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THE POTENTIAL POINT OF VIEW FOR RENORMALIZATION

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For the shift σ in $\Sigma = \{0,1\}^{\mathbb{N}}$, we define the renormalization for potentials by

$$\mathcal{R}(V) := V \circ \sigma \circ H + V \circ H, \quad \text{with } \sigma^2 \circ H = H \circ \sigma.$$

We show that for a good H, there is a unique fixed point for $\mathcal{R}$. It is the Hofbauer potential V^*.

We show that the stable set of the Hofbauer potential, i.e. the set of potentials V such that $\mathcal{R}^n(V)$ converges to V^* is characterized by the germ of these potentials close to $0^\infty = 000\ldots$.

Then, we make connections with the Manneville–Pomeau map $f : [0,1] \circlearrowleft$. In particular we show that the lift in Σ of $\log f'$ is in the stable set of V^*.

In the second part, we characterize "good" H, such that $\sigma^2 \circ H = H \circ \sigma$.

In the last part, we study the thermodynamic formalism for some special potentials in the stable set of V^*. They are called virtual Manneville–Pomeau maps.

Keywords: Renormalization; the shift; non-uniformly hyperbolic dynamics; Manneville–Pomeau map; symbolic dynamics; thermodynamic formalism; phase transition.

AMS Subject Classification: 27E05, 37E20, 37F25, 37D25

1. Introduction

This paper is the first in a series to come. The goal of the series is to study relations between renormalization and phase transition, or, equivalently, between the mathematical and the physical points of view on Renormalization. Beyond the fact that the two theories have the same name, several hints (see e.g., [5]) suggest that there are a more strong connection between them. In particular, we show here that for the Manneville–Pomeau case this connection can be made precise.

Renormalization has indeed different meanings in mathematics or in physics. From the mathematical point of view, the Renormalization operator acts on dynamics. The prototype is the period doubling renormalization operator as introduced by Feigenbaum and by Coullet and Tresser in the context of the quadratic map (see [4, 7, 8, 21, 22, 10]). For $f:[0,1]\circlearrowleft$, we set

$$\widetilde{\mathcal{R}}(f)(x) = h^{-1}\circ f^2\circ h(x), \tag{1.1}$$

where h is an affine map. Then, the point is to study the existence of fixed points for $\widetilde{\mathcal{R}}$ and the hyperbolicity of the operator at these fixed points.

On the other hand, the renormalization in physics is associated to phase transitions[a] and decays of correlations (see [9, 12, 13, 26, 27]). It acts on potentials and not on maps (see for instance [9, 6] or [11]). It is also sometimes presented as a way to rescale the action of a potential. In particular, it is defined for a fixed and single dynamics.

To make connections between the two theories, we first need to determine a class of dynamics and deal with it. Here, we have chosen to work with dynamics on the interval conjugated or semi-conjugated to the full 2-shift $\Sigma=\{0,1\}^{\mathbb{N}}$. The choice is motivated by some facts:

- In Chap. 5 of [20], the renormalization is associated to Manneville–Pomeau like maps. There, the justifications of several of the statements are difficult to be understood from the pure mathematical point of view. One of our motivations was thus to improve our understanding of these results and provide proofs.
- The dynamics in the shift Σ is a toy model to the study of the problems which arise in statistical mechanics on a one-dimensional lattice (see [9, 26]): one can see 0 as a positive spin and 1 as a negative spin.
- Even in this simple situation, it is relatively difficult to find potentials which exhibit a phase transition.

[a]See p. 11 for a definition.

If $V : \Sigma \to \mathbb{R}$ is a continuous[b] function, we set $\mathcal{R}(V) := V \circ H + V \circ \sigma \circ H$ for some "good" $H : \Sigma \to \Sigma$; $\mathcal{R}$ is the renormalization operator acting on potentials. H satisfies

$$\sigma^2 \circ H = H \circ \sigma.$$

Note that any dynamics $f : [0,1] \circlearrowleft$ in our class produces a potential $\log f' \circ \theta$ (where θ is the (semi)-conjugacy between (Σ, σ) and $([0,1], f)$). Then, roughly speaking, the connection between the two points of view is the following:

A fixed point for $\mathcal{R}$ has a phase transition. Its stable set contains some $\log f' \circ \theta$, *where f is a fixed point for $\widetilde{\mathcal{R}}$ (for some good h).*

Of course, this statement is far from being precise. In particular, we did not say neither what "good" H means, nor what "stable set" means. One of the purposes of this paper is to make these two points precise for the Manneville–Pomeau case. These are the goals, respectively, of Theorem B (see Sec. 4) and Theorem A (see Sec. 2).

Due to the fact that all the dynamics we consider are (semi-)conjugated to Σ, we can restrict our study to Σ. There, a heuristic way to understand Theorems A and B is that for a "good" H we have a unique fixed point for the renormalization operator $\mathcal{R}$ on potentials. This fixed point generates a stable set (for the renormalization operator). The last result (Theorem C, see Sec. 6) studies the thermodynamic formalism for some specific potentials which are in this stable set.

Since some results (particularly Theorem C) require quite technical statements, we have dedicated specific sections to them. Sections 2, 4 and 6 are respectively devoted to the statements of Theorems A, B and C. Sections 3, 5 and 7 are respectively devoted to the proofs of Theorems A, B and C.

We conclude this introduction with some words about the existence of phase transitions. In the shift Σ, conditions yielding to the existence and uniqueness of equilibrium states are well known. It is sufficient that the potential is Hölder continuous, or satisfies the Walters condition or the Bowen's condition to ensure existence and uniqueness of the equilibrium state (see [23, 24]).

Consequently, this means that potentials which present phase transitions have low regularity. On the other hand, it is quite difficult to study the thermodynamic formalism for potentials with low regularity, hence to prove that they have a phase transition. Thus, a difficult task nowadays is to get a way to exhibit potentials with phase transitions. Several recent works go in that direction (see e.g., [25] for a larger class of potentials, and [14] for potentials with phase transitions). Concerning our work, our strategy is to exhibit such potentials via the "good" H's. We mention here that in a second paper of the series (see [2]), Bruin *et al.*, emphasize that the "good" H can be chosen among the substitutions.

[b]The set Σ is a compact and metric space with $d((x_n),(y_n)) = 2^{-\min(n, x_n \neq y_n)}$.

2. Statement of Theorem A: Connection between the Renormalizations for the Manneville–Pomeau Map

2.1. *Manneville–Pomeau map and renormalizations*

We set

$$\begin{cases} f(x) = \dfrac{x}{1-x}, & \text{if } 0 \le x \le \dfrac{1}{2}, \\[2ex] f(x) = 2 - \dfrac{1}{x}, & \text{if } \dfrac{1}{2} < x \le 1. \end{cases}$$

Note that one branch above is obtained from the other by the change of coordinate $x \to (1-x)$. The first branch $f(x) = \frac{x}{1-x}$ can be considered as a translation by -1 in the variable $s = 1/x$. Seeing it as a shift helps to understand the canonical partition in fundamental domains close to the fixed point 0: $(\frac{1}{3}, \frac{1}{2}), \dots, (\frac{1}{k}, \frac{1}{(k+1)}), \dots$ (see also p. 153 in [16]).

The semi-conjugacy between $([0,1], f)$ and (Σ, σ) is very simple: taking x in $[0,1]$, we create the word $\underline{x} = x_0, x_1, x_2, \dots$ by setting

$$x_j = \begin{cases} 0 & \text{if } f^j(x) \in \left[0, \dfrac{1}{2}\right[, \\[2ex] 1 & \text{if } f^j(x) \in \left]\dfrac{1}{2}, 1\right]. \end{cases}$$

This is well-defined (one-to-one) except for dyadic points, which explains the semi-conjugacy.

We point out that f is a fixed point for the renormalization $\widetilde{\mathcal{R}}$:

$$\forall\, x \in \left[0, \frac{1}{2}\right], \quad f^2\left(\frac{x}{2}\right) = (f \circ f)\left(\frac{x}{2}\right) = \frac{1}{2} f(x). \tag{2.1}$$

In other words, setting $h(x) = \frac{x}{2}$ we have $f^2 \circ h = h \circ f$.

Then, taking the logarithm of the derivative in (2.1), and keeping in mind that h is affine, we get

$$\log f' \circ f \circ h + \log f' \circ h = \log f'.$$

Now, we are interested in dynamical systems semi-conjugated to Σ, hence we are naturally led to study the maps $H : \Sigma \to \Sigma$ satisfying

$$\sigma^2 \circ H = H \circ \sigma$$

and to study the fixed points for the operator

$$\mathcal{R} : V \mapsto \mathcal{R}(V) := V \circ \sigma \circ H + V \circ H.$$

In Σ, any point is a alternation of sequences of 0's and sequences of 1's. Therefore the point $x := (\underbrace{0, \dots, 0}_{c_1}, \underbrace{1, \dots, 1}_{c_2}, \underbrace{0, \dots, 0}_{c_3}, 1, \dots)$ will simply be denoted by

$0^{c_1}1^{c_2}0^{c_3}\cdots.0^\infty$ and 1^∞ are easily understandable. The cylinders [0] and [1] denote points starting with 0 or 1. This is trivially extended to any finite word in 0's and 1's.

We denote by $M_n \subset \Sigma$, for $n \geq 1$, the cylinder set $[0^n 1]$ and by M_0 the cylinder set [1]. The collection $(M_n)_{n=0}^{\infty}$ is a partition of Σ. It corresponds to the partition into fundamental domains for the Manneville–Pomeau map f (and M_0 corresponds to $[\frac{1}{2}, 1]$). Finally we denote by θ the conjugacy between (Σ, σ) and $([0,1], f)$. It is proved in [16] that θ is the usual continued fraction expansion:

$$\theta(0^{n_0}1^{n_1}0^{n_2}\cdots) = \frac{1}{1 + n_0 + \frac{1}{n_1 + \frac{1}{n_2 + \cdots}}}.$$

Then we set $V_f := \log f' \circ \theta$. Following what was explained above, this potential represents the dynamics f.

2.2. *Statement of Theorem A*

Definition 2.1. We define the renormalization operator in the following way:
For $x := 0^{n_0}1^{n_1}0^{n_2}\cdots$ we set $H(x) := 0^{2n_0+1}1^{n_1}0^{n_2}$.

We left it to the reader to check $\sigma^2 \circ H = H \circ \sigma$ on [0].

For $x = 0^{n_0}1^{n_1}\cdots$, we set $n_0 =: \lfloor x \rfloor$. Hence $x \to 0^\infty$ means $\lfloor x \rfloor \to +\infty$. In the following, $\mathcal{F}$ denotes the set of continuous functions $V : \Sigma \to \mathbb{R}$ such that there exists $a > 0$ such that $\lim_{x \to 0^\infty} \lfloor x \rfloor^a V(x)$ exists and is not null if $a < 1$. For convenience we denote by $\mathcal{F}_<$ the set of functions in $\mathcal{F}$ such that $a < 1$, $\mathcal{F}_>$ if $a > 1$ and $\mathcal{F}_1$ if $a = 1$.

Theorem A. *The operator $\mathcal{R}$ has a unique*[c] *fixed point in $\mathcal{F}$. It is the Hofbauer potential defined by*

$$V^*(\underline{x}) = \log\left(\frac{n+1}{n}\right) \quad \text{if } \underline{x} \in M_n, \ \ n > 0.$$

Moreover, for $V \in \mathcal{F}$,

$$\mathcal{R}^k(V) \xrightarrow{\|\ \|_\infty}{}_{k \to +\infty} V^*, \tag{2.2}$$

if and only if $V : \Sigma \to \mathbb{R}$ satisfies $V(\underline{x}) = \frac{1}{n} + o(\frac{1}{n})$ for $\underline{x} \in M_n$ $(n > 0)$.
The potential V_f satisfies the condition $V_f(\underline{x}) = \frac{2}{n} + o(\frac{1}{n})$ for $\underline{x} \in M_n$.

Equation (2.2) justifies the following definition:

Definition 2.2. The set of potentials $V \in \mathcal{F}$ such that $\mathcal{R}^k(V) \to_{k \to +\infty} V^*$ is called the stable set of the Hofbauer potential.

[c]Note that uniqueness only holds in the "basin of attraction" of the renormalization defined by H, namely the cylinder [0]. Due to linearity, uniqueness is also up to a multiplicative constant!

3. Proof of Theorem A

We leave it to the reader to check that V^* is a fixed point for $\mathcal{R}$. Uniqueness of the fixed point follows from a simple computation (Sec. 3.1). This computation also proves that $V \in \mathcal{F}$ belongs to the stable set of V^* if and only if V belongs to $\mathcal{F}_1$.

Section 3.2 is devoted to the proof that V_f belongs to the stable set of the Hofbauer potential. In Sec. 3.3, we discuss the restrictive condition $V \in \mathcal{F}$.

3.1. *Uniqueness and the stable set of Hofbauer potential*

Let x be the sequence $0^{n_0}1^{n_1}0^2\cdots$, with $n_0 > 0$. By induction, we easily show

$$\mathcal{R}^k(V)(x) = S_{2^k}(V)(x_k), \tag{3.1}$$

where $x_k := H^k(x) = (0^{2^k n_0 + 2^k - 1}, 1^{n_1}0^{n_2}, \ldots)$ and $S_j(V)$ is the Birkhoff sum $V(\cdot) + V\circ\sigma(\cdot) + \cdots + V\circ\sigma^{j-1}(\cdot)$.

For $0 \le j \le 2^k - 1$, $\sigma^j(x_k) = 0^{2^k(n_0+1)-1-j}1\cdots$. Assumption $n_0 \ge 1$, means that every $\sigma^j(x_k)$ starts with at least 2^k 0's. Therefore $\lfloor\sigma^j(x_k)\rfloor$ goes to $+\infty$ as k increases (and for every $j \le 2^k - 1$).

Let us pick some V in $\mathcal{F}$. Let $a > 0$ and α be such that $\lim_{x\to 0^\infty}\lfloor x\rfloor^a V(x) = \alpha$. The property $\lfloor\sigma^j(x_k)\rfloor \to +\infty$ and Eq. (3.1) yield

$$\begin{aligned}\mathcal{R}^k(V)(x) &= \sum_{j=0}^{2^k-1}\left(\frac{\alpha}{(2^k(n_0+1)-(j+1))^a} + o\left(\frac{1}{(2^k(n_0+1)-(j+1))^a}\right)\right)\\ &= 2^{k(1-a)}\left(\frac{1}{2^k}\sum_{j=1}^{2^k}\frac{\alpha}{(n_0+1-\frac{j}{2^k})^a} + o\left(\frac{1}{2^k}\sum_{j=1}^{2^k}\frac{1}{(n_0+1-\frac{j}{2^k})^a}\right)\right).\end{aligned}$$

The first term into the brackets on the right-hand side is a Riemann sum, and converges, as $k \to \infty$, to $\int_0^1 \frac{\alpha}{(n_0+1-t)^a}dt$. Therefore, the second term goes to zero (it is a small "o" of a constant).

Therefore, if $a < 1$, then $\alpha > 0$ and $\mathcal{R}^k(V)$ goes to $+\infty$ as k goes to $+\infty$. On the contrary, if $a > 1$, $\mathcal{R}^k(V)$ goes to 0.

The convergence to a nonzero function can only occur if $a = 1$, namely for V in $\mathcal{F}_1$. In that case $\mathcal{R}^k(x)$ converges to

$$\int_0^1 \frac{\alpha}{(n_0+1-t)}dt = \alpha[\log(n_0+1-t)]_1^0 = \alpha\log\left(1+\frac{1}{n_0}\right).$$

In other words, $\mathcal{R}^k(x)$ converges to $\alpha V^*(x)$.

This proves that if V belongs to $\mathcal{F}_1$, then $\mathcal{R}^k(V)(x)$ converges to $V^*(x)$ (pointwise convergence). Moreover, for $V \in \mathcal{F}$, if V does not belong to $\mathcal{F}_1$ (up to a multiplicative constant), then our computation shows that for every x, either $\mathcal{R}^k(V)(x)$ goes to $+\infty$ or to 0 (depending on $V \in \mathcal{F}_<$ or $V \in \mathcal{F}_>$). In other words, $V \in \mathcal{F}$ belongs to the stable set of V^* if and only if it belongs to $\mathcal{F}_1$.

This also proves the uniqueness of the fixed point in $\mathcal{F}$: if $\widetilde{V}$ is any nonzero fixed point, equality $\widetilde{V} = \mathcal{R}^k(\widetilde{V})$ shows that $\widetilde{V}$ belongs to $\mathcal{F}_1$. Then $\mathcal{R}^k(\widetilde{V})$ converges to V^* (up to a multiplicative constant).

The uniform convergence (with respect to x) of $\mathcal{R}^k(V)(x)$ to $V^*(x)$ for V in $\mathcal{F}_1$ follows from the control of the convergence of a Riemann sum.

It is well known that

$$\left|\frac{1}{n}\sum_{j=0}^{n-1}\varphi\left(\frac{j}{n}\right) - \int_0^1 \varphi(t)dt\right| \leq \frac{C}{n}\|\varphi''\|_\infty$$

holds for some universal constant C. Here, we have to consider functions φ of the form

$$\varphi(t) = \frac{1}{n_0 + 1 - t}.$$

They all satisfy (uniformly with respect to n_0)

$$\|\varphi''\|_\infty \leq 1.$$

3.2. *Manneville–Pomeau is in the stable set of the Hofbauer potential*

For x in $[0, \frac{1}{2}[$, $f(x) = x + x^2 + x^3 + \cdots$, and the series converges uniformly. Standard calculus show that

$$f'(x) = 1 + 2x + O(x^2).$$

Hence $\log f'(x) = 2x + O(x^2)$. Now, the partition in fundamental domains shows that $x = \theta(\underline{x})$ with $\underline{x}$ in M_n (and $n > 1$) if and only if x belongs to $[\frac{1}{n+2}, \frac{1}{n+1}[$. This shows $x = \frac{1}{\lfloor \underline{x} \rfloor} + o(\frac{1}{\lfloor \underline{x} \rfloor})$. Hence we have

$$\log f' \circ \theta(\underline{x}) = \frac{2}{\lfloor \underline{x} \rfloor} + o\left(\frac{1}{\lfloor \underline{x} \rfloor}\right).$$

Remark 3.1. It is usual to associate the Manneville–Pomeau map and the Hofbauer potential. Namely, the existence of the phase transition for $-t \log f'$ is sometimes only proved for the Hofbauer potential in the shift. The fact that $\log f' \circ \theta$ belongs to the stable set of the Hofbauer potential (up to a multiplicative constant) gives consistence to that proof.

3.3. *Discussion on the condition $V \in \mathcal{F}$*

The point 0^∞ is fixed by H. If $\widetilde{V}$ is a fixed point for $\mathcal{R}$,

$$\widetilde{V}(0^\infty) = \mathcal{R}(\widetilde{V})(0^\infty) = \widetilde{V} \circ \sigma \circ H(0^\infty) + \widetilde{V} \circ H(0^\infty) = 2\widetilde{V}(0^\infty).$$

This immediately shows $\widetilde{V}(0^\infty) = 0$.

Therefore, if we are interested in the stable set of any fixed point for $\mathcal{R}$, we have to restrict our study to continuous potentials vanishing at 0^∞. It thus makes sense to determine the germ close to 0^∞.

Then, the condition $V \in \mathcal{F}$ means that we restrict our study to potentials which have a specific kind of germ close to 0^∞.

The proof of the uniqueness (Sec. 3.1) also works if we weaken the hypothesis. Indeed, the same proof works for potentials satisfying just one of the conditions below:

- there exists $a < 1$, such that the interval $[\liminf_{\underline{x}\to 0^\infty} \lfloor \underline{x} \rfloor^a V(\underline{x}), \limsup_{\underline{x}\to 0^\infty} \lfloor \underline{x} \rfloor^a V(\underline{x})]$ does not contain 0,
- $\lim_{\underline{x}\to 0^\infty} \lfloor \underline{x} \rfloor V(\underline{x})$ exists,
- there exists $a > 1$ such that $\limsup_{\underline{x}\to 0^\infty} |\lfloor \underline{x} \rfloor^a V(\underline{x})| < +\infty$.

Very roughly speaking, the hypothesis $V \in \mathcal{F}$ means $V = \log g' \circ \theta$, with $g(x) = x + \frac{\alpha}{a+1} x^{a+1} + h.o.t.$ close to 0. This means that we restrict our study to dynamics with some minimal but fixed regularity close to 0.

4. Statement of Theorem B: Good H's

Previously we studied the renormalization induced by the map

$$H : 0^{n_0} 1^{n_1} 0^{n_2} \cdots \mapsto 0^{2n_0+1} 1^{n_1} 0^{n_2} \cdots.$$

Following our strategy, we are interested in finding other "good" H. Here, "good" would mean that H satisfies

$$\sigma^2 \circ H = H \circ \sigma$$

and the renormalization operator

$$\mathcal{R} : V \mapsto \mathcal{R}(V) := V \circ \sigma \circ H + V \circ H$$

has a (unique) fixed point which exhibits a phase transition.

On the other hand, we also want to understand why physicists sometime present renormalization as a way to rescale the action of the potential: as an example, for the two-dimensional lattice in statistical mechanics (see for instance [6]), one takes a square box, and then consider a new renormalized box such that each side is scaled by a factor of 2. The old potential is also rescaled in the new box.

Theorem B. *Let H be an increasing function on the shift Σ (for the lexicographic order), such that*

(1) *for every* $\underline{x} = (1, x_2, x_3, \ldots)$, $H(\underline{x}) = (\underbrace{0, \ldots, 0}_{a \text{ terms}}, 1, x_2, x_3, \ldots)$, *where* $a \geq 1$;
(2) $\sigma^2 \circ H = H \circ \sigma$,
(3) $H(\underline{0}^\infty) = \underline{0}^\infty$.

Then, for every $\underline{x} = (\underbrace{0, \ldots, 0}_{n_0 \text{ terms}}, 1, x_{n_0+2}, \ldots)$, *we have* $H(x) = (\underbrace{0, \ldots, 0}_{2n_0+a \text{ terms}}, 1, x_{n_0+2}, \ldots)$.

In other words, Theorem B shows that there exists a unique type of maps $H : \Sigma \circlearrowleft$, and (as a consequence) a unique type of "good" potential V which satisfy the fixed point property

$$\mathcal{R}(V) = V.$$

Here, the special importance of the Hofbauer potential appears.

We want here to emphasize that the assumptions on H are very natural if we consider the rescaling procedure described above. The lexicographic order is a good way to consider blocks at different scales. The assumption "$H([1]) = [0^a 1]$" is a good way to send blocks onto blocks.

Note that 0^∞ is a fixed point for σ. Hence, if H satisfies $\sigma^2 \circ H = H \circ \sigma$, $H(0^\infty)$ is fixed by σ^2. Moreover, $0^\infty < 10101\cdots$, and monotonicity of H yields $H(0^\infty) = 0^\infty$. In [2], the authors consider H such that $H(0^\infty) = (01)^\infty$, which is fixed by σ^2. The main conclusion here is that a "good" H generates a basin of attraction of the renormalization procedure. In this paper the basin is $\{0^\infty\}$. In [2] it is the uniquely ergodic compact set generated by the Thue–Morse substitution.

5. Proof of Theorem B

Let H be an increasing function on the shift Σ (for the lexicographic order), such that

(1) for every $\underline{x} = (1, x_2, x_3, \ldots)$, $H(\underline{x}) = (\underbrace{0, \ldots, 0}_{a \text{ terms}}, 1, x_2, x_3, \ldots)$, where $a \geq 1$;
(2) $\sigma^2 \circ H = H \circ \sigma$,
(3) $H(\underline{0}^\infty) = \underline{0}^\infty$

We want to prove that for every $\underline{x} = (\underbrace{0, \ldots, 0}_{n_0 \text{ terms}}, 1, x_{n_0+2}, \ldots)$, we have

$$H(x) = (\underbrace{0, \ldots, 0}_{2n_0+a \text{ terms}}, 1, x_{n_0+2}, \ldots).$$

Note that by assumption, this is already proved for every $\underline{x}$ in the cylinder [1].

First consider the case $a \geq 2$.

Let us pick $\underline{x} = 0^{n_0} 1^{n_1} 0^{n_2} \cdots$. We assume $n_0 > 1$. We point out that $\sigma(\underline{x}) \geq \underline{x}$, because a "1" appears sooner in $\sigma(\underline{x})$ than in $\underline{x}$. Therefore we must have

$$H(\sigma(\underline{x})) > H(\underline{x}), \quad \text{if } x \neq \underline{0}^\infty, \underline{1}^\infty. \tag{5.1}$$

Now, $\sigma^{n_0}(\underline{x})$ belongs to the cylinder [1], hence $H(\sigma^{n_0}(\underline{x})) = [0^a\underline{x}]$. Equality $\sigma^2 \circ H = H \circ \sigma$ yields $\sigma^{2n_0} \circ H = H \circ \sigma^{n_0}$. Therefore

$$H(\underline{x}) = (\underbrace{?, \ldots, ?}_{2n_0 \text{ terms}}, \underbrace{0, \ldots, 0}_{a \text{ terms}}, 1, x_{n_0+2}, \ldots), \tag{5.2}$$

where the first $2n_0$ digits are unknown.

As H is increasing, its image is in the cylinder [0], and the first digit in (5.2) is 0. Equality $\sigma^2 \circ H = H \circ \sigma$ shows that every $\sigma^{2k}(H(\underline{x}))$ (with $k \leq n_0$) belongs to the image of H, hence starts with 0. This means that every odd unknown digit in (5.2) is 0.

Now, we prove that no even unknown digit can be 1. Let us assume that the second digit is 1. Doing the same work for $\sigma(\underline{x})$ (here we use $n_0 > 1$), we have

$$H \circ \sigma(\underline{x}) = (\underbrace{0, ?, \ldots, 0, ?}_{2n_0-2 \text{ terms}}, \underbrace{0, \ldots, 0}_{a \text{ terms}}, 1, x_{n_0+2}, \ldots). \tag{5.3}$$

Moreover, $\sigma^2 \circ H = H \circ \sigma$ shows that each unknown digit at position $2p$ in (5.3) is the same digit as the digit in position $2p + 2$ in (5.2).

If the second digit in (5.2) is "1", then (5.1) shows that the second digit in (5.3) must also be "1". Therefore, the cascade rule yields that each even unknown digit must be 1, in (5.2) and in (5.3). In that case, and as we assumed $a \geq 2$, there will be "1" in $H(\underline{x})$ in position $2n_0$, and "0" for $H \circ \sigma(\underline{x})$, and the two words coincide before that position. Hence, $H(\sigma(\underline{x})) < H(\underline{x})$, which is impossible by (5.1). This proves that the assumption is false, and the second unknown digit in (5.2) must be "0".

Note that this also holds if $n_0 = 1$. Indeed, in that case we completely know $H \circ \sigma(\underline{x})$, by assumption (1) in Theorem B. Therefore the above discussion means that for every $\underline{\xi} = (0, \ldots)$, $H(\underline{\xi})$ starts with three symbols "0". Here again, the cascade rule between (5.2) and (5.3) yields that every even unknown digit is "0".

The case $a = 1$. In that case, the assumption "the second unknown digit in (5.2) in 1" yields

$$H(\underline{x}) = (\underbrace{0, 1, \ldots, 0, 1, 0, 1}_{2n_0 \text{ terms}}, \underbrace{0}_{a=1}, 1, x_{n_0+2}, \ldots),$$

$$H \circ \sigma(\underline{x}) = (\underbrace{0, 1, \ldots, 0, 1}_{2n_0-2 \text{ terms}}, \underbrace{0}_{a=1}, 1, x_{n_0+2}, \ldots).$$

Hence, the only possibility which respects the increasing property for H would be to alternate "0" and "1" for the tail of $\underline{x}$. But even in that case, this will be in contradiction with (5.1). This finishes the proof.

Remark 5.1. The potential defined by $\log \frac{k+a}{k+a-1}$ on M_k ($k \geq 1$) is invariant by $\mathcal{R}$. It is a "Hofbauer-like" potential.

We leave it to the reader to check that this potential exhibits a phase transition. Indeed, the usual proof for the well knwon Hofbauer potential works directly.

6. Statement of Theorem C: Thermodynamic Formalism for Virtual Manneville–Pomeau Maps

6.1. *Recall on equilibrium states*

We recall that, given a function ϕ, a probability measure μ is said to be ϕ-conformal if there exists a positive real number λ_ϕ with the following property: for every Borel set A such that $\sigma : A \to \sigma(A)$ is a homeomorphism, then

$$\mu(\sigma(A)) = \lambda_\phi \int_A e^{-\phi} d\mu.$$

If ϕ is continuous, there necessarily exists a ϕ-conformal measure. Indeed the Transfer Operator

$$\mathcal{P}(\psi)(x) := \sum_{y, \sigma(y)=x} e^{\phi(y)} \psi(y)$$

acts on continuous functions, hence its adjoint acts on measures. We then use the Schauder–Tychonoff theorem to get an eigen-measure. This measure is a ϕ-conformal measure. The question is then to study the existence (and uniqueness) of a σ-invariant probability measure equivalent to the ϕ-conformal measure. Such a measure is said to be ϕ-*quasi-conformal*. We shall simply say quasi-conformal when there is no confusion about the function ϕ.

We denote by h_μ the Kolmogorov entropy of the invariant probability μ. We recall that given a function $\phi : \Sigma \to \mathbb{R}$, an invariant *probability* measure μ is called an equilibrium state for the *potential* ϕ if it satisfies

$$h_\mu + \int \phi d\mu = \sup_\nu \left\{ h_\nu + \int \phi d\nu \right\}.$$

In "good" cases, given a potential ϕ, there exists a unique ϕ-quasi-conformal probability; it is also the unique equilibrium state for ϕ.

In the literature, phase transition means several things. On the one hand, it means that the map $\gamma \mapsto \mathcal{P}(\gamma \cdot \phi)$ loses analyticity at some point (see e.g., [17, 19]). On the other hand, it means that for some γ_0, $\gamma_0 \cdot \phi$ has several equilibrium states or loses the equilibrium state (this is e.g., the case for the Hofbauer potential). There are of course connections between these two definitions, but it is not clear at all (at least for the authors) that in *every setting* these two definitions coincide. In this paper we are more likely to use the second definition.

In statistical mechanics, the parameter γ is positive and is the inverse of the temperature. Roughly speaking, one gets a potential and a system "at equilibrium". One studies the variations of equilibrium as the temperature changes.

As we mentioned above, Hölder continuity for ϕ ensures the existence and uniqueness of the equilibrium state. It is also well known (see e.g., [18]) that in this case and with our settings, $\mathcal{P}(\gamma)$ is analytic. Then, the two definitions coincide and there is no phase transition (whichever the definition we choose).

Therefore, if one wants to produce potentials with phase transitions, it is necessary to find low regular functions. On the other hand, these functions have to be sufficiently regular in order to be studied. In [25], Walters defines a class $R(X)$ of such potentials. They are constant on cylinders of the form $[0^n1]$, $[10^n1]$, $[1^n0]$ or $[01^n0]$. This essentially means they do not distinguish points by the time their orbit spent in the two laminar regimes (cylinder [0] and cylinder [1]).

6.2. *Virtual Manneville–Pomeau maps and Theorem C*

Here we study a special family of potentials. They all belong to the stable set of the Hofbauer potential (up to a multiplicative constant) and, contrarily to [25], they are not constant on cylinders of the form $[0^n1]$ or $[10^n1]$ or $[1^n0]$ or $[01^n0]$. Namely, they are obtained as a specific perturbation of $-\log f' \circ \theta$, where f is the Manneville–Pomeau map already defined.

We consider real numbers, α in $[1, +\infty[$, β in $]0, 1]$, and a natural number $a \geq 0$. We assume that these parameters satisfy

$$\frac{1}{(\frac{3}{2})^\beta - 1} - \frac{1}{2^\beta - 1} = \left(1 + \frac{1}{a+1}\right)^\alpha - 1. \tag{6.1a}$$

$$\frac{1}{\alpha} = 2^\beta - 1. \tag{6.1b}$$

This system of conditions is referred to as (6.1). We shall prove in Lemma 7.4 that for each choice of one parameter, (6.1) gives a unique value for the other two parameters (except that a may not be an integer). Hence, for each positive integer value of a we have the corresponding values α_a and β_a. In this way, several renormalization operators, with different values $a \in \mathbb{N}$, can be considered as in Theorem B. In the following, we however prefer to keep β as parameter.

Given $\underline{x} = (0^{n_0}1^{n_1}0^{n_2}1, \ldots) \in \Sigma = \{0, 1\}^{\mathbb{N}}$ we define a real number in the following way:

$$\theta_\beta(\underline{x}) = \cfrac{1}{\cfrac{(n_0+1)^\beta}{(n_0+2)^\beta - (n_0+1)^\beta} + \cfrac{1}{\cfrac{(n_1+a)^\alpha}{(n_1+a+1)^\alpha - (n_1+a)^\alpha} + \cfrac{1}{\cfrac{n_2^\beta}{(n_2+1)^\beta - n_2^\beta} + \cfrac{1}{\cfrac{(n_3+a)^\alpha}{(n_3+a+1)^\alpha - (n_3+a)^\alpha} + \cdots}}}}$$

With these notations, the potential ϕ_β is defined by:

$$\phi_\beta(x) = \begin{cases} -2\log\left(\dfrac{\theta_\beta \circ \sigma(x)}{\theta_\beta(x)}\right) & \text{if } x \in [0], \\ -2\log\left(\dfrac{2^\beta - 1 - \theta_\beta \circ \sigma(x)}{2^\beta - 1 - \theta_\beta(x)}\right) & \text{if } x \in [1]. \end{cases}$$

Heuristically speaking, the potential ϕ_β should be seen as what one should expect to be the $-\log$ of the derivative of a "global" Manneville–Pomeau map $\widehat{f}_\beta$ defined for the Bernoulli space after the "change of coordinates" θ_β. We are studying existence of γ-conformal measures for our virtual Manneville–Pomeau maps.

Theorem C. *For any $\gamma \in]0, \frac{1}{2}]$ and for any β there exist a unique $\gamma\phi_\beta$-conformal measure and a unique quasi-conformal probability.*

For $\gamma \in]\frac{1}{2}, 1]$, there exists a critical value $\beta_c := \beta_c(\gamma) > 0$, which is maximal with this property, such that for any $\beta < \beta_c$ there exist a unique $\gamma\phi_\beta$-conformal measure and a unique quasi-conformal probability.

In both cases the quasi-conformal probability is the unique equilibrium state associated to the potential $\gamma\phi_\beta$.

It is left to the reader to check that $\phi_\beta(0^{n_0}1\cdots) = \frac{1}{n_0} + o(\frac{1}{n_0})$ for $n_0 \to +\infty$. This means that all the ϕ_β belong to the stable set of the Hofbauer potential V^*. Similarly, $\phi_\beta(1^{n_1}0\cdots) = \frac{1}{n_1+a} + o(\frac{1}{n_1+a})$ if $n_1 \to +\infty$. Following Theorem B, this means that $\gamma \cdot \phi_\beta$ belongs to the stable set of the Hofbauer-like potential but for the renormalization close to 1^∞.

Regarding this problem of phase transition, several questions are still unsolved. The case $\beta = \gamma = 1$ should indicate that for $\gamma > \frac{1}{2}$, there exists another critical value $\bar{\beta}_c = \bar{\beta}_c(\gamma)$ such that for $\beta > \bar{\beta}_c$ there exists no $\gamma\phi_\beta$-quasi-conformal probability.

Similarly and probably consequently, it is expected that for fixed β, the one family of potentials $\gamma \cdot \phi_\beta$ presents a phase transition: for γ sufficiently large, the pressure of $\gamma \cdot \phi_\beta$ is affine.

Nevertheless, our computations do not yet prove these expected results. For this, we should have better bounds in the proof of Proposition 7.1.

6.3. *Construction of Gibbs states*

In this section we recall the method of construction of Gibbs measures presented in [15] and developed further in later works of Leplaideur. This is the method we shall use to prove uniqueness of the equilibrium state for $\gamma\phi_\beta$. In the following, $\mathcal{P}(\gamma, \beta)$ denotes the associated pressure.

We consider the first return map g in the cylinder [01]. For y in [01], $r(y)$ denotes the first return time in [01] of y by iterations of σ. For a real number Z, for x in [01] and for ψ a continuous function from [01] to $\mathbb{R}$, we define

$$\mathcal{L}_{Z,\gamma,\beta}(\psi)(x) \sum_{y, g(y)=x} e^{S_{r(y)}(\gamma\phi_\beta)(y) - Zr(y)} \psi(y).$$

This is the transfer operator for the map g associated to the potential $S_{r(\cdot)}(\gamma\phi_\beta)(\cdot) - Zr(\cdot)$. We study this operator, for fixed γ and β and for large enough Z. Namely, we set

$$Z_c = Z_c(\gamma, \beta) := \limsup_{n \to +\infty} \frac{1}{n} \log \left(\sum_{x=g(y), r(y)=n} e^{S_n(\gamma\phi_\beta)(y)} \right).$$

Even if Z_c *a priori* depends on x, it actually does not (see Proposition 7.1) and Proposition 7.2 shows that for every γ and β, $Z_c = 0$. From here on, we may omit γ and β when they are not necessary.

Therefore we have, on the one hand the dynamical system (Σ, σ) with the potential $\gamma \cdot \phi_\beta$ (referred to as the global system), and on the other hand, the dynamical system $([01], g)$ and the family of potentials $\gamma \cdot S_{r(\cdot)}(\phi_\beta) - Zr(\cdot)$ (referred to as the local system). The point is that it is possible to deduce the existence and uniqueness of the equilibrium state for the global system from the thermodynamic formalism for the local one. The advantage to consider the local system is that, $S_{r(\cdot)(\phi_\beta)}$ satisfies the Bowen property (for g) when ϕ_β does not (for σ).

Proposition 6.1. *For every $Z > Z_c$ there exists a unique equilibrium state for the local dynamical system. It is obtained as*

$$d\mu_Z := h_Z d\nu_Z,$$

where ν_Z is the probability measure satisfying $\mathcal{L}_Z^*(\nu_Z) = \lambda_Z \nu_Z$, h_Z is the normalized eigenfunction $\mathcal{L}_Z(h_Z) = \lambda_Z h_Z$ and λ_Z is a positive real number.

A sketch of the proof is given in Sec. 7.2.2.

Proposition 6.2. *If there exists $Z > 0$ such that $\lambda_Z = 1$, then the global system admits a unique equilibrium state for $\gamma \cdot \phi_\beta$ and the pressure is Z. It is a quasi-conformal measure, and its restriction to* [01] *is the unique equilibrium state for the local system, μ_Z. There exists a unique conformal measure, whose restriction to* [01] *is the measure ν_Z.*

Remark 6.1. The "0" in the condition $Z > 0$ is important. It comes from the fact that $Z_c = 0$ and is also related to the fact that $\phi_\beta(0^\infty) = \phi_\beta(1^\infty) = 0$.

7. Proof of Theorem C

In Sec. 7.1 we fix the claims concerning the parameters. In Sec. 7.2 we prove Propositions 6.1 and 6.2. For this we actually prove that the local system satisfies the Bowen condition. In Sec. 7.3 we complete the proof of Theorem C.

7.1. *Properties for θ_β, parameters and virtual Manneville–Pomeau maps*

7.1.1. *Convergence of the continued fraction expansion defined by θ_β*

Here, we define a generalization of the continued fraction expansion. We consider real numbers, α in $[1, +\infty[$, β in $]0, 1]$, and the natural number $a \geq 0$. These parameters are not supposed to satisfy (6.1).

Lemma 7.1. *Let $(a_k)_{k\in\mathbb{N}}$ be a sequence of real numbers such that $a_0 = 0$, each a_{2k+1} is larger than* 1, *and all the even terms a_{2k}, $k > 0$, are positive and uniformly* *bounded away from zero. Then, the sequence of real numbers* (r_k) *defined by*

$$r_k = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_k}}}},$$

converges to a real number denoted by $[0, a_1, a_2, a_3, \ldots]$.

Proof. Let $(a_k)_{k\in\mathbb{N}}$ be as in the assumptions. We define two new sequences $(p_k)_{k\in\mathbb{N}}$ and $(q_k)_{k\in\mathbb{N}}$, by induction:

$$p_0 = 0, \quad p_1 = 1, \quad q_0 = 1, \quad q_1 = a_1$$

$$\forall\, k \in \mathbb{N}, \quad p_{k+2} = a_{k+2}p_{k+1} + p_k, \quad q_{k+2} = a_{k+2}q_{k+1} + q_k.$$

By induction, $q_k \geq 1$ for every $k > 0$. Using $a_{2k+1} \geq 1$, we get $q_{2k+1} \geq k$, and then $q_{2k} \geq A \cdot k$, where A is a positive lower bound for all the a_{2j}'s. Therefore, q_k goes to $+\infty$ as k increases to $+\infty$.

If we set $u_k = p_{k+1}q_k - p_kq_{k+1}$, then $u_{k+1} = -u_k$ for every k. We claim that $r_k = \dfrac{p_k}{q_k}$. Then, the two subsequences (r_{2k}) and (r_{2k+1}) are mutually adjacent and converge to the same limit. We left it to the reader to check that the even sequence (r_{2k}) increases and the odd sequence (r_{2k+1}) decreases. □

Let $\delta > 0$ be a real number. We define $g\colon (0, \infty) \to \mathbb{R}$, given by

$$g_\delta(z) = \frac{1}{(1+\frac{1}{z})^\delta - 1} = \frac{z^\delta}{(z+1)^\delta - z^\delta}.$$

We have for every $z \in (0, +\infty)$, $g'_\delta(z) = \frac{\delta}{z^2}\frac{1}{((1+\frac{1}{z})^\delta - 1)^2}(1+\frac{1}{z})^{\delta-1}$, hence g_δ is increasing. Moreover, $\lim_{z\to 0} g_\delta(z) = 0$ and $\lim_{z\to+\infty} g_\delta(z) = +\infty$.

Also, $g_\delta(z) = z^\delta + o(z^\delta)$ when z is close to 0, and, $g_\delta(z) = \frac{z}{\delta} - \frac{\delta-1}{2\delta} + O(\frac{1}{z})$, when z is close to $+\infty$.

Lemma 7.2. *The map* g_δ *is convex for* $\delta > 1$ *and concave for* $\delta < 1$.

Proof. To prove this lemma, first note that $g'(z) = \frac{\delta}{z^2+z}(g(z) + g^2(z))$. This yields

$$g''(z) = -\delta\frac{2z+1}{(z^2+z)^2}(g(z) + g^2(z)) + \frac{\delta}{z^2+z}(g'(z) + 2g'(z)g(z)).$$

If we replace in this last expression the value of $g'(z)$ as a function of z and $g(z)$, we get

$$g''(z) = 2\delta^2\frac{g(z) + g^2(z)}{(z^2+z)^2}\left(g(z) - \left(\frac{z}{\delta} - \frac{\delta-1}{2\delta}\right)\right).$$

Note that $\frac{z}{\delta} - \frac{\delta-1}{2\delta}$ is the asymptote of g close to $+\infty$. Then, the convexity of the map depends on the position of the graph with respect to the asymptote: it is convex when the graph is above the asymptote, and it is concave when the graph is below the asymptote. Now, remind that a convex map has a non-decreasing derivative, and a concave map has a non-increasing derivative.

Therefore, easy considerations on the relative position of the graph with respect to the asymptote prove that the graph cannot cross the asymptote. Hence the map is convex for $\delta > 1$, and concave for $\delta < 1$. □

Moreover, $g_\delta(1) = \frac{1}{2^\delta - 1}$. Therefore, $g_\alpha(1) < 1$, for $\alpha > 1$, and $g_\beta(1) > 1$, for $\beta < 1$. This shows that for a given sequence $n_0 \geq 0, n_1 > 0, n_2 > 0, \ldots$ of integers, the sequence defined by $a_{2k} = g_\alpha(n_{2k-1} + a)$ and $a_{2k+1} = g_\beta(n_{2k})$ satisfies the properties of Lemma 7.1. Therefore the real number $[0, a_1, a_2, \ldots]$ is well-defined. In other words, for α, β and a satisfying (6.1), and for $\underline{x} = (0^{n_0}1^{n_1}0^{n_2}\cdots) \in \Sigma = \{0,1\}^{\mathbb{N}}$,

$$\begin{aligned}\theta_\beta(\underline{x}) &= [0, g_\beta(n_0+1), g_\alpha(n_1+a), g_\beta(n_2), g_\alpha(n_3+a), \ldots] \\ &= \cfrac{1}{\frac{(n_0+1)^\beta}{(n_0+2)^\beta - (n_0+1)^\beta} + \cfrac{1}{\frac{(n_1+a)^\alpha}{(n_1+a+1)^\alpha - (n_1+a)^\alpha} + \cfrac{1}{\frac{n_2^\beta}{(n_2+1)^\beta - n_2^\beta} + \cfrac{1}{\frac{(n_3+a)^\alpha}{(n_3+a+1)^\alpha - (n_3+a)^\alpha} + \cdots}}}}\end{aligned}$$

is well defined.

We claim that $\theta_\beta(\underline{x})$ belongs[d] to $[0, 2^\beta - 1]$. Indeed, the odd subsequence (r_{2k+1}) decreases and the even subsequence (r_{2k}) increases. To minimize the value of $\theta_\beta(\bar{x})$, it is necessary and sufficient to maximize n_0. On the other hand, to maximize the value of $\theta_{\beta,}(\underline{x})$, it is necessary and sufficient to minimize n_0 and to maximize n_1. Therefore, for every $\underline{x}$,

$$0 = \theta_\beta(0^\infty) \leq \theta_\beta(\underline{x}) \leq \theta_\beta(1^\infty) = 2^\beta - 1.$$

Remark 7.1. The fact that the sequences (r_{2k}) and (r_{2k+1}) are mutually adjacent shows that for $n_0 \geq 0$:

$$\begin{aligned}\theta_\beta(0^{n_0}1^{n_1}\cdots 0^{n_{2p}}10^\infty) &\leq \theta_\beta(0^{n_0}1^{n_1}\cdots 0^{n_{2p}}1^{n_{2p+1}}0\cdots) \\ &\leq \theta_\beta(0^{n_0}1^{n_1}\cdots 0^{n_{2p}}1^\infty) \\ \theta_\beta(0^{n_0}1^{n_1}\cdots 0^{n_{2p}}1^{n_{2p+1}}0^\infty) &\leq \theta_\beta(0^{n_0}1^{n_1}\cdots 0^{n_{2p}}1^{n_{2p+1}}0\cdots) \\ &\leq \theta_\beta(0^{n_0}1^{n_1}\cdots 0^{n_{2p}}1^{n_{2p+1}}01^\infty).\end{aligned}$$

Remark 7.2. The number a does not need to be in $\mathbb{N}$ to define θ_β, but in $\mathbb{R}^+$. This restriction is due to the fact that we want to see a as a parameter of the renormalization.

[d]It is actually possible to prove that θ_β is onto but not one-to-one except for $\beta = 1$.

7.1.2. *Lexicographic order and values for θ_β*

Here, we present a technical lemma which gives inequalities with respect to the lexicographic order in Σ.

Lemma 7.3. *Let $a_n = a_n(\beta) := -2\log\frac{g_\beta(n+1)}{g_\beta(n)}$ and $b_n = b_n(\beta) := -2\log\frac{g_\beta(n+1)+(1+\frac{1}{1+a})^\alpha-1}{g_\beta(n)+(1+\frac{1}{1+a})^\alpha-1}$. Then for $(0^n\ 10^\infty) \le w < (0^n\ 1^\infty)$ and $n > 0$ we have*

$$a_n \le \phi_\beta(w) \le b_n.$$

Let $u_m = u_m(\beta) := -2\log\frac{g_\alpha(m+a)+2^\beta-1}{g_\alpha(m-1+a)+2^\beta-1}$ and $v_m = v_m(\beta) := -2\log\frac{g_\alpha(m+a)+2(2^\beta-1)}{g_\alpha(m-1+a)+2(2^\beta-1)}$. Then for $(1^m\ 0^\infty) \le w < (1^m\ 01^\infty)$ and $m > 1$ we have

$$u_m \le \phi_\beta(w) \le v_m.$$

If $m = 1$ we have

$$\begin{aligned} u_1 &:= -2\log\left(1+\frac{(\frac{3}{2})^\beta-1}{2^\beta-(\frac{3}{2})^\beta}\right) \le \phi_\beta(w) \\ &\le -2\log\left(\left(1+\left(\left(\frac{3}{2}\right)^\beta-1\right)\frac{2.2^\beta-(\frac{3}{2})^\beta-1}{(2^\beta-(\frac{3}{2})^\beta)(2^\beta-1)}\right)\left(\frac{2^\beta-(\frac{3}{2})^\beta}{2^\beta-1}\right)\right) =: v_1. \end{aligned}$$

Proof. For w satisfying $(0^n\ 10^\infty) \le w < (0^n\ 1^\infty)$ and $n > 0$, we have

$$\phi_\beta(w) = -2\log\frac{\theta_\beta\circ\sigma(w)}{\theta_\beta(w)}.$$

We set $\theta(w) = \frac{1}{g_\beta(n+1)+r}$ and we have $\theta\circ\sigma(w) = \frac{1}{g_\beta(n)+r}$. Here we use $n > 0$. We thus have to give bounds for

$$\frac{g_\beta(n+1)+r}{g_\beta(n)+r} = 1 + \frac{g_\beta(n+1)-g_\beta(n)}{g_\beta(n)+r}.$$

A bound from above is obtained when $r = 0$ and a bound from below is obtained for $r = (1+\frac{1}{1+a})^\alpha - 1$. Since $-2\log$ decreases, this finally reverses the order.

For w satisfying $(1^m\ 0^\infty) \le w < (1^m\ 01^\infty)$ and $m > 1$ we first recall that we have $\theta(w) = \frac{1}{\frac{1}{2^\beta-1}+\frac{1}{g_\alpha(m+a)+r}}$. Hence,

$$2^\beta - 1 - \theta(w) = \frac{1}{\frac{1}{2^\beta-1}} - \theta(w) = \frac{(2^\beta-1)^2}{g_\alpha(m+a)+r+2^\beta-1}.$$

For $m > 1$, we want to give bounds for

$$\frac{g_\alpha(m+a)+r+2^\beta-1}{g_\alpha(m-1+a)+r+2^\beta-1} = 1 + \frac{g_\alpha(m+a)-g_\alpha(m-1+a)}{g_\alpha(m-1+a)+r+2^\beta-1}. \tag{7.1}$$

Note that the last expression shows that the function is decreasing in r. Again, a bound from above is obtained for $r = 0$ and a bound from below for $r = 2^\beta - 1$ (remember that we have to compose with $-2\log$).

If $m = 1$, we want to give bounds for

$$\frac{(2^\beta - 1 - r)}{(2^\beta - 1)^2}(g_\alpha(1+a) + r + 2^\beta - 1).$$

This is a decreasing function of r on the interval $[0, (\frac{3}{2})^\beta - 1]$. □

7.1.3. *Choices for parameters α, β and a*

We first check that conditions (6.1) are compatible with our assumptions $\alpha \geq 1$ and $\beta \leq 1$. Remember that (6.1) means:

$$\frac{1}{\left(\frac{3}{2}\right)^\beta - 1} - \frac{1}{2^\beta - 1} = \left(1 + \frac{1}{a+1}\right)^\alpha - 1. \quad (6.1a)$$

$$\frac{1}{\alpha} = 2^\beta - 1. \quad (6.1b)$$

Note that $\beta \leq 1$ yields $2^\beta - 1 \leq 1$, and then, we actually have $\alpha \geq 1$.

We now want to solve a (from the two equations) as a function of β. For this we have to consider the map

$$\beta \mapsto a(\beta) + 1 := \frac{1}{\left(\frac{1}{(\frac{3}{2})^\beta - 1} - \frac{1}{2^\beta - 1} + 1\right)^{2^\beta - 1} - 1}.$$

Lemma 7.4. *The map $A : x \to \frac{1}{(\frac{1}{(\frac{3}{2})^x - 1} - \frac{1}{2^x - 1} + 1)^{2^x - 1} - 1} - 1$ is a decreasing bijection from $]0,1[$ onto $]1, +\infty]$.*

Proof. We first prove that the function A is one-to-one.

Let us pick some $a > 0$, and set $C := 1 + \frac{1}{1+a}$. Note that C belongs to the interval $]1, 2[$.

We set $\varphi(x) = C^{\frac{1}{2^x - 1}} - 1 - \frac{1}{(\frac{3}{2})^x - 1} + \frac{1}{2^x - 1}$. Hence we have

$$A(x) = a \Leftrightarrow \varphi(x) = 0.$$

We thus want to prove that there exists a unique x in $]0, 1[$ such that $\varphi(x) = 0$. Note that $\varphi(1) = C - 2 < 0$. Moreover $\frac{1}{2^x - 1} = \frac{1}{x \log 2} + o(\frac{1}{x})$ close to 0.

Therefore for x close to 0 we have $\varphi(x) = e^{\log C(\frac{1}{x\log 2} + o(\frac{1}{x}))} - 1 - \frac{1}{x\log(\frac{3}{2})} + \frac{1}{x \log 2} + o(\frac{1}{x})$. This yields

$$\lim_{x \to 0^+} \varphi(x) = +\infty.$$

As the function is continuous on the interval $]0, 1]$, there exists at least one x such that $\varphi(x) = 0$. We thus want to prove the uniqueness of this solution.

Claim 1. *The function φ is either decreasing on $]0,1]$ or there exists $c\in]0,1[$ such that φ is decreasing on $]0,c[$ and increasing on $]c,1[$.*

We first explain why Claim 1 gives our result: indeed, the variations of φ and the fact that $\varphi(1) < 0$ imply that there can be at most one solution for the equation

$$\varphi(x) = 0.$$

Now, we prove Claim 1. Note that φ is C^∞ and we have

$$\varphi'(x) = \frac{\log(\frac{3}{2})\,(\frac{3}{2})^x}{\left((\frac{3}{2})^x - 1\right)^2} - \frac{\log 2\, 2^x}{(2^x-1)^2}(1 + \log Ce^{\frac{1}{2^x-1}\log C}).$$

Thus we want to know where we have

$$\frac{\log(\frac{3}{2})(2^x-1)^2(\frac{3}{2})^x}{\log 2((\frac{3}{2})^x-1)^2 2^x} \leq 1 + \log Ce^{\frac{1}{2^x-1}\log C}. \tag{7.2}$$

Claim 2. *The function $x \mapsto 1 + \log Ce^{\frac{1}{2^x-1}\log C}$ is decreasing.*

Indeed, $x \mapsto \frac{1}{2^x-1}$ is decreasing, $x \mapsto e^x$ is increasing and C is larger than 1.

Claim 3. *The function $x \mapsto \frac{(2^x-1)^2(\frac{3}{2})^x}{((\frac{3}{2})^x-1)^2 2^x}$ is increasing.*

We first explain how these two claims prove that Claim 1 is correct. Note that for $x = 1$

$$\frac{(2^x-1)^2(\frac{3}{2})^x}{((\frac{3}{2})^x-1)^2 2^x} = 3.$$

On the other hand, note that $1+\log Ce^{\frac{1}{2^x-1}\log C} \sim \log Ce^{\frac{\log C}{x\log 2}}$ for x close to 0 and then

$$\lim_{x\to 0^+} 1 + \log Ce^{\frac{1}{2^x-1}\log C} = +\infty.$$

We remind that $\sim$ means that the quotient goes to 1. Hence, Claims 2 and 3 yield that there exists at most one real number $c\in]0,1]$ such that for

$$\frac{\log(\frac{3}{2})(2^x-1)^2(\frac{3}{2})^x}{\log 2((\frac{3}{2})^x-1)^2 2^x} - (1 + \log Ce^{\frac{1}{2^x-1}\log C})$$

is negative for $x < c$ and positive for $x > c$. If such a c exist, φ is decreasing on $[0,c[$ and increasing on $]c,1]$. If c does not exist, then (7.2) holds for every $x\in]0,1]$. This proves that Claim 1 is correct.

We now prove Claim 3. It is sufficient to prove that $\psi := x \mapsto \log \frac{(2^x-1)^2(\frac{3}{2})^x}{((\frac{3}{2})^x-1)^2 2^x}$ increases. Equivalently, we want to prove that ψ' is positive on $]0,1]$. We have

$$\psi'(x) = \frac{2\log 2\, 2^x}{2^x-1} - \frac{2\log(\frac{3}{2})\,(\frac{3}{2})^x}{(\frac{3}{2})^x-1} - \log 2 + \log\left(\frac{3}{2}\right).$$

Hence $\psi'(x) > 0$ is equivalent to

$$\log 2\left(\frac{2\,2^x}{2^x-1} - 1\right) > \log\left(\frac{3}{2}\right)\left(\frac{2(\frac{3}{2})^x}{(\frac{3}{2})^x-1} - 1\right)$$

$$\Updownarrow$$

$$\log 2 \frac{2^x+1}{2^x-1} > \log\left(\frac{3}{2}\right)\frac{(\frac{3}{2})^x+1}{(\frac{3}{2})^x-1}$$

$$\Updownarrow$$

$$\frac{x\log 2}{2}\left(\frac{e^{x\log 2}+1}{e^{x\log 2}-1}\right) > \frac{x\log(\frac{3}{2})}{2}\left(\frac{e^{x\log(\frac{3}{2})}+1}{e^{x\log(\frac{3}{2})}-1}\right) \tag{7.3}$$

$$\Updownarrow$$

$$\frac{x\log 2}{2}\left(\frac{e^{\frac{x\log 2}{2}}+e^{-\frac{x\log 2}{2}}}{e^{\frac{x\log 2}{2}}-e^{-\frac{x\log 2}{2}}}\right) > \frac{x\log(\frac{3}{2})}{2}\left(\frac{e^{\frac{x\log(\frac{3}{2})}{2}}+e^{-\frac{x\log(\frac{3}{2})}{2}}}{e^{\frac{x\log(\frac{3}{2})}{2}}-e^{-\frac{x\log(\frac{3}{2})}{2}}}\right)$$

$$\Updownarrow$$

$$\frac{x\log 2}{2}\coth\frac{x\log 2}{2} > \frac{x\log(\frac{3}{2})}{2}\coth\frac{x\log(\frac{3}{2})}{2}.$$

Now, we leave it to the reader to check that the function $x \mapsto x\coth x$ is increasing on $\mathbb{R}_+$. Therefore (7.3) holds and Claim 3 is correct. This finishes the proof that the function A is one-to-one.

We let the reader check that, close to 1, we have $A(x) = (x-1)(-2\log 2 - 4\log^2(2) - 6\log\frac{3}{2}) + O((x-1)^2)$.

On the other hand, close to 0 we have $A(x) = \frac{\frac{1}{\log\frac{3}{2}} - \frac{1}{\log 2}}{x\log x\log 2}[1+\kappa\cdot\frac{1}{\log x}] + O(1)$.

The function A is one-to-one and the limits on the boundaries yield that it is a decreasing bijection from $]0,1[$ on its image $]0,+\infty[$. □

From the lemma above, we get the property that each positive integer value of a can be reached. In this way, several renormalization operators, with different values $a \in \mathbb{N}$, can be considered in our future reasoning. For each such value a, we have the corresponding values α_a and β_a. We point out, however, that it also has meaning to consider real values of a (any positive real is possible) in several of our results (which are not related to the renormalization operator for the shift).

7.2. *Local thermodynamic formalism*

7.2.1. *Distortion on cylinders*

We recall that the potential ϕ_β is defined as follows:

$$\phi_\beta(x) = \begin{cases} -2\log\left(\dfrac{\theta_\beta\circ\sigma(x)}{\theta_\beta(x)}\right) & \text{if } x\in[0],\\ -2\log\left(\dfrac{2^\beta-1-\theta_\beta\circ\sigma(x)}{2^\beta-1-\theta_\beta(x)}\right) & \text{if } x\in[1].\end{cases}$$

The theory of equilibrium states has been developed for many types of dynamics and different kinds of potentials. It is however noteworthy that in every case, one of the main points is to control the distortion of Birkhoff sum of the potential on cylinders.

Proposition 7.1. *There exists a positive real number $\mathfrak{A}$ such that for every k in $\mathbb{N}^*$, for every w and w' in $01^{m_1}0^{n_1}1^{m_2}0^{n_2}\cdots1^{m_k}0^{n_k}1$ (with $0<m_i,n_i<+\infty$) and for every β,*

$$|S_{|\mathbf{m}|+|\mathbf{n}|}(\phi_\beta)(w)-S_{|\mathbf{m}+\mathbf{n}|}(\phi_\beta)(w')|\le\mathfrak{A},$$

where $|\mathbf{m}|+|\mathbf{n}|:=\sum_i m_i+n_i$.

Proof. The proof consists of three steps. In the first step we recall some simple analytical facts. In the second step we do explicit computations to get an upper bound for the difference $S_{|\mathbf{m}|+|\mathbf{n}|}(\phi_\beta)(w)-S_{|\mathbf{m}+\mathbf{n}|}(\phi_\beta)(w')$. In the last step we show that all these upper bounds (depending on w, w', k and β) are uniformly bounded from above. This shall gives a value for $\mathfrak{A}$.

Some usual analysis arguments. Given $A_0>0$, R_1 and R_2 non-negative, then

$$\left|\frac{1}{A_0+R_1}-\frac{1}{A_0+R_2}\right|\le\frac{|R_2-R_1|}{A_0^2}.$$

Repeated use of this fact yields (assuming that any A_i is positive)

$$\left|\frac{1}{A_0+\frac{1}{A_1+\cdots+\frac{1}{A_n+R_1}}}-\frac{1}{A_0+\frac{1}{A_1+\cdots+\frac{1}{A_n+R_2}}}\right|\le\frac{1}{A_0^2A_1^2\cdots A_n^2}|R_2-R_1|. \tag{7.4}$$

We shall use several times estimates of the form

$$\left|\log\frac{X}{Y}\right|=|\log X-\log Y|\le\frac{1}{\min(X,Y)}|X-Y|, \tag{7.5}$$

with X and Y positive real numbers. In particular, to get a bound from above for (7.5) we need to get a bound from below for X and Y.

Computation. We set

$$w = (01^{m_1}0^{n_1}1^{m_2}0^{n_2}\cdots 1^{m_k}0^{n_k}1W)$$

and

$$w' = (01^{m_1}0^{n_1}1^{m_2}0^{n_2}\cdots 1^{m_k}0^{n_k}1W').$$

We want to estimate

$$|\Delta S_{\mathbf{m}+\mathbf{n}}| := |S_{m_1+n_1+m_2+n_2+\cdots+m_k+n_k}(\phi_\beta)(w) \\ - S_{m_1+n_1+m_2+n_2+\cdots+m_k+n_k}(\phi_\beta)(w')|.$$

For simplicity we drop the indices β. Note that we have

$$\begin{aligned} S_{\mathbf{m}+\mathbf{n}}(\phi)(w) = {} & \underbrace{\phi(w)}_{\text{word starting with initial 0}} + \underbrace{S_{m_1}(\phi)(\sigma(w))}_{\text{first series of 1's}} + \underbrace{S_{n_1}(\phi)(\sigma^{m_1+1}(w))}_{\text{first series of 0's}} \\ & + \underbrace{S_{m_2}(\phi)(\sigma^{m_1+n_1+1}(w))}_{\text{second series of 1's}} + \underbrace{S_{n_2-}(\phi)(\sigma^{m_1+n_1+m_2+1}(w))}_{\text{second series of 0's}} + \cdots \\ & + \underbrace{S_{m_k}(\phi)(\sigma^{m_1+n_1+\cdots+m_{k-1}+n_{k-1}+1}(w))}_{\text{last series of 1's}} \\ & + \underbrace{S_{n_k-1}(\phi)(\sigma^{m_1+n_1+\cdots+m_{k-1}+n_{k-1}+m_k+1}(w))}_{\text{last series of 0's}}. \end{aligned}$$

Due to the definition of ϕ we thus get

$$\begin{aligned} S_{\mathbf{m}+\mathbf{n}}(\phi)(w) = {} & 2\log\frac{\theta\circ\sigma(w)}{\theta(w)} + 2\log\frac{2^\beta - 1 - \theta\circ\sigma^{m_1+1}(w)}{2^\beta - 1 - \theta\circ\sigma(w)} \\ & + 2\log\frac{\theta\circ\sigma^{m_1+n_1+1}(w)}{\theta\circ\sigma^{m_1+1}(w)} + 2\log\frac{2^\beta - 1 - \theta\circ\sigma^{m_1+n_1+m_2+1}(w)}{2^\beta - 1 - \theta\circ\sigma^{m_1+n_1+1}(w)} \\ & + 2\log\frac{\theta\circ\sigma^{m_1+n_1+m_2+n_2+1}(w)}{\theta\circ\sigma^{m_1+n_1+m_2+1}(w)} + \cdots \\ & + 2\log\frac{\theta\circ\sigma^{m_1+n_1+\cdots+m_{k-1}+n_{k-1}+1}(w)}{\theta\circ\sigma^{m_1+\cdots+m_{k-1}+1}(w)} \\ & + 2\log\frac{2^\beta - 1 - \theta\circ\sigma^{m_1+n_1+\cdots m_{k-1}+n_{k-1}+m_k+1}(w)}{2^\beta - 1 - \theta\circ\sigma^{m_1+n_1+\cdots+m_{k-1}+n_{k-1}+1}(w)} \\ & + 2\log\frac{\theta\circ\sigma^{m_1+n_1+\cdots+m_k+n_k}(w)}{\theta\circ\sigma^{m_1+n_1+\cdots+m_{k-1}+n_{k-1}+m_k+1}(w)}. \end{aligned}$$

If we just rewrite the first terms we get

$$\begin{aligned} S_{\mathbf{m}+\mathbf{n}}(\phi)(w) = {} & 2\log\frac{\theta\circ\sigma(w)}{\theta(w)} + 2\log\frac{2^\beta - 1 - \theta\circ\sigma^{m_1+1}(w)}{2^\beta - 1 - \theta\circ\sigma(w)} \\ & + 2\log\frac{\theta\circ\sigma^{m_1+n_1+1}(w)}{\theta\circ\sigma^{m_1+1}(w)} + \cdots \end{aligned}$$

$$= 2\log\frac{\theta\circ\sigma(w)}{\theta(w)} + 2\log\frac{2^\beta - 1 - \theta\circ\sigma^{m_1+1}(w)}{2^\beta - 1 - \theta\circ\sigma(w)}$$
$$+ 2\log\frac{\theta\circ\sigma^{m_1+n_1}(w)}{\theta\circ\sigma^{m_1+1}(w)} + 2\log\frac{\theta\circ\sigma^{m_1+n_1+1}(w)}{\theta\circ\sigma^{m_1+n_1}(w)} + \cdots$$

and $\sigma^{m_1+n_1}(w)$ belongs to [01]. Therefore, the terms which involve the first series of 1's and 0's (before the first return in $[0,1]$) are the ones listed in this last equality, except the last one, which already concerns the second series of 1's.

Due to the chain rule, the term in $\sigma^{m_1+n_1}(w)$ finally disappears. Therefore, each pair of series of 1's and 0's will produce four terms to compute:

(1) $2\log\theta\circ\sigma^{\sum_{i\leq j} m_i+n_i+1}(w) - 2\log\theta\circ\sigma^{\sum_{i\leq j} m_i+n_i+1}(w')$,
(2) $2\log(2^\beta - 1 - \theta\circ\sigma^{\sum_{i\leq j} m_i+n_i+1}(w)) - 2\log(2^\beta - 1 - \theta\circ\sigma^{\sum_{i\leq j} m_i+n_i+1}(w'))$,
(3) $2\log(2^\beta - 1 - \theta\circ\sigma^{\sum_{i\leq j} m_i+n_i+m_{j+1}+1}(w)) - 2\log(2^\beta - 1 - \theta\circ\sigma^{\sum_{i\leq j} m_i+n_i+m_{j+1}+1}(w'))$,
(4) $2\log(\theta\circ\sigma^{\sum_{i\leq j} m_i+n_i+m_{j+1}+1}(w)) - 2\log(\theta\circ\sigma^{\sum_{i\leq j} m_i+n_i+m_{j+1}+1}(w'))$.

Terms of the form (1) or (4) are called type A, and terms of the forms (2) and (3) are called type B.

Terms of the forms (1) and (2) deal with points starting with 1's (namely the $j+1$ series of 1's), and terms of the forms (3) and (4) deal with points starting with 0's (namely the $j+1$ series of 0's).

In addition, the initial term in the Birkhoff sum produces the difference $2\log\theta(w) - 2\log\theta(w')$. Similarly, the last one produces the difference $2\log(\theta\circ\sigma^{m_1+n_1+\cdots+m_k+n_k}(w)) - 2\log(\theta\circ\sigma^{m_1+n_1+\cdots+m_k+n_k}(w'))$.

We set

$$\theta(w) = [g_\beta(1), g_\alpha(m_1+a), g_\beta(n_1), \ldots, g_\alpha(m_k+a), g_\beta(n_k), R] \quad \text{and}$$
$$\theta(w') = [g_\beta(1), g_\alpha(m_1+a), g_\beta(n_1), \ldots, g_\alpha(m_k+a), g_\beta(n_k), R'].$$

We recall that $[a_1, a_2, \ldots, a_p]$ means $\frac{1}{a_1+\frac{1}{a_2+\frac{1}{\ddots+\frac{1}{a_p}}}}$.

Using (7.4), (7.5), Lemma 7.3 and also (7.1), we let the reader check that the following inequalities hold:

$$\left|\log\left(\frac{\theta\circ\sigma^{\sum_{i\leq j} m_i+n_i+1}(w)}{\theta\circ\sigma^{\sum_{i\leq j} m_i+n_i+1}(w')}\right)\right|$$
$$\leq \frac{1}{((\frac{3}{2})^\beta - 1)(g_\beta(1))^2}\prod_{j+1}^{k}\frac{1}{(g_\alpha(m_i+a))^2(g_\beta(n_i))^2}|R - R'|, \tag{7.6}$$

$$\left|\log\left(\frac{2^\beta-1-\theta\circ\sigma^{\sum_{i\le j}m_i+n_i+1}(w)}{2^\beta-1-\theta\circ\sigma^{\sum_{i\le j}m_i+n_i+1}(w')}\right)\right|$$

$$\le\frac{2(2^\beta-1)+g_\alpha(m_{j+1}+a)}{(2^\beta-1)^2(g_\beta(1))^2}\prod_{j+1}^{k}\frac{1}{(g_\alpha(m_i+a))^2(g_\beta(n_i))^2}|R-R'|, \tag{7.7}$$

$$\left|\log\left(\frac{2^\beta-1-\theta\circ\sigma^{\sum_{i\le j}m_i+n_i+m_{j+1}+1}(w)}{2^\beta-1-\theta\circ\sigma^{\sum_{i\le j}m_i+n_i+m_{j+1}+1}(w')}\right)\right|$$

$$\le\frac{1}{(2^\beta-(\frac{3}{2})^\beta)(g_\beta(n_{j+1}))^2}\prod_{j+2}^{k}\frac{1}{(g_\alpha(m_i+a))^2(g_\beta(n_i))^2}|R-R'|, \tag{7.8}$$

$$\left|\log\left(\frac{\theta\circ\sigma^{\sum_{i\le j}m_i+n_i+m_{j+1}+1}(w)}{\theta\circ\sigma^{\sum_{i\le j}m_i+n_i+m_{j+1}+1}(w')}\right)\right|$$

$$\le\frac{\frac{1}{(\frac{3}{2})^\beta-1}-\frac{1}{2^\beta-1}+g_\beta(n_{j+1})}{(g_\beta(n_{j+1}))^2}\prod_{j+2}^{k}\frac{1}{(g_\alpha(m_i+a))^2(g_\beta(n_i))^2}|R-R'|. \tag{7.9}$$

Let us set

$$U_1:=\frac{1}{(\frac{3}{2})^\beta-1}-\frac{1}{2^\beta-1}+\frac{1}{2^\beta-(\frac{3}{2})^\beta},$$

$$U_2:=\frac{1}{(2^\beta-1)^2(g_\beta(1))^2},$$

$$U_3:=\frac{1}{((\frac{3}{2})^\beta-1)^2(g_\beta(1))^2}+\frac{2}{(2^\beta-1)(g_\beta(1))^2}.$$

Then, summing term by term the four inequalities (7.6), (7.7), (7.8), (7.9), the left-hand side term is the contribution of the $(j+1)$th series of 1's and 0's, and the right-hand side term can be written as

$$\frac{1}{g_\beta(n_{j+1})}\left[1+\frac{1}{g_\beta(n_{j+1})}\left[U_1+\frac{1}{g_\alpha(m_{j+1}+a)}\left[U_2+\frac{1}{g_\alpha(m_{j+1}+a)}U_3\right]\right]\right]$$

$$\times\prod_{j+2}^{k}\frac{1}{(g_\alpha(m_i+a)g_\beta(n_i))^2}|R-R'|. \tag{7.10}$$

Note that the contribution of the jth series of 1's and 0's has the same bound from above, exchanging $j+1$ by j. It thus has an extra factor

$$\frac{1}{(g_\alpha(m_j+a)g_\beta(n_j))^2}.$$

If we want to add the contribution of the jth series of 1's and 0's to the contribution of the $(j+1)$th series, the expression of the upper bound can be factorized. The common term is

$$\prod_{j+2}^{k} \frac{1}{(g_\alpha(m_i+a)g_\beta(n_i))^2}|R-R'|.$$

Thus, the introduction into (7.10) of the upper bound for the jth series, has to be done by replacing the term U_3 by:

$$U_3 \to U_3 + \frac{1}{g_\beta(n_j)}\left[1+\frac{1}{g_\beta(n_j)}\left[U_1+\frac{1}{g_\alpha(m_j+a)}\left[U_2+\frac{1}{g_\alpha(m_j+a)}U_3\right]\right]\right].$$

Let us set

$$F(t) := U_3 + t, \quad G_{m,1}(t) := \frac{t}{g_\alpha(m+a)} + U_1, \quad G_{m,2}(t) := \frac{t}{g_\alpha(m+a)} + U_2,$$

$$G_{n,1}(t) := \frac{t}{g_\beta(n)} + 1, \quad G_{n,2}(t) := \frac{t}{g_\beta(n)},$$

and then $H_i = H_{m_i,n_i} = G_{n_i,2} \circ G_{n_i,1} \circ G_{m_i,1} \circ G_{m_i,2} \circ F$.

With these notations, we finally get

$$|\Delta S_{\mathbf{m}+\mathbf{n}}| \le [A + H_k \circ H_{k-1} \circ \cdots \circ H_2 \circ H_1(A)]|R-R'|, \tag{7.11}$$

where $A = (\frac{2}{(\frac{3}{2})^\beta - 1} - \frac{1}{2^\beta - 1})\frac{1}{(g_\beta(2))^2} > 0$ is an upper bound for both terms, the initial one, due to $|\log \frac{\theta(w)}{\theta(w')}|$ and the last one due to $|\log(\frac{\theta\circ\sigma^{m_1+n_1+\cdots+m_k+n_k}(w)}{\theta\circ\sigma^{m_1+n_1+\cdots+m_k+n_k}(w')})|$.

Obviously, $|R-R'| \le 2^\beta - 1$. Moreover, all the $G_{n,i}$ and U_j are positive real numbers, and

$$\frac{1}{g_\beta(n)} \le \left(\frac{3}{2}\right)^\beta - 1, \quad \frac{1}{g_\alpha(m+a)} \le \left(1+\frac{1}{1+a}\right)^\alpha - 1.$$

This shows that there exists $F_\beta : x \mapsto b(\beta)x + c(\beta)$ such that for every i and for every $x \ge 0$

$$0 \le H_i(x) \le F_\beta(x).$$

Hence, (7.11) yields

$$|\Delta S_{\mathbf{m}+\mathbf{n}}| \le (A + F_\beta^k(A))(2^\beta - 1). \tag{7.12}$$

It thus remains to prove that the sequence $(F_\beta^k(A))$ is bounded from above. This point is a simple consequence of the computation of $b(\beta)$:

$$b(\beta) = \left(1 - \frac{(\frac{3}{2})^\beta - 1}{2^\beta - 1}\right)^2 < 1,$$

hence the sequence $(F_\beta^k(A))$ is bounded (depending on β).

End of the proof. Existence and value for $\mathfrak{A}$. Now, we prove that the terms of the form $(F_\beta^k(A))$ are uniformly bounded from above in k and also in β.

First, remember that

$$A = \left(\frac{2}{(\frac{3}{2})^\beta - 1} - \frac{1}{2^\beta - 1}\right)\frac{1}{(g_\beta(2))^2} \leq \left(\frac{2}{(\frac{3}{2})^\beta - 1} - \frac{1}{2^\beta - 1}\right)\left(\left(\frac{3}{2}\right)^\beta - 1\right)^2.$$

For β close to 0, $a^\beta - 1 = \beta \log a + o(\beta)$. Thus, the family of terms $A = A(\beta)$ is uniformly bounded from above.

Arithmetic–geometric sequences are well known, and we have for every β and every k,

$$\left|F_\beta^k(A) - \frac{c(\beta)}{1 - b(\beta)}\right| \leq b^k(\beta)\left|A - \frac{c(\beta)}{1 - b(\beta)}\right|.$$

We have already seen that A is uniformly bounded (in β). It thus remains to check that this also holds for $\frac{c(\beta)}{1-b(\beta)}$. Note that $b(\beta)$ is uniformly bounded away from 1 (in β), hence, it is sufficient to prove that $c(\beta)$ is uniformly bounded from above in β. As everything is continuous, it holds in any compact in $]0, 1]$. The problem is when β goes to 0.

Note that for β close to 0, $\frac{1}{g_\beta(n)} \sim O(\beta)$, $\frac{1}{g_\alpha(m)} \sim O(\frac{1}{\beta})$, $U_1 \sim O(\frac{1}{\beta})$, $U_2 \sim O(1)$, $U_3 \sim O(\beta)$, and $A \sim O(\beta)$. This yields

$$\begin{aligned} O(1) \stackrel{F}{\longleftarrow} U_3 + (A) = O(\beta) &\stackrel{G_{m,2}}{\longleftarrow} \frac{\beta}{\beta} + 1 = O(1) \stackrel{G_{m,1}}{\longleftarrow} \frac{1}{\beta} + \frac{1}{\beta} \\ &= O(1/\beta) \stackrel{G_{n,1}}{\longleftarrow} \frac{\beta}{\beta} + 1 = O(1) \stackrel{G_{n,2}}{\longleftarrow} O(\beta). \end{aligned}$$

This proves that $c(\beta)$ goes to 0 as β goes to 0, thus the terms $F_\beta^k(A)$ are all uniformly bounded from above in k and β.

We can thus find some $\mathfrak{A}$ such that for every k, for every β and for every w and w' coinciding for $\sum_{i \leq k} m_i + n_i$

$$|\Delta S_{\mathbf{m}+\mathbf{n}}| \leq \mathfrak{A}.$$

This finishes the proof of the proposition. □

7.2.2. *Proof of Proposition* 6.1

We study the existence and uniqueness of equilibrium state for the system $([01], g)$ and the potential $\gamma \cdot S_{r(\cdot)}(\phi_\beta) - Z \cdot r(\cdot)$. Moreover, we prove that for every $Z > Z_c$, the unique equilibrium state μ_Z is the restriction and renormalization to the cylinder $[01]$ of some σ-invariant probability measure $\widehat{\mu}_Z$.

This section deeply relies on results from [15] and developed in further later works of Leplaideur.

Lemma 7.5. *For every x and x' in* $[01]$

$$\limsup_{n\to+\infty}\frac{1}{n}\log\left(\sum_{x=g(y),r(y)=n}e^{S_n(\gamma\phi_\beta)(y)}\right)$$
$$=\limsup_{n\to+\infty}\frac{1}{n}\log\left(\sum_{x'=g(y'),r(y')=n}e^{S_n(\gamma\phi_\beta)(y')}\right).$$

Proof. For any $k+m=n$, if $y=[01^k0^mx]$ is a preimage of x, then $y'=[01^k0^mx']$ is a preimage for x'. Then Proposition 7.1 shows that

$$|S_{n+1}(\gamma\phi_\beta)(y')-S_{n+1}(\gamma\phi_\beta)(y)|\le\gamma\mathfrak{A}.$$ □

A direct consequence of this lemma is that Z_c is independent of the choice of the initial point x. Moreover, Proposition 7.1 means that the potential $\gamma\cdot S_{r(\cdot)}(\phi_\beta)-Z\cdot r(\cdot)$ satisfies the Bowen property (see [25]). It is then well known that this property is the minimal one to use the ordinary operator theory and to build equilibrium states (see e.g., [1]). In our case we also have to check that the operator is well-defined, namely that the series converges (for every continuous function ψ). This holds as soon as $Z>Z_c$.

From there, and for every $Z>Z_c$, the spectral radius λ_Z of the two adjoint operators $\mathcal{L}_Z$ and $\mathcal{L}_Z^*$ is a simple and dominating eigenvalue. If ν_Z is the associated eigen-measure and if h_Z is the associated eigenfunction (characterized by $\int_{[01]}h_Zd\nu_Z=1$), then the measure μ_Z defined by

$$d\mu_Z:=h_Zd\nu_Z,$$

is the unique equilibrium state associated to $S_{r(\cdot)}(\gamma\phi_\beta)(\cdot)-Zr(\cdot)$ for the dynamical system $(01,g)$ (see Propositions 4.5, 4.8, 5.7 and 5.9 in [15]). The pressure of the equilibrium state is $\log\lambda_Z$ (see Proposition 5.9 in [15]). By construction, the eigen-measure ν_Z is a conformal measure for g and $\gamma\cdot S_{r(\cdot)}(\phi_\beta)-Zr(\cdot)$.

Now, we want to "open out" these measures. It is still true for every $Z>Z_c$ that there exists a unique σ-invariant probability measure $\widehat{\mu}_Z$ such that its restriction (correctly normalized) to the cylinder $[01]$ is the measure μ_Z (see Proposition 6.8 in [15]). Then, a simple computation gives (see again Proposition 6.8 in [15])

$$h_{\widehat{\mu}_Z}(\sigma)+\int\gamma\phi_\beta d\widehat{\mu}_Z=Z+\widehat{\mu}_Z([01])\log\lambda_Z.\tag{7.13}$$

7.2.3. *Proof of Proposition* 6.2

We shall see later that $Z_c=0$ (Proposition 7.2). Now, the potential ϕ_β satisfies $\phi_\beta(0^\infty)=\phi_\beta(1^\infty)=0$. Hence the pressure (for the global system) is bigger than

$$\mathcal{P}(\gamma,\beta)\ge h_{\delta_{0^\infty}}(\sigma)+\gamma\cdot\phi_\beta(0^\infty)=0.$$

In other words, we get $Z_c\le\mathcal{P}(\gamma,\beta)$.

Let us assume that there exists $Z_0 > Z_c$ such that $\lambda_{Z_0} = 0$. We prove now that necessarily, $Z_0 = \mathcal{P}(\gamma, \beta)$ and $\widehat{\mu}_{Z_0}$ is the unique equilibrium state for $\gamma \cdot \phi_\beta$.

First, for every $Z > \mathcal{P}(\gamma, \beta) \geq 0 = Z_c$, equality (7.13) shows that $\lambda_Z \leq 1$. For $Z = Z_0$, the equality (7.13) also shows

$$\mathcal{P}(\gamma, \beta) \geq h_{\widehat{\mu}_{Z_0}}(\sigma) + \int \gamma\phi_\beta d\widehat{\mu}_{Z_0} = Z_0 > 0.$$

Now, the potential $\gamma \cdot \phi_\beta$ is continuous, and the variational principle shows that there exists an equilibrium state for the global system and the potential $\gamma \cdot \phi_\beta$. We have just seen that it has positive pressure, hence neither the Dirac measure δ_{0^∞} nor the Dirac measure δ_{1^∞} can be equilibrium states for $\gamma \cdot \phi$. Moreover, $\mathcal{P}(\gamma, \beta) > 0$ also means that we have measures μ_Z and $\widehat{\mu}_Z$ for $Z = \mathcal{P}(\gamma, \beta)$.

Due to the dynamics in Σ, every invariant probability different from δ_{0^∞} and δ_{1^∞} gives positive weight to the cylinder [01], and can thus be induced on this cylinder. Let $\widehat{\mu}$ be a σ-invariant probability, and let μ be its renormalized restriction to [01]. Remember that $\int r d\mu = \frac{1}{\widehat{\mu}([01])}$. Then we have

$$h_{\widehat{\mu}}(\sigma) + \int \gamma\phi_\beta d\widehat{\mu} - \mathcal{P}(\gamma, \beta) \leq 0$$

$$\Updownarrow$$

$$\widehat{\mu}([01]) \left(h_\mu(g) + \int \gamma \cdot S_{r(x)}(\phi_\beta)(x) - \mathcal{P}(\gamma, \beta) r(x) d\mu(x) \right) \leq 0,$$

with equality if and only if $\widehat{\mu}$ is an equilibrium state. The last inequality yields that the pressure for the local system and for $Z = \mathcal{P}(\gamma, \beta)$ is non-negative. It is known (see [3]) that $Z \mapsto \lambda_Z$ is decreasing. Then, $\log \lambda_{\mathcal{P}(\gamma,\beta)} \geq 0 = \log \lambda_{Z_0}$ yields $\mathcal{P}(\gamma, \beta) \leq Z_0$. As we already had $Z_0 \leq \mathcal{P}(\gamma, \beta)$, we finally have equality. Uniqueness of the equilibrium state follows from uniqueness of the equilibrium state for the local system.

The fact that $\widehat{\mu}_Z$ is a quasi-conformal measure follows from the fact that μ_Z is equivalent to ν_Z. We leave to the reader to check that ν_Z can be extended in a unique way as a conformal measure.

7.3. *End of the proof*

Proposition 7.2. *For every γ and β we have $Z_c(\gamma, \beta) = 0$.*

Proof. We recall that the transfer operator is defined by

$$\mathcal{L}_{Z,\gamma,\beta}(\mathbb{1}_{[01]})(w) := \sum_{v \in [01], g(v) = w} e^{S_{r(v)}(\gamma\phi_\beta)(v) - r(v)Z}, \quad w \in [01].$$

The point v is of the form $v = 01^m0^{n-1}w$. In that case we have $r(v) = 1 + m + n - 1 + 1 = m + n + 1$. Therefore we get

$$\mathcal{L}_{Z,\gamma,\beta}(\mathbb{1}_{[01]})(w) = \sum_{n \geq 1} \sum_{m \geq 1} e^{S_{1+m+n}(\gamma\phi_\beta)(01^m0^{n-1}w) - (n+m+1)Z}.$$

Now we have

$$S_{1+m+n}(\phi_\beta)(01^m0^{n-1}w) = \phi_\beta(01^m0^{n-1}w) + S_m(\phi_\beta)(1^m0^{n-1}w) + \phi_\beta(10^n w) + S_{n-1}(\phi_\beta)(0^{n-1}w),$$

and using Lemma 7.3 we get for $m \geq 1$

$$\begin{aligned} S_{1+m+n}(\phi_\beta)(01^m0^{n-1}w) &\geq a_1 + \sum_{k=1}^{m} u_k + \sum_{k=2}^{n-1} a_k = \sum_{k=1}^{n-1} a_k + \sum_{k=1}^{m} u_k \\ &\geq -2\log(g_\beta(n+1)) - 2\log\left(\frac{g_\alpha(m+a)+2^\beta-1}{g_\alpha(1+a)+2^\beta-1}\right) \\ &\quad + 2\log g_\beta(1) - 2\log\left(1+\frac{(\frac{3}{2})^\beta-1}{2^\beta-(\frac{3}{2})^\beta}\right), \end{aligned} \tag{7.14}$$

$$\begin{aligned} S_{1+m+n}(\phi_\beta)(01^m0^{n-1}w) &\leq \sum_{k=1}^{n-1} b_k + \sum_{k=1}^{m} v_k \\ &\leq -2\log\left(g_\beta(n+1) + \left(1+\frac{1}{1+a}\right)^\alpha - 1\right) \\ &\quad - 2\log\left(\frac{g_\alpha(m+a)+2(2^\beta-1)}{g_\alpha(1+a)+2(2^\beta-1)}\right) \\ &\quad + 2\log\left(g_\beta(1) + \left(1+\frac{1}{1+a}\right)^\alpha - 1\right) + v_1. \end{aligned} \tag{7.15}$$

Let us set

$$A(\beta) := \frac{g_\beta(1)}{\left(1+\frac{(\frac{3}{2})^\beta-1}{2^\beta-(\frac{3}{2})^\beta}\right)} = \frac{1}{(2^\beta-1)\left(1+\frac{(\frac{3}{2})^\beta-1}{2^\beta-(\frac{3}{2})^\beta}\right)} > 0,$$

$$B(\beta) := \frac{1}{\left(1+((\frac{3}{2})^\beta-1)\frac{2.2^\beta-(\frac{3}{2})^\beta-1}{(2^\beta-(\frac{3}{2})^\beta)(2^\beta-1)}\right)\left(\frac{2^\beta-(\frac{3}{2})^\beta}{2^\beta-1}\right)}\frac{1}{(\frac{3}{2})^\beta-1} > 0.$$

Then, (7.14) and (7.15) yield

$$\begin{aligned} &\mathcal{L}_{Z,\gamma,\beta}(\mathbb{1}_{[01]})(w) \\ &\quad \geq A^{2\gamma}(\beta)e^{-Z}\left[\sum_{n=1}^{+\infty}\left(\left(1+\frac{1}{n+1}\right)^\beta - 1\right)^{2\gamma} e^{-nZ}\right] \\ &\quad\quad \times \left[1 + \left(1+\frac{(\frac{3}{2})^\beta-1}{2^\beta-(\frac{3}{2})^\beta}\right)^{2\gamma}\sum_{m=2}^{+\infty}\left(\frac{\left(1+\frac{1}{m+a}\right)^\alpha-1}{\frac{1}{2^\beta-1}+\left(1+\frac{1}{m+a}\right)^\alpha-1}\right)^{2\gamma} e^{-mZ}\right], \end{aligned} \tag{7.16}$$

$$\begin{aligned}&\mathcal{L}_{Z,\gamma,\beta}(\mathbb{1}_{[01]})(w)\\ &\quad\le B^{2\gamma}(\beta)e^{-Z}\left[\sum_{n=1}^{+\infty}\left(\frac{\left(1+\frac{1}{n+1}\right)^{\beta}-1}{1+\frac{\left(1+\frac{1}{n+1}\right)^{\beta}-1}{g_\alpha(1)}}\right)^{2\gamma}e^{-nZ}\right]\\ &\qquad\times\left[1+\left(2+\frac{(\frac{3}{2})^{\beta}-1}{2^{\beta}-(\frac{3}{2})^{\beta}}\right)^{2\gamma}\sum_{m=2}^{+\infty}\left(\frac{\left(1+\frac{1}{m+a}\right)^{\alpha}-1}{\frac{1}{2^{\beta}-1}+2\left(1+\frac{1}{m+a}\right)^{\alpha}-2}\right)^{2\gamma}e^{-mZ}\right].\end{aligned} \tag{7.17}$$

Now, the four series in (7.16) and (7.17) have a general term equivalent to $\frac{1}{n^{2\gamma}}e^{-nZ}$ or $\frac{1}{m^{2\gamma}}e^{-mZ}$ when n or m go to $+\infty$. Hence, we get $Z_c = 0$ and the proposition is proved. □

Proposition 7.3. *For any $\gamma \le \frac{1}{2}$, for any $\beta \le 1$ and for any w in* 01 *we have*

$$\lim_{Z\downarrow 0}\lambda_Z = +\infty.$$

For any $\gamma > \frac{1}{2}$, there exists $\beta_c = \beta_c(\gamma)$ such that for any $\beta < \beta_c$ and for any w in 01 *we have*

$$\lim_{Z\downarrow 0}\lambda_Z > 1.$$

Proof. The function $x \mapsto (1+x)^{\beta} - 1 - \frac{\beta}{2}x$ is increasing on the interval $[0, 2^{\frac{1}{1-\beta}} - 1]$. This interval contains $[0,1]$. Therefore, for every $\beta < 1$ and for every $n \ge 1$,

$$\left(1+\frac{1}{n}\right)^{\beta} - 1 \ge \frac{\beta}{2n}.$$

Therefore, we get

$$\begin{aligned}&A^{2\gamma}(\beta)\left[\sum_{n=1}^{+\infty}\left(\left(1+\frac{1}{n+1}\right)^{\beta}-1\right)^{2\gamma}\right]\\ &\qquad\ge\left(\frac{\beta}{2^{\beta}-1}\right)^{2\gamma}\frac{1}{2^{2\gamma}\left(2+\frac{(\frac{3}{2})^{\beta}-1}{2^{\beta}-(\frac{3}{2})^{\beta}}\right)^{2\gamma}}(\zeta(2\gamma)-1),\end{aligned} \tag{7.18}$$

where $\zeta(z) = 1 + 1/2^{z} + 1/3^{z} + \cdots$. Then, if $\gamma \le \frac{1}{2}$, for every w, $\lim_{Z\downarrow 0}\mathcal{L}_{Z,\gamma,\beta}(\mathbb{1}_{[01]})(w) = +\infty$. This proves the proposition for the case $\gamma \le \frac{1}{2}$.

We now deal with the case $\gamma > \frac{1}{2}$.

All the terms from the right-hand side in (7.18) are bounded from below away from 0 when β goes from 0 to 1. This also holds for $\frac{(\frac{3}{2})^{\beta}-1}{2^{\beta}-(\frac{3}{2})^{\beta}}$.

Let us set $H(Z) := \sum_{m=1}^{+\infty}\left(\frac{(1+\frac{1}{m+a+1})^{\alpha}-1}{\frac{1}{2^{\beta}-1}+(1+\frac{1}{m+a+1})^{\alpha}-1}\right)^{2\gamma}e^{-mZ}$. Note that $H(0)$ converges (for fixed β).

Our strategy is to show that $H(0)$ goes to $+\infty$ when β goes to 0. This yields that for every β sufficiently small, say $\beta \leq \beta_c$,

$$\left[1+\left(1+\frac{(\frac{3}{2})^\beta-1}{2^\beta-(\frac{3}{2})^\beta}\right)H(0)\right]\left(\frac{\beta}{2^\beta-1}\right)^{2\gamma}\frac{1}{2^{2\gamma}\left(2+\frac{(\frac{3}{2})^\beta-1}{2^\beta-(\frac{3}{2})^\beta}\right)^{2\gamma}}(\zeta(2\gamma)-1)>1.$$

Then, inequality (7.16) shows that for every w in [01],

$$\lim_{Z\downarrow 0}\mathcal{L}_{Z,\gamma,\beta}(\mathbb{1}_{[01]})(w)>1.$$

In other words, the spectral radius λ_Z is larger than 1 for every Z small enough (and $\beta \leq \beta_c$).

Hence, we now analyze for $\beta > 0$ the function

$$S(\beta,\gamma)=\sum_{m=1}^{\infty}\left(\frac{(1+\frac{1}{m+a+1})^\alpha-1}{(\frac{1}{2^\beta-1}+[(1+\frac{1}{m+a+1})^\alpha-1])}\right)^{2\gamma},$$

for fixed values of β, α, γ.

We remind the reader that when $\beta \to 0$ we have that $\alpha \to \infty$ and $a \to \infty$.

We are interested now in the upper bound.

Note that

$$S(\beta,\gamma)=\sum_{m=1}^{\infty}\left(1-\frac{1}{1+(2^\beta-1)\big[\big(1+\frac{1}{m+a+1}\big)^\alpha-1\big]}\right)^{2\gamma}.$$

Consider

$$u(\alpha,m,a)=\left(1+\frac{1}{m+a+1}\right)^\alpha-1=e^{\alpha\log(1+\frac{1}{m+a+1})}-1.$$

As $\log(x)\geq 1-\frac{1}{x}$, we get

$$u(\alpha,m,a)=e^{\alpha\log(1+\frac{1}{m+a+1})}-1\geq e^{\alpha(1-\frac{1+m+a}{2+m+a})}-1\geq e^{\alpha\frac{1}{2+m+a}}-1.$$

Then,

$$S(\beta,\gamma)\geq\sum_{m=1}^{\infty}\left(1-\frac{1}{1+(2^\beta-1)\big[e^{\alpha(\frac{1}{2+m+a})}-1\big]}\right)^{2\gamma}.$$

From elementary calculus we get that last summation is, up to a multiplicative constant, of the same order as the integral

$$\int_0^\infty\left(1-\frac{1}{1+(2^\beta-1)\big[e^{\alpha(\frac{1}{2+t+a})}-1\big]}\right)^{2\gamma}dt.$$

Consider the change of variable $s=e^{\alpha(\frac{1}{2+t+a})}-1$. Then,

$$ds=-\frac{\alpha}{(2+t+a)^2}e^{\alpha(\frac{1}{2+t+a})}dt=-\frac{1}{\alpha}(s+1)\log^2(s+1)dt.$$

Note that $s \to 0$ as $t \to \infty$, and $s \to e^{\frac{\alpha}{2+a}} - 1$, as $t \to 0$. We claim that $e^{\frac{\alpha}{2+a}} - 1 \sim \frac{C}{\beta}$ as $\beta \to 0$, where C is some universal constant (we remind that $\sim$ means that the quotient goes to 1).

Indeed, $(1 + \frac{1}{1+a})^\alpha - 1 = \frac{1}{(\frac{3}{2})^\beta - 1} - \frac{1}{2^\beta - 1}$ behaves like $\frac{\log(2) - \log(\frac{3}{2})}{\log(2)\log(\frac{3}{2})}\frac{1}{\beta} = \frac{1}{\beta}1.02361\ldots$, as β goes to 0.

As $\alpha\log(1+\frac{1}{1+a}) \sim \frac{\alpha}{1+a}$, when a and α are large, then $e^{\frac{\alpha}{1+a}} - 1 \sim \frac{C}{\beta}$. Finally, from $\frac{\alpha}{2+a} = \frac{\alpha}{1+1+a} = \frac{\alpha}{1+a}\frac{1}{\frac{1}{1+a}+1}$, we get the claim.

We return to our main estimation. After the change of variables we get for some fixed constants $0 < C' < C$

$$\int_0^{\frac{C}{\beta}} \left(1 - \frac{1}{1+(2^\beta-1)s}\right)^{2\gamma} \left[\frac{1}{\alpha}(s+1)\log^2(s+1)\right]^{-1} ds$$
$$\geq \int_{\frac{C'}{\beta}}^{\frac{C}{\beta}} \left(1 - \frac{1}{1+(2^\beta-1)s}\right)^{2\gamma} \left[\frac{1}{\alpha}(s+1)\log^2(s+1)\right]^{-1} ds.$$

For any fixed $0 \leq \gamma \leq 1$, and any s such that $C'/\beta \leq s \leq C/\beta$, the expression

$$\left(1 - \frac{1}{1+(2^\beta-1)s}\right)^{2\gamma},$$

is bounded from above and from below far away from zero (uniformly in β). Therefore, there exists a universal positive constant K such that

$$\int_{\frac{C'}{\beta}}^{\frac{C}{\beta}} \left(1 - \frac{1}{1+(2^\beta-1)s}\right)^{2\gamma} \left[\frac{1}{\alpha}(s+1)\log^2(s+1)\right]^{-1} ds$$
$$\geq K\int_{\frac{C'}{\beta}}^{\frac{C}{\beta}} \left[\frac{1}{\alpha}(s+1)\log^2(s+1)\right]^{-1} ds$$
$$= K\alpha\left[\frac{-1}{\log(s+1)}\right]_{\frac{C'}{\beta}}^{\frac{C}{\beta}} = K\alpha\left[\frac{1}{\log(\frac{C'}{\beta}+1)} - \frac{1}{\log(\frac{C}{\beta}+1)}\right]$$
$$= K\alpha\left[\frac{\log\left(\frac{(\frac{C}{\beta}+1)}{(\frac{C'}{\beta}+1)}\right)}{\log(\frac{C'+\beta}{\beta})\log(\frac{C+\beta}{\beta})}\right]$$
$$\sim K\frac{1}{\beta}\frac{C_3}{(A_1 - \log(\beta))(A_2 - \log(\beta))} \sim \frac{C_4}{\beta\log^2(\beta)},$$

for some constant A_1, A_2, $C_3 > 0$ and $C_4 > 0$ (remember $\alpha = \frac{1}{2^\beta - 1}$). Therefore, for fixed γ, we have that $S(\beta,\gamma) \to \infty$ when $\beta \to 0$. □

Now, Propositions 6.2 and 7.3 prove Theorem C.

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