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NONLINEAR PHYSICS AND MECHANICS

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The Possibility of Introducing of Metric Structure in Vortex Hydrodynamic Systems

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Geometrization of the description of vortex hydrodynamic systems can be made on the basis of the introduction of the Monge – Clebsch potentials, which leads to the Hamiltonian form of the original Euler equations. For this, we construct the kinetic Lagrange potential with the help of the flow velocity field, which is preliminarily determined through a set of scalar Monge potentials, and thermodynamic relations. The next step is to transform the resulting Lagrangian by means of the Legendre transformation to the Hamiltonian function and correctly introduce the generalized impulses canonically conjugate to the configuration variables in the new phase space of the dynamical system. Next, using the Hamiltonian function obtained, we define the Hamiltonian space on the cotangent bundle over the Monge potential manifold. Calculating the Hessian of the Hamiltonian, we obtain the coefficients of the fundamental tensor of the Hamiltonian space defining its metric. Next, we determine analogs of the Christoffel coefficients for the N-linear connection. Considering the Euler – Lagrange equations with the connectivity coefficients obtained, we arrive at the geodesic equations in the form of horizontal and vertical paths in the Hamiltonian space. In the case under study, nontrivial solutions can have only differential equations for vertical paths. Analyzing the resulting system of equations of geodesic motion from the point of view of the stability of solutions, it is possible to obtain important physical conclusions regarding the initial hydrodynamic system. To do this, we investigate a possible increase or decrease in the infinitesimal distance between the geodesic vertical paths (solutions of the corresponding system of Jacobi – Cartan equations). As a result, we can formulate very general criterions for the decay and collapse of a vortex continual system.

Keywords: vortex dynamics, geodesic deviation, Monge manifold, metric tensor

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1. Introduction

The possibility of introducing a metric on manifolds associated with hydrodynamic flows of various types has been repeatedly discussed in the literature (see, for example, [1, 2]), however, the results to date have either an excessive generality (which makes development of techniques suitable for describing flows of specific types) [3, 10] or require the introduction of significant restrictions and problem-oriented problems with an a priori rigidly defined setting of additional conditions, the physical content of which requires special analysis [5, 6]. If we consider the hydrodynamic flow as a statistical system in a state close to equilibrium, then its geometric properties can be investigated using the general Amari – Weinhold technique, using the possibility of introducing of a Riemannian topology on Gibbs manifolds (these manifolds are determined with the help of the relations between the thermodynamic potentials $p = p(\rho, T)$ and $s = s(u, \rho)$). In particular, if we turn to the model of a real flow in the form of a set of point vortices of Onsager type, we can obtain meaningful differential-functional relations that are expressions for the specific heats of the vortex population and the geodesics equations relating the states of equilibrium levels of the system [4].

In [7], the possibility of a transition in the Lamb – Kozlov hydrodynamic equation [11] to new canonical coordinates (representing the generalized Monge potentials) is considered. This approach can be applied to the geometric representation of the Hamiltonian dynamical system on the basis of Lagrangian and Hamiltonian geometry methods [8]. In this paper, we consider the fundamentals of the geometrodynamic Hamiltonian formalism in relation to the stability of hydrodynamic structures.

2. Lagrangian and Hamiltonian formalism of the description of the hydrodynamic system in terms of Monge potentials

Locally, the state of the hydrodynamic system may be described by density, velocity and density of entropy (in the general case of a compressible medium) at a given point $(\mathbf{x}, t) \in K_{N+1} \subset \mathbb{R}^{N+1}$ ($\mathbf{x} \in \mathbb{R}^{N\leqslant 3}$, $t \in \mathbb{R}^1_+$), that is, by the set of quantities $\{\rho(\mathbf{x}, t); \mathbf{v}(\mathbf{x}, t); s(\mathbf{x}, t)\}$. The hydrodynamic equations that describe their variation in time have the form

$$\rho_t + (\rho\mathbf{v})_\mathbf{x} = 0, \qquad (\rho\mathbf{v})_t + (\rho\mathbf{v}^2)_\mathbf{x} = -p_\mathbf{x}, \qquad (\rho s)_t + (\rho\mathbf{v}s)_\mathbf{x} = 0, \tag{2.1}$$

where $p = p(\mathbf{x}, t)$ is a scalar pressure field. We will use the representation of velocity fields with the help of the Monge potentials $\{M_\alpha\}|_{\alpha=\overline{1,m}} \in Y_m$, separating in the expansion of the velocity field gradient term and a set of quasi-solenoidal ones:

$$\mathbf{v} = -(M^4) \cdot (M^1)_\mathbf{x} - s \cdot (M^2)_\mathbf{x} - (M^3)_\mathbf{x} \quad \text{if } m = 4,$$

where $M^\alpha(\mathbf{x}, t)$ are some scalar variables. For isentropic flows with $s = \text{const}$, we must change $M^3 + s \cdot M^2 \to M^2$, $M^4 \to M^3$ (in this case $m = 3$). For the uniqueness of the choice of these fields it is necessary to introduce additional conditions; following [9], we select the following conditions as the first three ones: $\widehat{D}(M^4) = \widehat{D}(M^1) \equiv 0$, $\widehat{D}(M^2) \equiv T$, where $\widehat{D}(\ldots) \equiv (\ldots)_t + \mathbf{v} \cdot (\ldots)_\mathbf{x}$ is the substantial derivative operator, $T = T(\mathbf{x}, t)$ is the field of thermodynamic temperature. We take for the last additional condition the relation defining the specific enthalpy of the flow: $w(\mathbf{x}, t) = (M^4) \cdot (M^1)_t + s \cdot (M^2)_t + (M^3)_t - \mathbf{v}^2/2$. In accordance with the results of [9], the hydrodynamic equations (2.1) with the above additional conditions are equivalent to the corollaries from the Seliger – Whitham variational principle $\delta \iint \pi \, d\mathbf{x} \, dt = 0$, where the density of the kinetic potential (pressure in the Monge representation) can be represented in the following form (the basic thermodynamic relation $u = (w - p/\rho)|_{p=\pi}$ is used):

$$\begin{aligned} &\pi\big(\{M^\alpha\};\{(M^\alpha)_\mathbf{x}\},\{(M^\alpha)_t\}\big) = \rho(w-u) = \rho\big((M^4)\cdot(M^1))_t + s\cdot(M^2)_t + \\ &\qquad + (M^3)_t - \frac{1}{2}\big(-M^4\cdot(M^1)_\mathbf{x} - s\cdot(M^2)_\mathbf{x} - (M^3)_\mathbf{x}\big)^2 - u(\rho,s)\big) \quad (m=4), \\ &\pi\big(\{M^\alpha\};\{(M^\alpha)_\mathbf{x}\},\{(M^\alpha)_t\}\big) = \rho\big((M^3)\cdot(M^1))_t + (M^2)_t - \\ &\qquad - \frac{1}{2}\big(-M^3\cdot(M^1)_\mathbf{x} - (M^2)_\mathbf{x}\big)^2 - u(\rho,s)\big) \quad (m=3). \end{aligned} \tag{2.2}$$

Here $u(\rho, s)$ is the specific internal energy of the flow (the caloric equation of the state of the medium is assumed to be known).

We consider a Hamiltonian representation for which we define the “Monge representation impulses” $\{P_\alpha\}$ conjugated to quasi-velocity variables $\{(M^\alpha)_t\}$:

$$P_1 = \frac{\partial \pi}{\partial((M^1)_t)} = \rho M^4, \quad P_2 = \frac{\partial \pi}{\partial((M^2)_t)} = \rho s, \quad P_3 = \frac{\partial \pi}{\partial((M^3)_t)} = \rho \quad (m=4);$$

$$P_1 = \frac{\partial \pi}{\partial((M^1)_t)} = \rho M^3, \qquad P_2 = \frac{\partial \pi}{\partial((M^2)_t)} = \rho \quad (m=3).$$

We introduce a Hamiltonian function $H_m\big(\{M^\alpha\},\{P_\alpha\}\big)$ with the help of Legendre transformation of Lagrangian ($H_m \equiv \sum_{\alpha=1}^{m-1} P_\alpha \cdot (M^\alpha)_t - \pi\big(\{M^\alpha\};\{(M^\alpha)_\mathbf{x}\}\big)$):

$$H_{m=4} = \frac{1}{2P_3}\left(\sum_{\alpha=1}^{3} P_\alpha \cdot (M^\alpha)_\mathbf{x}\right)^2 + P_3 \cdot u(P_2, P_3),$$

$$H_{m=3} = \frac{1}{2P_2}\left(\sum_{\alpha=1}^{2} P_\alpha \cdot (M^\alpha)_\mathbf{x}\right)^2 + P_2 \cdot u(P_2).$$

We consider the Hamiltonian space $W^m = (Y_m, H_m)$, $H_m \colon T^*Y_m \to \mathbb{R}^1$ with fundamental tensor $g_m^{\alpha\beta}\big(\{M^\mu\},\{P_\mu\}\big) = \frac{1}{2}\,\partial^2 H_m/\partial P_\alpha \partial P_\beta$; the corresponding Riemannian element of the interval is $d\sigma^2_{W^m} = \sum_{\alpha,\beta}(g_{\alpha\beta})_m \, dM^\alpha \otimes dM^\beta$. As an example, we give an explicit form of the contravariant metric coefficients for the simplest case of $m = 3$, $N = 2$ (index m omits):

$$g^{11} = \frac{(M^1)^2_{x_1} + (M^1)^2_{x_2}}{P_2}, \qquad g^{12} = g^{21} = -P_1\frac{(M^1)^2_{x_1} + (M^1)^2_{x_2}}{P_2^2},$$

$$g^{22} = (P_1)^2\frac{(M^1)^2_{x_1} + (M^1)^2_{x_2}}{(P_2)^3} + 2\frac{du(P_2)}{dP_2} + P_2\frac{d^2u(P_2)}{dP_2{}^2},$$

and the determinant $\det\big(g^{\alpha\beta}\big) = |(M^1)_\mathbf{x}|^2(P_2)^{-1}(2u' + P_2u'') \not\equiv 0$. It should be noted that the coefficients $g^{\alpha\beta}$ (as well as their analogs for $m = 4$) do not depend directly on the Monge potentials (new “configuration variables”). This greatly simplifies further analysis of the geometrodynamic properties of vortex motion of the hydrodynamic medium.

3. The canonical connections of the Hamiltonian space W^m and geodesic equations

The N-linear connection on T^*Y_m is characterized by a pair of d-tensor fields $D\Gamma(N) = \left(H^{\alpha}_{\beta\gamma}, C^{\beta\gamma}_{\alpha}\right)$, that is, the system of generalized Christoffel coefficients (H, C), which in the general case are functions of Monge potentials $\{M^{\mu}\}$ and pseudo-impulses $\{P_{\mu}\}$ canonically conjugate to them. However, one should pay attention to the specific structure of the dependence only on the impulses of the components of the fundamental tensor $g^{\alpha\beta} = g^{\alpha\beta}(\{P_{\mu}\})$, obtained in Section 2. In this case, there is a nullification of coefficients $H^{\alpha}_{\beta\gamma} \equiv \frac{1}{2}g^{\alpha\eta}(\delta_{\zeta}g_{\eta\gamma}+\delta_{\gamma}g_{\beta\zeta}-\delta_{\zeta}g_{\beta\gamma})$, where $\delta_{\mu} \equiv \partial/\partial M^{\mu} + N_{\mu\nu}\partial/\partial P_{\nu}$ is an element of adapted basis for direct decomposition $T_uT^*Y_m = N_u \oplus V_u$ $(\forall u \in T^*Y_m)$. Here $N_{\mu\nu} = \frac{1}{4}\{g_{\mu\nu}, H\} - \frac{1}{4}\left(g_{\mu\xi}\partial^2 H/\partial M^{\nu}\partial P_{\xi} + g_{\nu\eta}\partial^2 H/\partial M^{\mu}\partial P_{\eta}\right)$ are coefficients of the nonlinear connection of the Hamiltonian space W^m (the notation $\{g_{\mu\nu}, H\}$ is used for Poisson brackets on the T^*Y_m). Thus, the horizontal paths of the N-linear connection D are described by a system of differential equations $(M^{\alpha})_{tt} + H^{\alpha}_{\beta\gamma}(M^{\beta})_t(M^{\gamma})_t = 0$, $(P_{\alpha})_t - N_{\mu\alpha}(M^{\mu})_t = 0$; however, these equations obtain the confluent forms because of the absence of a direct functional relationship of variables M^{α} in the Hamiltonian and the components of metric tensor (the solutions of horizontal paths equations are trivial: $M^{\alpha} = c_1t+c_2$, $P_{\beta} = c_3$).

Accordingly, we will analyze physically more interesting vertical paths (v-paths) (at a fixed point $(M_{\alpha})_0 \in Y_m$) with respect to the N-linear connection $D\Gamma$. These v-paths are characterized by a system of differential equations that are analogs of the (nonconfluent) Euler – Lagrange equations:

$$\frac{d^2P_{\alpha}}{dt^2} - C^{\beta\gamma}_{\alpha}\left(\{M^{\eta}\}, \{P_{\eta}\}\right)\Big|_{M_{\alpha}=(M_{\alpha})_0}\frac{dP_{\beta}}{dt}\frac{dP_{\gamma}}{dt} = 0, \tag{3.1}$$

$$C^{\beta\gamma}_{\alpha} = -\frac{1}{2}g_{\alpha\zeta}\left(\frac{\partial g^{\zeta\gamma}}{\partial P_{\beta}} + \frac{\partial g^{\beta\zeta}}{\partial P_{\gamma}} - \frac{\partial g^{\beta\gamma}}{\partial P_{\zeta}}\right).$$

The values of generalized Christoffel coefficients (for $m = 3$) $C^{\beta\gamma}_{\alpha}$ are

$$C^{11}_1 = -\frac{P_1|(M^1)_{\mathbf{x}}|^2}{(P_2)^4} - \frac{(P_1)^2|(M^1)_{\mathbf{x}}|^2}{(P_2)^5}, \qquad C^{21}_1 = \frac{|(M^1)_{\mathbf{x}}|^2}{(P_2)^3} + \frac{(P_1)^2|(M^1)_{\mathbf{x}}|^2}{(P_2)^5},$$

$$C^{22}_1 = -\frac{2P_1|(M^1)_{\mathbf{x}}|^2}{(P_2)^4} + \frac{P_1(M^1)_{\mathbf{x}}}{2(P_2)^2}\left(\frac{3(M^2)_{\mathbf{x}}}{(P_2)^2} - \frac{3(P_2)^2(M^2)_{\mathbf{x}} + 3(P_1)^2(M^1)_{\mathbf{x}}}{(P_2)^4} + \right.$$
$$\left. + 3u''(P_2) + P_2u'''(P_2)\right) - \frac{(M^1)^2_{\mathbf{x}}}{2(P_2)^3},$$

$$C^{12}_1 = \frac{(M^1)^2_{\mathbf{x}}}{2(P_2)^3} + \frac{2(P_1)^2(M^1)^2_{\mathbf{x}}}{(P_2)^5} - \frac{P_1(M^1)_{\mathbf{x}}}{(P_2)^2} \times$$
$$\times\left(\frac{3(M^2)_{\mathbf{x}}}{(P_2)^2} - \frac{3(P_2)^2(M^2)_{\mathbf{x}} + 3(P_1)^2(M^1)_{\mathbf{x}}}{(P_2)^4} + 3u''(P_2) + P_2u'''(P_2)\right).$$

The above Eqs. (3.1) can be considered as the basic equations in the study of the stability of the dynamics of the vortex flow of a fluid, described initially by the system of Euler equations.

The basic information contained in their solutions relates to the form of the geodetic trajectory in an "impulse space" (really, on a manifold of densities of scalar flow characteristics). Of special interest are closed trajectories that are naturally associated with periodic hydrodynamic structures of different scales.

4. The deviation of geodesics on the Monge manifolds and its relation to the evolution of coherent hydrodynamic systems

The question naturally arises of the stability of periodic orbits in an impulse space. To analyze deviation from the geodesic motion described by Eq. (3.1), we represent $P_\alpha = (P_\alpha)_0 + \epsilon\Pi_\alpha + O(\epsilon^2)$, where $(P_\alpha)_0$ is the solution of (3.1), ϵ is a small parameter, Π_α is the deviation from the exact solution (depending on the geodetic parameter or time). Substituting the above equation into (3.1), we get

$$0 = \frac{d^2(P_\alpha)_0}{dt^2} - C_\alpha^{\beta\gamma}\big(\{M^\eta\},\{(P_\eta)_0\}\big)\Big|_{M^\alpha=(M^\alpha)_0}\frac{d(P_\beta)_0}{dt}\frac{d(P_\gamma)_0}{dt} +$$
$$+\,\epsilon\bigg(\frac{d^2\Pi_\alpha}{dt^2} - 2C_\alpha^{\beta\gamma}\big(\{(M^\alpha)_0\},\{(P_\alpha)_0\}\big)\frac{d(P_\beta)_0}{dt}\frac{d\Pi_\gamma}{dt} -$$
$$-\,C_{\alpha,\eta}^{\beta\gamma}\big(\{(M^\zeta)_0\},\{(P_\zeta)_0\}\big)\frac{d(P_\beta)_0}{dt}\frac{d(P_\gamma)_0}{dt}\Pi_\eta(t)\bigg) + O(\epsilon^2).$$

Making a transformation of the expression in parentheses with the factor ϵ, we obtain an analog of the Jacobi equation (it can be called the Jacobi – Cartan equation) for the deviation vector with the components Π_α:

$$\frac{D_p^2(\Pi_\alpha)}{dt^2} + \left(\frac{d(P_\beta)_0}{dt}\right)\left(\frac{d(P_\gamma)_0}{dt}\right)S_\alpha^{\beta\gamma\eta}(\Pi_\eta) = 0, \tag{4.1}$$

where $D_p\Pi_\alpha/dt \equiv d\Pi_\alpha/dt - C_\alpha^{\beta\gamma}\big(\{(M^\xi)_0\},\{P_\xi\}\big)(\Pi_\beta)(d(P_\gamma)_0/dt)$, $S_\alpha^{\beta\gamma\eta}$ is the ("third") curvature d-tensor of path:

$$S_\alpha^{\beta\gamma\eta} = \frac{\partial C_\alpha^{\beta\gamma}}{\partial P_\eta} - \frac{\partial C_\alpha^{\beta\eta}}{\partial P_\gamma} + C_\alpha^{\mu\beta}C_\mu^{\beta\eta} - C_\alpha^{\mu\eta}C_\mu^{\beta\gamma}.$$

Equation (4.1) describes the evolution of the deviation vector from the geodesic motion and, when considering the (λ,ϵ)-congruence of closed trajectories (λ being the affine parameter along the streamline proportional to time t), it is possible to trace the change in the density characteristics $(\rho,\rho M^4,\rho s)|_{m=4}$ or $(\rho,\rho M^3)|_{m=3}$ of the hydrodynamic structure, containing this congruence. At the same time, it is not assumed that the system has strict limitations in the spatial sense, that is, this system has nonlocal correlation properties (which is characteristic of coherent structures of different genesis). If the solution of the Jacobi – Cartan system has stable limit cycles, then this allows us to state that the system has a set of certain pseudo-stationary states (associated with these cycles). It seems very interesting to investigate the properties of these pseudo-stationary states and to reveal the relationship with QSS (quasi-stationary states), associated with coherent structures in multi-vortex dynamics. If the norm of the solution of the Jacobi – Cartan equation grows with time, this obviously indicates the decay of the coherent hydrodynamic system. If the equation of geodesics has a singularity at some point and the deviation vector tends to zero in the norm, then we have a collapse of the vortex system.

Example. We consider the simple special cases of Eq. (4.1), which demonstrate the main features of the physical processes described with the help of Jacobi – Cartan equation. We assume that the values of $(M^{1,2})_{\mathbf{x}} = \widetilde{M}^{1,2}$ are constants, then the coefficients of the connection $C_{\gamma}^{\alpha\beta}$ and components of $S_{\alpha}^{\beta\gamma\eta}$ depend only on $(P_{1,2})_0$: $C_1^{11} = -(P_1)_0\widetilde{M}^1/(P_2)_0^4 - (P_1)_0^2\widetilde{M}^2/(P_2)_0^3$ $(m = 3)$, etc. Respectively, in the case under study, Eqs. (4.1) take the form of a system of ordinary differential equations of the 2nd order: $d^2\Pi_\alpha/dt^2 + K_1\big((P_1)_0, (P_2)_0\big)d\Pi_\alpha/dt + {}$ $+ K_2\big((P_1)_0, (P_2)_0\big) \cdot \Pi_\alpha + \big(d(P_\beta)_0/dt\big)\big(d(P_\gamma)_0/dt\big)S_\alpha^{\beta\gamma\eta}\big((P_1)_0, (P_2)_0\big) \cdot \Pi_\eta = 0$ with coefficients depending on variables $(P_\alpha)_0$, $\big((P_\alpha)_0\big)_t$ explicitly and being implicit functions of variable t. In fact, these coefficients may be considered constant for a fixed point in time. Thus, considering this equation as an equation for the vector variable P_α with “frozen” coefficients, one can see its analogy with the system of equations (with dissipative terms) describing the dynamics of coupled oscillators (in principle, it is possible to create conditions for the occurrence of self-oscillations in the system described by Eq. (4.1)). If we additionally set $P_2 = \mathrm{const}$, then for the simplest case $m = 3$ we will get one dissipative oscillator ordinary differential equation for the variable $P_1(\equiv \rho M^3)$ (without singularities in the coefficients); its solutions, according to general criteria, are oscillations decreasing in magnitude.

5. Conclusion

We have considered the possibility of formulating the dynamics of large-scale hydrodynamic structures in terms of geometric objects associated with the Hamiltonian system derived from Euler’s equations of hydrodynamics. This line of research seems extremely promising, since all the existing coherence criteria are either descriptive (they allow us to determine only the qualitative level of the situation) or they are purely specific and cannot be extended to flows with similar properties. The geometrodynamic approach developed in the present article is universal, it subtracts the deep essence of coherence as a congruence of geodesic streamlines, and also allows predicting the behavior of the system (and, possibly, influencing the creation and destruction of coherent structures by local impact methods).

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