The Poisson Multiple-Access Channel

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***Abstract*—The Poisson multiple-access channel (MAC) models many-to-one optical communication through an optical fiber or in free space. For this model we compute the capacity region for the two-user case as a function of the allowed peak power. Focusing on the maximum throughput we generalize our results to the case where the users are subjected to an additional average-power constraint and to the many-users case. We show that contrary to the Gaussian MAC, in the Poisson MAC the maximum throughput is bounded in the number of users. We quantify the loss that is incurred when Time-Division Multiple Access (TDMA) is employed and show that while in the two-user case and in the absence of dark current the penalty is rather mild, the penalty can be quite severe in the many-users case in the presence of large dark current. We introduce a generalized TDMA technique that mitigates this loss to a large extent.**

***Index Terms*—Capacity region, infrared, multiple-access channels, multiuser, optical CDMA, optical TDMA, Poisson.**

I. Introduction

THE Poisson channel attracts much interest as it serves as the standard model for optical communications [1]–[3]. Its conceptual simplicity and the advent of many uncoded and coded communications techniques [1]–[4] have propelled an extensive information-theoretic study of communication over this channel in an effort to identify and quantify the ultimate limits and the ultimate potential of this channel. The overwhelming majority of these papers [4]–[12] treat the single-user channel only. In this model, which is depicted in Fig. 1(a), the channel output $y(t),\ t \in [0, T]$ is a doubly stochastic Poisson process with instantaneous rate $x(t) + \lambda_0$, where $x(t) \geq 0$ is the channel input, and $\lambda_0 \geq 0$ is a constant. The output $y(t)$ corresponds to the number of counts registered by the direct detection device (usually a p-i-n diode) in the interval $[0, t]$; the input $x(t)$ is proportional to the squared magnitude of the optical field impinging on the detector at time t integrated over its active surface; and the constant λ_0 stands for "dark current" and accounts for spontaneous emissions due to background radiation.

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The input signal $x(t)$ is often peak- and average-power limited [5]–[8] so that

$$\begin{aligned} 0 \leq x(t) \leq A \\ \mathbb{E}\Big(\tfrac{1}{T}\int_0^T x(\tau)\,d\tau\Big) \leq B \end{aligned} \tag{1.1}$$

where A stands for the peak power and B denotes the allowed average power. Here $\mathbb{E}$ denotes the expectation operator, and subscripts, if attached, denote the random variables over which the expectation is taken. The time T stands for the transmission duration and is usually assumed to approach infinity. The capacity C_1 in nats per second under these constraints is given by [5]–[7]

$$\begin{aligned} C_1 = A[&p_{\text{opt}}(1+\lambda_o/A)\log(1+\lambda_o/A) \\ &+ (1-p_{\text{opt}})\,\lambda_o/A\,\log(\lambda_o/A) \\ &- (p_{\text{opt}}+\lambda_o/A)\log(p_{\text{opt}}+\lambda_o/A)] \end{aligned} \tag{1.2a}$$

where

$$p_{\text{opt}} = \min\left(B/A,\ p_o(\lambda_o/A)\right) \tag{1.2b}$$

and where

$$p_o(u) = \frac{(1+u)^{1+u}}{u^u e} - u. \tag{1.2c}$$

The capacity of the single-user Poisson channel is maximized in the absence of dark current ($\lambda_0 = 0$) and when the average-power constraints are relaxed. In this case, the capacity is given by A/e. Thus

$$C_1 \leq A/e. \tag{1.3}$$

To achieve capacity, input signals of infinite bandwidth are required, and the capacity is typically reduced if the input is subjected to bandwidth-like constraints [10]–[12].

The Poisson single-user channel is one of the few channels for which, in addition to the channel capacity, the reliability function at all rates below capacity is also known [5]. In fact, in the absence of dark current and under capacity-reducing average-power constraints, the reliability function is even known in the presence of a noiseless feedback link from the receiver to the transmitter [13].

In recent years optical multiuser communication systems were introduced and intensively investigated [14], [15]. A variety of multiple-access techniques such as Wavelength-Division Multiplexing (WDM), Time-Division Multiple Access (TDMA), and Code-Division Multiple Access (CDMA) are commonly considered [14], [15]. While these accessing methods have natural counterparts in the radio channel, the

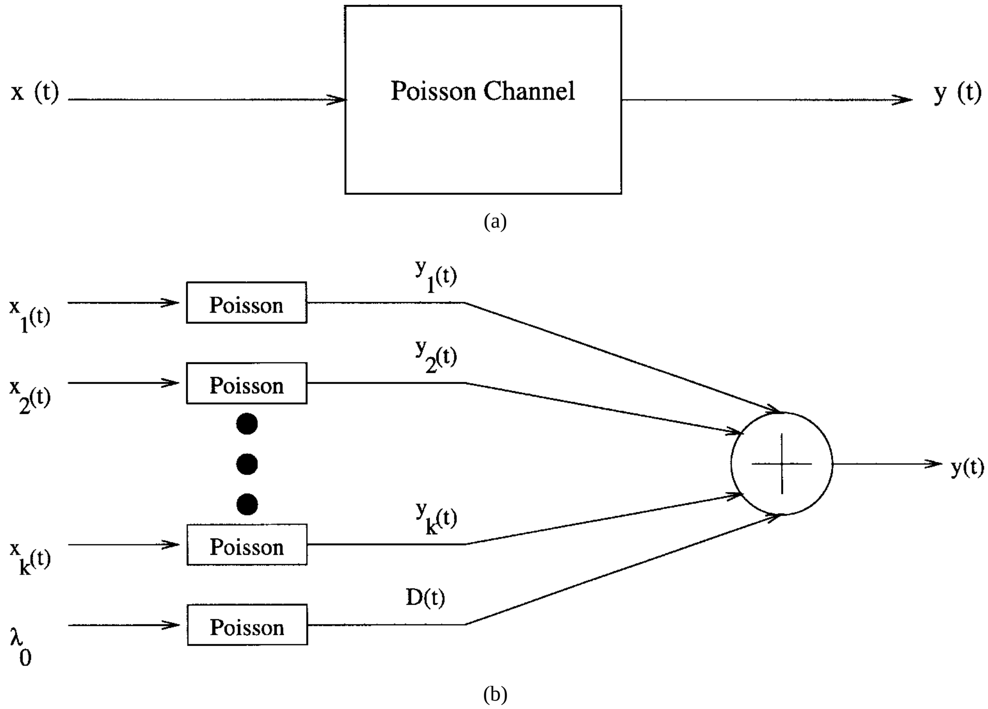

Fig. 1. Schematic diagram of the single- and multiple-access Poisson channel. (a) The single-user channel. $y(t)$ is a conditional Poisson process with instantaneous rate $x(t) + \lambda_0$. (b) The multiple-access Poisson channel. $y(t)$ is the observed Poisson process combined of the Poisson processes $\{y_k(t)\}$, which correspond to the individual rates of the independent users $\{x_k(t)\}$, $k = 1, 2, \cdots, K$. $D(t)$ is the dark-current Poisson process with rate λ_0.

Poisson channel is unique in that the channel input must be nonnegative.

Multiuser optical channels with a variety of single-user and multiuser detection methods were studied [16]; optical CDMA was particularly studied in [17]–[29] and in references therein. The constraints of having nonnegative inputs fundamentally impacts the design of good spreading sequences [26]–[28]. In fact, TDMA can be viewed as a special case of synchronous CDMA where the disjoint time slots of the different users are determined by properly selecting the spreading sequences. Most of the reported studies examine uncoded, possibly spread, communication systems; but see [29]–[32], where coding is addressed in the context of multiuser optical communication and in particular in combination with CDMA-based methods.

The model for the Poisson multiple-access channel (MAC) that we study is shown in Fig. 1(b). The input of the kth user $x_k(t) \geq 0$ determines the rate of the corresponding doubly stochastic Poisson process $y_k(t)$ while the overall observation

$$y(t) = \sum_{k=1}^{K} y_k(t) + D(t)$$

is also a doubly stochastic Poisson process with instantaneous rate

$$\lambda_0 + \sum_{k=1}^{K} x_k(t).$$

Here $D(t)$ is a homogeneous Poisson process of rate λ_0 (the dark current), and K designates the number of users. This channel model is equivalent to having an input

$$\sum_{k=1}^{K} x_k(t)$$

to the single-user Poisson channel. Clearly, this multiuser channel model accounts for any possible CDMA or TDMA multiuser optical system and, therefore, motivates an information-theoretic investigation in an effort to identify the ultimate possible reliable transmission rates.

The literature on this topic is at best scarce. In [33] a somewhat loose upper bound on the overall information throughput is given in terms of the total photon count of all users in the case of no dark current. In [34] a somewhat different model for the two-user Poisson channel is investigated in terms of cutoff rates. The channel model in [34] is different from our model in that our model assumes that the rates, rather than the optical fields, combine additively. The model in [34] is appropriate when the surface area of the p-i-n diode is small compared to the wavelength and when the optical fields produced at the detector by the different users can be individually controlled. For the model studied in [34] and [16] it has been shown [34] that in the average-power dominated regime, a TDMA strategy of both users optimizes the cutoff rates.

In this paper, we address the Poisson MAC and investigate its capacity region and the overall throughput in an effort to determine its ultimate limitations as predicted by multiuser Shannon theory [35], [36]. In the next section we show that for the K-users case, the capacity region is not reduced if the users are limited to the use of binary waveforms taking on the extreme values of zero and the peak power A. The full-capacity region is treated in Section III and is determined in the two-user case and peak-power constrained inputs. The total throughput is discussed in Section IV where it is investigated for the many-user case both with and without average-power constraints. No further limitations on the input signal such as bandwidth and the like are imposed. We show that contrary to the Gaussian MAC, where maximum throughput increases logarithmically with the number of users [36], in the Poisson regime maximum throughput is bounded in the number of users. This result significantly sharpens the conclusion in [33]. In the concluding Section V, we quantify the loss incurred when TDMA is employed. We show that the loss is fairly mild in the two-users case with low dark current, but that the loss is quite severe in the many-users case with high dark current. We then introduce a generalized TDMA scheme where more than one user may transmit at a given time slot, but where single-user detection is employed. This generalized TDMA mitigates to a large extent the loss that is incurred by the standard TDMA scheme.

II. Optimal Input Distributions

In this section, we show that the capacity region of a Poisson MAC is not reduced if the inputs are restricted to the set $\{0, A\}$, where A denotes the peak allowed power. The inputs shall be assumed throughout to be subjected to the peak- and average-power constraints

$$0 \le x_i(t) \le A, \; \forall t, \qquad i = 1, 2, \cdots, K \tag{2.1}$$

$$\frac{1}{T}\int_0^T m_i(\tau)\, d\tau \le B, \qquad i = 1, 2, \cdots, K \tag{2.2}$$

where

$$m_i(t) = \mathbb{E}\, x_i(t).$$

Here, as in (1.1), A and B stand for the peak- and average-power constraints, respectively, and $T\,(T \to \infty)$ designates the transmission time.

The capacity region of the MAC is intimately related to all possible sets of conditional (and unconditional) average mutual information expressions [35], [36]

$$I^{\text{av}}_{\boldsymbol{x}_S; \boldsymbol{y} | \boldsymbol{x}_{S^c}} \triangleq (1/T)\, I\left(\bigcup_{i \in S} x_{i,0}^T\, ; \, y_0^T \,|\, \bigcup_{j \in S^c} x_{j,0}^T\right)$$

where S stands for any subset of $\{1, 2, \cdots, K\}$, S^c is the complementary subset, $\boldsymbol{x}_S$, $\boldsymbol{x}_{S^c}$ stand for a vector with components indexed by the elements in set S and S^c, respectively, and the abbreviation "av" stands for average. The notation u_0^T designates the sample path of a process $u(t)$, $0 < t \le T$.

It should also be noted that in the synchronous (frame [37], and symbol [38]) multiple access channel all the users $\{x(t)\}$ are conditionally independent given the time axis, which means here that they can choose their instantaneous average power $\mathbb{E}(x_i(t))$ $i = 1, 2, \cdots, K$, arbitrarily and in synchronism provided that the peak- and average-power constraint (2.1) and (2.2) are satisfied. The time-varying strategy of each user employed in (time) synchronism but otherwise independently is equivalent to the independence of the user given the auxiliary (time-sharing) variable used to characterize the capacity region of an input constrained MAC [35]–[39].

By Kabanov [6] and Davis [7] we then have

$$I^{\text{av}}_{\boldsymbol{x}_S; \boldsymbol{y} | \boldsymbol{x}_{S^c}} = \frac{1}{T}\int_0^T dt\, \mathbb{E}\left\{\phi\left(\sum_{i \in S} x_i(t), \sum_{i \in S^c} x_i(t)\right) - \phi\left(\sum_{i \in S} \hat{x}_i(t), \sum_{i \in S^c} x_i(t)\right)\right\} \tag{2.3}$$

where

$$\hat{x}_i(t) = \mathbb{E}\left(x_i(t) \,|\, y_0^t, \bigcup_{j \in S^c} x_{j,0}^t\right), \qquad i \in S \tag{2.4}$$

and where

$$\phi(\alpha, \beta) = (\alpha + \beta + \lambda_0)\log(\alpha + \beta + \lambda_0) - (\beta + \lambda_0)\log(\beta + \lambda_0), \qquad \alpha, \beta, \lambda_0 \ge 0. \tag{2.5}$$

Hereafter, natural logarithms are used.

We now upper-bound the relevant average mutual information expressions with a bound that will later be shown to be tight for "quickly varying" inputs. By the convexity of $\phi(\alpha, \beta)$ with respect to α, the conditional independence of $x_i(t)$, $i = 1, 2, \cdots, K$ and Jensen's inequality and using

$$\mathbb{E}_{y_0^t}\hat{x}_i(t) = \mathbb{E}\big(x_i(t) | \cup_{j \in S^c} x_{j,0}^t\big) = \mathbb{E}(x_i(t)) = m_i(t), \qquad i \in S$$

it follows that

$$I^{\text{av}}_{\boldsymbol{x}_S; \boldsymbol{y} | \boldsymbol{x}_{S^c}} \le \frac{1}{T}\int_0^T dt \cdot \mathbb{E}\,\psi\left(\sum_{i \in S} x_i(t), \sum_{i \in S^c} x_i(t); \sum_{i \in S} m_i(t)\right) \tag{2.6}$$

where

$$\psi(\alpha, \beta; c) = \phi(\alpha, \beta) - \phi(c, \beta) \tag{2.7}$$

is a function of the indeterminates α and β, and it is parameterized by a nonnegative constant c.

For the time being, we omit the time dependence of the integrand in the right-hand side in (2.6) and opt to maximize

$$\max_{\{x_i\}} \mathbb{E}\psi\left(\sum_{i \in S} x_i, \sum_{i \in S^c} x_i; \sum_{i \in S} m_i\right) \tag{2.8}$$

over all independent random variables $\{x_i\}_{i=1}^K$ satisfying the peak- and average-power constraints

$$0 < x_i < A \qquad \mathbb{E}(x_i) = m_i. \tag{2.9}$$

To this end, the following assertion will be useful.

Assertion 1:

a) The function $\psi(\alpha, \beta; c)$ is strictly convex with respect to α for each β and constant c, and hence $\mathbb{E}_{\boldsymbol{\beta}}(\alpha, \boldsymbol{\beta}; c)$ is a strictly convex function with respect to α for each c and for any distribution on the random variable $\boldsymbol{\beta}$.

b) $\mathbb{E}_{\boldsymbol{\alpha}}\psi(\boldsymbol{\alpha}, \beta; \mathbb{E}(\boldsymbol{\alpha}))$ is convex with respect to β, where $\boldsymbol{\alpha}$ is assumed to be a random variable.

Proof: Part a) follows immediately by the strict convexity of $\phi(\alpha, \beta)$ with respect to α for each β. To prove part b) we write

$$\mathbb{E}_{\boldsymbol{\alpha}}\psi(\boldsymbol{\alpha}, \beta; \mathbb{E}(\boldsymbol{\alpha})) = \mathbb{E}_{\boldsymbol{\alpha}}(\boldsymbol{\alpha}+\beta+\lambda_0)\log(\boldsymbol{\alpha}+\beta+\lambda_0) - (\mathbb{E}(\boldsymbol{\alpha})+\beta+\lambda_0)\log(\mathbb{E}(\boldsymbol{\alpha})+\beta+\lambda_0). \tag{2.10}$$

Differentiating twice with respect to β (switching the order of expectation and differentiation) yields

$$\frac{\partial^2 \mathbb{E}_{\boldsymbol{\alpha}}\psi(\boldsymbol{\alpha}, \beta; \mathbb{E}(\boldsymbol{\alpha}))}{\partial \beta^2} = \mathbb{E}\boldsymbol{\alpha}\left(\frac{1}{\boldsymbol{\alpha}+\beta+\lambda_0}\right) - \frac{1}{\mathbb{E}(\boldsymbol{\alpha})+\beta+\lambda_0} \geq 0 \tag{2.11}$$

where the inequality in the above is due to the convexity of the function $1/x,\ x > 0$, and Jensen's inequality. □

We now state Assertion 2 which limits the optimizing distributions for (2.8) to binary.

Assertion 2: The optimizing independent random variables $\{x_i\},\ i = 1, 2, \cdots, K$ in the maximization problem stated in (2.8) and (2.9) are binary, taking on the values 0 and A with the probability function

$$\Pr(x_i = A) = 1 - \Pr(x_i = 0) = p_i = m_i/A, \qquad 1 \leq i \leq K. \tag{2.12}$$

Proof: Consider the following random variables:

$$\begin{aligned} a_l &= \sum_{i \in S,\, i \neq l} x_i & s &= \sum_{i \in S} x_i = x_l + a_l \\ b_l &= \sum_{i \in S^c,\, i \neq l} x_i & w &= \sum_{i \in S^c} x_i = x_l + b_l. \end{aligned} \tag{2.13}$$

Assume first that $l \in S$. The expectation in (2.8) is then given by

$$\mathbb{E}_{x_l}\mathbb{E}_{a_l}\mathbb{E}_w\psi(x_l + a_l,\ w; \mathbb{E}(s)).$$

Note that, by Assertion 1, the function

$$\psi_1(\alpha) = \mathbb{E}_w\psi(\alpha,\ w; \mathbb{E}(s))$$

is a strictly convex function of α, and, therefore, the function

$$\psi_2(x) = \mathbb{E}_{a_l}\psi_1(x + a_l) = \int d\mu_{a_l}\,\psi_1(x + a_l)$$

where μ_{a_l} stands for the probability measure of a_l is also a strictly convex function of x. Now fix the probability measures μ_{x_i} for all $i \in S,\ i \in S^c$ but $i \neq l \in S$ (i.e., the probability measures of w and a_l are fixed). The optimization with respect to x_l boils down to

$$\begin{aligned} &\max \mathbb{E}_{x_l}\psi_2(x_l) = \int_0^A d\mu_{x_l}\,\psi_2(x_l) \\ &0 \leq x_l \leq A \qquad\qquad l \in S \\ &\mathbb{E}(x_l) = \int_0^A d\mu_{x_l}\,x_l = m_l \end{aligned} \tag{2.14}$$

i.e., the maximization of a strictly convex function over all finite support probability measures with a given first moment. The solution is achieved by a distribution of two mass points—one at 0 and the other at A—and the maximizing probability measure μ_{x_l} is given by (2.12), with $i = l$. The result holds for any $l \in S$. The precise result from [40] that is needed here can be also found in [12, Lemma 1].

Now, let $l \in S^c$ and, in this case, the optimization problem in (2.8) boils down to

$$\max_{\{x_i\}\ i=1,2,\cdots,K} \mathbb{E}_{x_l}\,\mathbb{E}_{b_l}\,\mathbb{E}_s\psi(s,\ x_l + b_l; \mathbb{E}(s))$$

under the constraints in (2.9).

Fix now the probability measures of all $x_i,\ i \in S, S^c$, except for $x_l,\ l \in S^c$ (i.e., the probability measures of s and b_l are fixed). The optimization problem with respect to x_l is then given by

$$\begin{aligned} &\max \mathbb{E}_{x_l}(\mathbb{E}_{b_l}\mathbb{E}_s\psi(s,\ x_l + b_l; \mathbb{E}(s))) \\ &\qquad = \int_0^A d\mu_{x_l}\,\mathbb{E}_{b_l}\,\mathbb{E}_s\psi(s,\ x_l + b_l; \mathbb{E}(s)) \\ &0 \leq x_l \leq A \qquad\qquad l \in S^c \\ &\mathbb{E}(x_l) = \int_0^A d\mu_{x_l}\,x_l = m_l. \end{aligned} \tag{2.15}$$

The function $\mathbb{E}_s\psi(s, \beta; \mathbb{E}(s))$ is by Assertion 1 strictly convex with respect to β and hence the function $\mathbb{E}_{b_l}\{\mathbb{E}_s(\psi(s,\ b_l + x; \mathbb{E}(s)))\}$ is a strictly convex function of x. Thus the maximization in (2.15) is of a strictly convex function over the probability measures of μ_{x_l} of finite support $[0, A]$ and of a given expectation. The conclusion about the optimality of the binary $(x_l = 0, A)$ measure now follows as in the previous case by [40]. Since the result is valid for all $l \in S^c$ and for all $l \in S$, the assertion is established. □

So far, we have examined an upper bound on the relevant mutual information expression (2.6). This bound, however, can be made arbitrarily tight by selecting the time-varying inputs $x_i(t),\ 1 \leq i \leq K$ to be "infinitely fast" (infinite bandwidth) Markov processes. This follows directly from the result of [41] and is also evident by the results of [5] and [7].

The rational behind this phenomenon is that the bounding step leading to (2.6) is the replacement of $\hat{x}_i(t)$ in (2.4) by $\mathbb{E}(x_i(t)) = m_i(t)$. Now selecting $x_i(t)$ to be an infinitely fast varying process with expanding unrestricted bandwidth, renders y_0^t useless in the conditional estimation of $x_i(t)$ and, therefore,

$$\begin{aligned} \hat{x}_i(t) &= \mathbb{E}\left(x_i(t)|y_0^t,\ \bigcup_{j \in S^c} x_{j,0}^t\right) \xrightarrow{B \to \infty} \mathbb{E}\left(x_i(t)|\cup_{j \in S^c} x_{j,0}^t\right) \\ &= \mathbb{E}(x_i(t)) = m_i(t) \end{aligned}$$

where the sign $\overset{B\to\infty}{\longrightarrow}$ denotes the limit of the process $x(t)$ at infinite bandwidth, and where we resort to [7] and [41] for the precise definitions of this limiting process.

We can now state the following lemma, which is fundamental in the determination of the capacity region of the Poisson MAC.

Lemma 1: The capacity-region achieving distributions of the K-user Poisson MAC under peak (2.1) and average (2.2) power constrained inputs are binary. The independent inputs $x_i(t),\ i = 1, \cdots, K$, assume the values 0 and A only. □

The Lemma follows directly by examining the expression in (2.6) and invoking Assertion 2.

Lemma 1 can also be proved using the approximation technique of [5]. One first approximates the signals in the codebooks by piecewise-constant functions and then demonstrates that the effect of an input that is constant over an infinitesimal time interval can be attained using binary pulsewidth modulation. These approximations typically result in input signals of fast variations and are thus applicable only when no spectral restrictions are imposed on the input [10]–[12] (as we assume throughout). General results on sufficiency of binary inputs can be found in [42].

The supremization problem of $I^{\text{av}}_{\boldsymbol{x}_S;\boldsymbol{y}|\boldsymbol{x}_{S^c}}$ under the input peak- and average-power constraints (2.1), (2.2) is equivalent to supremizing

$$\frac{1}{T}\int_0^T dt\, \mathbb{E}\,\psi\left(\sum_{i\in S} x_i(t),\ \sum_{i\in S^c} x_i(t);\ \sum_{i\in S} m_i(t)\right)$$

under these input constraints, because for processes of infinite bandwidth (2.6) holds with equality [41]. By direct application of Assertion 2 it follows that the latter supremization is achieved by binary signals $x_i(t) \in \{0, A\},\ \forall t$. Note, however, that Lemma 1 does not imply stationarity in the sense that $\Pr\,(x_i(t) = A) = p_{i,t}$ is independent of t. This possible time dependence allows for time-sharing strategies [8], [39]. Nevertheless, in the following sections we will show that in a variety of interesting cases time-sharing is superfluous.

III. The Boundary of the Capacity Region: Two Users

In this section, we study the capacity region of the Poisson multiple-access channel when only two users access the channel. The signal transmitted by each user is peak-power limited, with the peak power being identical for the two users. Thus

$$0 \le x_1(t),\, x_2(t) \le A, \qquad \forall t. \tag{3.1}$$

Throughout this section, we shall assume that no additional average-power constraints are in effect, corresponding to setting $B = A$ in (2.2).

By Lemma 1 we may assume without loss in optimality that the signals transmitted by the two users take on the values 0 and A only. With this observation in mind we define, for any pair $0 \le p,\, q \le 1$, two independent random variables $X_1,\, X_2$ by

$$\Pr\{X_1 = A\} = 1 - \Pr\{X_1 = 0\} = p \tag{3.2}$$
$$\Pr\{X_2 = A\} = 1 - \Pr\{X_2 = 0\} = q. \tag{3.3}$$

By choosing the signal $X_1(t)$ to be stationary with marginal distribution identical to that of X_1 but otherwise of ever increasing bandwidth and likewise for $X_2(t)$, we can attain (2.6) with equality [41], and we can thus deduce that for every $0 \le p,\, q \le 1$ the pentagon $\mathcal{R}_{p,\,q}$ consisting of all pairs $(R_1,\, R_2)$ satisfying

$$R_1 \le I_{X_1;Y|X_2}(p,\, q) \tag{3.4}$$
$$R_2 \le I_{X_2;Y|X_1}(p,\, q) \tag{3.5}$$
$$R_1 + R_2 \le I_{X_1,\, X_2;Y}(p,\, q) \tag{3.6}$$

is achievable. The notation we adopt here makes the dependence of the average mutual informations on $p,\, q$ explicit with

$$I_{X_1;Y|X_2}(p,\, q) = \mathbb{E}\psi(X_1,\, X_2; Ap) \tag{3.7}$$
$$I_{X_2;Y|X_1}(p,\, q) = \mathbb{E}\psi(X_2,\, X_1; Aq) \tag{3.8}$$
$$I_{X_1,\, X_2;Y}(p,\, q) = \mathbb{E}\psi(X_1 + X_2,\, 0; A(p+q)) \tag{3.9}$$

and where all expectations are with respect to the independent random variables X_1 and X_2 satisfying (3.2) and (3.3).

By (2.6), we conclude that the capacity region $\mathcal{C}$ of the two-user Poisson multiple-access channel is given by

$$\mathcal{C} = \text{convex closure of } \mathcal{R} \tag{3.10}$$

where

$$\mathcal{R} = \bigcup_{0\le p,\, q\le 1} \mathcal{R}_{p,\,q}. \tag{3.11}$$

Notice that by (1.3) the pentagons $\mathcal{R}_{p,\,q}$ are compact in the two-dimensional Euclidean space with $\mathcal{R}_{p,\,q} \subset [0,\, A/e] \times [0,\, A/e]$. The convex closure of $\mathcal{R}$ is thus equal to the convex hull of the closure of $\mathcal{R}$, and it is also equal to the closure of the convex hull of $\mathcal{R}$.

As mentioned above, in this section we only consider the case where no average-power constraints are placed on the transmitted signals. Average-power constraints cannot be generally treated simply by limiting the pairs $(p,\, q)$ over which the union in (3.11) is taken to those pairs that satisfy the average-power constraint: the capacity region may be larger than that, see [39] and [42].

We next demonstrate that the region $\mathcal{R}$ is compact, and that we can therefore replace (3.10) with

$$\mathcal{C} = \text{convex hull of } \mathcal{R}. \tag{3.12}$$

This easily follows by noting that $\mathcal{C} \subset [0,\, A/e] \times [0,\, A/e]$, and by noting that the functions

$$I_{X_1;Y|X_2}(p,\, q) \qquad I_{X_2;Y|X_1}(p,\, q) \qquad I_{X_1,\, X_2;Y}(p,\, q)$$

are all continuous[1] on the compact $[0,\, 1] \times [0,\, 1]$. Indeed, assume that $(R_1^n,\, R_2^n) \in \mathcal{R}_{p_n,\, q_n},\ R_1^n \to R_1,\ R_2^n \to R_2$. It then follows by the compactness of $[0,\, 1] \times [0,\, 1]$ that there exists a subsequence n_k and a pair $(p^*,\, q^*)$ such that $p_{n_k} \to p^*$, $q_{n_k} \to q^*$. The continuity of $I_{X_1;Y|X_2}(\cdot,\, \cdot),\ I_{X_2;Y|X_1}(\cdot,\, \cdot)$, and $I_{X_1,\, X_2;Y}(\cdot,\, \cdot)$ now demonstrates that $(R_1,\, R_2) \in \mathcal{R}_{p^*,\, q^*}$, and $\mathcal{R}$ is thus closed.

[1] In the definition of the function $\phi(\alpha,\, \beta)$, see (2.5), we define $0\ \log\ 0 = 0$. With this definition, the function $\phi(\alpha,\, \beta)$ becomes continuous.

To continue our study of the region $\mathcal{R}$, we now compute the maximum throughput R_Σ, which is defined as

$$\begin{aligned} R_\Sigma &= \max_{(R_1,R_2)\in\mathcal{C}} R_1 + R_2 \\ &= \max_{(R_1,R_2)\in\mathcal{R}} R_1 + R_2 \end{aligned} \tag{3.13}$$

where the second equality follows from (3.12). In fact,

$$R_\Sigma = \max_{0\le p,\, q\le 1} I_{X_1,\, X_2; Y}(p,\, q) \tag{3.14}$$

as can be verified by noting that if the maximum in (3.14) is achieved by $(p^*,\, q^*)$ then the pair $(R_1^*,\, R_2^*)$, where

$$\begin{aligned} R_1^* &= I_{X_1; Y|X_2}(p^*,\, q^*) \\ R_2^* &= I_{X_1,\, X_2; Y}(p^*,\, q^*) - R_1^* \end{aligned}$$

is achievable since

$$I_{X_2; Y|X_1}(p,\, q) \ge I_{X_1,\, X_2; Y}(p,\, q) - I_{X_1; Y|X_2}(p,\, q).$$

The following lemma demonstrates that R_Σ can be attained at a point of the form $(p^*,\, p^*)$, thus reducing the calculation of R_Σ from a two-dimensional optimization problem to a one-dimensional optimization problem. It should be noted that this cannot, in general, be deduced directly from the symmetry of the channel and from the concavity of the mutual information functional, because a convex combination of two product distributions is not a product distribution and thus cannot be used as a valid input distribution to the multiple-access channel.

Lemma 2: Let $\{X_i\}_{i=1}^K$ be independent random variables distributed as

$$\Pr(X_i = A) = 1 - \Pr(X_i = 0) = p_i, \qquad i = 1, \cdots, K$$

then the function

$$\mathbb{E}\left[\psi\left(\sum_{i=1}^K X_i,\ 0;\ A\sum_{i=1}^K p_i\right)\right]$$

is a Schur concave [43] function of $p_1, \cdots, p_K$ and, in particular,

$$\mathbb{E}\left[\psi\left(\sum_{i=1}^K X_i,\ 0;\ A\sum_{i=1}^K p_i\right)\right] \le \mathbb{E}\left[\psi\left(\sum_{i=1}^K X_i',\ 0;\ AKp'\right)\right]$$

where $\{X_i'\}$ are independent and identically distributed (i.i.d.) with

$$\Pr(X_i' = A) = 1 - \Pr(X_i' = 0) = p'$$

and

$$p' = \frac{1}{K}\sum_{i=1}^K p_i.$$

Remark: A real-valued function $\phi(\cdot)$ defined over $\mathcal{A} \subseteq \mathbb{R}^k$ is Schur-concave if

$$x = yP \Rightarrow \phi(x) \ge \phi(y)$$

for any $k \times k$ doubly stochastic matrix P and for any pair of row vectors x, y in $\mathcal{A}$. An important consequence that we shall use repeatedly is that if $\phi(\cdot)$ is Schur-concave then $\phi(x_1, \cdots, x_k) \le \phi(\bar{x}, \cdots, \bar{x})$, where $\bar{x} = (x_1 + \cdots + x_k)/k$.

Proof: To prove that the mapping

$$(p_1, \cdots, p_K) \mapsto \mathbb{E}\left[\psi\left(\sum_{i=1}^K X_i,\ 0;\ A\sum_{i=1}^K p_i\right)\right]$$

is Schur-concave [43] for all $(p_1, \cdots, p_K) \in [0, 1]^K$, we define the function

$$\varphi(x) = \phi(x, 0) = (x + \lambda_0)\log(x+\lambda_0) - \lambda_0 \log \lambda_0 \tag{3.15}$$

where, as before, we define $0 \log 0 = 0$. Note that

$$\begin{aligned} &\mathbb{E}\left[\psi\left(\sum_{i=1}^K X_i,\ 0;\ A\sum_{i=1}^K p_i\right)\right] \\ &\qquad = \mathbb{E}\left[\varphi\left(\sum_{i=1}^K X_i\right)\right] - \varphi\left(\mathbb{E}\left[\sum_{i=1}^K X_i\right]\right). \end{aligned}$$

The function $\varphi(\cdot)$ will play an important role in this paper, and for future reference we list its derivatives here.

$$\varphi'(x) = 1 + \log(x + \lambda_0) \tag{3.16}$$

$$\varphi''(x) = \frac{1}{x + \lambda_0} \tag{3.17}$$

$$\varphi'''(x) = -\frac{1}{(x+\lambda_0)^2} \tag{3.18}$$

$$\varphi^{(iv)}(x) = \frac{2}{(x+\lambda_0)^3}. \tag{3.19}$$

The proof can be now concluded by noting that by (3.17) the function $\varphi(\cdot)$ is convex in $[0, \infty)$, and the lemma now follows from [43, Proposition F.1., p. 360] and [44]. □

Continuing our computation of R_Σ in the two user case $(K = 2)$, we conclude from Lemma 2 that

$$R_\Sigma = \max_{0\le p\le 1} I_2(p) \tag{3.20}$$

where

$$\begin{aligned} I_2(p) &= I_{X_1,\, X_2; Y}(p,\, p) \\ &= \varphi(0)(1-p)^2 + 2\varphi(A)p(1-p) + \varphi(2A)p^2 - \varphi(2Ap) \\ &= 2\varphi(A)p(1-p) + \varphi(2A)p^2 - \varphi(2Ap). \end{aligned} \tag{3.21}$$

One can readily verify from (3.18) that the third derivative $I_2'''(p) = -8A^3\varphi'''(2Ap)$ is positive in the interval $(0, 1)$, and that $I_2(0) = I_2(1) = 0$. These facts and the positivity of $I_2(p)$ in the interval $(0, 1)$ guarantee that in this interval $I_2(p)$ has a unique extremum, which is a global maximum. We thus conclude that the maximum throughput R_Σ in the two-user case is given by

$$R_\Sigma = I_{X_1,\, X_2; Y}(p^*,\, p^*) \tag{3.22}$$

where p^* is the unique solution in the interval $(0, 1)$ to the equation

$$\frac{d}{dp} I_{X_1,\, X_2; Y}(p,\, p) = 0. \tag{3.23}$$

Having determined the point of maximum throughput, we now continue our investigation of the region $\mathcal{R}$. By the symmetry of the channel with respect to the two users it follows that

$\mathcal{R}$ is symmetric about the line $R_1 = R_2$. It thus suffices to study the set

$$\mathcal{R} \cap \{(R_1, R_2) : R_1 \geq R_2\}.$$

In fact, it suffices to study the even smaller set $\mathcal{D}$ defined by

$$\mathcal{D} = \mathcal{R} \cap \{(R_1, R_2) : R_1 \geq R_2,\, R_2 < I_{X_2;Y}(p^*, p^*)\}$$

where (p^*, p^*) achieves the maximum throughput, and

$$I_{X_2;Y}(p, q) = I_{X_1, X_2;Y}(p, q) - I_{X_1;Y|X_2}(p, q). \tag{3.24}$$

This observation follows by noticing that if maximum throughput is achieved by (p^*, p^*) then the boundary segment of $\mathcal{R}_{p^*, p^*}$ that is of slope -1 must be on the boundary of $\mathcal{R}$.

The region $\mathcal{D}$ will be determined once we compute its boundary ∂D. The parts of ∂D that are of least interest to us are those for which R_1 or R_2 are zero. We thus define $\mathcal{E}$ to be the interesting part $\partial \mathcal{D}$, i.e.,

$$\mathcal{E} = \{(R_1, R_2) \in \partial\mathcal{D} : R_1, R_2 > 0\}.$$

Inspecting (3.11), we see that for some pairs (p, q) the pentagon $\mathcal{R}_{p,q}$ may not touch (intersect) $\mathcal{E}$ and for others it may. The following lemma characterizes the point at which $\mathcal{R}_{p,q}$ could touch $\mathcal{E}$.

Lemma 3: If for some pair $(\tilde{p}, \tilde{q}) \in [0, 1] \times [0, 1]$

$$\mathcal{R}_{\tilde{p}, \tilde{q}} \cap \mathcal{E} \neq \emptyset \tag{3.25}$$

then $\mathcal{R}_{\tilde{p}, \tilde{q}} \cap \mathcal{E}$ consists of only one point, and

$$\mathcal{R}_{\tilde{p}, \tilde{q}} \cap \mathcal{E} = \{(I_{X_1;Y|X_2}(\tilde{p}, \tilde{q}), I_{X_2;Y}(\tilde{p}, \tilde{q}))\}. \tag{3.26}$$

Proof: We shall prove that (3.26) follows from (3.25) using a perturbation argument. Let

$$(\tilde{R}_1, \tilde{R}_2) \in \mathcal{R}_{\tilde{p}, \tilde{q}} \cap \mathcal{E}.$$

By the definition of $\mathcal{E}$, it follows that $\tilde{R}_2 \neq 0$ and thus $\tilde{q} \neq 0$. It can be easily verified that $I_{X_1;Y|X_2}(p, \cdot)$ is monotonically decreasing, and it follows that (3.25) (and in particular $\tilde{R}_2 > 0$) implies

$$\frac{\partial}{\partial q} I_{X_1;Y|X_2}(p, q)\,|_{(\tilde{p}, \tilde{q})} < 0. \tag{3.27}$$

It follows that

$$\tilde{R}_1 + \tilde{R}_2 = I_{X_1, X_2;Y}(\tilde{p}, \tilde{q}) \tag{3.28}$$

for otherwise we would have

$$\tilde{R}_1 + \tilde{R}_2 < I_{X_1, X_2;Y}(\tilde{p}, \tilde{q})$$

and we could slightly decrease $\tilde{q}$ and in this way achieve a point $(\tilde{R}_1 + \delta, \tilde{R}_2)$ for some positive δ.

It follows from (3.28) and the definition of $\mathcal{E}$ that

$$(\tilde{p}, \tilde{q}) \neq (p^*, p^*) \tag{3.29}$$

where (p^*, p^*) attains the maximum throughput. Condition (3.29) implies that the point $(\tilde{p}, \tilde{q})$ is not a local maximum for $I_{X_1, X_2;Y}(\cdot, \cdot)$, i.e., that there is some direction in which $I_{X_1, X_2;Y}(\cdot, \cdot)$ is strictly increasing. Indeed, if $\tilde{p} \neq \tilde{q}$ then this observation follows from the strict Schur concavity of $I_{X_1, X_2;Y}(\cdot, \cdot)$, and if $\tilde{p} = \tilde{q}$ this observation follows from our observation that the only zero in the interval $(0, 1)$ of the derivative of $I_2(p) = I_{X_1, X_2;Y}(p, p)$ with respect to p is p^*, see (3.23).

With this observation we can readily deduce that

$$\tilde{R}_1 = I_{X_1;Y|X_2}(\tilde{p}, \tilde{q}) \tag{3.30}$$

for, otherwise, we would have

$$\tilde{R}_1 < I_{X_1;Y|X_2}(\tilde{p}, \tilde{q})$$

and we would be able to achieve $(\tilde{R}_1, \tilde{R}_2 + \delta)$ for some positive δ by slightly perturbing $(\tilde{p}, \tilde{q})$ in the direction that increases $I_{X_1, X_2;Y}$ without violating (3.4). Equations (3.28) and (3.30) combine to prove the lemma. □

Lemma 3 establishes that an achievable pentagon $\mathcal{R}_{p,q}$ can intersect the boundary $\mathcal{E}$ at most at a single point, and that this point must be a vertex point of the form

$$(I_{X_1;Y|X_2}(p, q), I_{X_2;Y}(p, q)).$$

The following lemma determines a relationship between p and q that must be satisfied if $\mathcal{R}_{p,q}$ is to touch $\mathcal{E}$.

Lemma 4: For condition (3.25) to hold, the pair $(\tilde{p}, \tilde{q})$ must satisfy

$$\frac{\partial I_{X_1,X_2;Y}(p,q)}{\partial p}\,\frac{\partial I_{X_1;Y|X_2}(p,q)}{\partial q} - \frac{\partial I_{X_1,X_2;Y}(p,q)}{\partial q} \cdot \frac{\partial I_{X_1;Y|X_2}(p,q)}{\partial p}\,|_{(\tilde{p}, \tilde{q})} = 0. \tag{3.31}$$

Proof: First note that by the definition of $\mathcal{E}$ it follows that (3.25) implies that $(\tilde{p}, \tilde{q})$ must be in the interior of $[0, 1] \times [0, 1]$. In particular, this implies that we can perturb $(\tilde{p}, \tilde{q})$ in any direction. Clearly, a necessary condition for a pair $(\tilde{p}, \tilde{q})$ to satisfy (3.25) is that in any direction we perturb $(\tilde{p}, \tilde{q})$ we cannot have both $I_{X_1;Y|X_2}(\cdot, \cdot)$ and $I_{X_1, X_2;Y}(\cdot, \cdot)$ increase. This implies that the gradients of these two functions must be antipodal, which implies that the cross product of these gradients must be zero. □

Using Lemmas 3 and 4, we can obtain a description of $\mathcal{E}$ and thus determine the set $\mathcal{R}$. This can be done by allowing $\tilde{q}$ to vary freely between 0 and p^* and by solving for $\tilde{p}(\tilde{q})$ from (3.31). The curve

$$(I_{X_1;Y|X_2}(\tilde{p}(\tilde{q}), \tilde{q}), I_{X_2;Y}(\tilde{p}(\tilde{q}), \tilde{q})), \qquad 0 < \tilde{q} < p^* \tag{3.32}$$

then traces $\mathcal{E}$.

The final step in the computation of the capacity region $\mathcal{C}$ is to compute the convex hull of $\mathcal{R}$, see (3.12). If $\mathcal{R}$ is convex then $\mathcal{C} = \mathcal{R}$ and there is no need for further computation. To check whether $\mathcal{R}$ is convex one needs to check whether the trajectory $\mathcal{E}$ has negative curvature, but the calculation of this curvature is quite messy.

While we conjecture that $\mathcal{R}$ is indeed convex, we have been unable to verify this analytically using the above approach. However, in the absence of dark current $(\lambda_0 = 0)$ we were able to compute and plot the curvature of $\mathcal{E}$ and to verify that $\mathcal{R}$ is indeed convex.

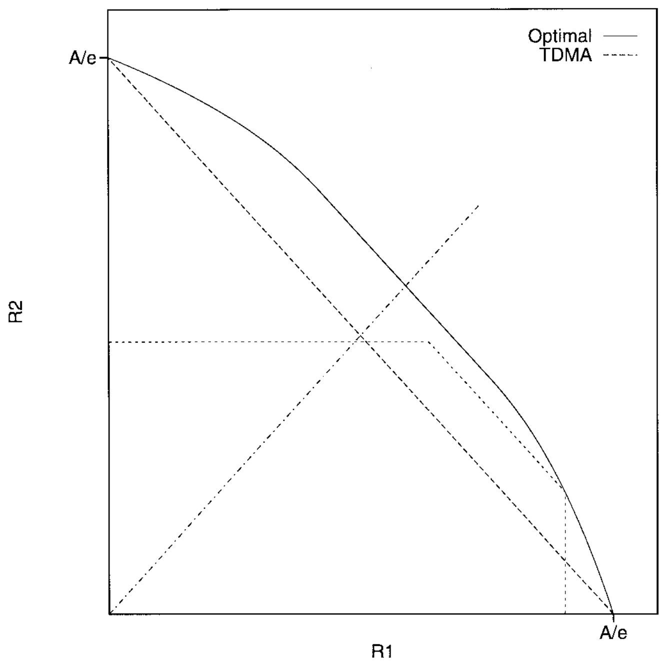

Fig. 2. The capacity region of the Poisson multiple-access channel in the absence of dark current. Also shown is the suboptimal TDMA region, a pentagon (corresponding to some pair of input distributions) touching the boundary of the region, and the region's symmetry line.

In the absence of dark current we have that

$$A^{-1}I_{X_1,X_2;Y}(p,q) = 2pq\log 2 - (p+q)\log(p+q)$$
$$A^{-1}I_{X_1;Y|X_2}(p,q) = 2pq\log 2 - q(1+p)\log(1+p) - (1-q)p\log p.$$

Solving (3.23) numerically we obtain that $p^* \approx 0.2659$, which corresponds to $R_\Sigma \approx 0.434A$. Equation (3.31) reduces to

$$(2q\log 2 - \log(p+q) - 1)(2p\log 2 - (1+p)\log(1+p) + p\log p) = (2p\log 2 - \log(p+q) - 1)(2q\log 2 - q\log(1+p) - (1-q)\log p - 1)$$

and the capacity region can be obtained by solving for $\tilde{p}(\tilde{q})$, $0 < \tilde{q} < p^*$, and mapping $(\tilde{p}(\tilde{q}), \tilde{q})$ according to (3.32). The results are depicted in Fig. 2. In Fig. 2, we also show an example of a pentagon $\mathcal{R}_{p,q}$ touching the capacity region, and the single-user-based time-sharing capacity region, whose boundary is the straight line connecting the point $(A/e, 0)$ and $(0, A/e)$. For reference, we also show the symmetry line of the region.

At the other extreme, when the dark current is very large, one can also verify that $\mathcal{R}$ is convex. Indeed, for very large dark current the capacity region tends to an empty set, but if we properly normalize the rates the limiting capacity region is a rectangle.

IV. K USERS: MAXIMUM THROUGHPUT

A. Peak-Power Constraints Only

In this section, we consider the case where more than two users access the channel and study the maximal achievable throughput. We only consider the symmetric case where all users are subjected to the same peak-power constraint A.

Denoting the maximum throughput for K users by $R_\Sigma^{(K)}$ we have by Lemma 2 that

$$R_\Sigma^{(K)} = \max_{p\in[0,1]} I_K(p) \tag{4.1}$$

where

$$I_K(\cdot) = \mathbb{E}\left[\varphi\left(\sum_{i=1}^K X_i\right)\right] - \varphi\left(\mathbb{E}\left[\sum_{i=1}^K X_i\right]\right) \tag{4.2}$$

and where $X_1, \cdots, X_K$ are i.i.d. with

$$\Pr(X_i = A) = 1 - \Pr(X_i = 0) = p \tag{4.3}$$

and $\varphi(\cdot)$ is defined in (3.15).

Maximum throughput can be thus achieved when all users transmit at the same rate, without the need for time-division multiple accessing. It should be noted that this result does not hold true for a general multiple-access channel, where time division (and, hence, synchronization) may be required to achieve maximum throughput at equal rates [45], [46].

Lemma 5: The sequence

$$\left\{R_\Sigma^{(K)}\right\}_{K=1}^\infty$$

corresponding to the maximum throughput achievable by K users, is monotonically increasing and bounded by the peak power A.

Proof: Monotonicity is a simple consequence of the Schur-concavity, which was proved in Lemma 2. Indeed, setting one of $K+1$ users to be deterministically zero demonstrates that the throughput achievable with $K+1$ users is at least as high as the throughput achievable with K users. In fact, the strict Schur-concavity of $I_{X_1,\dots,X_{K+1};Y}$ demonstrates that $R_\Sigma^{(K+1)}$ is strictly bigger than $R_\Sigma^{(K)}$

$$I_{X_1,\dots,X_{K+1};Y}(0, p, \cdots, p) < I_{X_1,\dots,X_{K+1};Y}(p', p', \cdots, p')$$

where $p' = K/(K+1)p$.

We now turn to proving that

$$R_\Sigma^{(K)} \le A. \tag{4.4}$$

Note that since dark current cannot increase throughput,[2] for the purposes of proving (4.4), we may assume the absence of dark current, i.e., $\lambda_0 = 0$. Let p be fixed and set

$$Z = \sum_{i=1}^K X_i \tag{4.5}$$

where $\{X_i\}_{i=1}^K$ are independent random variables satisfying (4.3). The random variable Z satisfies

$$0 \le Z \le KA \tag{4.6}$$

$$\mathbb{E}[Z] = KpA \tag{4.7}$$

$$\mathbb{E}[Z^2] = \left(Kp(1-p) + K^2p^2\right)A^2. \tag{4.8}$$

[2] Dark current cannot increase throughput because in its absence the receiver can always add an independent homogeneous Poisson process to the received process and thus in effect introduce dark current.

We upper-bound

$$I_K(p) = \mathbb{E}\varphi(Z) - \varphi(\mathbb{E}(Z))$$

by maximizing

$$\mathbb{E}\varphi(Z') - \varphi(\mathbb{E}(Z')) \tag{4.9}$$

over all random variables Z' that satisfy (4.6)–(4.8). Since $\varphi(\cdot)$ has a strictly negative third derivative for all positive arguments (3.18), it follows that the solution to this maximization problem is to have Z' take on only two values, one of which is 0 [40], see also [12, Lemma 1]. Denoting the second of the two values by ξ and its corresponding mass by q, we can solve for ξ and q

$$\xi = A(Kp + (1-p))$$
$$q = \frac{Kp}{Kp + (1-p)}.$$

Computing (4.9) and noting that $\varphi(0) = 0$, we have

$$\begin{aligned} R_\Sigma^{(K)} &\le q\varphi(\xi) - \varphi(q\xi) \\ &= \frac{Kp}{Kp+(1-p)}\,\varphi(A(Kp+(1-p))) - \varphi(Kpa) \\ &= KpA \log\left(\frac{Kp+1-p}{Kp}\right) \qquad (4.10) \\ &\le A(1-p) \qquad (4.11) \\ &\le A \end{aligned}$$

concluding the proof of the lemma. Here the inequality before last follows from the inequality $\log(1+x) \le x$.

Having established that the sequence $R_\Sigma^{(K)}$ converges we now study its limit.

Lemma 6:

a) Irrespective of the strength of the dark current λ_0

$$\lim_{K\to\infty} R_\Sigma^{(K)} \ge A/2. \tag{4.12}$$

b) In the absence of dark current

$$\lim_{K\to\infty} R_\Sigma^{(K)} \approx 0.58A \tag{4.13}$$

and

$$0 < \limsup_{K\to\infty} Kp_K^* < \infty \tag{4.14}$$

where p_K^* is the argument that achieves the maximum in (4.1).

Proof: To simplify notation we normalize the peak power and assume that $A = 1$. We begin by proving part a) of the lemma. To this end, we define the random variable Z as in (4.5), where $\{X_i\}_{i=1}^K$ are independent random variables satisfying (4.3). We next define the zero-mean unit variance random variable N by

$$N = \frac{Z - Kp}{\sqrt{Kp(1-p)}} \tag{4.15}$$

and note that by the Central Limit Theorem, as K tends to infinity, the distribution of N tends to a zero-mean, unit-variance Normal distribution. We now have

$$\begin{aligned} I_K(p) &= \mathbb{E}[\varphi(Z)] - \varphi(\mathbb{E}[Z]) \\ &= \int \left[\varphi\left(\sqrt{Kp(1-p)}\nu + Kp\right) - \varphi(Kp)\right] d\mu_N(\nu) \end{aligned}$$

where $\mu_N(\nu)$ is the probability distribution of N. Noting that the fourth derivative of $\varphi(x)$ is positive for positive x (3.19) it follows from Taylor's expansion of the function $\varphi(x)$ about Kp that

$$\begin{aligned} &\varphi\left(\sqrt{Kp(1-p)}\nu + Kp\right) - \varphi(Kp) \\ &\quad\ge \sqrt{Kp(1-p)}\nu\varphi'(Kp) \\ &\quad+ \tfrac{1}{2}(\sqrt{Kp(1-p)}\nu)^2\varphi''(Kp) \\ &\quad+ \tfrac{1}{3!}(\sqrt{Kp(1-p)}\nu)^3\varphi'''(Kp), \qquad \forall \nu > 0. \quad (4.16) \end{aligned}$$

Similarly, for negative ν we note that the third derivative of $\varphi(x)$ is negative for all x (3.18), and hence by a second-order Taylor expansion we obtain

$$\begin{aligned} &\varphi\left(\sqrt{Kp(1-p)}\nu + Kp\right) - \varphi(Kp) \\ &\quad\ge \sqrt{Kp(1-p)}\nu\varphi'(Kp) \\ &\quad+ \tfrac{1}{2}(\sqrt{Kp(1-p)}\nu)^2\varphi''(Kp), \qquad \forall \nu < 0. \quad (4.17) \end{aligned}$$

Recalling that N is of zero mean and unit variance we obtain from (4.16) and (4.17)

$$\begin{aligned} I_K(p) &\ge \tfrac{1}{2}\,Kp(1-p)\varphi''(Kp) \\ &\quad+ \tfrac{1}{6}\,(Kp(1-p))^{3/2}\varphi'''(Kp)\int_0^\infty \nu^3 d\mu_N(\nu). \end{aligned}$$

Upon substitution of the derivatives of φ from (3.17) and (3.18) we obtain

$$\begin{aligned} I_K(p) &\ge \frac{Kp(1-p)}{2(Kp+\lambda_0)} - \frac{(Kp(1-p))^{3/2}}{6(Kp+\lambda_0)^2}\int_0^\infty \nu^3 d\mu_N(\nu) \\ &\to \frac{1-p}{2} \end{aligned}$$

where the limiting behavior as K tends to infinity follows from the Central Limit Theorem, which guarantees[3] that

$$\int_0^\infty \nu^3 d\mu_N(\nu) \to \sqrt{\frac{2}{\pi}}.$$

Choosing p arbitrarily small demonstrates (4.12) and thus concludes the proof of a).

To prove part b) we must consider three cases corresponding to the $\limsup$ of Kp_K^* being equal to zero, a constant, or infinity. The first case is ruled out by (4.10) as it leads to zero throughput.

We next consider the second case corresponding to

$$\limsup_{K\to\infty} Kp_K^* = \infty.$$

[3] Strictly speaking, this does not follow directly from the Central Limit Theorem since the function $f(\nu) = \nu^3$ is unbounded. Nevertheless, standard techniques, possibly using (4.18), guarantee this limiting behavior.

To simplify notation, we shall normalize the peak power and assume that $A = 1$. We define Z via (4.5) where $\{X_i\}$ are i.i.d. Bernoulli random variables with probability of success p_K. We also define N as in (4.15). To upper-bound the resulting throughput we need the four moments of N and their limiting behavior as Kp_K tends to infinity. Those are given by

$$\begin{aligned} \mathbb{E}[N] &= 0 \\ \mathbb{E}[N^2] &= 1 \\ \mathbb{E}[N^3] &= \frac{Kp_K(1-p_K)(1-2p_K)}{(Kp_K(1-p_K))^{3/2}} \\ &= (Kp_K(1-p_K))^{-1/2}(1-2p_K) \longrightarrow 0. \end{aligned}$$

To compute $\mathbb{E}[N^4]$ note that

$$\begin{aligned} \mathbb{E}[(Z-Kp_K)^4] &= 3\sum_{(i,j)i\neq j}\mathbb{E}[(X_i-p_K)^2(X_j-p_K)^2] \\ &\quad + \sum_i \mathbb{E}[(X_i-p_K)^4] \\ &\le 3(Kp_K(1-p_K))^2 + K\mathbb{E}[(X_1-p_K)^4] \\ &\le 3(Kp_K(1-p_K))^2 + Kp_K(1-p_K) \end{aligned}$$

where the last inequality follows from

$$\mathbb{E}[(X_1-p_K)^4] = p_K(1-p_K)^4 + (p_K)^4(1-p_K) \le p_K(1-p_K).$$

We thus have

$$\mathbb{E}[N^4] \le \frac{3(Kp_K(1-p_K))^2 + Kp_K(1-p_K)}{(Kp_K(1-p_K))^2} \longrightarrow 3 \tag{4.18}$$

where the limiting behavior holds when p_K does not converge to 1, which is the only case of interest by (4.11). We can now upper-bound the maximum throughput as follows:

$$\begin{aligned} I_K(p_K) &= \mathbb{E}\varphi(Z) - \varphi(\mathbb{E}[Z]) \\ &= \mathbb{E}\left[\left(\sqrt{K\sigma^2}N + Km\right)\log\left(\sqrt{K\sigma^2}N + Km\right)\right] \\ &\quad - Km\log(Km) \\ &= \mathbb{E}\left[\left(\sqrt{K\sigma^2}N + Km\right)\log\left(1 + \frac{\sqrt{K\sigma^2}N}{Km}\right)\right] \end{aligned}$$

where $\sigma^2 = p_K(1-p_K)$, $m = p_K$, and the last equality holds because $\mathbb{E}[N] = 0$. Since $\sigma^2 \le m$ we have

$$I_K(p_K) \le \mathbb{E}\left[(\sqrt{Km}N + Km)\log\left(1 + (Km)^{-1/2}N\right)\right].$$

We now use the inequality

$$\log(1+x) \le x - \frac{x^2}{2} + \frac{x^3}{3}$$

to get

$$\begin{aligned} I_K(p_K) &\le \mathbb{E}\Bigg[\left((Km)^{1/2}N + Km\right)\Bigg((Km)^{-1/2}N \\ &\qquad - \frac{1}{2}(Km)^{-1}N^2 + \frac{1}{3}(Km)^{-3/2}N^3\Bigg)\Bigg] \\ &= \frac{1}{2} - \frac{1}{6}(Km)^{-1/2}\mathbb{E}[N^3] + (Km)^{-1}\mathbb{E}[N^4] \to \frac{1}{2}. \end{aligned}$$

We can thus conclude that

$$\lim Kp_K = \infty \;\Rightarrow\; \limsup I_K(p_K) \le 1/2.$$

The monotonicity of the maximum throughput in the number of users now establishes

$$\limsup Kp_K^* = \infty \;\Rightarrow\; \lim R_\Sigma^K \le A/2.$$

To conclude the discussion, we now examine that case where $Kp_K \to \lambda$ for some $0 < \lambda < \infty$. In this case, the distribution of Z converges to Poisson with parameter λ. One can now numerically compute

$$e^{-\lambda}\sum_{k=0}^{\infty}\frac{\lambda^k}{k!}\varphi(k) - \varphi(\lambda)$$

and verify that this is maximized at $\lambda \approx 1.35$ with corresponding mutual information that satisfies (4.13).

We thus see that of the three cases originally considered, the case that yields the highest throughput is the third case where the sum of the channel inputs obeys a Poisson Limit Theorem. □

B. Peak- and Average-Power Constraints

We now consider the case where in addition to the peak-power constraints the users are also average-power limited. We treat only the case where the peak powers and average powers of all users are identical. The peak powers are denoted by A and the average powers by B, according to (2.1) and (2.2).

Accounting for average-power constraints in a multiple-access channel is generally more complicated than in the single-user case [39], [42]. The capacity region in the constrained case could be larger than the convex hull of the union of all pentagons corresponding to pairs of input distributions that satisfy the average-power constraint. To simplify the analysis, we shall not study the entire capacity region but only the maximum throughput, which we denote by $R_\Sigma^{(K)}(B)$, where K denotes the number of users accessing the channel, and B is the highest allowed average power for each user.

It follows from [39] and [42] that the set of achievable rates for the constrained Poisson channel is the closure of the set of all tuples $(R_1, \cdots, R_K)$ of the form

$$(R_1, \cdots, R_K) = \sum_{l=1}^{L}\alpha_l(R_1^{(l)}, \cdots, R_K^{(l)})$$

where

$$\sum_{l=1}^{L}\alpha_l = 1 \qquad \alpha_l \ge 0 \qquad l = 1, \cdots, L$$

and the tuple $(R_1^{(l)}, \cdots, R_K^{(l)})$ is achievable with some product input distribution

$$\Pr(X_k = A) = 1 - \Pr(X_k = 0) = p_k^{(l)} \quad k = 1, \cdots, K$$

and where

$$\sum_{l=1}^{L}\alpha_l p_k^{(l)} \le B/A, \qquad 1 \le k \le K. \tag{4.19}$$

Here by Carathéodory's Theorem[4] [36] L can be taken to be $2K+1$, but in fact $K+2$ is enough [47], [48].

It is clear that if

$$p_k^{(l)}, \qquad l=1,\cdots,L, \;\; k=1,\cdots,K$$

satisfy (4.19), then so do

$$\tilde{p}_k^{(l)}, \qquad l=1,\cdots,L, \;\; k=1,\cdots,K$$

where

$$\tilde{p}_k^{(l)} = \frac{1}{K}\sum_{k=1}^{K} p_k^{(l)}.$$

We can now conclude from the Schur-concavity of the maximum throughput (Lemma 2) that

$$R_\Sigma^K(B) = \max_{\substack{(\alpha_1,\cdots,\alpha_L)\\ (\bar{p}^{(1)},\cdots,\bar{p}^{(L)})}} \sum_{l=1}^{L} \alpha_l I_K(\bar{p}^{(l)}) \tag{4.20}$$

where the maximum is over all nonnegative α_l that sum to one, and all tuples $(\bar{p}^{(1)},\cdots,\bar{p}^{(L)})$ with entries between zero and one that satisfy

$$\sum_{l=1}^{L} \alpha_l \bar{p}^{(l)} \le B/A. \tag{4.21}$$

We have thus proved the following assertion.

Assertion 3: If the function $I_K(p)$, which is defined in (4.2) as the maximum throughput achievable with the input distribution $\Pr(X_i = A) = p$ for all $1 \le i \le K$, is concave in the interval $[0, p_K^*]$, where p_K^* is the argument that maximizes $I_K(\cdot)$, then the maximal throughput under an average power constraint B is given by

$$R_\Sigma^{(K)}(B) = I_K(\hat{p})$$

where

$$\hat{p} = \min\{B/A, p_K^*\}.$$

The significance of this lemma is in demonstrating that under the above concavity conditions, maximum throughput can be achieved in the presence of average power constraints without the need to resort to time-division multiaccessing, and that synchronization is thus not needed. The analogous result in the absence of average power constraints follows, of course, from Lemma 2. While we conjecture that the function $I_K(p)$ is indeed concave in the interval $[0, p_K^*]$ irrespective of the number of users K and of the dark current λ_0, we have been unable to prove this in general. Note, however, that for a particular number of users K and a particular value of the dark current λ_0 this condition can be easily checked numerically. In the two-users case, it is particularly simple to show that $I_2(\cdot)$ is concave in the interval $[0, p_2^*]$, see the discussion leading to (3.22).

We next show that in the absence of dark current, time sharing is not required to achieve maximum throughput in the three-users case, i.e., that our conjecture holds for $K=3$ and $\lambda_0 = 0$. To simplify notation we assume normalized peak power $A=1$. By the definition of $I_3(p)$ we have

$$\begin{aligned} I_3(p) &= p^3\varphi(3) + 3p^2(1-p)\varphi(2) + 3p(1-p)^2\varphi(1) - \varphi(3p) \\ &= p^3[\varphi(3) - 3\varphi(2) + 3\varphi(1)] + p^2[3\varphi(2) - 6\varphi(1)] \\ &\quad + p[3\varphi(1)] - \varphi(3p). \end{aligned} \tag{4.22}$$

Note that

$$I_3''(p)|_{p=1} > 0 \tag{4.23}$$

as can be verified by evaluating

$$\begin{aligned} I_3''(p) &= 6p[\varphi(3) - 3\varphi(2) + 3\varphi(1)] + 2[3\varphi(2) - 6\varphi(1)] \\ &\quad - \frac{9}{3p+\lambda_0} \end{aligned} \tag{4.24}$$

at $p=1$ (and $\lambda_0 = 0$). Next, note that

$$I_3''(p)|_{p=0} < 0 \tag{4.25}$$

which can be verified by evaluating (4.24) at $p=0$. Consider now the function $I_3''(p)$ for $\lambda_0 = 0$. It starts negative at $p=0$ and ends positive at $p=1$ and must therefore have an odd number of zeroes in $(0, 1)$. If $I_3''(p)$ has more than one zero in $(0, 1)$ it must have at least three and $I_3'''(p)$ must have at least two zeros. This would contradict the fact that $I_3''''(p) < 0$, which can be easily verified. We thus conclude that $I_3''(p)$ starts negative, and then goes positive and remains positive until $p=1$. The zero of $I_3''(p)$, which we denote by $\tilde{p}$ must satisfy

$$\tilde{p} > p^*$$

where p^* is the zero of $I_3'(p)$, which exists because $I_3(0) = I_3(1) = 0$. This easily follows by noting that $I_3'(1) < 0$. Indeed, suppose, by contradiction that $\tilde{p} < p^*$. Since $I_3'(0) > 0$ this would imply that $I_3'(\tilde{p}) > 0$. But for $p > \tilde{p}$, we have that $I_3''(p) > 0$ which implies that for $p > \tilde{p}$ we have that $I_3'(p)$ is monotonically increasing in $(\tilde{p}, 1)$, and hence, $I_3'(\tilde{p}) > 0$ implies $I_3'(1) > 0$ which is a contradiction.

V. Summary and Conclusions

In this paper, we have studied the capacity region of a Poisson multiple-access channel. In the case where only two users access the channel we have demonstrated how the capacity region can be computed when both users are subjected to the same peak-power constraint, but are otherwise unlimited in their average transmit power. The computation relies on the optimality of binary signaling (Lemma 1) and on a perturbation argument that leads to a characterization of the input distributions that achieve points on the boundary of the capacity region (Lemmas 3 and 4). The perturbation argument leading to this characterization may well find uses in the computation of the capacity regions of other multiple-access channels.

We next considered the maximum throughput achievable on the Poisson multiple-access channel, and demonstrated that in the absence of average-power constraints, maximum throughput can be achieved at equal rates without the need

[4] Note that L larger than $K+1$ may be required here due to the average power constraints [39], [42], [47], [48].

for time-division multiaccessing (Lemma 2). This result does not hold for all multiple-access channels as demonstrated in [45] and [46]. We have also demonstrated that the maximum throughput is monotonically increasing with the number of users, but bounded by the peak power A (Lemma 5) (or more precisely, by roughly $0.58A$—Lemma 6). This should be contrasted with the Gaussian channel where throughput increases logarithmically with the number of users [36].

Notice that if we allowed full cooperation between the users by assuming that the messages to be transmitted by each of the users is known to all other users, a maximum throughput of KA/e could have been achieved (in the absence of dark current and average-power constraints). This throughput can be achieved by viewing this situation as a single-user Poisson channel with peak power KA. Maximum throughput thus increases linearly in the number of users if full cooperation is allowed, whereas it is bounded in the number of users if each user is ignorant of the other users' messages.

In the absence of dark current, the maximum throughput achievable using time-division multiple-access (TDMA) is $A/e \approx 0.368A$ irrespective of the number of users, while the maximum achievable throughput with optimal coding and decoding is $0.434A$ in the two user case, and converges to $0.58A$ as the number of users tends to infinity. We can thus conclude that in the absence of dark current, the loss in throughput due to time division is at most a factor of 1.58.

The situation changes dramatically in the presence of a large dark current. TDMA achieves a throughput that does not depend on the number of users and which decreases to zero with the dark current (1.2). In contrast, with optimal signaling, throughput increases with the number of users, and in the limit where the number of users tends to infinity one can achieve a throughput of $0.5A$, irrespective of the dark current (Lemma 6).

Time-division multiaccessing has the advantage that it does not require joint decoding, and the receiver complexity is thus significantly reduced. A natural question is thus whether one can find a channel-accessing scheme that would not require joint decoding and yet achieve a positive throughput in the limit of large dark current and many users. A positive answer to this question is given in the appendix where we describe a "generalized TDMA" scheme that does not require joint detection and yet achieves a throughput of $A/4$ in this limit.

In the generalized TDMA scheme, K-time zones are specified and in different time zones the strategies of users are cyclically shifted. As opposed to standard TDMA, in each time zone more than one user can be active, but each user is decoded by treating all other users as background radiation. It is shown that with this scheme one can achieve a throughput of

$$R_{\text{GTDMA}} = \sum_{k=1}^{K} \left[q^{(k)} \phi\left(A,\, A \sum_{j \neq k} q^{(j)} \right) - \phi\left(q^{(k)} A,\, A \sum_{j \neq k} q^{(j)} \right) \right] \tag{5.1}$$

where $0 \le q^{(1)}, \cdots, q^{(K)} \le 1$ are arbitrary, and where $\phi(\cdot, \cdot)$ is defined in (2.5).

Standard TDMA results when all but one of $\{q^{(k)}\}_{k=1}^{K}$ are zero. A throughput of $A/4$ results when the dark current is large, $K \to \infty$, and

$$q^{(k)} = \frac{1}{2}, \qquad \forall 1 \le k \le K.$$

A different approach to achieving high throughput with single-user detection can be based on the rate-splitting approach [49]. This approach allows one to achieve the entire capacity region of the asynchronous channel using single-user detection (and without requiring synchronization). While rate splitting was originally proposed for discrete memoryless multiple-access channels, it also applys to the Poison MAC as the latter can be viewed as a limit of discrete memoryless multiple-access channels by finely discretizing the time axis [5]. Note also that for various scenarios we have demonstrated that the maximal throughput in the asynchronous Poisson MAC is identical to the maximum throughput in the synchronous case.

In this paper, we have also treated average-power constraints. If average-power constraints are present, the computation of the capacity region becomes much more elaborate. We have, therefore, focused on maximum throughput and derived the maximum throughput under average-power constraints for the two-user case as well as for the three-users case in the absence of dark current. For these cases, time division is not necessary, and maximum throughput can be achieved without synchronization. We conjectured that this behavior holds for more users too, and gave a numerical algorithm for checking this conjecture for a given number of users and a given level of dark current.

Our model did not account for any spectral (bandwidth) constraints. Bandwidth constraints are of practical interest and an investigation of the Poisson MAC subjected to such additional constraints is called for, thus extending the single-user results reported in [10]–[12].

APPENDIX
GENERALIZATION OF TDMA: SINGLE-USER DECODING

Here, we generalize the standard optical TDMA technique by allowing more than one user to be active in a given time slot. Only single-user detection is, however, allowed, and each user is thus decoded by treating the other active users as background radiation (noise). The scheme depends on a parameter vector $(q^{(1)}, \cdots, q^{(K)})$ whose components are in the interval $[0, 1]$. If average power constraints (2.2) are in effect, we shall require that the vector additionally satisfy

$$\sum_{k=1}^{K} q^{(k)} \le B/A.$$

The proposed accessing scheme can be described as follows. The time axis is divided into K slots, and in slot m user l transmits using a stationary binary signaling scheme with the probability of transmitting A being $p_{m,l}$. To achieve symmetry we shall assume

$$p_{m,l} = q^{(m+l \bmod K)}$$

so that the signaling schemes are cyclically shifted from slot to slot. Decoding is assumed to be single-user decoding treating other users as noise.

Using a random coding argument one can demonstrate that for the purposes of computing the achievable rates for a given user one can treat all other active users in the slot as background radiation. Since the scheme is symmetric we can obtain the maximum throughput by summing over the achievable rates of the active users in a given slot to yield (5.1).

Throughput is maximized by optimizing over $\{q^{(k)}\}$. In the two-user case, and in the absence of dark current, the optimal parameter vector is $(1/e, 0)$ corresponding to regular TDMA. However, when dark current is high and many users access the channel the vector $(1/2, \cdots, 1/2)$ outperforms TDMA to achieve a throughput of $A/4$ in the limit of high dark current and many users.

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