The Presentation of Lagrange's Equations in Introductory Robotics Courses

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***Abstract*—The topic of Lagrange's dynamic equations is presented in a fashion suitable for introductory robotics courses. The development of the material does not rely on either principles of virtual work or variational calculus. The presentation assumes the students have previously taken an introductory course in dynamics. Depending upon the exact background of the students, this material can be covered in one or two lectures.**

Introduction

THE subject of robotics is a discipline which spans the fields of electrical, industrial, and mechanical engineering. The lecture material in a first course in robotics usually falls into one of two nonexclusive categories. The first category emphasizes a technological point of view and treats the manipulator arm as an integral part of a much larger picture consisting of the manufacturing environment. A course devoted to developing the dynamics, kinematics, and control of the manipulator determines the second category.

The second category has roots primarily in electrical and mechanical engineering. The topics of kinematics, dynamics, and hydraulics are courses well developed in a mechanical engineering curriculum while electrical actuators, continuous and discrete control, noise reduction, and microprocessors with the supporting circuitry are topics of interest to the electrical engineering student. An introductory robotics course of the second category will use one of several popular texts, e.g., Snyder [1], Paul [2], and Craig [3], regardless of who is teaching the class. The emphasis of the material might be weighted according to the department through which the course is offered.

The topics of arm dynamics and especially Lagrange's equations are not easy subjects to present to students whose dynamics background consists solely of the required introductory course. Even if the robotics course is intended to emphasize topics thought to be "exclusively" electrical engineering subjects, the kinematic and dynamic description of the arm must be covered at some point if only to appreciate the control strategies. During a recent workshop concerning robotics instruction [4], several electrical engineering faculty members brought up the question of how to present Lagrange's equations without resorting to virtual work or variational calculus. It is frustrating to the student to merely present the equations with the exclusive justification that the technique works.

This paper contains material presented to a robotics class consisting of mechanical engineering seniors and first-year graduate students, about half of which had only one undergraduate course in dynamics. This is significant since electrical engineering students within the same college have only the same undergraduate dynamics course as their total dynamics background at this point in the curriculum. Thus, an instructor faces the same problem of presenting Lagrange's equations regardless of the students' major department.

The conventional approaches to developing Lagrange's equations consist of virtual work together with D'Alembert's principle and variational calculus together with Hamilton's principle. Examples of these developments are contained in both intermediate and advanced dynamics texts such as Greenwood [5] and Goldstein [6]. By using the principle of virtual work, one is able to demonstrate that the work expression can be manipulated into a form consisting of the sum of products of the virtual displacement of a generalized coordinate and a term which is later shown to be the dynamic equation of motion expressed in terms of the Lagrangian of the system. In the variational derivation, one starts with Hamilton's principle and seeks the stationary point with respect to the generalized coordinates (or state variables) of the time integral of the Lagrangian using the algebra of variational calculus. The equations which constitute the stationary point are seen to be the dynamic equations of motion.

In order to present either of these derivations sufficient background material must be developed and presented to the students. The success of such a demonstration can depend upon the inclination of the students toward this subject area, the amount of time allotted for the development, and the instructor's major field of expertise.

It is the intention of this paper to convince the students that Lagrange's method does work without resorting to virtual work or variational calculus arguments. It is not the claim of the authors that the material to be presented constitutes a rigorous proof. It will, however, present

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enough justification to accept the technique on something more than faith.

This paper consists of four examples of dynamic systems common to both introductory dynamics and robotics courses. In each example, the equations of motion are derived by determining the time rate of change of linear and angular momentum. Having obtained the equations of motion it is then demonstrated that the dynamic equations can be manipulated back into the form of Lagrange's equations. The first three examples each pertain to specific dynamic systems while the last example consists of the rigid body dynamics of a system of n particles which can be generalized into the rigid body dynamics of solids as demonstrated in the example.

The presentation of the material in this paper is tutorial in tone. While developing the relation between the Lagrangian of the system and the dynamic equations, mechanics fundamentals are illustrated and explained as they are required in the development. The authors believe that the presentation of Lagrange's equations is facilitated by simultaneously reinforcing important concepts of dynamics. The material contained herein is intended to constitute two 1 h lectures. The level of detail used in deriving the dynamics equations could be reduced for those classes having stronger mechanics backgrounds. That the generalized coordinates for a problem also constitute a set of state variables is a point which may ease the theoretical development for those students more familiar with state variable theory.

Examples

The $r - \theta$ Manipulator

Consider the $r - \theta$ manipulator shown in Fig. 1 which consists of a frictionless revolute joint followed by a frictionless prismatic joint. This 2° of freedom arm is the same manipulator presented by Snyder [1]. The mass of the shoulder m_1 is concentrated at the centroid which is located a constant distance r_1 from the axis of revolution. The centroid of the prismatic joint is located at a distance r from the axis of revolution and the mass of this joint m_2 is concentrated at this point. The gravitational acceleration vector is consistent with the coordinate system and in this situation the vector is aligned with the negative Y_0 axis. Numerically, the gravitational acceleration g is a negative number.

The dynamic equations for this body can be developed by Newton's laws of motion and Euler's equation. For the revolute joint, we set the time derivative of the angular momentum about the axis of revolution, namely $(m_1 r_1^2 + m_2 r^2)\dot{\theta}$, equal to the sum of the applied moments which consist of the applied torque T and that owing to the gravitational field. This application of Euler's equation produces

$$\frac{d}{dt}\left[(m_1 r_1^2 + m_2 r^2)\dot{\theta}\right] = T + (m_1 r_1 + m_2 r) g \cos\theta. \tag{1}$$

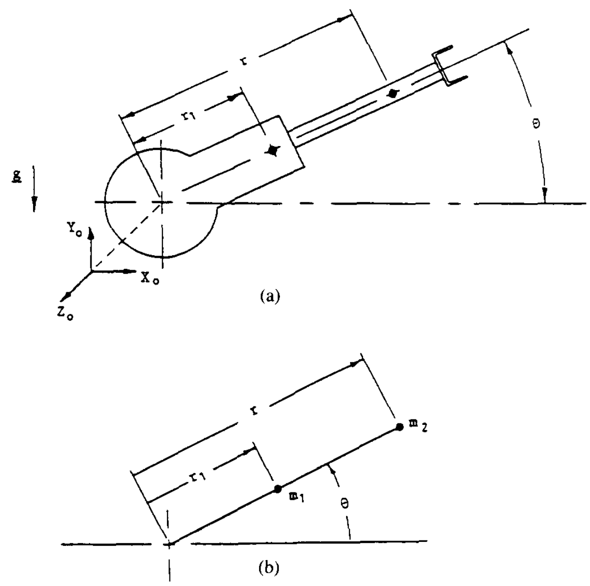

Fig. 1. r—θ manipulator.

The dynamic equation for the prismatic joint is obtained by setting the time derivative of the linear momentum $m_2\dot{r}$ equal to the applied forces which consist of the actuator force F, the centrifugal force $m_2 r\dot{\theta}^2$, and the gravitational force $m_2 g \sin\theta$. Thus, by Newton's law we arrive at

$$\frac{d}{dt}(m_2\dot{r}) = F + m_2 r\dot{\theta}^2 + m_2 g \sin\theta. \tag{2}$$

As a matter of direction at this point, we state that we seek a scalar L involving the mechanical energies of the system shown in Fig. 1 from which we can obtain the equations of motion. The development will depend on the following concepts:

a) Potential mechanical energy is a function of the displacements of a system and is not a function of any of the velocities (time derivative of displacement).

b) Kinetic energy is a function of the velocities of the system as well as displacements.

c) Angular and linear momenta can be functions of both the velocities and displacements of the system.

d) Integration followed by differentiation with respect to the same variable will return the original expression. (Note that differentiation followed by integration may not return the original expression; a constant may be lost.)

In order to obtain this scalar function L, we will manipulate both (1) and (2) to demonstrate that the dynamic equations can indeed be obtained from a scalar quantity. This scalar function will later be shown to be the Lagrangian of this system.

The manipulation will consist of integrating and differentiating parts of the equations of motion with respect to the dynamic variables. The potential and kinetic energies of the system can always be represented in terms of the dynamic variables. The dynamic variables, therefore, consist of a set of state variables for the system.

In performing the following manipulations to the equations of motion we will follow some general guidelines. These guidelines are as follows:

1) All momentum terms will be integrated and differentiated with respect to their associated velocities. The integration process yields a contribution to the total kinetic energy.

2) All terms which are either constants or exclusive functions of displacements will be integrated and differentiated with respect to their associated displacements. This operation yields terms which contribute to the total potential energy.

3) All remaining terms in the equations will be integrated and differentiated with respect to either a velocity or a displacement provided that the result of the integration can be demonstrated to be either a kinetic or potential energy term, respectively.

In order to show this correspondence, the momentum terms in (1) and (2) will be partially integrated and partially differentiated with respect to the dynamic variables. This operation yields for the angular momentum the result

$$\frac{d}{dt}\left[(m_1 r_1^2 + m_2 r^2)\dot{\theta}\right] = \frac{d}{dt}\left\{\frac{\partial}{\partial\dot{\theta}}\left[(m_1 r_1^2 + m_2 r^2)\frac{\dot{\theta}^2}{2}\right]\right\} \tag{3}$$

while for the linear momentum we get the result

$$\frac{d}{dt}(m_2\dot{r}) = \frac{d}{dt}\left(\frac{\partial}{\partial\dot{r}} m_2 \frac{\dot{r}^2}{2}\right). \tag{4}$$

Note that the partial, rather than ordinary, integration and differentiation allow the variable r in the momentum expression of (3) to be treated as a constant.

The next step of the demonstration involves partial integration and differentiation with respect to θ of the remaining excitations in (1) which are not produced by an actuator. A similar task is accomplished in (2) with respect to r. Performing this procedure for (1) with respect to θ yields

$$(m_1 r_1 + m_2 r) g \cos\theta = \frac{\partial}{\partial\theta}\left[(m_1 r_1 + m_2 r) g \sin\theta\right] \tag{5}$$

while the corresponding terms of (2) provide

$$m_2 r\dot{\theta}^2 + m_2 g \sin\theta = \frac{\partial}{\partial r}\left(m_2 r^2 \frac{\dot{\theta}^2}{2} + m_2 r g \sin\theta\right). \tag{6}$$

Combining the results of (3)–(6) shows that

$$\frac{d}{dt}\left\{\frac{\partial}{\partial\dot{\theta}}\left[(m_1 r_1^2 + m_2 r^2)\frac{\dot{\theta}^2}{2}\right]\right\} - \frac{\partial}{\partial\theta}\left[(m_1 r_1 + m_2 r) g \sin\theta\right] = T \tag{7}$$

and

$$\frac{d}{dt}\left(\frac{\partial}{\partial\dot{r}} m_2 \frac{\dot{r}^2}{2}\right) - \frac{\partial}{\partial r}\left(m_2 r^2 \frac{\dot{\theta}^2}{2} + m_2 r g \sin\theta\right) = F. \tag{8}$$

At this point, we examine the individual mechanical energies of the system. The rotational and translational kinetic energy is

$$\mathrm{KE} = \tfrac{1}{2}(m_1 r_1^2 + m_2 r^2)\dot{\theta}^2 + \tfrac{1}{2} m_2 \dot{r}^2 \tag{9}$$

while the gravitational potential energy is

$$\mathrm{PE} = -g(m_1 r_1 + m_2 r)\sin\theta. \tag{10}$$

The minus sign in (10) is necessary because g is defined as a negative number consistent with the coordinate system. Note that (10) increases with increasing displacement in the positive Y_o direction.

The term involving the time derivative in (7) contains the kinetic energy stemming from rotational motion while the corresponding term in (8) contains the kinetic energy from pure translation of the prismatic joint. The translational kinetic energy depends only on the time derivative of displacement r and the only velocity term in the rotational kinetic energy is the angular velocity $\dot{\theta}$. If we were to add the translational kinetic energy to the rotational kinetic energy already present in (7), the resulting dynamic equation is the same owing to the partial differentiation with respect to $\dot{\theta}$. Similarly, if we were to add the rotational kinetic energy to the translational kinetic energy already present in (8) the resulting dynamic equation is again unchanged since the partial differentiation is with respect to $\dot{r}$, a term absent in the rotational kinetic energy. Thus, we can replace the individual kinetic energies in (7) and (8) with the total kinetic energy. (In our example, the direction of motion of translation and rotation are always at right angles. However, the total kinetic energy may be used instead of the individual kinetic energies even if the directions are not orthogonal.) Performing this total kinetic energy substitution yields

$$\frac{d}{dt}\left(\frac{\partial}{\partial\dot{\theta}}\mathrm{KE}\right) - \frac{\partial}{\partial\theta}\left[(m_1 r_1 + m_2 r) g \sin\theta\right] = T \tag{11}$$

and

$$\frac{d}{dt}\left(\frac{\partial}{\partial\dot{r}}\mathrm{KE}\right) - \frac{\partial}{\partial r}\left(m_2 r^2 \frac{\dot{\theta}^2}{2} + m_2 r g \sin\theta\right) = F. \tag{12}$$

The second terms of (11) and (12) primarily involve the potential energy. Again, owing to the partial derivatives with respect to θ and r, the dynamic equations produced by (11) and (12) will be unchanged if the total potential energy is included in both equations. Performing this substitution gives

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{\theta}}\,\mathrm{KE}\right) - \frac{\partial}{\partial \theta}(-\mathrm{PE}) = T \qquad (13)$$

and

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{r}}\,\mathrm{KE}\right) - \frac{\partial}{\partial r}\left(m_2 r^2 \frac{\dot{\theta}^2}{2} - \mathrm{PE}\right) = F \qquad (14)$$

where the negative of the potential energy is necessary to provide consistency of signs.

The remaining unidentified term in (14) is the rotational kinetic energy of the mass m_2 as it revolves around the joint 1 axis. If this term is replaced by the total kinetic energy, the dynamic equation will be unchanged owing to the partial differentiation with respect to r. Performing this step produces for both (13) and (14) the result

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{\theta}}\,\mathrm{KE}\right) - \frac{\partial}{\partial \theta}(\mathrm{KE} - \mathrm{PE}) = T \qquad (15)$$

and the result

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{r}}\,\mathrm{KE}\right) - \frac{\partial}{\partial r}(\mathrm{KE} - \mathrm{PE}) = F. \qquad (16)$$

Finally, note that the potential energy does not contain either the radial or angular velocities. Subtracting the potential energy from the kinetic energy within the first term of both (15) and (16) leaves us with

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{\theta}}L\right) - \frac{\partial}{\partial \theta}(L) = T \qquad (17)$$

and

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{r}}L\right) - \frac{\partial}{\partial r}(L) = F \qquad (18)$$

where L is the Lagrangian given by

$$L = \mathrm{KE} - \mathrm{PE}. \qquad (19)$$

Equations (17) and (18) are called Lagrange's equations which provide the description of motion for rigid body dynamics. The variables r, $\dot{r}$, θ, and $\dot{\theta}$ are used to describe the energies of the system, both potential and kinetic, and, therefore, constitute a set of state variables for this mechanical system.

Spring-Mass System

Fig. 2 illustrates a spring-mass oscillator with an applied force F, mass m, gravitational load mg, and spring stiffness K. The coordinate system is aligned such that positive displacement occurs downward. As a result of this choice of reference, the gravitational acceleration g, which also points downward, is a positive number consistent with the coordinate system.

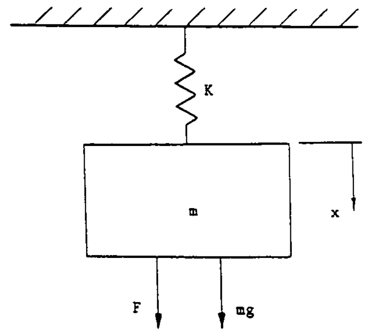

Fig. 2. Spring-mass system.

The equation of motion for this system is obtained by setting the time derivative of the linear momentum $m\dot{x}$ equal to the net force consisting of the applied force, the gravitational load, and the stiffness force. This operation produces the familiar result

$$\frac{d}{dt}(m\dot{x}) = -Kx + F + mg. \qquad (20)$$

Integrating and differentiating the momentum with respect to $\dot{x}$ and performing the same operation on the stiffness and gravitational forces with respect to the displacement x yields

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}}\frac{m\dot{x}^2}{2}\right) - \frac{\partial}{\partial x}\left[-\left(\frac{Kx^2}{2} - mgx\right)\right] = F. \qquad (21)$$

The use of the double negative signs in the second term of (21) is intentional. The quantity in the first term of (21) is the kinetic energy while the quantity in the second term is the negative of the total potential energy, strain plus gravitational potential. Altering (21) to reflect this fact gives us

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}}\,\mathrm{KE}\right) - \frac{\partial}{\partial x}(-\mathrm{PE}) = F. \qquad (22)$$

Since the potential energy only depends upon x and the kinetic energy is an exclusive function of the velocity in this problem, the dynamic equation will be unaltered if we subtract the potential energy from the kinetic energy in the first term and add the kinetic energy to the potential energy of the second term. Equation (22) then becomes

$$\frac{d}{dt}\left[\frac{\partial}{\partial \dot{x}}(\mathrm{KE} - \mathrm{PE})\right] - \frac{\partial}{\partial x}(\mathrm{KE} - \mathrm{PE}) = F \qquad (23)$$

or by use of (19) we get

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}}L\right) - \frac{\partial}{\partial x}L = F. \qquad (24)$$

Equation (24) is Lagrange's equation for this system while x and $\dot{x}$ constitute a set of state variables for this oscillator.

The Double Pendulum

In this more complicated example, we will show that by starting with the dynamic equations and working backwards we can recover Lagrange's equations for this system. The system under consideration is shown in Fig. 3. The double pendulum consists of two very light rods of lengths d_1 and d_2 supporting the point masses m_1 and m_2. The configuration of the pendulum is described by the angles θ_1 and θ_2. A Cartesian inertial reference frame is fixed to the support point of joint 1. The positions of the mass m_1 and mass m_2 with respect to the reference frame are described by the vectors $\underline{r}_1$ and $\underline{r}_2$, respectively. The velocities of the masses are $\underline{v}_1$ and $\underline{v}_2$. The position of the mass m_2 with respect to the mass m_1 is described by the vector $\underline{r}_{12}$. These vectors are illustrated in Fig. 3. The angle θ_{12}, also shown in Fig. 3, is the sum of the angles θ_1 and θ_2.

The equation of motion for joint 1 is obtained by setting the time derivative of the angular momentum equal to the sum of the applied external moments. The angular momentum $\underline{H}_1$ about the support is given by

$$\begin{aligned} \underline{H}_1 &= \underline{r}_1 \times m_1\underline{v}_1 + \underline{r}_2 \times m_2\underline{v}_2 \\ &= \left\{m_1 d_1^2\dot{\theta}_1 + m_2\left[(d_1^2 + d_2^2 + 2d_1d_2C_2)\,\dot{\theta}_1 \right.\right. \\ &\quad \left.\left. + (d_2^2 + d_1d_2C_2)\,\dot{\theta}_2\right]\right\}\underline{k} \end{aligned} \tag{25}$$

where C_2 is the cosine of θ_2 and $\underline{k}$ is a unit vector along the Z_0 axis. The shorthand notation for trigonometric functions used by such authors as Paul [2] and Snyder [1] has been adopted here. The external torques are given by

$$\begin{aligned} &T_1\underline{k} + \underline{r}_1 \times m_1\underline{g} + \underline{r}_2 \times m_2\underline{g} \\ &\quad = \left[T_1 + m_1d_1S_1g + m_2g(d_1S_1 + d_2S_{12})\right]\underline{k} \end{aligned} \tag{26}$$

where, owing to the choice of the coordinate system, the gravitational acceleration g is a negative number. From (25) and (26) we have

$$\begin{aligned} &\frac{d}{dt}\left\{m_1d_1^2\dot{\theta}_1 + m_2\left[(d_1^2 + d_2^2 + 2d_1d_2C_2)\,\dot{\theta}_1 \right.\right. \\ &\qquad \left.\left. + (d_2^2 + d_1d_2C_2)\,\dot{\theta}_2\right]\right\} \\ &\qquad = T_1 + m_1d_1S_1g + m_2g(d_1S_1 + d_2S_{12}). \end{aligned} \tag{27}$$

The equation for joint two will be found from the Newton–Euler method described by Asada and Slotine [7] and originally presented by Luh, Walker, and Paul [8]. If the second link is considered as a rigid body moving through space we note that the mass center is located at the extreme end where the mass m_2 is attached to the link. By summing the forces acting on the link and equating this with the rate of change of the linear momentum of the link $m_2\dot{\underline{r}}_2$ we get

$$\frac{d}{dt}(m_2\dot{\underline{r}}_2) = m_2\underline{g} + \underline{f}_{1,2} \tag{28}$$

where $\underline{f}_{1,2}$ is the force of constraint which keeps the two links joined together. The force $\underline{f}_{1,2}$ is exerted by link one and acts on link two. Summing the moments about the location of the mass m_2 gives

$$T_2\underline{k} - \underline{r}_{1,2} \times \underline{f}_{1,2} = 0 \tag{29}$$

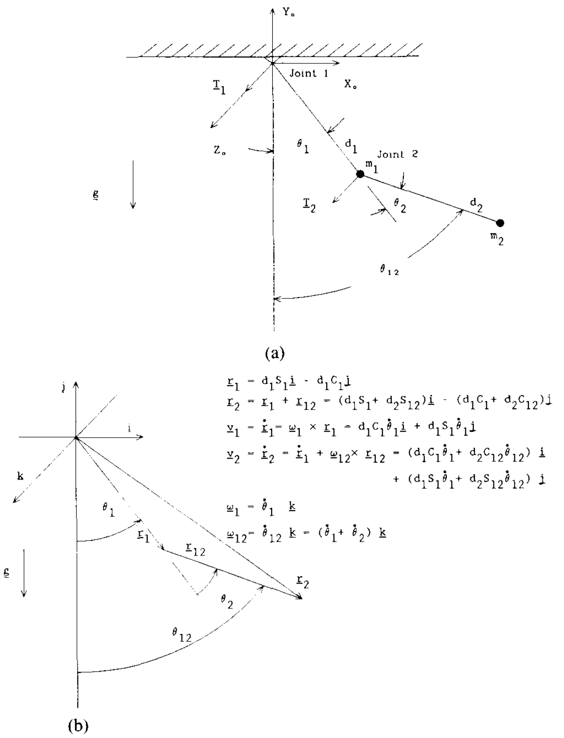

Fig. 3. Double pendulum.

where it is seen that the inertia of the link about the location of m_2 is zero owing to the assumption that m_2 is a point mass. Taking the cross product of (28) with the vector $\underline{r}_{1,2}$ and substituting for $\underline{f}_{1,2}$ from (29) produces

$$\underline{r}_{1,2} \times \frac{d}{dt}(m_2\dot{\underline{r}}_2) = \underline{r}_{1,2} \times m_2\underline{g} + T_2\underline{k}. \tag{30}$$

Substituting for the position vectors in terms of the angles θ_1 and θ_2 and simplifying the result shows that the equation of motion is

$$\begin{aligned} &m_2d_1d_2C_2\frac{d^2}{dt^2}\theta_1 + m_2d_1d_2S_2\dot{\theta}_1^2 + m_2d_2^2\frac{d^2}{dt^2}\theta_{12} \\ &\qquad = T_2 + m_2gd_2S_{12}. \end{aligned} \tag{31}$$

Equation (31) can be further modified into a form expedient to the later analysis by adding and subtracting $m_2d_1d_2S_2\dot{\theta}_1\dot{\theta}_2$ and combining those terms which constitute perfect differentials in time to get

$$\begin{aligned} &\frac{d}{dt}(m_2d_1d_2C_2\,\dot{\theta}_1 + d_2^2m_2\dot{\theta}_{12}) + m_2d_1d_2S_2(\dot{\theta}_1^2 + \dot{\theta}_1\dot{\theta}_2) \\ &\qquad = T_2 + m_2gd_2S_{12}. \end{aligned} \tag{32}$$

In order to show that (27) and (32) can be determined from Lagrange's equations we will integrate and differentiate the angular momentum in (27) with respect to $\dot{\theta}_1$. We will also integrate and differentiate the term within the time derivative in (32) with respect to $\dot{\theta}_2$. The gravitational moments in (27) and (32) will be integrated and differentiated with respect to θ_1 and θ_2, respectively. Performing these operations produces the results

$$\frac{d}{dt}\left\{\frac{\partial}{\partial\dot{\theta}_1}\left[m_1 d_1 \frac{\dot{\theta}_1^2}{2} + m_2(d_1^2 + d_2^2 + 2d_1 d_2 C_2) \cdot \frac{\dot{\theta}_1^2}{2} + m_2(d_2^2 + d_1 d_2 C_2)\dot{\theta}_1\dot{\theta}_2\right]\right\} = T_1 + \frac{\partial}{\partial\theta_1}\left[-m_1 d_1 C_1 g - m_2 g(d_1 C_1 + d_2 C_{12})\right] \tag{33}$$

and

$$\frac{d}{dt}\left\{\frac{\partial}{\partial\dot{\theta}_2}\left[m_2(d_2^2 + d_1 d_2 C_2)\dot{\theta}_1\dot{\theta}_2 + m_2 d_2^2 \frac{\dot{\theta}_2^2}{2}\right]\right\} + m_2 d_1 d_2(\dot{\theta}_1^2 + \dot{\theta}_1\dot{\theta}_2) S_2 = T_2 + \frac{\partial}{\partial\theta_2}(-m_2 g d_2 C_{12}). \tag{34}$$

For this system, we note that the kinetic and potential energies are given by

$$\begin{aligned} \text{KE} &= \frac{1}{2} m_1 \underline{v}_1 \cdot \underline{v}_1 + \frac{1}{2} m_2 \underline{v}_2 \cdot \underline{v}_2 \\ &= \left[m_1 d_1^2 + m_2(d_1^2 + d_2^2 + 2d_1 d_2 C_2)\right]\frac{\dot{\theta}_1^2}{2} \\ &\quad + \frac{1}{2} m_2\left[d_2^2\dot{\theta}_2^2 + 2(d_2^2 + d_1 d_2 C_2)\dot{\theta}_1\dot{\theta}_2\right] \end{aligned} \tag{35}$$

and

$$\begin{aligned} \text{PE} &= -\underline{r}_1 \cdot m_1 \underline{g} - \underline{r}_2 \cdot m_2 \underline{g} \\ &= m_1 d_1 C_1 g + m_2 g(d_1 C_1 + d_2 C_{12}). \end{aligned} \tag{36}$$

Because of the partial differentiation, the terms involving derivatives with respect to angular velocity in (33) and (34) can be replaced with partials of the kinetic energy with respect to the same angular velocities without altering the original dynamic equations. The rightmost term in (33) and (34) can be replaced with the partial of the negative potential energy with respect to the angles θ_1 and θ_2 by use of the same argument used for the kinetic energy. These substitutions leave us with

$$\frac{d}{dt}\left[\frac{\partial}{\partial\dot{\theta}_1}(\text{KE})\right] = T_1 + \frac{\partial}{\partial\theta_1}(-\text{PE}) \tag{37}$$

and

$$\frac{d}{dt}\left[\frac{\partial}{\partial\dot{\theta}_2}(\text{KE})\right] + m_2 d_1 d_2(\dot{\theta}_1^2 + \dot{\theta}_1\dot{\theta}_2) S_2 = T_2 + \frac{\partial}{\partial\theta_2}(-\text{PE}). \tag{38}$$

The second term in (38) is the negative of the partial derivative of the kinetic energy with respect to θ_2. Equation (38) can be rewritten to reflect this fact as

$$\frac{d}{dt}\left[\frac{\partial}{\partial\dot{\theta}_2}(\text{KE})\right] - \frac{\partial}{\partial\theta_2}(\text{KE} - \text{PE}) = T_2. \tag{39}$$

The potential energy does not include the angular velocities nor does the kinetic energy contain the angle θ_1. Equations (37) and (39) then reduce to the form

$$\frac{d}{dt}\left(\frac{\partial}{\partial\dot{\theta}_1} L\right) - \frac{\partial}{\partial\theta_1}(L) = T_1 \tag{40}$$

and

$$\frac{d}{dt}\left(\frac{\partial}{\partial\dot{\theta}_2} L\right) - \frac{\partial}{\partial\theta_2}(L) = T_2 \tag{41}$$

which are Lagrange's equations for this system. Through the use of the particular kinematic description of Fig. 3 we see that the angles θ_1 and θ_2, together with their first time derivatives, make up a set of state variables for this system.

A System of Particles

As a final example, we examine a system of n point masses shown in Fig. 4. Each particle is connected to the center of mass through a rigid, massless link (not shown). While the system is free to translate and rotate, the position of each particle remains fixed with respect to the remaining particles. This system of point masses is subjected to an externally applied force $\underline{F}$ and moment $\underline{T}$ in addition to the gravitational forces.

The configuration of the body of particles is described by the position vector $\underline{R}$ from the reference frame to the center of mass and by the orientation of the body described by the roll, pitch, and yaw angles ϕ_x, ϕ_y, and ϕ_z as described by Craig [3]. Let $\underline{\phi}$ denote the vector containing these orientation angles. These orientation angles are measured with respect to the fixed frame. The velocity of the center of mass is $\dot{\underline{R}}$ while the rate of rotation is described by the vector $\underline{\omega}$ defined by

$$\underline{\omega} = \dot{\underline{\phi}}. \tag{42}$$

The vector $\underline{R} + \underline{r}_i$ describes the position of each particle, consisting of mass m_i, in the fixed frame. The velocity of each particle is

$$\frac{d}{dt}(\underline{R} + \underline{r}_i) = \dot{\underline{R}} + \underline{\omega} \times \underline{r}_i. \tag{43}$$

To determine the equations of motion, we first calculate the linear and angular momenta. The linear momentum is

$$\sum_{i=1}^{n} m_i(\dot{\underline{R}} + \underline{\omega} \times \underline{r}_i) = \dot{\underline{R}} \sum_{i=1}^{n} m_i + \underline{\omega} \times \sum_{i=1}^{n} m_i \underline{r}_i = M\dot{\underline{R}} \tag{44}$$

where M is the sum of the masses of the individual particles. The cross product term of (44) vanishes owing to

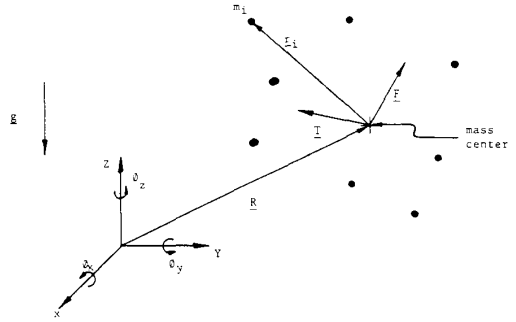

Fig. 4. System of n particles.

the definition of the center of mass. The angular momentum of the particles about the reference frame is

$$\sum_{i=1}^{n} (\underline{R} + \underline{r}_i) \times m_i(\dot{\underline{R}} + \underline{\omega} \times \underline{r}_i)$$
$$= \underline{R} \times \dot{\underline{R}} \sum_{i=1}^{n} m_i + \underline{R} \times \left(\underline{\omega} \times \sum_{i=1}^{n} m_i \underline{r}_i\right) + \sum_{i=1}^{n} m_i \underline{r}_i \times \dot{\underline{R}} + \sum_{i=1}^{n} \underline{r}_i \times m_i(\underline{\omega} \times \underline{r}_i)$$
$$= \underline{R} \times M\dot{\underline{R}} + \sum_{i=1}^{n} \underline{r}_i \times m_i(\underline{\omega} \times \underline{r}_i) \tag{45}$$

where the definition of the center of mass has again been used to simplify the expression.

The system of particles has 6° of freedom being the three spatial components of the center of mass and the three orientation angles. The equations of motion will be written in terms of these quantities. The dynamic equation for the translational motion is determined by setting the time derivative of the linear momentum equal to the external forces and is seen to be

$$\frac{d}{dt}(M\dot{\underline{R}}) = \underline{F} + \sum_{i=1}^{n} m_i g \underline{k} = \underline{F} + Mg\underline{k}. \tag{46}$$

The moments produced about the origin are

$$\underline{T} + \underline{R} \times \underline{F} + \sum_{i=1}^{n} (\underline{R} + \underline{r}_i) \times m_i g \underline{k}$$
$$= \underline{T} + \underline{R} \times \underline{F} + \underline{R} \times Mg\underline{k} \tag{47}$$

where, again, the definition of the center of mass has been used to simplify the expression. Setting the time rate of change of the angular momentum equal to the applied moments produces

$$\frac{d}{dt}\left[\underline{R} \times M\dot{\underline{R}} + \sum_{i=1}^{n} \underline{r}_i \times m_i(\underline{\omega} \times \underline{r}_i)\right]$$
$$= \underline{T} + \underline{R} \times \underline{F} + \underline{R} \times Mg\underline{k}. \tag{48}$$

This last expression can be rearranged into the form

$$\dot{\underline{R}} \times M\dot{\underline{R}} + \underline{R} \times \left[\frac{d}{dt}(M\dot{\underline{R}}) - \underline{F} - Mg\underline{k}\right] + \frac{d}{dt}\left[\sum_{i=1}^{n} \underline{r}_i \times m_i(\underline{\omega} \times \underline{r}_i)\right] = \underline{T}. \tag{49}$$

The first term in (49) vanishes by definition of the cross product. The second term of (49) also vanishes since the bracketed term is an equation of motion. We arrive at the result

$$\frac{d}{dt}\left[\sum_{i=1}^{n} \underline{r}_i \times m_i(\underline{\omega} \times \underline{r}_i)\right] = \underline{T} \tag{50}$$

as the equation of motion for the orientation angles.

Equations (46) and (50) can be derived from Lagrange's equations. To demonstrate this, first integrate and differentiate the momentum term in (46) with respect to $\dot{\underline{R}}$. Simultaneously, we will perform the same operation on the gravitational force with respect to $\underline{R}$. We now have the result

$$\frac{d}{dt}\left[\frac{\partial}{\partial \dot{\underline{R}}}\left(\frac{M}{2}\dot{\underline{R}} \cdot \dot{\underline{R}}\right)\right] - \frac{\partial}{\partial \underline{R}}(Mg\underline{k} \cdot \underline{R}) = \underline{F}. \tag{51}$$

We can perform a similar operation on (50) with respect to $\underline{\omega}$ but first it is convenient to rewrite the angular momentum about the mass center through the use of a vector identity as

$$\sum_{i=1}^{n} \underline{r}_i \times m_i(\underline{\omega} \times \underline{r}_i) = \sum_{i=1}^{n} m_i[\underline{\omega}(\underline{r}_i \cdot \underline{r}_i) - (\underline{\omega} \cdot \underline{r}_i)\underline{r}_i]. \tag{52}$$

Integrating and differentiating the angular momentum of (50) with respect to $\underline{\omega}$ yields

$$\frac{d}{dt}\left\{\frac{\partial}{\partial \underline{\omega}} \sum_{i=1}^{n} \frac{1}{2} m_i[(\underline{\omega} \cdot \underline{\omega})(\underline{r}_i \cdot \underline{r}_i) - (\underline{\omega} \cdot \underline{r}_i)^2]\right\} = \underline{T}. \tag{53}$$

The bracketed term in (53) can be rewritten, again by vector identity, to produce the result

$$\frac{d}{dt}\left[\frac{\partial}{\partial \underline{\omega}} \sum_{i=1}^{n} \frac{1}{2} m_i(\underline{\omega} \times \underline{r}_i) \cdot (\underline{\omega} \times \underline{r}_i)\right] = \underline{T}. \tag{54}$$

Equations (51) and (54) can be simplified by determining the total kinetic energy which is

$$\mathrm{KE} = \sum_{i=1}^{n} \tfrac{1}{2} m_i(\dot{\underline{R}} + \underline{\omega} \times \underline{r}_i) \cdot (\dot{\underline{R}} + \underline{\omega} \times \underline{r}_i)$$
$$= \sum_{i=1}^{n} \tfrac{1}{2} m_i[\dot{\underline{R}} \cdot \dot{\underline{R}} + 2\dot{\underline{R}} \cdot (\underline{\omega} \times \underline{r}_i) + (\underline{\omega} \times \underline{r}_i) \cdot (\underline{\omega} \times \underline{r}_i)]$$
$$= \tfrac{1}{2} M\dot{\underline{R}} \cdot \dot{\underline{R}} + \sum_{i=1}^{n} \tfrac{1}{2} m_i(\underline{\omega} \times \underline{r}_i) \cdot (\underline{\omega} \times \underline{r}_i). \tag{55}$$

Equation (55) is seen to be the sum of the translational kinetic energy depending upon the velocity of the center of mass and the rotational kinetic energy, depending only upon the rate of rotation. The potential energy is given by

$$\begin{aligned} \mathrm{PE} &= -\sum_{i=1}^{n} m_i g \underline{k} \cdot \underline{R} - \sum_{i=1}^{n} \left[(\underline{R} + \underline{r}_i) \times m_i g \underline{k} \right] \cdot \underline{\phi} \\ &= -Mg\underline{k} \cdot \underline{R}. \end{aligned} \tag{56}$$

Again, the simplification in (56) arises by invoking the definition of the center of mass. Substituting the kinetic and potential energy expressions into (51) and (54) shows that the equations of motion now become

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{\underline{R}}} \mathrm{KE}\right) - \frac{\partial}{\partial \underline{R}}(-\mathrm{PE}) = \underline{F} \tag{57}$$

and

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{\underline{\phi}}} \mathrm{KE}\right) = \underline{T}. \tag{58}$$

Owing to the partial derivatives we may alter these equations by substituting the Lagrangian defined by (19) into these equations to obtain

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{\underline{R}}} L\right) - \frac{\partial}{\partial \underline{R}}(L) = \underline{F} \tag{59}$$

and

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{\underline{\phi}}} L\right) - \frac{\partial}{\partial \underline{\phi}}(L) = \underline{T}. \tag{60}$$

The partial with respect to $\underline{\phi}$ in (60) vanishes since L is not an explicit function of $\underline{\phi}$; this term was added merely to demonstrate the symmetry of the equations.

The generalization of the foregoing development to rigid body dynamics is accomplished by replacing the summations over the point masses m_i with integrals of the material density ρ [mass/unit volume] over the volume. The location of the differential volumes inside the body is described by the vector $\underline{r}$ which now varies continuously. Using integrals in place of the summations, it is an easy process to show that (52) provides the familiar definition of the inertia tensor.

Conclusions

A technique for presenting Lagrange's equations has been developed which neither uses variational calculus nor virtual work. The thrust of the demonstration is that given dynamic equations for a rigid body one can always manipulate the equations to show that the equations depend exclusively on the difference between the total kinetic and potential energies of the system.

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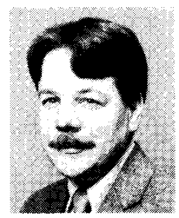

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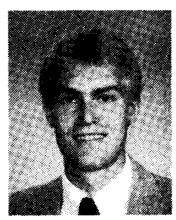

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