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# The possibility of non-synchronism of convective secondaries in close binary stars

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**Summary.** We calculate the non-synchronous velocity field of a low mass convective secondary in a close binary system, taking rotation into account. We find that, contrary to previous belief, the velocity tends to zero as the $L_1$ point is approached. We also find that the use of tidal lobes is inappropriate when the secondary is asynchronous.

We consider the action of a turbulent viscosity on the velocity field and find that, when convection is inefficient, synchronization times can approach the lifetime of the system.

## 1 Introduction

In the study of close interacting binary systems, such as the cataclysmic systems, it is usually assumed that the mass transferring component is in a circular orbit and in a state of synchronous rotation (see Shu & Lubow 1982 and references therein). If the companion is compact, this leads to a picture in which matter overflows the Roche lobe of the mass transferring component and an accretion disc is formed (Lubow & Shu 1975).

In view of the fact that the cataclysmic binary systems show a complex range of outburst behaviour, a study of the possible effects of non-synchronism is of interest. Lack of synchronism has been suggested by Vogt (1980), and the presence of a small eccentricity by Papaloizou & Pringle (1979), as a possible explanation of the superoutburst phenomenon.

A study of non-synchronous motion is also of interest in order to assess the applicability of tidal lobes. The assumption here is that the stellar surface is a total potential surface, the total potential being a combination of gravitational and centrifugal potentials. It has been used in the study of X-ray binary systems (Davidson & Ostriker 1973).

Finally, if the non-synchronous flow can be calculated, the synchronization time may be found if some dissipation mechanism is assumed. In order that there should be incomplete tidal relaxation, it is necessary that the synchronization time be at least comparable to the age of the system.

The nature of non-synchronous motion in radiative envelopes has been considered by Lubow (1979). Scharlemann (1981) has considered the case of convective envelopes, but with particular reference to RS CVn binary systems. These systems are detached so that the tidal distortion is weak. A highly condensed secondary was assumed.

In Section 2 of this paper we calculate the non-synchronous velocity field for the case of a completely convective star corresponding to a full polytrope with $n = 1.5$. This is relevant to the cataclysmic systems in which the mass transferring star is likely to be a low mass main-sequence star, and so largely, if not fully convective. We assume the degree of non-synchronism to be small and both tidal distortion and rotation are taken into account. We show that, as the star approaches the Roche lobe, the non-synchronous motion is slowed down near the $L_1$ point. This has the consequence that mass transfer in the non-synchronous case is probably not too different from the synchronous case. Our results do not confirm previous work which assumed that matter would flow round the critical surface with essentially uniform velocity. Our solution also shows that tidal lobes cannot be presumed to apply to close binary systems.

In Section 3 we show how the synchronization rate may be derived from our solution. Zahn (1977), has estimated this for a non-rotating spherical model. He assumed that turbulent viscosity as a result of convection provided the dissipative mechanism. We assume the same mechanism here. However, because the orbital periods for cataclysmic binaries are much shorter than the typical convection time-scale, the efficiency of the viscosity should be reduced. We follow the procedure of Goldreich & Keeley (1977). Some reduction in efficiency is necessary to ensure that the dissipation rate from synchronization does not exceed the luminosity of the star so resulting in an inconsistency.

In Section 4 we evaluate the synchronization rate for the case when the tidal distortion is weak. We argue from a variational principle that an extrapolation of our results to the case of strong tidal distortion should not underestimate the synchronization rate. Finally, in Section 5 we discuss our results. We find that, given the uncertainties in this type of calculation, if the degree of non-synchronism is moderate, then the synchronization time may approach the lifetime.

## 2 Calculation of the tidal velocity field

We consider a secondary of mass $M_s$ and a compact primary of mass $M_p$, separated by distance $D$. We work in a frame corotating with the orbit which has period $2\pi/\Omega$. We take the origin of the cylindrical polar coordinates $(\varpi, \phi, z)$ at the centre of mass of the secondary, with the $z$-axis perpendicular to the orbital plane. The primary is at $\phi = 0$.

For a polytrope with $n = 1.5$, the equation of motion describing the evolution of a velocity field $\mathbf{u}$ is

$$\frac{\partial \mathbf{u}}{\partial t} + (2\boldsymbol{\Omega} + \nabla \wedge \mathbf{u}) \wedge \mathbf{u} = -\nabla H + \mathbf{F}_v \quad (1)$$

where $H = 5P/2\rho + \psi + \tfrac{1}{2}\mathbf{u}^2$, $P$ is the pressure, $\rho$ the density, $\psi$ the total gravitational plus centrifugal potential and $\mathbf{F}_v$ is the viscous force. We are looking for solutions of equation (1) that correspond to a non-synchronous rotation. However, in a tidally distorted star it is not possible to find a motion corresponding to rigid rotation. If the degree of non-synchronism is small, quadratic terms in $\mathbf{u}$ may be neglected. In this case, the time-dependence may be separated out by assuming an $\exp(-\lambda t)$ dependence.

Equation (1) then becomes

$$-\lambda \mathbf{u} + 2\boldsymbol{\Omega} \wedge \mathbf{u} = -\nabla W + \mathbf{F}_v \tag{2}$$

where $W$ is the perturbed form of $H$ resulting from the imposition of the velocity field $\mathbf{u}$ on the synchronous polytrope. We have also the perturbed form of the continuity equation which may be written,

$$\lambda\rho' = \nabla\cdot(\rho\mathbf{u}) \tag{3}$$

where $\rho$ is the density of the synchronous polytrope and $\rho'$ is the perturbed density. In general the decay rate $\lambda$ and the viscous force $\mathbf{F}_v$ are small so that, as a first approximation, they may be neglected in equations (2) and (3). In this case the components of equation (2) are

$$2\Omega u_\phi = \frac{\partial W}{\partial \varpi} \tag{4}$$

$$2\Omega u_\varpi = -\frac{1}{\varpi}\frac{\partial W}{\partial \phi} \tag{5}$$

$$\frac{\partial W}{\partial z} = 0. \tag{6}$$

Equation (3) gives

$$\frac{\partial}{\partial z}(\rho u_z) = \frac{1}{2\Omega\varpi}\left(\frac{\partial \rho}{\partial \varpi}\cdot\frac{\partial W}{\partial \phi} - \frac{\partial \rho}{\partial \phi}\cdot\frac{\partial W}{\partial \varpi}\right). \tag{7}$$

We may integrate equation (7) through the star with respect to $z$, then, using the fact that the density vanishes at the surface, we find

$$\frac{\partial \Sigma}{\partial \varpi}\cdot\frac{\partial W}{\partial \phi} = \frac{\partial \Sigma}{\partial \phi}\cdot\frac{\partial W}{\partial \varpi} \tag{8}$$

where the surface density $\Sigma$ is defined by

$$\Sigma = 2\int_0^{z_s} \rho\, dz,$$

and $z_s(\varpi, \phi)$ is the equation for the stellar surface. From equation (8), it follows immediately that $W$ is a function of $\Sigma$ alone. From this result, some important properties of the flow emerge. From equations (4) and (5) we find that in the orbital plane where $u_z = 0$

$$|\mathbf{u}| = \frac{1}{2\Omega}|\nabla W|. \tag{9}$$

In the absence of non-synchronous motion, it follows from equation (1) that $H = 5P/2\rho + \psi$ is constant. Thus if $\psi_s$ is the potential at the stellar surface, we have $P/\rho = \frac{2}{5}(\psi_s - \psi)$. Using the polytropic equation of state, we may then write

$$\rho = \left[\frac{2}{5K}(\psi_s - \psi)\right]^{3/2} \tag{10}$$

where $K$ is the polytropic constant. If we are interested in a region of the star near the orbital plane, we may expand $\psi$ in a Taylor series, thus

$$\psi = \psi_0 + fz^2 \tag{11}$$

where

$$f = \frac{1}{2}\left(\frac{\partial^2 \psi}{\partial z^2}\right)_{z=0}$$

and $\psi_0$ is the potential in the orbital plane.

We may use equations (10) and (11) to evaluate the surface density, we find

$$\Sigma = \left(\frac{2}{5K}\right)^{3/2}\left(\frac{1}{f^{1/2}}\right)\int_{\psi_0}^{\psi_s}(\psi_s - \psi)^{3/2}\cdot(\psi - \psi_0)^{-1/2}\,d\psi = \frac{3\pi}{8}\left(\frac{2}{5K}\right)^{3/2}\frac{1}{f^{1/2}}(\psi_s - \psi_0)^2. \tag{12}$$

Equation (12) gives behaviour of the surface density in regions near to the bounding curve where the star intersects the orbital plane. We may use equation (12) together with equation (9) to work out how the velocity behaves on the bounding curve. Of course, on the bounding curve, we have $\psi_0 = \psi_s$ and $\Sigma = 0$. Using equations (9) and (12), we find that

$$|\mathbf{u}| = \frac{3\pi}{16\Omega}\left(\frac{2}{5K}\right)^{3/2}\left|\frac{2(\psi_s - \psi_0)\nabla\psi_0}{f^{1/2}} + \frac{(\psi_s - \psi_0)^2\nabla f}{2f^{3/2}}\right|\cdot\frac{dW}{d\Sigma}$$

and close to the bounding curve

$$|\mathbf{u}| \propto \frac{dW}{d\Sigma}\cdot\frac{(\psi_s - \psi_0)}{f^{1/2}}|\nabla\psi_0|. \tag{13}$$

If the velocity, is not everywhere zero or infinite as $\psi_0 \to \psi_s$, we require $dW/d\Sigma \propto \Sigma^{-1/2}$ as $\Sigma$ approaches zero. Thus

$$|\mathbf{u}| \propto f^{-1/4}\cdot|\nabla\psi_0|. \tag{14}$$

If $r$ is the distance to the centre of the secondary and $r_p$ the distance to the primary, then

$$2f = \left(\frac{\partial^2\psi}{\partial z^2}\right)_{z=0} = \frac{GM_s}{r^3} + \frac{GM_p}{r_p^3}.$$

We find that the factor $f^{-1/4}$ is slowly varying on the critical equipotential surface in the orbital plane. For most mass ratios this factor varies by less than 20 per cent. So that equation (14) essentially states that the magnitude of the velocity is proportional to the magnitude of the local gravity.

Equation (14) leads to some, at first sight, surprising results concerning the non-synchronous flow field on the stellar surface where that surface intersects the orbital plane. As the bounding curve approaches the critical surface, we see that $|\mathbf{u}| \to 0$ at the $L_1$ point. Thus the flow is decelerated as it approaches the region from where mass transfer may occur, and as a consequence, we may expect that mass transfer would proceed more or less as in the synchronized case.

It has usually been assumed in previous work (Kopal 1959, Sparks & Stecher 1974, Batten 1973, Morris 1981), that matter on the surface of a non-synchronous contact component moves with essentially uniform velocity, so being ejected from the critical $L_1$ point because it is a conical point. Our result shows that such a mass ejection situation does not occur in non-synchronous contact components, when the hydrodynamic equations are used to derive the flow. Although we made the approximation of linearizing the equations, our effects should also operate in the non-linear regime. Matter slows down as the critical point is reached because the equipotential surfaces, isobaric surfaces and stream lines all diverge as that point is approached. This forces the magnitude of the velocity to decrease.

Our results also have some bearing on the use of tidal lobes for non-synchronous binaries (see Davidson & Ostriker 1973). In the absence of viscosity it is easy to show from equation (1) that $\psi + \frac{1}{2}\mathbf{u}^2$ is constant on the stellar surface. From this, it follows that if $\mathbf{u}$ is zero at the $L_1$ point, but non-zero elsewhere, then the surface must lie below the critical equipotential surface whatever the sign of $\mathbf{u}$. This contradicts the result from the use of tidal lobes that the critical surface should diminish in volume monotonically with the stellar rotation speed (Davidson & Ostriker 1973). Accordingly, tidal lobes cannot be used in the study of non-synchronous binary systems.

From equations (4), (5) and (7) we may write expressions for the velocity field at any point

$$u_\phi = \frac{1}{2\Omega}\frac{dW}{d\Sigma}\cdot\frac{\partial\Sigma}{\partial\varpi}, \tag{15}$$

$$u_\varpi = \frac{-1}{2\Omega\varpi}\frac{dW}{d\Sigma}\cdot\frac{\partial\Sigma}{\partial\phi}, \tag{16}$$

$$u_z = \frac{1}{2\Omega\varpi\rho}\frac{dW}{d\Sigma}\cdot\left(\frac{\partial F}{\partial\varpi}\cdot\frac{\partial\Sigma}{\partial\phi} - \frac{\partial F}{\partial\phi}\cdot\frac{\partial\Sigma}{\partial\varpi}\right), \tag{17}$$

where

$$F = \int_0^z \rho\, dz.$$

We note that equations (15)–(17) contain an unspecified function of $\Sigma$. This is because, in the absence of viscosity, we are free to associate an arbitrary amount of differential rotation with the non-synchronism. We can localize it to particular regions of the star. However, when viscosity is taken into account this freedom disappears.

### 2.1 THE CASE OF WEAK TIDAL DISTORTION

If the secondary is detached, so that the mean surface radius is less than 80 per cent of the value pertaining to the critical surface, the tidal distortion is weak. Under these conditions it may be treated by perturbation theory. The same is also true of the internal regions for lobe filling secondaries. The theory of weakly distorted polytropes is well known (Chandrasekhar 1933, Schwarzschild 1958, Kopal 1959). One defines a coordinate $r_\psi$ constant on equipotential surfaces such that

$$r_\psi = r + \sum_{l,\,m} a_{lm}(r) P_l^m(\mu) \cos m\phi. \tag{18}$$

$r_\psi$ is a mean radius that can be thought of as the radius of the equivalent spherical volume. For our problem only the term with $l = m = 2$ is significant so we drop the subscripts $l$ and $m$. The density $\rho$ is a function of $r_\psi$ alone and takes on the same functional form as in the undistorted spherical case. We then write the surface density as

$$\Sigma = 2\int_0^{z_s} \rho\, dz = 2\int_{\varpi}^{R_s} \frac{\rho(r_\psi) r\, dr}{(r^2 - \varpi^2)^{1/2}}$$

where $R_s$ is the surface value of $r$. Developing to first order in $a$, we write

$$\Sigma = \Sigma_0 + \Sigma_1,$$

where

$$\Sigma_0 = 2\int_{\varpi}^{R_s} \frac{\rho(r) r\, dr}{(r^2 - \varpi^2)^{1/2}},$$

and

$$\Sigma_1 = 6\varpi^2 \cos 2\phi \int_{\varpi}^{R_s} \frac{d\rho}{dr} \cdot \frac{a(r)\, dr}{r(r^2 - \varpi^2)^{1/2}}. \tag{19}$$

The function $a$ satisfies the second order differential equation

$$\frac{d^2 a}{dr^2} + 2\left(\frac{4\pi r^2 \rho}{M} - \frac{1}{r}\right)\frac{da}{dr} - \frac{4}{r^2}\, a = 0 \tag{20}$$

where $M$ is the mass at an internal point of the secondary. In addition $a$ a must be regular near the centre and satisfy the surface boundary condition

$$\left(\frac{da}{dr} + \frac{a}{r}\right)_{r = R_s} = -\frac{5}{4}\frac{M_p}{M_s}\left(\frac{R_s}{D}\right)^3. \tag{21}$$

It is a simple matter to determine $a$ for any spherical model. We write $a = ca_0$, where $a_0$ is the solution of equation (20) which satisfies $a_0 = r + O(r^3)$ near the centre. Then

$$c = -\frac{5}{4}\frac{M_p}{M_s}\left(\frac{R_s}{D}\right)^3 \Big/ \left(\frac{da_0}{dr} + \frac{a_0}{r}\right)_{r=R_s}$$

contains all the mass ratio dependence. We may now determine the velocity components from equations (15)–(17) and (19). We note first of all that in the limit of no tidal distortion, the only non-zero component is $u_\phi$. We have equation (15) and $\Sigma = \Sigma_0(\varpi)$ giving a general non-uniform cylindrical rotation law

$$u_\phi = g(\varpi) = \frac{1}{2\Omega}\frac{dW}{d\Sigma_0}\cdot\frac{d\Sigma_0}{d\varpi}.$$

We may use this result to determine $u_\varpi$ correct to first order from equations (16) and (19). After some simplifications we find

$$u_\varpi = \frac{6cg(\varpi)I_1}{I_2}\sin 2\phi \tag{22}$$

where

$$I_1 = \int_\varpi^{R_s}\frac{d\rho}{dr}\frac{a_0(r)\,dr}{r(r^2-\varpi^2)^{1/2}}$$

and

$$I_2 = \int_\varpi^{R_s}\frac{d\rho}{dr}\frac{dr}{(r^2-\varpi^2)^{1/2}}.$$

We may then find the first order non-axisymmetric contribution $v_\phi$ to $u_\phi$. This is most easily done by using the fact that

$$\frac{\partial u_\phi}{\partial\phi} + \frac{\partial}{\partial\varpi}(\varpi u_\varpi) = 0.$$

We find that

$$v_\phi = \frac{3c}{I_2}\left[\frac{d}{d\varpi}(\varpi g)\cdot I_1 + \varpi^2 g(I_3 - I_1\cdot I_4/I_2)\right]\cdot\cos 2\phi \tag{23}$$

where

$$I_3 = \int_\varpi^{R_s}\frac{d}{dr}\left(\frac{a_0}{r^2}\frac{d\rho}{dr}\right)\cdot\frac{dr}{(r^2-\varpi^2)^{1/2}}$$

$$I_4 = \int_\varpi^{R_s}\frac{d}{dr}\left(\frac{1}{r}\frac{d\rho}{dr}\right)\cdot\frac{dr}{(r^2-\varpi^2)^{1/2}}.$$

Finally we calculate $u_z$ from equation (17), and obtain

$$u_z = \frac{6c\varpi g}{\rho}(I_5 - I_1 \cdot I_6/I_2)\sin 2\phi \tag{24}$$

where

$$I_5 = \int_{\varpi}^{r} \frac{d\rho}{dr} \cdot \frac{a_0\, dr}{r(r^2-\varpi^2)^{1/2}}$$

and

$$I_6 = \int_{\varpi}^{r} \frac{d\rho}{dr} \cdot \frac{dr}{(r^2-\varpi^2)^{1/2}}.$$

If one knows the density as a function of radius it is a simple matter to compute the integrals $I_1$–$I_6$ which are needed to specify the non-axisymmetric part of the velocity field correct to first order. We note the function $g(\varpi)$ enables the non-synchronous rotation to be non-uniform. It cannot be specified without consideration of viscous forces. When the non-synchronous rotation is uniform the appropriate form of $g(\varpi)$ is

$$g(\varpi) = \varpi \cdot \Delta\Omega \tag{25}$$

where the period of rotation as seen in the corotating frame is equal to $2\pi/\Delta\Omega$.

## 3 The effect of viscosity

The principal effect of introducing conventional viscous dissipation is that, as a result of energy loss, the magnitude of the velocity decreases with time, and synchronism is achieved. In general we may expect the decay constant $\lambda$, appearing in equation (2), to be an eigenvalue corresponding to a decay mode. The unknown function $W(\Sigma)$, corresponding to $g(\varpi)$, will be prescribed by this eigenvalue problem as an eigenfunction. If we take the scalar product of equation (2) with $\rho\mathbf{u}$ and integrate over the volume of the star, we find that

$$-\lambda \int \rho \mathbf{u}^2\, d\tau = \int W \nabla \cdot (\rho \mathbf{u})\, d\tau + \int \rho \mathbf{u} \cdot \mathbf{F}_{\mathrm{v}}\, d\tau, \tag{26}$$

where $W$ is the perturbed form of $H$.

$$H = \frac{5P}{2\rho} + \psi + \frac{1}{2}\mathbf{u}^2.$$

If we neglect the variation of the gravitational potential of the secondary and use the polytropic equation of state, then $W = 5P\rho'/3\rho^2$. Using this, and the equation of continuity, we may find an expression for $\lambda$ in the form

$$\lambda = \frac{-\int \rho \mathbf{u} \cdot \mathbf{F}_{\mathrm{v}} d\tau}{\int (\rho \mathbf{u}^2 + 3W^2\rho^2/5P)\, d\tau}. \tag{27}$$

Equation (27) is a convenient expression from which the decay rate may be calculated. If a conventional form of viscous force is used, after an integration by parts, the numerator of equation (27) may be rewritten as

$$-\int \rho \mathbf{u} \cdot \mathbf{F}_v \, d\tau = \frac{1}{2} \sum_{i,j} \int \rho \nu e_{ij} e_{ij} \, d\tau \tag{28}$$

where $\nu$ is the coefficient of kinematic viscosity and, in Cartesian coordinates,

$$e_{ij} = \frac{\partial u_i}{\partial x^j} + \frac{\partial u_j}{\partial x^i} - \frac{2}{3} \nabla \cdot \mathbf{u} \delta^i_j.$$

In this case it may be shown that equations (27) and (28) yield a variational problem for $\lambda$ and the eigenfunction $W$, (see the Appendix). If equations (15)–(17) are substituted for $\mathbf{u}$ and the integral expression for $\lambda$ minimized, then the smallest decay rate and corresponding eigenfunction are specified. However, one can only use the standard form of viscous force when the viscosity is molecular in origin. In our problem we are interested in low mass stars which are likely to be fully convective. As a result it is generally believed that viscous effects will be caused primarily by turbulence. In this case the form of the stress tensor is not known and it is likely to be affected by the anisotropy of the turbulence (see Wasiutyński 1946, Kippenhahn 1963 and Tayler 1973). If the stress tensor does depart from the standard form then there may not be a tendency for uniform rotation to be achieved. Hence, synchronization is impossible, and some compromise between tidal effects and the tendency of turbulence to generate non-uniform rotation will result in a steady state. However, in the work presented here we shall ignore this possibility and assume the stress tensor takes the standard form. The relaxation times we calculate should be estimates of the time required to achieve a steady state in the more general case.

### 3.1 THE TURBULENT VISCOSITY

It is well known that there is no good theory of convection available to give a convenient estimate of the coefficient of viscosity in the presence of turbulence. For simplicity, we adopt the approach of mixing length theory (see Cox & Giuli 1968), which gives the simplest estimate of $\nu$ as

$$\nu = \frac{1}{3} \Lambda \cdot \bar{V} \tag{29}$$

where $\bar{V}$ is the mean convective velocity and $\Lambda$ the mixing length. The convective heat flux is given by

$$F_c = 10 \rho \bar{V}^3 \lambda_p / \Lambda$$

where $\lambda_p$ is the pressure scale height, so we may write

$$\nu = \nu_0 = \frac{1}{3} \lambda_p \left(\frac{\Lambda}{\lambda_p}\right)^{4/3} \cdot \left(\frac{L}{40 \pi r^2 \rho}\right)^{1/3} \tag{30}$$

where $L$ is the luminosity.

We may adopt equation (30) for the viscosity provided that the time-scale for convection $t_c = \Lambda/\bar{V}$ is short compared to a relative rotation period. In general, the typical convective time-scale in the stellar interior is found to be $\sim 10^7$ s. This means that for binaries with short periods of order a hundred minutes there will be many cases where the above will not be satisfied, so that equation (30) should not be used unmodified. We use the modification adopted by Goldreich & Keeley (1977). If the dissipating motion has a frequency $\omega$ associated with it, we write

$$\nu = \nu_0 G_R \tag{31}$$

where if $\omega t_c < 1$, then $G_R = 1$. If $\omega t_c > 1$, then $G_R = (\omega t_c)^{-2}$. We use $\omega = 2\Delta\Omega$, which is the relative forcing frequency of the tide as seen in a frame which on average corotates with the star. It is less clear that this is the relevant forcing frequency in the surface layers of a strongly distorted star, but the convective time-scale is so short in these layers that no modification is required. For large $\omega$, the rate of viscous dissipation in the interior is sensitive to the way $G_R$ approaches zero, and different results may be obtained by altering that presciption. In this regard, we note that Zahn (1977) uses a linear rather than quadratic fall-off for $G_R$. However, we feel that the form of $G_R$ should be constrained by the requirement that the power generated by synchronization should be less than the stellar luminosity (see Discussion). In addition we note that convection is likely to be inhibited by the effect of rotation in the synchronous state, so that the viscosity could well be smaller than that derived by using the mixing length theory assuming spherical symmetry as we have done. The mixing length theory is a local theory and is used taking the mixing length to be a multiple, $\alpha$, of the pressure scale height. The theory does not apply near the centre of the star where the scale height diverges. To avoid this difficulty we take the mixing length to be a constant equal to the value where $\alpha\lambda_p = r$, for radii less than that at which equality first occurs. We also assume $\bar{V}$ to be a constant in the central regions and equal to the minimum value predicted using simple mixing length theory with a constant luminosity, $L_s$.

## 4 The synchronization rate

With the above presciption for the viscosity, we are in a position to calculate the synchronization rate $\lambda$. From equations (27)–(29), we have

$$\lambda = \frac{\int \rho\nu e_{ij} e_{ij}\, d\tau}{2\int (\rho \mathbf{u}^2 + 3\rho^2 W^2/5P)\, d\tau}. \tag{32}$$

When the tidal distortion is small, we may use equations (22)–(24) for the velocity components and equations (29)–(31) for the viscosity to evaluate $\lambda$. We note that in this limit, the variational principle governing the determination of $\lambda$, shows that the correct form of $g(\varpi)$ to use is given by equation (25) corresponding to uniform non-synchronous rotation, whatever the spatial behaviour of the viscosity. This follows essentially because the standard form of viscosity tends to result in uniform rotation. Furthermore, the dominant term in the denominator of equation (32) comes from the axisymmetric component of $u_\phi$, and other terms may be neglected for weak tidal distortion.

We have calculated $\lambda$ from equation (32) after making the above simplications. The calculation is rather long because inclusion of the $z$-component of velocity requires a two-dimensional numerical integration over $r$ and $\varpi$. We used up to 300 grid points for each dimension. Because we are working with a simple polytropic model most of the dimensional dependence can be removed from the results. It is easy to show that $\lambda$ may be written in the form

$$\lambda = \left(\frac{M_p}{M_s}\right)^2 \left(\frac{R_s}{D}\right)^6 \left(\frac{L_s}{M_s R_s^2}\right)^{1/3} \cdot f(\eta, \alpha). \tag{33}$$

The function $f$ depends on the quantity

$$\eta = 2\left(\frac{\Delta\Omega}{\Omega}\right)\Omega\left(\frac{M_s R_s^2}{L_s}\right)^{1/3} \tag{34}$$

and $\alpha$, the mixing length to pressure scale height ratio. To fix ideas; if we consider a low mass main-sequence star with $M_s = 0.3\,M_\odot$, $R_s = 1.7\times 10^{10}$ cm and $L_s = 10^{-2}\,L_\odot$, then $(MR^2/L_s)^{1/3} = 1.6\times 10^7$ s. If such a star fills its Roche lobe in a binary system in which $M_p \gg M_s$, we expect in the synchronous state that $\Omega = \Omega_c = (0.1\,GM_s/R_s^3)^{1/2}$ (Kopal 1959). We then find $\Omega_c(M_s R_s^2/L_s)^{1/3} = 1.5\times 10^4$ to be a characteristic value of the dimensionless quantity $\eta$, which may be expected to be large for any reasonable degree of non-synchronism.

In Table 1, we give numerical values of $f(\eta, \alpha)$ for various values of $\eta$ and $\alpha$. From these $\lambda$ may be calculated. Of course, the synchronization time is just equal to $\lambda^{-1}$. We also give $(\Delta\Omega/\Omega)_c$ which is the degree of non-synchronism pertaining to the value of $\eta$ if the results are extrapolated to apply to a lobe filling $0.3\,M_\odot$ star. For values of $\eta$ less than about 10 or $(\Delta\Omega/\Omega)_c < 10^{-4}$, the value of $f$ is constant, for a given $\alpha$, because the convective turnover time is shorter than $(\Delta\Omega)^{-1}$ and turbulent viscosity operates at the highest efficiency. But for the larger values of $\eta$ we have approximately $f \sim 30(\eta)^{-2}$ and the synchronization time-scale increases rapidly. We may summarize the results for $\lambda$, when $\alpha = 1$ in the approximate form

$$\lambda = 0.4\left(\frac{M_p}{M_s}\right)^2\left(\frac{R_s}{D}\right)^6\left(\frac{L_s}{M_s R_s^2}\right)^{1/3}, \qquad \eta < 9, \tag{35}$$

$$\lambda = 30\left(\frac{M_p}{M_s}\right)^2\left(\frac{R_s}{D}\right)^6\left(\frac{L_s}{M_s R_s^2}\right)^{1/3}\frac{1}{\eta^2}, \qquad \eta > 9. \tag{36}$$

Equation (35) is in reasonable agreement with the order of magnitude estimate of Zahn (1977) which neglected rotation. Our results are such that the power generated from synchronization is less than the luminosity of the star. The power generated $\bar{W}$ may be written as $\bar{W} = K_g M_s R_s^2 (\Delta\Omega)^2 \lambda$, where $K_g$ is the radius of gyration equal to 0.2. Using equations (36) and (34), we find

$$\bar{W} = 1.5\,L_s\left(\frac{M_p}{M_s}\right)^2\left(\frac{R_s}{D}\right)^6,$$

and $\bar{W}$ is always less than $L_s$ even if the star is lobe filling.

Equations (35) and (36) apply to detached binaries with mean radii less than 80 per cent of the critical mean radius and it is not clear that they can be extrapolated to lobe filling

**Table 1.** The function $f(\alpha, \eta)$.

| | $\eta = 0$, $(\Delta\Omega/\Omega)_c = 0$ | | | $\eta = 30$, $(\Delta\Omega/\Omega)_c = 10^{-3}$ | | |
|---|---|---|---|---|---|---|
| $\alpha$ | 1 | 1/3 | 3 | 1 | 1/3 | 3 |
| $f$ | 0.4 | 0.1 | 1.7 | $1.4\times 10^{-2}$ | $9.8\times 10^{-3}$ | $1.5\times 10^{-2}$ |

| | $\eta = 300$, $(\Delta\Omega/\Omega)_c = 10^{-2}$ | | | $\eta = 3\times 10^3$, $(\Delta\Omega/\Omega)_c = 10^{-1}$ | | |
|---|---|---|---|---|---|---|
| $\alpha$ | 1 | 1/3 | 3 | 1 | 1/3 | 3 |
| $f$ | $3.2\times 10^{-4}$ | $2.5\times 10^{-4}$ | $3.4\times 10^{-4}$ | $6.4\times 10^{-6}$ | $5.1\times 10^{-6}$ | $8.2\times 10^{-6}$ |

conditions. However, if equation (36) is extrapolated to these conditions, when $(M_p/M_s)(R_s/D)^3 \sim 0.1$, one finds

$$\lambda \sim 0.3\left(\frac{L_s}{M_s R_s^2}\right)^{1/3} \frac{1}{\eta^2}. \tag{37}$$

This is just the characteristic viscous diffusion rate through the entire star. It is clear from the variational principle governing equation (32) for $\lambda$ that the correct value should not be much greater than this. Therefore we feel the error in extrapolating to lobe filling conditions should not be greater than other uncertainties in the problem.

Two other features of the results are worthy of note. It will be seen from Table 1 that for large $\eta$, the results are not very sensitive to the value of the mixing length. Also the distribution of the dissipation through the star changes as $\eta$ is increased. For small values most dissipation occurs in the centre, while for large values it occurs near the surface.

If we apply equations (34)–(37) to a lobe filling main-sequence star with $M_s = 0.3\,M_\odot$ and $M_p \gg M_s$, we see that the synchronization time varies strongly with the degree of non-synchronism $\Delta\Omega/\Omega$. If $\Delta\Omega = 0$, the synchronization time is short and about 120 yr but if $\Delta\Omega/\Omega$ is as large as 0.1, then the synchronization time is as large as $10^7$ yr.

## 5 Discussion

We have found a solution for the fluid velocity field associated with the non-synchronous rotation of a member of a close binary system. We considered the case when a polytropic equation of state applies so our results should apply when the component is largely convective, as is the case for the secondary of the cataclysmic binary systems.

Our results, when applied to the lobe filling situation, show that the velocity approaches zero as the $L_1$ point is approached. This indicates that mass transfer may proceed in much the same way as in the synchronous case, at least if the degree of non-synchronism is not too large. Furthermore, because of the slowing down, no definite period can be associated with the flow of matter round the surface in the equatorial plane.

We have also shown how our solution may be used to determine the synchronization rate. However, this is rather uncertain because there is no good theory of turbulent viscosity. We find that, for close binary systems, if the degree of non-synchronism $\Delta\Omega/\Omega$ is greater than about $10^{-4}$, then the mean convective turnover time is long compared to the relative rotation period. Adopting the same treatment of the viscosity as Goldreich & Keeley (1977), we find that, if $\Delta\Omega/\Omega$ is very small, the synchronization time is short and of order $10^2$ yr, but if $\Delta\Omega/\Omega$ is $10^{-1}$ or greater then the synchronization time is greater than $10^7$ yr. Also, we note that a long synchronization time for large $\Delta\Omega/\Omega$ is needed if the power generated from synchronization is not to exceed the stellar luminosity. If the lifetime of the system is less than $10^8$ yr, then it is possible for some significant non-synchronism to exist, although this would depend on the starting conditions.

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## References

Batten, A. H., 1973. *Binary and Multiple Systems of Stars,* Pergamon Press, Oxford.
Chandrasekhar, S., 1933. *Mon. Not. R. astr. Soc.,* **93**, 390.
Cox, J. P. & Giuli, R. T., 1968. *Principles of Stellar Structure,* Vol. I, Gordon & Breach.
Davidson, K. & Ostriker, J. P., 1973. *Astrophys. J.,* **179**. 585.
Goldreich, P. & Keeley, D. A., 1977. *Astrophys. J.,* **211**, 934.

Kippenhahn, R., 1963. *Astrophys. J.*, **137**, 664.
Kopal, Z., 1959. *Close Binary Systems*, Chapman & Hall, London.
Lubow, S. H., 1979. *Astrophys. J.*, **229**, 1008.
Lubow, S. H. & Shu, F. H., 1975. *Astrophys. J.*, **198**, 383.
Morris, M., 1981. *Astrophys. J.*, **249**, 572.
Papaloizou, J. & Pringle, J. E., 1979. *Mon. Not. R. astr. Soc.*, **189**, 293.
Scharlemann, E. T., 1981. *Astrophys. J.*, **246**, 292.
Schwarzschild, M., 1958. *Structure and Evolution of the Stars*, Princeton University Press.
Shu, F. H. & Lubow, S. H., 1981. *A. Rev. Astr. Astrophys.*, **19**, 277.
Sparks, W. M. & Stecher, T. P., 1974. *Astrophys. J.*, **188**, 149.
Tayler, R. J., 1973. *Mon. Not. R. astr. Soc.*, **165**, 39.
Vogt, N., 1980. *Astr. Astrophys.*, **88**, 66.
Wasiutyński, J., 1946. *Hydrodynamics and Structure of Stars and Planets, Astrophysica Norvegica*, Vol. 4.
Zahn, J. P., 1977. *Astr. Astrophys.*, **57**, 383.

## Appendix

In this appendix we show how to formulate a variational principle for the decay rate $\lambda$. We first formulate the eigenfunction problem for the function $W(\Sigma)$ as described in Section 3. First, we take the $z$-component of the curl of equation (2) multiplied by $\rho$. This is

$$-\lambda\left[\frac{1}{\varpi}\frac{\partial}{\partial\varpi}(\rho\varpi u_\phi)-\frac{1}{\varpi}\frac{\partial}{\partial\phi}(\rho u_\varpi)\right]+2\Omega\left[\frac{1}{\varpi}\frac{\partial}{\partial\phi}(\rho u_\phi)+\frac{1}{\varpi}\frac{\partial}{\partial\varpi}(\rho\varpi u_\varpi)\right]$$

$$=\frac{1}{\varpi}\left(\frac{\partial\rho}{\partial\phi}\frac{\partial W}{\partial\varpi}-\frac{\partial\rho}{\partial\varpi}\frac{\partial W}{\partial\phi}\right)+\frac{1}{\varpi}\frac{\partial}{\partial\varpi}(\rho\varpi F_{v\phi})-\frac{1}{\varpi}\frac{\partial}{\partial\phi}(\rho F_{v\varpi}). \tag{A1}$$

From the $z$-component of equation (2) it is easy to show that

$$-\lambda\left[\frac{\partial}{\partial\phi}\left(\frac{u_z}{\varpi}\frac{\partial F}{\partial\varpi}\right)-\frac{1}{\varpi}\frac{\partial}{\partial\varpi}\left(u_z\frac{\partial F}{\partial\phi}\right)\right]=\frac{\partial}{\partial z}\left[\left(\frac{1}{\varpi}\frac{\partial W}{\partial\varpi}\frac{\partial F}{\partial\phi}-\frac{1}{\varpi}\frac{\partial F}{\partial\varpi}\frac{\partial W}{\partial\phi}\right)\right]+\frac{1}{\varpi}\left(\frac{\partial\rho}{\partial\varpi}\frac{\partial W}{\partial\phi}-\frac{\partial W}{\partial\varpi}\frac{\partial\rho}{\partial\phi}\right)$$

$$+\frac{\partial}{\partial\phi}\left(\frac{F_{vz}}{\varpi}\frac{\partial F}{\partial\varpi}\right)-\frac{1}{\varpi}\frac{\partial}{\partial\varpi}\left(F_{vz}\frac{\partial F}{\partial\phi}\right) \tag{A2}$$

where $F$ is defined by equation (17) and is such that

$$F=\int_0^z\rho\,dz$$

and it is assumed that $W$ is an even function of $z$.

If we add (A1) and (A2) and integrate through the star from $z_s$ to $-z_s$ along a line of constant $(\varpi, \phi)$ we obtain after using the continuity equation (3):

$$-\lambda\int\left[\frac{1}{\varpi}\frac{\partial}{\partial\varpi}(\rho\varpi u_\phi)-\frac{1}{\varpi}\frac{\partial}{\partial\phi}(\rho u_\varpi)+\frac{\partial}{\partial\phi}\left(\frac{u_z}{\varpi}\frac{\partial F}{\partial\varpi}\right)-\frac{1}{\varpi}\frac{\partial}{\partial\varpi}\left(u_z\frac{\partial F}{\partial\phi}\right)\right]dz$$

$$=-2\Omega\lambda\int\rho'\,dz+\int\left[\frac{1}{\varpi}\frac{\partial}{\partial\varpi}(\rho\varpi F_{v\phi})-\frac{1}{\varpi}\frac{\partial}{\partial\phi}(F_{v\varpi}\rho)+\frac{\partial}{\partial\phi}\left(\frac{F_{vz}}{\varpi}\frac{\partial F}{\partial\varpi}\right)-\frac{1}{\varpi}\frac{\partial}{\partial\varpi}\left(F_{vz}\frac{\partial F}{\partial\phi}\right)\right]dz$$

$$-\frac{1}{\varpi}\left(\frac{\partial\Sigma}{\partial\varpi}\frac{\partial W}{\partial\phi}-\frac{\partial W}{\partial\varpi}\frac{\partial\Sigma}{\partial\phi}\right)_{z_s}. \tag{A3}$$

At this point it is convenient to introduce an orthogonal coordinate system $(\Sigma, \chi)$ instead of $(\varpi, \phi)$. For any quantity $Q$ we have

$$\frac{1}{\varpi}\left(\frac{\partial \Sigma}{\partial \varpi}\frac{\partial Q}{\partial \phi} - \frac{\partial Q}{\partial \varpi}\frac{\partial \Sigma}{\partial \phi}\right) = |\nabla\Sigma||\nabla\chi|\frac{\partial Q}{\partial \chi}. \tag{A4}$$

If we multiply equation (A3) by $1/(|\nabla\Sigma||\nabla\chi|)$, and integrate with respect to $\chi$ round a loop of constant $\Sigma$, we find

$$-\lambda\iint J\frac{dz\,d\chi}{|\nabla\Sigma||\nabla\chi|} + 2\Omega\lambda\iint\frac{3}{5}\rho^2\frac{W}{P}\frac{dz\,d\chi}{|\nabla\Sigma||\nabla\chi|} = \iint K\frac{dz\,d\chi}{|\nabla\Sigma||\nabla\chi|} + \int\left(\frac{\partial W}{\partial z}\right)_s\frac{\partial z_s}{\partial \chi}d\chi, \tag{A5}$$

where $J$ is an abbreviation for the integrand on the left-hand side of equation (A3) and $K$ is an abbreviation for the integrand contained in the second term on the right-hand side of equation (A3). In equation (A5), $\partial W/\partial z$, by use of the $z$-component of equation (2), may be replaced by $\lambda u_z + F_{vz}$. In the limit of zero viscosity the velocity components are given by equations (15)–(17), which contain the arbitrary function $W(\Sigma)$. We see that if these are substituted into (A5) as a first approximation, (A5) becomes a differential equation for the single function $W(\Sigma)$ and eigenvalue $\lambda$. In this way the arbitrariness in the inviscid solution is removed. We now want to show that (A5) is equivalent to a variational problem derived from equations (27) and (28). This may be written as

$$\lambda\int\left\{\rho\mathbf{u}^2 + \frac{3}{5}W^2\rho^2/P\right\}d\tau = \frac{1}{2}\sum_{ij}\int\rho\nu e_{ij}e_{ij}\,d\tau. \tag{A6}$$

Equations (15)–(17) for $\mathbf{u}$ have to be substituted into (A6) and the result varied with respect to $W(\Sigma)$. We first vary (A6) with respect to $\mathbf{u}$ and $W$ in complete generality. This gives

$$2\lambda\int\left(\rho\mathbf{u}\cdot\delta\mathbf{u} + \frac{3}{5}\rho^2\frac{W\delta W}{P}\right)d\tau = -2\int\delta\mathbf{u}\cdot\mathbf{F}_v\rho\,d\tau \tag{A7}$$

and we will assume from now on that $\nu$ vanishes at the surface. The next step is to use equations (15)–(17), which when perturbed give a relation between the components of $\delta\mathbf{u}$ and $\delta W$. After performing several integrations by parts, remembering that the volume element may be written $d\tau = dz\,d\chi\,d\Sigma/(|\nabla\Sigma||\nabla\chi|)$, (A7) may be worked into the form

$$-\frac{\lambda}{\Omega}\int\delta WJ\frac{d\chi\,dz\,d\Sigma}{|\nabla\chi||\nabla\Sigma|} + 2\lambda\int\frac{3}{5}\frac{\rho^2}{P}W\delta W\frac{d\chi\,dz\,d\Sigma}{|\nabla\Sigma||\nabla\chi|} + \frac{1}{\Omega}\int\left(\frac{\partial F}{\partial\chi}\right)_s(\lambda u_z + F_{vz})\delta W\,dz\,d\chi$$

$$= \frac{1}{\Omega}\int\delta WK\frac{d\chi\,dz\,d\Sigma}{|\nabla\chi||\nabla\Sigma|}, \tag{A8}$$

where the last term on the left-hand side arises from an integration by parts with respect to $\Sigma$ remembering that the lower limit corresponds to the stellar surface. This is a surface integral. On the surface $\Sigma$ is a function of $z_s$ and $\chi$,

$$\left(\frac{\partial F}{\partial \chi}\right)_s = \frac{1}{2}\left(\frac{\partial \Sigma}{\partial \chi}\right)_{z_s}$$

and

$$\left(\frac{\partial \Sigma}{\partial \chi}\right)_{z_s} = -\left(\frac{\partial z_s}{\partial \chi}\right)_\Sigma \left(\frac{\partial \Sigma}{\partial z_s}\right)_\chi .$$

Thus

$$\frac{1}{\Omega}\int\left(\frac{\partial F}{\partial \chi}\right)_s \delta W\, dz\, d\chi\,(\lambda u_z + F_{vz}) = -\frac{1}{\Omega}\int (\lambda u_z + F_{vz})\,\delta W \frac{\partial z_s}{\partial \chi}\, d\Sigma\, d\chi,$$

where we remember there are equal contributions from above and below the orbital plane if $W$ is an even function of $z$. We then see that (A8) is obtained from (A5) by multiplying by $\delta W/\Omega$ and integrating with respect to $\Sigma$. Thus if (A5) is satisfied for a given $W(\Sigma)$, $\lambda$ as specified by (A6) will be stationary with respect to variations of the velocity satisfying equations (15)–(17). The problem has the trivial solution $\lambda = 0$ corresponding to $W$ constant. However, this is eliminated from consideration by insisting that

$$\int \rho'\, d\tau = \int \frac{3}{5}\rho^2 \frac{W}{P}\, d\tau = 0,$$

so restricting trial functions to those which conserve the total mass of the star.