac{d}{dx^{-2}}\left(\frac{dL}{dx^{-1}}\right)\\
&= -\frac{d}{dx^{-2}}\left(\frac{\partial L}{\partial x^{-1}} + y^\nu_{-1}\frac{\partial L}{\partial y^\nu} + y^\nu_{-1,\alpha}\frac{\partial L}{\partial y^\nu_\alpha}\right)\\
&= -\frac{d}{dx^{-2}}\frac{\partial L}{\partial x^{-1}} - y^\nu_{-2,-1}\frac{\partial L}{\partial y^\nu} + (-1)^\nu y^\nu_{-1}\frac{d}{dx^{-2}}\frac{\partial L}{\partial y^\nu}\\
&\quad - y^\nu_{-2,-1,\alpha}\frac{\partial L}{\partial y^\nu_\alpha} - (-1)^{\nu+\alpha+1}y^\nu_{-1,\alpha}\frac{d}{dx^{-2}}\frac{\partial L}{\partial y^\nu_\alpha}.
\end{aligned}
\tag{6.6}
$$

In any case, neither the factors of $\partial L/\partial y^\mu$, $\partial L/\partial x^{-i}$ nor $d/dx^{-i}(\partial L/\partial y^\mu)$ contain 3-derivatives of the kind $y^\nu_{-1,-2,\alpha}$. Thus, by using (6.5), we have

$$\begin{aligned}\frac{\partial}{\partial y^\mu_{-j-ki}}\frac{d^2L}{dx^{-1}dx^{-2}} &= \frac{\partial}{\partial y^\mu_{-j-ki}}\left(y^\nu_{-1,-2,\alpha}\frac{\partial L}{\partial y^\nu_\alpha}\right)\\ &= \delta^{-1}_{-j}\delta^{-2}_{-k}\frac{\partial L}{\partial y^\mu_i}.\end{aligned}$$

Also, there are no terms like y^μ_{ij} in $L \in \mathcal{A}_{J^1_G(p)}$, neither in $d^2L/dx^{-1}dx^{-2}$ (as d/dx^{-1}, d/dx^{-2} just introduce derivatives with respect to odd indices), so that

$$\frac{\partial}{\partial y^\mu_{i\alpha}}\frac{d^2L}{dx^{-1}dx^{-2}} = \frac{\partial}{\partial y^\mu_{-ji}}\frac{d^2L}{dx^{-1}dx^{-2}}.$$

Now, comparing (6.3) and (6.4), we see that proving $\Theta^L = \tilde{\Theta}^L$ reduces to see whether

$$\begin{aligned}&\iota_{\frac{\partial}{\partial x^i}}\eta^G \wedge \theta^\mu_{-j}\cdot\frac{\partial}{\partial y^\mu_{-ji}}\frac{d^2L}{dx^{-1}dx^{-2}}\\ &\quad= \iota_{\frac{\partial}{\partial x^i}}\eta^G \wedge\left((-1)^\mu\theta^\mu_{-1}\cdot\frac{d}{dx^{-2}}\frac{\partial L}{\partial y^\mu_i} + (-1)^{\mu+1}\theta^\mu_{-2}\cdot\frac{d}{dx^{-1}}\frac{\partial L}{\partial y^\mu_i}\right),\end{aligned}$$

or, developing the left-hand side,

$$\begin{aligned}&\iota_{\frac{\partial}{\partial x^i}}\eta^G \wedge\left(\theta^\mu_{-1}\cdot\frac{\partial}{\partial y^\mu_{-1,i}} + \theta^\mu_{-2}\cdot\frac{\partial}{\partial y^\mu_{-2,i}}\right)\frac{d^2L}{dx^{-1}dx^{-2}}\\ &\quad= (-1)^\mu\iota_{\frac{\partial}{\partial x^i}}\eta^G \wedge\left(\theta^\mu_{-1}\cdot\frac{d}{dx^{-2}}\frac{\partial L}{\partial y^\mu_i} - \theta^\mu_{-2}\cdot\frac{d}{dx^{-1}}\frac{\partial L}{\partial y^\mu_i}\right).\end{aligned}$$

What we are going to see is

$$\left.\begin{aligned}\frac{\partial}{\partial y^\mu_{-1,i}}\frac{d^2L}{dx^{-1}dx^{-2}} &= (-1)^\mu\frac{d}{dx^{-2}}\frac{\partial L}{\partial y^\mu_i}\\ \frac{\partial}{\partial y^\mu_{-2,i}}\frac{d^2L}{dx^{-1}dx^{-2}} &= -(-1)^\mu\frac{d}{dx^{-1}}\frac{\partial L}{\partial y^\mu_i}\end{aligned}\right\}. \tag{6.7}$$

To prove the first formula in (6.7), we use (6.5). It is clear that the only terms containing factors like $y^\mu_{-1,i}$, $y^\mu_{-2,i}$ are those indicated:

$$\begin{aligned}\frac{\partial L}{\partial y^\mu_{-1,i}} &= \frac{\partial}{\partial y^\mu_{-1,i}}\frac{d^2L}{dx^{-1}dx^{-2}}\\ &= \frac{\partial}{\partial y^\mu_{-1,i}}\left(y^\nu_{-1,\alpha}\frac{\partial^2 L}{\partial y^\nu_\alpha\partial x^{-2}} - (-1)^\nu y^\nu_{-2}y^\xi_{-1,\alpha}\frac{\partial^2 L}{\partial y^\xi_\alpha\partial y^\nu}\right.\\ &\qquad\left. - (-1)^{\nu+\alpha}y^\nu_{-2,\alpha}y^\xi_{-1\beta}\frac{\partial^2 L}{\partial y^\xi_\beta\partial y^\nu_\alpha}\right)\\ &= \frac{\partial^2 L}{\partial y^\mu_i\partial x^{-2}} + (-1)^{\mu(\nu+1)}y^\nu_{-2}\frac{\partial^2 L}{\partial y^\mu_i\partial y^\nu} + (-1)^{\mu+\mu(\nu+\alpha)}y^\nu_{-2,\alpha}\frac{\partial^2 L}{\partial y^\mu_i\partial y^\nu_\alpha}\end{aligned}$$

$$= (-1)^{\mu} \left(\frac{\partial}{\partial x^{-2}} + y^{\nu}_{-2} \frac{\partial}{\partial y^{\nu}} + y^{\nu}_{-2,\alpha} \frac{\partial}{\partial y^{\nu}_{\alpha}} \right) \frac{\partial L}{\partial y^{\mu}_{i}}$$
$$= (-1)^{\mu} \frac{d}{dx^{-2}} \frac{\partial L}{\partial y^{\mu}_{i}}.$$

To prove the second formula in (6.7), one has just to repeat the preceding computations but using (6.6).

Remark 6.6. The proof of the lemmas and the proposition in Subsec. 6.1 is just a generalization (by induction) of the computations leading to equations (6.5), (6.6) and (6.7).

Thus, once a volume form has been chosen on the base manifold M, we have constructed a Poincaré–Cartan form,

$$\Theta^L = \mathcal{L}^G_{\mathcal{J}_{n+1}} \left(\frac{d^n L}{dx^{-1} \cdots dx^{-n}} \right) + \eta^G \cdot \frac{d^n L}{dx^{-1} \cdots dx^{-n}}$$

out of intrinsically defined objects. Moreover, we have proved the equivalence with the alternative expression

$$\Theta^L = \mathcal{L}^G_{\frac{d}{dx^{-1}}} \circ \cdots \circ \mathcal{L}^G_{\frac{d}{dx^{-n}}} \Theta^L_0,$$

which, as it does not involve higher-order operators, could be more appropriate for explicit computations.

7. Deduction of the Euler–Lagrange Equations from the Poincaré–Cartan Form

7.1. *The exterior derivative of the Poincaré–Cartan form*

According to the previous section, we have a well-defined procedure to obtain the Euler–Lagrange superequations for a superfield theory described by a first-order Berezinian density

$$\xi_L = \left[d^G x^1 \wedge \cdots \wedge d^G x^m \otimes \frac{d}{dx^{-1}} \circ \cdots \circ \frac{d}{dx^{-n}} \right] \cdot L, \quad L \in \mathcal{A}_{J^1_G(p)},$$

in a similar way to that of the classical case: First, we must consider the Poincaré–Cartan form Θ^L, then its differential $d^G\Theta^L$ and finally study the insertion of vertical superfields. The idea is to obtain a decomposition of $d^G\Theta^L$ as the product of the Euler–Lagrange operator by the graded contact 1-forms and/or their derivatives plus other terms, as expressed in the following proposition.

We make use of the decomposition $d^G = D + \partial$, where

$$\begin{aligned} D &= D_0 + D_1 \\ &= d^G x^{\alpha} \otimes \mathcal{L}^G_{\frac{d}{dx^{\alpha}}} \end{aligned}$$

is the graded horizontal differential (given as a sum of the horizontal differential with respect to even and odd coordinates on the base manifold) and $\partial = d^G - D$

is the graded vertical differential, which differentiates with respect to the fiber coordinates (recall Subsec. 3.3).

Proposition 7.1. *For every $L \in \mathcal{A}_{J^1_G(p)}$, we have*

$$d^G\Theta^L = \mathcal{L}^G_{\frac{d}{dx^{-1}}} \circ \cdots \circ \mathcal{L}^G_{\frac{d}{dx^{-n}}}(\alpha_L + \varpi_L + D_1(\Theta^L_0 - \eta^G \cdot L) + \partial(\Theta^L_0 - \eta^G \cdot L)),$$

where ϖ_L and α_L are the $(m+1)$-forms on $J^2_G(p)$, defined by

$$\varpi_L = (-1)^m \eta^G \wedge \left(\theta^\mu \left(\frac{\partial L}{\partial y^\mu} - \frac{d}{dx^i}\frac{\partial L}{\partial y^\mu_i}\right) + \theta^\mu_{-i}\frac{\partial L}{\partial y^\mu_{-i}}\right)$$

$$\alpha_L = (-1)^m \eta^G \wedge d^G x^\alpha \cdot \left(2\frac{dL}{dx^\alpha} - \frac{\partial L}{\partial x^\alpha}\right).$$

Proof. From the preceding section, recalling that the operators $\mathcal{L}^G_{d/dx^{-1}}$ and d^G commute, we obtain

$$\begin{aligned} d^G\Theta^L &= \mathcal{L}^G_{\frac{d}{dx^{-1}}} \circ \cdots \circ \mathcal{L}^G_{\frac{d}{dx^{-n}}} d^G\Theta^L_0 \\ &= \mathcal{L}^G_{\frac{d}{dx^{-1}}} \circ \cdots \circ \mathcal{L}^G_{\frac{d}{dx^{-n}}}(D_0(\Theta^L_0 - \eta^G \cdot L) \\ &\quad + D_1(\Theta^L_0 - \eta^G \cdot L) + \partial(\Theta^L_0 - \eta^G \cdot L) \\ &\quad + (-1)^m \eta^G \wedge d^G L). \end{aligned}$$

Let us concentrate in the terms $D_0(\Theta^L_0 - \eta^G \cdot L) + (-1)^m \eta^G \wedge d^G L$. On the one hand, we have

$$\begin{aligned} D_0(\Theta^L_0 - \eta^G \cdot L) &= d^G x^i \wedge \mathcal{L}^G_{\frac{d}{dx^i}}(\Theta^L_0 - \eta^G \cdot L) \\ &= (-1)^{m-1}\eta^G \wedge \left(\theta^\mu_i \frac{\partial L}{\partial y^\mu_i} + \theta^\mu \frac{d}{dx^i}\frac{\partial L}{\partial y^\mu_i}\right), \end{aligned}$$

and, on the other,

$$\eta^G \wedge d^G L = \eta^G \wedge \left(d^G x^\alpha \cdot \frac{dL}{dx^\alpha} + d^G y^\mu \cdot \frac{\partial L}{\partial y^\mu} + d^G y^\mu_\alpha \cdot \frac{\partial L}{\partial y^\mu_\alpha}\right).$$

Thus, substituting,

$$\begin{aligned} &D_0\big(\Theta^L_0 - \eta^G \cdot L\big) + (-1)^m \eta^G \wedge d^G L \\ &\quad = (-1)^m \eta^G \wedge \left(-\theta^\mu_i \frac{\partial L}{\partial y^\mu_i} - \theta^\mu \frac{d}{dx^i}\frac{\partial L}{\partial y^\mu_i}\right. \\ &\qquad \left. + d^G x^\alpha \cdot \frac{dL}{dx^\alpha} + d^G y^\mu \cdot \frac{\partial L}{\partial y^\mu} + d^G y^\mu_\alpha \cdot \frac{\partial L}{\partial y^\mu_\alpha}\right) \\ &\quad = (-1)^m \eta^G \wedge \left(-\theta^\mu_i \frac{\partial L}{\partial y^\mu_i} - \theta^\mu \frac{d}{dx^i}\frac{\partial L}{\partial y^\mu_i} + d^G x^\alpha \cdot \frac{dL}{dx^\alpha}\right. \\ &\qquad \left. + \big(\theta^\mu + d^G x^\alpha \cdot y^\mu_\alpha\big)\frac{\partial L}{\partial y^\mu} + \big(\theta^\mu_\alpha + d^G x^\beta \cdot y^\mu_{\beta\alpha}\big)\frac{\partial L}{\partial y^\mu_\alpha}\right) \end{aligned}$$

$$
\begin{aligned}
&= (-1)^m \eta^G \wedge \left(\theta^\mu_{-i} \frac{\partial L}{\partial y^\mu_{-i}} + \theta^\mu \left(\frac{\partial L}{\partial y^\mu} - \frac{d}{dx^i} \frac{\partial L}{\partial y^\mu_i} \right) \right. \\
&\quad \left. + d^G x^\alpha \cdot \frac{dL}{dx^\alpha} + d^G x^\alpha \cdot y^\mu_\alpha \frac{\partial L}{\partial y^\mu} + d^G x^\beta \cdot y^\mu_{\beta\alpha} \frac{\partial L}{\partial y^\mu_\alpha} \right) \\
&= (-1)^m \eta^G \wedge \left(\theta^\mu_{-i} \frac{\partial L}{\partial y^\mu_{-i}} + \theta^\mu \left(\frac{\partial L}{\partial y^\mu} - \frac{d}{dx^i} \frac{\partial L}{\partial y^\mu_i} \right) \right. \\
&\quad \left. + d^G x^\alpha \cdot \left(2 \frac{dL}{dx^\alpha} - \frac{\partial L}{\partial x^\alpha} \right) \right) \\
&= \varpi_L + \alpha_L.
\end{aligned}
$$

□

We should also remark that for every vector field X on $J^2_G(p)$ vertical over $(M, \mathcal{A})$, we have $\iota_X \alpha_L = 0$.

Now, we would like to extract the Euler–Lagrange superequations of field theory from the decomposition of the previous proposition. To this end, we first need the following technical lemma, whose proof reduces to a simple computation:

Lemma 7.2. *Let Σ_{-n} denote the group of permutations of $\{-1, \ldots, -n\}$. For any $A, B \in \Omega_G(J^1_G(p))$, we have*

$$
\begin{aligned}
&\mathcal{L}^G_{\frac{d}{dx^{-1}}} \circ \cdots \circ \mathcal{L}^G_{\frac{d}{dx^{-n}}} (A \wedge B) \\
&\quad = \sum_{\substack{\sigma = \sigma_1 \cup \sigma_2 \in \Sigma_{-n} \\ 0 \le |\sigma| \le n}} (-1)^{|\sigma_2||A| + \tau} \left(\mathcal{L}^G_{\frac{d}{dx^{\sigma_1(-1)}}} \circ \cdots \circ \mathcal{L}^G_{\frac{d}{dx^{\sigma_1(-|\sigma_1|)}}} A \right) \\
&\quad\quad \cdot \left(\mathcal{L}^G_{\frac{d}{dx^{\sigma_2(-|\sigma_1|+1)}}} \circ \cdots \circ \mathcal{L}^G_{\frac{d}{dx^{\sigma_2(-n)}}} B \right),
\end{aligned}
$$

where τ is the number of transpositions needed to reorder $(\sigma_1(-1), \ldots, \sigma_2(-n))$.

Proposition 7.3. *With the preceding notations, we have*

$$
\begin{aligned}
&\mathcal{L}^G_{\frac{d}{dx^{-1}}} \circ \cdots \circ \mathcal{L}^G_{\frac{d}{dx^{-n}}} (\varpi_L) \\
&\quad = \sum_{\substack{\sigma = \sigma_1 \cup \sigma_2 \in \Sigma_{-n} \\ 0 \le |\sigma_2| \le n \\ |\sigma_2|\mu + \tau}} (-1)^{|\sigma_2|\mu + \tau + m} \eta^G \wedge \theta^\mu_{\sigma_1(-1) \cdots \sigma_1(-|\sigma_1|)} \frac{d^{|\sigma_2|} \mathcal{E}(L)}{dx^{\sigma_2(-|\sigma_1|-1)} \ldots dx^{\sigma_2(-n)}},
\end{aligned}
$$

where $\mathcal{E}$ is the Euler–Lagrange operator,

$$
\mathcal{E}(L) = \frac{\partial L}{\partial y^\mu} - \frac{d}{dx^i} \frac{\partial L}{\partial y^\mu_i} - (-1)^\mu \frac{d}{dx^{-i}} \frac{\partial L}{\partial y^\mu_{-i}}.
$$

Proof. Let us write

$$
\varpi_L = (-1)^m \eta^G \wedge \left(\theta^\mu \omega_\mu + \theta^\mu_{-i} \frac{\partial L}{\partial y^\mu_{-i}} \right),
$$

where

$$
\omega_\mu = \frac{\partial L}{\partial y^\mu} - \frac{d}{dx^i} \frac{\partial L}{\partial y^\mu_i}.
$$

Then, we have

$$\begin{aligned}
&\mathcal{L}^G_{\frac{d}{dx^{-1}}} \circ \cdots \circ \mathcal{L}^G_{\frac{d}{dx^{-n}}}(\varpi_L) \\
&\quad = (-1)^m \eta^G \wedge \mathcal{L}^G_{\frac{d}{dx^{-1}}} \circ \cdots \circ \mathcal{L}^G_{\frac{d}{dx^{-n}}} \left(\theta^\mu \omega_\mu + \theta^\mu_{-i} \frac{\partial L}{\partial y^\mu_{-i}}\right),
\end{aligned}$$

and by applying Lemma 7.2, we obtain

$$\begin{aligned}
&\mathcal{L}^G_{\frac{d}{dx^{-1}}} \circ \cdots \circ \mathcal{L}^G_{\frac{d}{dx^{-n}}} \left(\theta^\mu \omega_\mu + \theta^\mu_{-i} \frac{\partial L}{\partial y^\mu_{-i}}\right) \\
&\quad = \sum_{\substack{\sigma=\sigma_1\cup\sigma_2\in\Sigma_{-n}\\ 0\le|\sigma_2|\le n}} \left((-1)^{|\sigma_2|\mu+\tau}\left(\mathcal{L}^G_{\frac{d}{dx^{\sigma_1(-1)}}} \circ \cdots \circ \mathcal{L}^G_{\frac{d}{dx^{\sigma_1(-|\sigma_1|)}}}\right)\theta^\mu\right. \\
&\qquad \cdot \left(\mathcal{L}^G_{\frac{d}{dx^{\sigma_2(-|\sigma_1|+1)}}} \circ \cdots \circ \mathcal{L}^G_{\frac{d}{dx^{\sigma_2(-n)}}}\right)\omega_\mu \\
&\qquad + (-1)^{|\sigma_2|(\mu+1)+\tau}\left(\mathcal{L}^G_{\frac{d}{dx^{\sigma_1(-1)}}} \circ \cdots \circ \mathcal{L}^G_{\frac{d}{dx^{\sigma_1(-|\sigma_1|)}}}\theta^\mu_{-i}\right) \\
&\qquad \left.\cdot \left(\mathcal{L}^G_{\frac{d}{dx^{\sigma_2(-|\sigma_1|+1)}}} \circ \cdots \circ \mathcal{L}^G_{\frac{d}{dx^{\sigma_2(-n)}}}\frac{\partial L}{\partial y^\mu_{-i}}\right)\right) \\
&\quad = \sum_{\substack{\sigma=(\sigma_1\cup\sigma_2)\in\Sigma_{-n}\\ 0\le|\sigma_2|\le n}} \left((-1)^{|\sigma_2|\mu+\tau}\theta^\mu_{\sigma_1(-1)\cdots\sigma_1(-|\sigma_1|)}\frac{d^{|\sigma_2|}}{dx^{\sigma_2(-|\sigma_1|-1)}\ldots dx^{\sigma_2(-n)}}\omega_\mu\right. \\
&\qquad \left.+ (-1)^{|\sigma_2|(\mu+1)+\tau}\theta^\mu_{\sigma_1(-1)\cdots\sigma_1(-|\sigma_1|),-i}\frac{d^{|\sigma_2|}}{dx^{\sigma_2(-|\sigma_1|-1)}\ldots dx^{\sigma_2(-n)}}\frac{\partial L}{\partial y^\mu_{-i}}\right) \\
&\quad = \sum_{\substack{\sigma=(\sigma_1\cup\sigma_2)\in\Sigma_{-n}\\ 0\le|\sigma_2|\le n}} (-1)^{|\sigma_2|\mu+\tau}\theta^\mu_{\sigma_1(-1)\cdots\sigma_1(-|\sigma_1|)}\left(\frac{d^{|\sigma_2|}}{dx^{\sigma_2(-|\sigma_1|-1)}\ldots dx^{\sigma_2(-n)}}\omega_\mu\right. \\
&\qquad \left.- (-1)^\mu\frac{d^{|\sigma_2|}}{dx^{\sigma_2(-|\sigma_1|-1)}\ldots dx^{\sigma_2(-n)}}\frac{d}{dx^{-i}}\frac{\partial L}{\partial y^\mu_{-i}}\right) \\
&\quad = \sum_{\substack{\sigma=\sigma_1\cup\sigma_2\in\Sigma_{-n}\\ 0\le|\sigma_2|\le n}} (-1)^{|\sigma_2|\mu+\tau}\theta^\mu_{\sigma_1(-1)\cdots\sigma_1(-|\sigma_1|)}\frac{d^{|\sigma_2|}}{dx^{\sigma_2(-|\sigma_1|-1)}\ldots dx^{\sigma_2(-n)}}(\mathcal{E}(L)).
\end{aligned}$$

□

7.2. *An example*

Again, let us clarify the notation by working out the example of $(m|2)$-superfield theory. Here we have

$$\begin{aligned}
&\mathcal{L}^G_{\frac{d}{dx^{-1}}}\mathcal{L}^G_{\frac{d}{dx^{-2}}}\left(\theta^\mu\left(\frac{\partial L}{\partial y^\mu} - \frac{d}{dx^i}\frac{\partial L}{\partial y^\mu_i}\right) + \theta^\mu_{-1}\frac{\partial L}{\partial y^\mu_{-1}} + \theta^\mu_{-2}\frac{\partial L}{\partial y^\mu_{-2}}\right) \\
&\quad = \mathcal{L}^G_{\frac{d}{dx^{-1}}}\left(\theta^\mu_{-2}\left(\frac{\partial L}{\partial y^\mu} - \frac{d}{dx^i}\frac{\partial L}{\partial y^\mu_i}\right) + (-1)^\mu\theta^\mu\frac{d}{dx^{-2}}\left(\frac{\partial L}{\partial y^\mu} - \frac{d}{dx^i}\frac{\partial L}{\partial y^\mu_i}\right)\right. \\
&\qquad \left.+\ \theta^\mu_{-2,-1}\frac{\partial L}{\partial y^\mu_{-1}} - (-1)^\mu\theta^\mu_{-1}\frac{d}{dx^{-2}}\frac{\partial L}{\partial y^\mu_{-1}} - (-1)^\mu\theta^\mu_{-2}\frac{d}{dx^{-2}}\frac{\partial L}{\partial y^\mu_{-2}}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \theta^{\mu}_{-1,-2}\left(\frac{\partial L}{\partial y^{\mu}} - \frac{d}{dx^i}\frac{\partial L}{\partial y^{\mu}_i}\right) - (-1)^{\mu}\theta^{\mu}_{-2}\frac{d}{dx^{-1}}\left(\frac{\partial L}{\partial y^{\mu}} - \frac{d}{dx^i}\frac{\partial L}{\partial y^{\mu}_i}\right)\\
&\quad + (-1)^{\mu}\theta^{\mu}_{-1}\frac{d}{dx^{-2}}\left(\frac{\partial L}{\partial y^{\mu}} - \frac{d}{dx^i}\frac{\partial L}{\partial y^{\mu}_i}\right) + \theta^{\mu}\frac{d}{dx^{-1}}\frac{d}{dx^{-2}}\left(\frac{\partial L}{\partial y^{\mu}} - \frac{d}{dx^i}\frac{\partial L}{\partial y^{\mu}_i}\right)\\
&\quad + (-1)^{\mu}\theta^{\mu}_{-2,-1}\frac{d}{dx^{-1}}\frac{\partial L}{\partial y^{\mu}_{-1}} + \theta^{\mu}_{-1}\frac{d}{dx^{-1}}\frac{d}{dx^{-2}}\frac{\partial L}{\partial y^{\mu}_{-1}}\\
&\quad - (-1)^{\mu}\theta^{\mu}_{-1,-2}\frac{d}{dx^{-2}}\ \frac{\partial L}{\partial y^{\mu}_{-2}} + \theta^{\mu}_{-2}\frac{d}{dx^{-1}}\frac{d}{dx^{-2}}\frac{\partial L}{\partial y^{\mu}_{-2}}.
\end{aligned}
$$

Next, grouping common factors of the contact 1-forms,

$$
\begin{aligned}
&\mathcal{L}^{G}_{\frac{d}{dx^{-1}}}\mathcal{L}^{G}_{\frac{d}{dx^{-2}}}\left(\theta^{\mu}\left(\frac{\partial L}{\partial y^{\mu}} - \frac{d}{dx^i}\frac{\partial L}{\partial y^{\mu}_i}\right) - \theta^{\mu}_{-1}\frac{\partial L}{\partial y^{\mu}_{-1}} - \theta^{\mu}_{-2}\frac{\partial L}{\partial y^{\mu}_{-2}}\right)\\
&\quad = \theta^{\mu}_{-1,-2}\left(\frac{\partial L}{\partial y^{\mu}} - \frac{d}{dx^i}\frac{\partial L}{\partial y^{\mu}_i} - (-1)^{\mu}\frac{d}{dx^{-1}}\frac{\partial L}{\partial y^{\mu}_{-1}} - (-1)^{\mu}\frac{d}{dx^{-2}}\frac{\partial L}{\partial y^{\mu}_{-2}}\right)\\
&\qquad + \theta^{\mu}_{-1}\left((-1)^{\mu}\frac{d}{dx^{-2}}\left(\frac{\partial L}{\partial y^{\mu}} - \frac{d}{dx^i}\frac{\partial L}{\partial y^{\mu}_i}\right) + \frac{d^2}{dx^{-1}dx^{-2}}\frac{\partial L}{\partial y^{\mu}_{-1}}\right)\\
&\qquad - \theta^{\mu}_{-2}\left((-1)^{\mu}\frac{d}{dx^{-1}}\left(\frac{\partial L}{\partial y^{\mu}} - \frac{d}{dx^i}\frac{\partial L}{\partial y^{\mu}_i}\right) + \frac{d^2}{dx^{-1}dx^{-2}}\frac{\partial L}{\partial y^{\mu}_{-2}}\right)\\
&\qquad + \theta^{\mu}\left(\frac{d^2}{dx^{-1}dx^{-2}}\left(\frac{\partial L}{\partial y^{\mu}} - \frac{d}{dx^i}\frac{\partial L}{\partial y^{\mu}_i}\right)\right),
\end{aligned}
$$

and an algebraic rearrangement finally gives,

$$
\begin{aligned}
&\mathcal{L}^{G}_{\frac{d}{dx^{-1}}}\mathcal{L}^{G}_{\frac{d}{dx^{-2}}}\left(\theta^{\mu}\left(\frac{\partial L}{\partial y^{\mu}} - \frac{d}{dx^i}\frac{\partial L}{\partial y^{\mu}_i}\right) - \theta^{\mu}_{-1}\frac{\partial L}{\partial y^{\mu}_{-1}} - \theta^{\mu}_{-2}\frac{\partial L}{\partial y^{\mu}_{-2}}\right)\\
&\quad = \theta^{\mu}_{-1,-2}\left(\frac{\partial L}{\partial y^{\mu}} - \frac{d}{dx^i}\frac{\partial L}{\partial y^{\mu}_i} - (-1)^{\mu}\frac{d}{dx^{-j}}\frac{\partial L}{\partial y^{\mu}_{-j}}\right)\\
&\qquad + (-1)^{\mu}\theta^{\mu}_{-1}\left(\frac{d}{dx^{-2}}\left(\frac{\partial L}{\partial y^{\mu}} - \frac{d}{dx^i}\frac{\partial L}{\partial y^{\mu}_i}\right) - (-1)^{\mu}\frac{d}{dx^{-2}}\frac{d}{dx^{-j}}\frac{\partial L}{\partial y^{\mu}_{-j}}\right)\\
&\qquad - (-1)^{\mu}\theta^{\mu}_{-2}\left(\frac{d}{dx^{-1}}\left(\frac{\partial L}{\partial y^{\mu}} - \frac{d}{dx^i}\frac{\partial L}{\partial y^{\mu}_i}\right) - (-1)^{\mu}\frac{d}{dx^{-1}}\frac{d}{dx^{-j}}\frac{\partial L}{\partial y^{\mu}_{-j}}\right)\\
&\qquad + \theta^{\mu}\left(\frac{d^2}{dx^{-1}dx^{-2}}\left(\frac{\partial L}{\partial y^{\mu}} - \frac{d}{dx^i}\frac{\partial L}{\partial y^{\mu}_i} - (-1)^{\mu}\frac{d}{dx^{-j}}\frac{\partial L}{\partial y^{\mu}_{-j}}\right)\right)\\
&\quad = \big(\theta^{\mu}_{-1,-2} + (-1)^{\mu}\theta^{\mu}_{-1} - (-1)^{\mu}\theta^{\mu}_{-2} + \theta^{\mu}\big)\mathcal{E}(L).
\end{aligned}
$$

7.3. *The Euler–Lagrange equations*

In view of Proposition 7.3, the term ϖ_L alone already gives us the Euler–Lagrange equations, so we must study the vanishing of the terms

$$D_1\big(\Theta^L_0 - \eta^G \cdot L\big) + \partial\big(\Theta^L_0 - \eta^G \cdot L\big).$$

Lemma 7.4. *With the preceding notations, we have*

$$\partial = \theta^\mu \wedge \mathcal{L}^G_{\frac{\partial}{\partial y^\mu}} + \theta^\mu_\alpha \wedge \mathcal{L}^G_{\frac{\partial}{\partial y^\mu_\alpha}} - d^G\theta^\mu \wedge \iota_{\frac{\partial}{\partial y^\mu}} - d^G\theta^\mu_\beta \wedge \iota_{\frac{\partial}{\partial y^\mu_\beta}}.$$

Proof. From the very definition we have

$$d^G = d^G x^\beta \wedge \mathcal{L}^G_{\frac{\partial}{\partial x^\beta}} + d^G y^\mu \wedge \mathcal{L}^G_{\frac{\partial}{\partial y^\mu}} + d^G y^\mu_\alpha \wedge \mathcal{L}^G_{\frac{\partial}{\partial y^\mu_\alpha}},$$

and also from the definition, $\partial = d^G - D$, where D is the horizontal differential.

Therefore,

$$\begin{aligned}\partial &= d^G - D\\ &= d^G - d^G x^\gamma \wedge \mathcal{L}^G_{\frac{d}{dx^\gamma}}\\ &= d^G x^\beta \wedge \mathcal{L}^G_{\frac{\partial}{\partial x^\beta}} + d^G y^\mu \wedge \mathcal{L}^G_{\frac{\partial}{\partial y^\mu}} + d^G y^\mu_\alpha \wedge \mathcal{L}^G_{\frac{\partial}{\partial y^\mu_\alpha}} - d^G x^\gamma \wedge \mathcal{L}^G_{\frac{d}{dx^\gamma}}.\end{aligned}$$

Furthermore, as

$$\frac{d}{dx^\gamma} = \frac{\partial}{\partial x^\gamma} + y^\mu_\gamma \frac{\partial}{\partial y^\mu} + y^\mu_{\gamma\alpha}\frac{\partial}{\partial y^\mu_\alpha},$$

taking the properties of the graded Lie derivative into account, we obtain

$$\begin{aligned}\partial &= d^G y^\mu \wedge \mathcal{L}^G_{\frac{\partial}{\partial y^\mu}} + d^G y^\mu_\alpha \wedge \mathcal{L}^G_{\frac{\partial}{\partial y^\mu_\alpha}} - d^G x^\alpha \wedge \mathcal{L}^G_{y^\mu_\alpha \frac{\partial}{\partial y^\mu}} - d^G x^\alpha \wedge \mathcal{L}^G_{y^\mu_{\alpha\beta}\frac{\partial}{\partial y^\mu_\beta}}\\ &= d^G y^\mu \wedge \mathcal{L}^G_{\frac{\partial}{\partial y^\mu}} + d^G y^\mu_\alpha \wedge \mathcal{L}^G_{\frac{\partial}{\partial y^\mu_\alpha}} - d^G x^\alpha \wedge \left(d^G \iota_{y^\mu_\alpha \frac{\partial}{\partial y^\mu}} + \iota_{y^\mu_\alpha\frac{\partial}{\partial y^\mu}} d^G\right)\\ &\quad - d^G x^\alpha \wedge \left(d^G \iota_{y^\mu_{\alpha\beta}\frac{\partial}{\partial y^\mu_\beta}} + \iota_{y^\mu_{\alpha\beta}\frac{\partial}{\partial y^\mu_\beta}} d^G\right)\\ &= d^G y^\mu \wedge \mathcal{L}^G_{\frac{\partial}{\partial y^\mu}} + d^G y^\mu_\alpha \wedge \mathcal{L}^G_{\frac{\partial}{\partial y^\mu_\alpha}} - d^G x^\alpha \cdot y^\mu_\alpha \wedge \mathcal{L}^G_{\frac{\partial}{\partial y^\mu}} - d^G x^\alpha \cdot y^\mu_{\alpha\beta} \wedge \mathcal{L}^G_{\frac{\partial}{\partial y^\mu_\beta}}\\ &\quad - d^G x^\alpha \wedge d^G y^\mu_\alpha \wedge \iota_{\frac{\partial}{\partial y^\mu}} - d^G x^\alpha \wedge d^G y^\mu_{\alpha\beta} \wedge \iota_{\frac{\partial}{\partial y^\mu_\beta}}.\end{aligned}$$

Finally, by grouping the correct terms and by noting that

$$d^G\theta^\nu_Q = d^G x^\alpha \wedge d^G y^\nu_{\alpha\star Q},$$

we arrive at the statement of the lemma. □

Lemma 7.5. *For every vector field X on $J^{n+1}_G(p)$, vertical over $(M, \mathcal{A})$, and for any local section s of p, we have*

$$(j^{n+1}s)^*\left(\iota_X\left(\mathcal{L}^G_{\frac{d}{dx^{-1}}} \circ \cdots \circ \mathcal{L}^G_{\frac{d}{dx^{-n}}}(D_1(\Theta^L_0 - \eta^G \cdot L) + \partial(\Theta^L_0 - \eta^G \cdot L))\right)\right) = 0.$$

Proof. As

$$D_1 = d^G x^{-i} \wedge \mathcal{L}^G_{\frac{d}{dx^{-i}}},$$

it is clear that

$$\mathcal{L}^G_{\frac{d}{dx^{-1}}} \circ \cdots \circ \mathcal{L}^G_{\frac{d}{dx^{-n}}} \circ D_1 = 0$$

(one of the $\frac{d}{dx^{-i}}$ factors appears twice). Now, let us see that

$$(j^{n+1}s)^*\Big(\iota_X\Big(\mathcal{L}^G_{\frac{d}{dx^{-1}}}\circ\cdots\circ\mathcal{L}^G_{\frac{d}{dx^{-n}}}(\partial(\Theta^L_0-\eta^G\cdot L))\Big)\Big)=0.$$

The bidegree of ∂ is $(1,0)$, so we have

$$\begin{aligned}\partial(\Theta^L_0-\eta^G\cdot L)&=\partial\bigg((-1)^{m-1}\iota_{\frac{\partial}{\partial x^j}}\eta^G\wedge\theta^\mu\frac{\partial L}{\partial y^\mu_j}\bigg)\\&=\iota_{\frac{\partial}{\partial x^j}}\eta^G\wedge\bigg(\partial\theta^\mu\cdot\frac{\partial L}{\partial y^\mu_j}-\theta^\mu\wedge\partial\bigg(\frac{\partial L}{\partial y^\mu_j}\bigg)\bigg).\end{aligned}$$

From Lemma 7.4, we know the explicit expression for ∂. Making use of it, along with the formulas

$$\begin{aligned}\mathcal{L}^G_{\frac{\partial}{\partial y^\mu}}\theta^\nu&=0,\\\mathcal{L}^G_{\frac{\partial}{\partial y^\mu_\alpha}}\theta^\nu&=-(-1)^{\alpha(\mu+\alpha)}d^Gx^\alpha\delta^\nu_\mu,\\\iota_{\frac{\partial}{\partial y^\mu}}\theta^\nu&=\delta^\nu_\mu,\\\iota_{\frac{\partial}{\partial y^\mu_\alpha}}\theta^\nu&=0,\end{aligned}$$

we obtain

$$\begin{aligned}&\partial(\Theta^L_0-\eta^G\cdot L)\\&\quad=\iota_{\frac{\partial}{\partial x^j}}\eta^G\wedge\bigg(d^Gx^\alpha\wedge\theta^\mu_\alpha\frac{\partial L}{\partial y^\mu_j}-d^G\theta^\mu\frac{\partial L}{\partial y^\mu_j}-\theta^\nu\wedge\theta^\mu\frac{\partial^2 L}{\partial y^\mu\partial y^\nu_j}-\theta^\nu\wedge\theta^\mu_\alpha\frac{\partial^2 L}{\partial y^\mu_\alpha\partial y^\nu_j}\bigg),\end{aligned}$$

and remarking that

$$d^Gx^\alpha\wedge\theta^\mu_\alpha-d^G\theta^\mu=-d^Gx^\alpha\wedge d^Gx^\beta\cdot y^\mu_{\beta\alpha},$$

we deduce

$$\begin{aligned}&\partial\big(\Theta^L_0-\eta^G\cdot L\big)\\&\quad=-\iota_{\frac{\partial}{\partial x^j}}\eta^G\wedge\bigg(d^Gx^\alpha\wedge d^Gx^\beta\cdot y^\mu_{\beta\alpha}+\theta^\nu\wedge\theta^\mu\frac{\partial^2 L}{\partial y^\mu\partial y^\nu_j}+\theta^\nu\wedge\theta^\mu_\alpha\frac{\partial^2 L}{\partial y^\mu_\alpha\partial y^\nu_j}\bigg).\end{aligned}$$

Here, the first term in the right-hand side vanishes when a vertical vector field is inserted. The other two, when the pull-back $(j^{n+1}s)^*$ is taken, as a contact form θ^μ remains even after the insertion of the vertical field. □

As a consequence of these results, we can see that the Euler–Lagrange equations for a superfield are those expected.

Theorem 7.6. *A local section s of p is a critical section for the Berezinian density $\xi_L=[d^Gx^1\wedge\cdots\wedge d^Gx^m\otimes\frac{d}{dx^{-1}}\circ\cdots\circ\frac{d}{dx^{-n}}]L$ with $L\in\mathcal{A}_{J^1_G(p)}$, if and only if the following equations holds*:

$$(j^{n+1}s)^*\big(\iota_X d^G\Theta^L\big)=0, \tag{7.1}$$

for every vector field X on $J^{n+1}_G(p)$ vertical over $(M,\mathcal{A})$.

Proof. As we have seen, Eq. (7.1) is equivalent to the Euler–Lagrange equations

$$(j^{n+1}s)^*\left(\frac{\partial L}{\partial y^\mu} - \frac{d}{dx^i}\frac{\partial L}{\partial y_i^\mu} - (-1)^\mu \frac{d}{dx^{-j}}\frac{\partial L}{\partial y_{-j}^\mu}\right) = 0,$$

and these are the well-known conditions on s to be a critical section (see [30, Theorem 6.3]). □

8. Some Applications

8.1. *Noether theorem*

Next, we consider the infinitesimal symmetries of Berezinian densities. The basic idea is to study under which conditions we can interchange ι_X with d^G in (7.1) to obtain the equation

$$d^G(j^{n+1}s)^*(\iota_X\Theta^L) = 0,$$

giving us an invariant, $\iota_X\Theta^L$. In classical mechanics, this is the case when the Lagrangian is invariant under the action of some group whose infinitesimal generator is precisely X; this observation motivates the following definitions.

A p-projectable vector field X on $(N, \mathcal{B})$ is said to be an *infinitesimal supersymmetry* of the Berezinian density

$$\xi_L = \left[d^G x^1 \wedge \cdots \wedge d^G x^m \otimes \frac{d}{dx^{-1}} \circ \cdots \circ \frac{d}{dx^{-n}}\right] \cdot L, \quad L \in \mathcal{A}_{J_G^1(p)},$$

if

$$\mathcal{L}^G_{X_{(n+1)}}\xi_L = 0,$$

where $X_{(n+1)}$ is the $(n+1)$-jet extension of X by graded contact infinitesimal transformations.

Now, the desired interchange amounts to have $\mathcal{L}^G_{X_{(n+1)}}\Theta^L = 0$. A basic result in this direction is the infinitesimal functoriality of the Poincaré–Cartan form, a concept which requires a previous definition.

A graded vector field X' on $(M, \mathcal{A})$ is said to have a *graded divergence* with respect to a graded volume m-form η^G on $(M, \mathcal{A})$ if there exists a function $f \in \mathcal{A}$ such that,

$$\mathcal{L}^G_{X'}\eta^G = \eta^G f.$$

In this case, we put $f = \mathrm{div}_G(X')$. A graded vector field X on $(N, \mathcal{B})$ is said to have graded divergence if it is p-projectable and if its projection X' has graded divergence.

Theorem 8.1 [Infinitesimal functoriality of Θ^L, [23]]. *Let $\eta^G \cdot L$ be a graded Lagrangian density on $p : (N, \mathcal{B}) \to (M, \mathcal{A})$ $(L \in \mathcal{A})$ and Θ^L the corresponding* *graded Poincaré–Cartan form. For every vector field X on $(N, \mathcal{B})$ with divergence, we have*

$$\mathcal{L}^G_{X_{(n+1)}} \Theta^L = \Theta^{L'}, \tag{8.1}$$

where $L' = X_{(n+1)}(L) + \mathrm{div}_G(X') \cdot L$.

According to this result, what we want ($\mathcal{L}^G_{X_{(n+1)}} \Theta^L = 0$) is equivalent to $\Theta^{L'} = 0$, that is, to $L' = 0$. Let us see under which conditions this is true for an infinitesimal supersymmetry. Let us write the Berezinian density as $\xi_L = [\xi]L$ and assume that X is such a supersymmetry. Then

$$0 = \mathcal{L}^G_{X_{(n+1)}} \xi_L = \left(\mathcal{L}^G_{X_{(n+1)}}[\xi]\right) L + (-1)^{|X_{(n+1)}||\xi|}[\xi] X_{(n+1)}(L). \tag{8.2}$$

As the Berezinian module plays a rôle akin to that of the volume forms (at least with respect to integration), we can use the concept of Berezinian divergence (see Sec. 4.3.3). We recall that if X' is a graded vector field on $(M, \mathcal{A})$ and ξ is a Berezinian density on $(M, \mathcal{A})$, we have $\mathcal{L}^G_{X'}[\xi] = (-1)^{|X'||\xi|}[\xi] \cdot \mathrm{div}_B(X')$.

Note that the graded divergence of a given graded vector field on $(M, \mathcal{A})$,

$$X' = (X')^i \frac{\partial}{\partial x^i} + (X')^{-j} \frac{\partial}{\partial x^{-j}},$$

does not necessarily exist. Indeed, the existence of the graded divergence requires,

$$\frac{\partial (X')^i}{\partial x^{-j}} = 0,$$

for any $i, -j$. On the other hand, the Berezinian divergence always exists.

If X on $(N, \mathcal{B})$ is p-projectable, we write

$$\mathcal{L}^G_X[\xi] = (-1)^{|X||\xi|}[\xi] \cdot \mathrm{div}_B(X).$$

This makes sense as long as X is projectable (with projection X'); then, if the Berezinian is given by $[\xi] = [\eta \otimes P]$ for some graded form $\eta \in \Omega_G((M, \mathcal{A}))$ and some differential operator $P \in \mathcal{D}(\mathcal{A})$, we extend the previous definition to

$$\begin{aligned} \mathcal{L}^G_X[\xi] &= (-1)^{|X||\omega \otimes P|+1}[\eta \otimes P \circ X'] \\ &= \mathcal{L}^G_{X'}[\xi]. \end{aligned}$$

In other words, the graded Lie derivative of $[\xi]$ with respect to X is that respect to its projection. The same observation (and definition) applies to a graded vector field on $(J^k_G(p), \mathcal{A}_{J^k_G(p)})$ projectable onto $(M, \mathcal{A})$.

Thus, Eq. (8.2) can be rewritten as

$$\begin{aligned} &(-1)^{|X_{(n+1)}||\xi|}\mathrm{div}_B(X_{(n+1)}) \cdot L + (-1)^{|X_{(n+1)}||\xi|} X_{(n+1)}(L) \\ &\quad = \mathrm{div}_B(X_{(n+1)}) \cdot L + X_{(n+1)}(L) \\ &\quad = 0, \end{aligned}$$

and this is the expression of $L' = 0$ except for the fact that we have two different divergences. In this way, we are led to the following result.

Theorem 8.2 [Noether]. *Assume X is an infinitesimal supersymmetry of the Berezinian density*

$$\xi_L = \left[d^G x^1 \wedge \cdots \wedge d^G x^m \otimes \frac{d}{dx^{-1}} \circ \cdots \circ \frac{d}{dx^{-n}} \right] \cdot L, \quad L \in \mathcal{A}_{J^1_G(p)},$$

such that,

(1) *The projection X' of X onto $(M, \mathcal{A})$ has a divergence with respect to*

$$d^G x^1 \wedge \cdots \wedge d^G x^m,$$

(2) $\operatorname{div}_B(X') = \operatorname{div}_G(X')$.

Then, for every critical section s of ξ_L we have

$$d^G \left[\left(j^{n+1} s \right)^* \left(\iota_{X_{(n+1)}} \Theta^L \right) \right] = 0.$$

Proof. If X is an infinitesimal supersymmetry of ξ_L, by (8.2) we have

$$\operatorname{div}_B(X_{(n+1)}) \cdot L + X_{(n+1)}(L) = 0$$

and by (1), (2), $L' = \operatorname{div}_G(X_{(n+1)}) \cdot L + X_{(n+1)}(L) = 0$. Moreover, from (8.1), we have

$$\Theta^{L'} = 0 = \mathcal{L}^G_{X_{(n+1)}} \Theta^L.$$

Thus,

$$(j^{n+1}s)^* (d^G \iota_{X_{(n+1)}} \Theta^L) + (j^{n+1}s)^* (\iota_{X_{(n+1)}} d^G \Theta^L) = 0,$$

and since s is a critical section,

$$(j^{n+1}s)^* (\iota_{X_{(n+1)}} d^G \Theta^L) = 0.$$

The statement now follows from the fact that d^G commutes with pullbacks. □

The superfunctions $\iota_{X_{(n+1)}} \Theta^L$ appearing in the statement, are called *Noether supercurrents*. Analogously, the graded vector fields X satisfying the conditions of the theorem (and, in general, those leading to Noether supercurrents; note that these conditions are sufficient, but not necessary) are called *Noether supersymmetries*.

Corollary 8.3. *Assume X is a p-vertical graded vector field which also is an infinitesimal supersymmetry of the Berezinian density* (6.2). *Then, for every critical section s of ξ_L we have*

$$d^G[(j^{n+1}s)^* (\iota_{X_{(n+1)}} \Theta^L)] = 0.$$

Proof. If X is vertical, its projection is 0 and so $\operatorname{div}_B(X') = \operatorname{div}_G(X') = 0$. □

8.2. *The case of supermechanics*

Consider the supermanifold $\mathbb{R}^{1|1} \doteq (\mathbb{R}, \Omega(\mathbb{R}))$ and the graded submersion

$$p : (N, \mathcal{B}) \to \mathbb{R}^{1|1}, \quad (N, \mathcal{B}) = \mathbb{R}^{1|1} \times \mathbb{R}^{1|1},$$

defined by the projection onto the first factor, which determines the graded bundle of 1-jets $(J^1_G(p), \mathcal{A}_{J^1_G(p)})$. This is the situation that would correspond to supermechanics (see [13, 14, 33, 35]). If (t, s) and (t, s, y, z) are supercoordinates for $\mathbb{R}^{1|1}$ and $(N, \mathcal{B})$, respectively (even y and odd z), we have a system $(s, t, y, z, y_t, y_s, z_t, z_s)$ for $(J^1_G(p), \mathcal{A}_{J^1_G(p)})$. These coordinates are defined through

$$\begin{aligned}
(j^1\sigma)^* t &= \sigma^*(t) = t\\
(j^1\sigma)^* s &= \sigma^*(s) = s\\
(j^1\sigma)^* y &= \sigma^*(y) = \varphi(t)\\
(j^1\sigma)^* z &= \sigma^*(z) = \psi(t)s\\
(j^1\sigma)^* y_t &= \frac{\partial}{\partial t}(j^1\sigma)^* y = \varphi'(t)\\
(j^1\sigma)^* y_s &= \frac{\partial}{\partial s}(j^1\sigma)^* y = 0\\
(j^1\sigma)^* z_t &= \frac{\partial}{\partial t}(j^1\sigma)^* z = \psi'(t)s\\
(j^1\sigma)^* z_s &= \frac{\partial}{\partial s}(j^1\sigma)^* z = \psi(t)
\end{aligned}$$

for a section $\sigma : \mathbb{R}^{1|1} \to (N, \mathcal{B})$ of p. Here, φ and ψ are just real functions. Note the particularity of the coordinate y_s, which evaluated on sections vanishes; this is a special feature of the $(1|1)$-dimension.

The traditional (physics oriented) notation would write $\partial y/\partial t$ instead of $\varphi'(t)$ and so on. In this way, the preceding observation about y_s is masked, so we prefer ours.

A graded Lagrangian is an element $L \in \mathcal{A}_{J^1_G(p)}$ (i.e., a "superfunction of $(s, t, y, z, y_t, y_s, z_t, z_s)$"). We are interested in determining the class of Lagrangians which admit a p-projectable graded vector field on $(N, \mathcal{B})$, of the particular form

$$D = f\frac{\partial}{\partial t} + g\frac{\partial}{\partial s},$$

as a Noether supersymmetry.

A priori, we should have $f = f(t, s)$ and $g = g(t, s)$, but the fact that D must be a supersymmetry imposes some restrictions which we now analyze. First of all, $\mathrm{div}_G(D)$ must exist, and this forces $f = f(t)$; hence

$$\frac{\partial f}{\partial s} = 0. \tag{8.3}$$

Moreover, it is immediate from the definition of graded divergence that

$$\mathrm{div}_G(D) = \frac{\partial f}{\partial t}. \tag{8.4}$$

Secondly, $\mathrm{div}_B(D)$ must coincide with $\mathrm{div}_G(D)$; from expression (4.4) in Sec. 4.3.3, we find that, necessarily,

$$\frac{\partial g}{\partial s} = 0. \tag{8.5}$$

With restrictions (8.3), and (8.5), the computation of the extension $D_{(2)}$ is relatively easy, and the result is

$$\begin{aligned} D_{(2)} = {} & f\frac{\partial}{\partial t} + g\frac{\partial}{\partial s} - \left(\frac{df}{dt}y_t + \frac{dg}{dt}y_s\right)\frac{\partial}{\partial y_t} - \left(\frac{df}{dt}z_t + \frac{dg}{dt}z_s\right)\frac{\partial}{\partial z_t} \\ & - \frac{df}{dt}y_{st}\frac{\partial}{\partial y_{st}} - \left(\frac{d^2 f}{dt^2}y_t + \frac{d^2 g}{dt^2}y_s + 2\frac{df}{dt}y_{tt} + 2\frac{dg}{dt}y_{ts}\right)\frac{\partial}{\partial y_{tt}} \\ & - \frac{df}{dt}z_{st}\frac{\partial}{\partial z_{st}} - \left(\frac{d^2 f}{dt^2}z_t + \frac{d^2 g}{dt^2}z_s + 2\frac{df}{dt}z_{tt} + 2\frac{dg}{dt}z_{ts}\right)\frac{\partial}{\partial z_{tt}}. \end{aligned} \tag{8.6}$$

Finally, the remaining condition for D to be a Noether supersymmetry is $\mathcal{L}^G_{D_{(2)}}\xi_L = 0$; that is,

$$\mathrm{div}_B(D)\cdot L + D_{(2)}L = 0,$$

or, in view of (8.4),

$$\frac{\partial f}{\partial t}L + D_{(2)}L = 0. \tag{8.7}$$

As $L \in \mathcal{A}_{J^1_G(p)}$, we have

$$\frac{\partial L}{\partial y_{tt}} = \frac{\partial L}{\partial y_{st}} = 0$$

and

$$\frac{\partial L}{\partial z_{tt}} = \frac{\partial L}{\partial z_{st}} = 0,$$

so the insertion of (8.6) into (8.7) gives

$$\frac{\partial f}{\partial t}L + f\frac{\partial L}{\partial t} + g\frac{\partial L}{\partial s} - \left(\frac{df}{dt}y_t + \frac{dg}{dt}y_s\right)\frac{\partial L}{\partial y_t} - \left(\frac{df}{dt}z_t + \frac{dg}{dt}z_s\right)\frac{\partial L}{\partial z_t} = 0.$$

Now, evaluating on a section σ we obtain

$$\begin{aligned} & \frac{\partial f}{\partial t}(j^1\sigma)^*L + f(j^1_G\sigma)^*\left(\frac{\partial L}{\partial t}\right) + g(j^1\sigma)^*\left(\frac{\partial L}{\partial s}\right) - \left(\frac{df}{dt}\varphi'(t)\right)(j^1\sigma)^*\left(\frac{\partial L}{\partial y_t}\right) \\ & \quad - \left(\frac{df}{dt}\psi'(t)s + \frac{dg}{dt}\psi(t)\right)(j^1\sigma)^*\left(\frac{\partial L}{\partial z_t}\right) = 0. \end{aligned} \tag{8.8}$$

Any $L \in \mathcal{A}_{J^1_G(p)}$ solution to this equation, is a superlagrangian admitting D as a Noether supersymmetry. Conversely, if we take a fixed $L \in \mathcal{A}_{J^1_G(p)}$, any pair of real functions, $f = f(t,s)$ and $g = g(t,s)$, satisfying (8.8) determines a graded vector field $D = f\partial/\partial t + g\partial/\partial s$, which is a Noether supersymmetry for L.

A trivial case is that of $f = g \equiv 1$. Then, Eq. (8.8) reduces to

$$\frac{\partial L}{\partial t} + \frac{\partial L}{\partial s} = D(L) = 0,$$

that is: *if L does not depend explicitly on (t, s), then the "supertime translation" $D = \partial/\partial t + \partial/\partial s$ is a Noether supersymmetry*, as in the classical setting (see [35]).

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References

[1] V. Aldaya and J. A. de Azcárraga, Variational principles on rth order jets of fibre bundles in field theory, *J. Math. Phys.* **19**(9) (1978) 1869–1875.

[2] ———, Vector bundles, rth order Noether invariants and canonical symmetries in Lagrangian field theory, *J. Math. Phys.* **19**(7) (1978) 1876–1880.

[3] C. Bartocci, U. Bruzzo and D. Hernández Ruipérez, The geometry of supermanifolds, in *Mathematics and its Applications*, Vol. 71 (Kluwer Academic Publishers, Dordrecht, The Netherlands, 1991).

[4] M. Batchelor, The structure of supermanifolds, *Trans. Amer. Math. Soc.* **253** (1979) 329–338.

[5] ———, Two approaches to supermanifolds, *Trans. Amer. Math. Soc.* **258** (1980) 257–270.

[6] F. A. Berezin, *The Method of Second Quantization* (in Russian). Izdat. (Nauka, Moscow, 1965), *Pure and Applied Physics (English translation)*, Vol. 24 (Academic Press, New York-London, 1966).

[7] ———, Introduction to superanalysis, in *Mathematical Physics and Applied Mathematics*, Vol. 9 (D. Reidel Publishing Co. Dordrecht, The Netherlands, 1987).

[8] C. P. Boyer and A. Sánchez-Valenzuela, Lie supergroup actions on supermanifolds, *Trans. Amer. Math. Soc.* **323**(1) (1991) 151–175.

[9] L. Corwin, Y. Ne'eman and S. Sternberg, Graded Lie algebras in mathematics and physics (Bose-Fermi symmetry), *Rev. Modern Phys.* **47** (1975) 573–603.

[10] P. Deligne and J. W. Morgan, Notes on supersymmetry (following Joseph Bernstein), in *Quantum Fields and Strings: A Course for Mathematicians* (Vol. 1, Part 1: Classical Fields and Supersymmetry) (American Mathematical Society, Providence, RI, 1999).

[11] M. Dubois-Violette, Lectures on graded differential algebras and noncommutative geometry, Noncommutative differential geometry and its applications to physics (Shonan, 1999), pp. 245–306. in *Math. Phys. Stud.*, Vol. 23 (Kluwer Academic Publishers Group, Dordrecht, The Netherlands, 2001).

[12] M. Ferraris and M. Francaviglia, Applications of the Poincaré–Cartan form in higher order field theories, in *Differential Geometry and its Applications* (Brno, 1986), pp. 31–52. in *Math. Appl.* (East European Ser.), Vol. 27 (Reidel, Dordrecht, The Netherlands, 1987).
[13] D. Freed, *Five Lectures on Supersymmetry* (American Mathematical Society, Providence, RI, 1999).
[14] P. G. O. Freund, *Introduction to Supersymmetry* (Cambridge University Press, Cambridge, 1998).
[15] P. L. García and J. Muñoz-Masqué, On the geometrical structure of higher order variational calculus, in *Proc. IUTAM-ISIMM Symposium on Modern Developments in Analytical Mechanics*, Vol. I (Torino, 1982), in *Atti Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur.* **117**(suppl. 1) (1983) 127–147.
[16] K. Gawdezki, Supersymmetries: mathematics of supergeometry, *Ann. Inst. Henri Poincaré XXVII* **4** (1977) 335–366.
[17] G. Giachetta, L. Mangiarotti and G. Sardanashvily, *New Lagrangian and Hamiltonian Methods in Field Theory* (World Scientific, River Edge, NJ, 1997).
[18] ———, *Gauge Mechanics* (World Scientific, River Edge, NJ, 1998).
[19] ———, Lagrangian supersymmetries depending on derivatives. Global analysis and cohomology, *Comm. Math. Phys.* **259**(1) (2005) 103–128.
[20] C. Godbillon, *Géométrie différentielle et mécanique analytique* (Hermann, Paris, 1969).
[21] D. Hernández Ruipérez and J. Muñoz Masqué, Higher order jet bundles for graded manifolds (Superjets), in *Proc. Int. Meeting on Geometry and Physics* (Florence, 1982), pp. 271–278. Pitagora Editrice, Bologna, 1983.
[22] ———, Global variational calculus on graded manifolds I, *J. Math. Pures Appl.* **63** (1984) 283–309.
[23] ———, Infinitesimal functoriality of graded Poincaré–Cartan forms, in *Differential Geometric Methods in Theoretical Physics* (Shumen, 1984), pp. 126–132 (World Sci. Publishing, Singapore, 1986).
[24] ———, Global variational calculus on graded manifolds II, *J. Math. Pures Appl.* **63** (1985) 87–104.
[25] ———, Construction intrinsèque du faisceau de Berezin d'une variété graduée, *Comptes Rendus Acad. Sci. Paris, Série I Math.* **301** (1985) 915–918.
[26] ———, Variational Berezinian problems and their relationship with graded variational problems, *Differential Geometric Methods in Mathematical Physics* (Salamanca, 1985), pp. 137–149. in *Lecture Notes in Mathematics*, Vol. 1251 (Springer, Berlin, 1987).
[27] B. Kostant, Graded manifolds, graded Lie theory, and prequantization, *Differential Geometrical Methods in Mathematical Physics* (Bonn, 1975), pp. 177–306, in *Lecture Notes in Mathematics*, Vol. 570 (Springer, Berlin, 1977).
[28] D. A. Leites, Introduction to the theory of supermanifolds, *Russ. Math. Surveys* **35** (1980) 1–64.
[29] Y. I. Manin, Gauge field theory and complex geometry, in *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*], Vol. 289 (Springer-Verlag, Berlin Heidelberg, 1988).
[30] J. Monterde, Higher order Graded and Berezinian Lagrangian densities and their Euler–Lagrange equations, *Ann. Inst. H. Poincaré Phys. Théor* **57**(1) (1992) 3–26.
[31] J. Monterde and J. Muñoz Masqué, Variational problems on graded manifolds, *Contemp. Math.* **132** (1992) 551–571.

[32] ———, Hamiltonian formalism for Berezinian variational problems in supermanifolds, *Group Theoretical Methods in Physics* (Salamanca, 1992), pp. 253–256. Anales de Física, Monografías 2. CIEMAT-Real Sociedad Española de Física, Madrid (Spain), 1993.

[33] ———, Hamiltonian formalism in supermechanics, *Int. J. Theoret. Phys.* **41**(3) (2002) 429–458.

[34] J. Monterde and O. A. Sánchez–Valenzuela, Calculus of variations in a simple superdomain without the Berezinian integral, *XXVIth National Congress of the Mexican Mathematical Society* (Spanish) (Morelia, 1993), pp. 313–318. Aportaciones Mat. Comun., Vol. 14, Soc. Mat. Mexicana, México, 1994.

[35] J. Monterde and J. A. Vallejo, The symplectic structure of Euler–Lagrange superequations and Batalin–Vilkovisky formalism, *J. Phys. A: Math. Gen.* **36** (2003) 4993–5009.

[36] A. N. Nersessian, On the geometry of supermanifolds with even and odd Kahlerian structures, *Theor. Math. Phys.* **96** (1993) 866–871 [Teor. Mat. Fiz. **96** (1993) 140–149].

[37] O. A. Sánchez-Valenzuela, On supervector bundles, *Comunicaciones Técnicas IIMAS-IMUNAM* (serie naranja) **457** (1986).

[38] ———, Matrix computations in linear superalgebra, *Linear Algebra and its Applications* **111** (1988) 151–181.

[39] ———, Linear supergroup actions. I: On the defining properties, *Trans. Amer. Math. Soc.* **307**(2) (1988) 569–595.

[40] ———, Remarks on Grassmannian supermanifolds, *Trans. Amer. Math. Soc.* **307**(2) (1988) 597–614.

[41] D. J. Saunders, The geometry of jet bundles, in *London Mathematical Society Lecture Note Series*, Vol. 142 (Cambridge University Press, Cambridge, UK, 1989).

[42] Th. Schmitt, Super differential geometry, Report MATH, 84–5. Akademie der Wissenschaften der DDR (Institut für Mathematik), Berlin, 1984.

[43] V. S. Varadarajan, Supersymmetry for mathematicians: an introduction, in *Lecture Notes CLN*, American Mathematical Society-Courant Institute of Mathematical Sciences, Vol. 11 (New York (NY) and Providence, RI, 2004).

[44] T. Voronov, Geometric integration theory on supermanifolds, in Soviet Scientific Reviews. Section C: Mathematical Physics Reviews, Vol. 9, Part 1 (Harwood Academic Publishers, Chur., 1991).

[45] J. Wess and J. Bagger, Supersymmetry and supergravity, 2nd edn., in *Princeton Series in Physics* (Princeton University Press, Princeton, NJ, 1992).