The Price of Anarchy of Finite Congestion Games*

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ABSTRACT

We consider the price of anarchy of pure Nash equilibria in congestion games with linear latency functions. For asymmetric games, the price of anarchy of maximum social cost is $\Theta(\sqrt{N})$, where N is the number of players. For all other cases of symmetric or asymmetric games and for both maximum and average social cost, the price of anarchy is $5/2$. We extend the results to latency functions that are polynomials of bounded degree. We also extend some of the results to mixed Nash equilibria.

Categories and Subject Descriptors

C.2.1 [**Computer-Communication Networks**]: Network Architecture and Design—*Network communications*; C.2.2 [**Computer-Communication Networks**]: Network Protocols—*Routing protocols*

General Terms

Theory, Performance

Keywords

Price of anarchy, Congestion games

1. INTRODUCTION

The price of anarchy [13, 19] measures the deterioration in performance of systems on which resources are allocated by selfish agents. It captures the lack of coordination between independent selfish agents as opposed to the lack of information (competitive ratio) or the lack of computational resources (approximation ratio).

The price of anarchy was originally defined [13] to capture the worst case selfish performance of a simple game of N players that compete for M parallel links. The question is what happens in more general networks or even in more general congestion games that have no underlying network. Roughgarden and Tardos [23, 24] gave the answer for the case where the players control a negligible amount of traffic. But what happens in the discrete case? This is the question that we address in this paper.

Congestion games, introduced by Rosenthal [20] and studied in [17], is a natural general class of games that provide a unifying thread between the two models studied in [13] and[23]. The parallel link model of [13] is a special case of congestion games (with singleton strategies but with weights) while the selfish routing model of [23] is the special case of congestion games of infinitely many players each one controlling a negligible amount of traffic. Congestion games have the fundamental property that a pure Nash equilibrium always exists. It is natural therefore to ask *What is the pure price of anarchy of congestion games?*

The price of anarchy depends not only on the game itself but also on the definition of the social (or system) cost. From the system's designer point of view, who cares about the welfare of the players, two natural social costs seem important: the maximum or the average cost among the players. For the original model of parallel links in [13], the social cost was the maximum cost among the players. For the Wardrop model studied by Roughgarden and Tardos [23], the social cost is the average player cost. Here we deal with both the maximum and the average social cost.

We also consider the price of anarchy of the natural subclass of symmetric congestion games. (Sometimes in the literature, the symmetric case is called single-commodity while the asymmetric or general case is called multi-commodity.)

1.1 Our results

We study the price of anarchy of *pure equilibria* in general congestion games with linear latency functions. The latency functions that we consider are of the form $f(x) = ax + b$ for nonnegative a and b, but for simplicity our proofs consider only the case $f(x) = x$; they directly extend to the general case.

We consider both the maximum and the average (sum) player cost as social cost. We also study both symmetric and asymmetric games. Our results (both lower and upper bounds) are summarized in Table 1. For the case of asymmetric games, the values hold also for network congestion games. We don't know if this is true for the symmetric case as well.

We extend these results to the case of latency functions that are polynomials of degree p with nonnegative coeffi-

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cients. The results (both lower and upper bounds) appear Table 2.

	SUM	MAX
Symmetric	$\frac{5N-2}{2N+1}$	$\frac{5N+1}{2N+2} \dots \frac{5}{2}$
Asymmetric	$\frac{5}{2}$	$\Theta(\sqrt{N})$

Table 1: Price of anarchy of pure equilibria for linear latencies. N is the number of the players.

	SUM	MAX
Symmetric	$p^{\Theta(p)}$	$p^{\Theta(p)}$
Asymmetric	$p^{\Theta(p)}$	$\Omega(N^{p/(p+1)}) \dots O(N)$

Table 2: Price of anarchy of pure equilibria for polynomial latencies of degree p. N is the number of the players.

We also extend our results on the average social cost to the case of mixed Nash equilibria (with price of anarchy at most $1 + \phi \approx 2.619$).

1.2 Related work

The study of the price of anarchy was initiated in [13], where (weighted) congestion games of m parallel links are considered. The price of anarchy for the maximum social cost, expressed as a function of m, is $\Theta(\log m / \log\log m)$—the lower bound was shown in [13] and the upper bound in [12, 6]. Furthermore, [6] extended the result to m parallel paths (which is equivalent to links with speeds) and showed that the price of anarchy is $\Theta(\log m / \log\log\log m)$. In [5], more general latency functions are studied, especially in relation to queuing theory. For the same model of parallel links, [9] and [14] consider the price of anarchy for other social costs.

In [25], the special case of congestion games in which each strategy is a singleton set is considered. They give bounds for the case of the average social cost. For the same class of congestion games and the maximum social cost, [10] showed that the price of anarchy is $\Theta(\log N / \log\log N)$ (a similar, perhaps unpublished, result was obtained by the group of [25]). On the other end where strategies have arbitrary size, we show here a $\Theta(\sqrt{N})$ upper bound. An interesting open question is how the price of anarchy goes from $\Theta(\log N / \log\log N)$ to $\Theta(\sqrt{N})$ as a function of the number of facilities in each strategy. The case of singleton strategies is also considered in [11] and [14].

In [8], they consider the mixed price of anarchy of symmetric network weighted congestion games, when the network is layered.

The non-atomic case of congestion games was considered in [23, 24] where they showed that for linear latencies the average price of anarchy is 4/3. They also extended this result to polynomial latencies. Furthermore, [22, 4] considered the social cost of maximum latency.

Paper [1], which appears in these proceedings, studies a similar problem with this work. They consider the price of anarchy of general congestion games, but they study the social cost of the total latency, which is the sum of the square of facilities loads. For pure equilibria of unweighted games the total latency and the average cost are the same and therefore some of the results are common in both papers. They also deal with both linear and polynomial latencies but they also consider weighted games. For linear latencies and for the weighted case they show price of anarchy 2.618, while for the unweighted case and for pure equilibria they show a price of anarchy of 2.5.

Recently, using similar techniques, we extended some of the results to the case of correlated equilibria [3]. Surprisingly, the price of anarchy is the same with the case of Nash equilibria: 2.5 for unweighted games and 2.618 for weighted ones. The techniques were also helpful to bound the price of stability (the optimistic price of anarchy of the best Nash equilibrium as opposed to the worst Nash equilibrium [2]).

2. THE MODEL

A congestion game is a tuple $(N, E, (\Sigma_i)_{i \in N}, (f_e)_{e \in M})$ where $N = \{1, \dots, n\}$ is the set of players, E is a set of facilities, $\Sigma_i \subseteq 2^E$ is a collection of pure strategies for player i: a pure strategy $A_i \in \Sigma_i$ is a set of facilities, and finally f_e is a cost (or latency) function associated with facility j.

Most of this work is concerned with linear cost functions: $f_e(k) = a_e \cdot k + b_e$ for nonnegative constants a_e and b_e. For simplicity, we will only consider the identity latency functions $f_e(k) = k$. We can ignore the factor a_e because we can obtain a similar game when we appropriately replace the facility e with a set of a_e facilities. When a_e is not an integer, we can use a similar trick. Also, in some cases, such as the asymmetric-max case, we can ignore the term b_e by adding additional players who play only on the facility e. For the rest of the results, it can be verified that our proofs work for nonzero b_e's as well. We leave the details for the full version.

A pure strategy profile $A = (A_1, \dots, A_n)$ is a vector of strategies, one for each player. The cost of player i for the pure strategy profile A is given by $c_i(A) = \sum_{e \in A_i} f_e(n_e(A))$, where $n_e(A)$ is the number of the players using e in A. A pure strategy profile A is a Nash equilibrium if no player has any reason to unilaterally deviate to another pure strategy: $\forall i \in N, \forall S \in (\Sigma_i) \quad c_i(A) \leq c_i(A_{-i}, S)$, where (A_{-i}, S) is the strategy profile produced if just player i deviates from A_i to S.

The *social cost* of A is either the maximum cost of a player $\textsc{Max}(A) = \max_{i \in N} c_i(A)$ or the average of the players' costs. For simplicity, we consider the sum of all costs (which is N times the average cost) $\textsc{Sum}(A) = \sum_{i \in N} c_i(A)$.

A congestion game is *symmetric* (or single-commodity) if all the players have the same strategy set: $\Sigma_i = \Sigma$. We use the term "asymmetric" (or multi-commodity) to refer to all games (including the symmetric ones).

A *mixed* strategy p_i for a player i, is a probability distribution over his pure strategy set Σ_i. The above definitions extend naturally to this case (with expected costs, of course).

For a class of congestion games, the pure price of anarchy of the average social cost is the worst-case ratio, among all pure Nash equilibria, of the social cost over the optimum social cost, $opt = \min_{P \in \Sigma} \textsc{Sum}(P)$.

$$PA = \sup_{A \text{ is a Nash eq.}} \frac{\textsc{Sum}(A))}{opt}$$

Similarly, we define the price of anarchy for the maximum social cost or for mixed Nash equilibria.

3. LINEAR LATENCY FUNCTIONS

In this section we prove theorems that fill Table 1. It should be clear that the values of each symmetric case are no greater than the corresponding asymmetric case. Similarly, the price of anarchy for average social cost is no greater than the corresponding price of anarchy for the maximum social cost. This is useful because we don't have to give upper and lower bounds for each entry. For example, a lower bound for the symmetric average case holds for every other case.

3.1 Asymmetric games - Average social cost

The following is a simple fact which will be useful in the proof of the next theorem.

LEMMA 1. *For every pair of nonnegative integers α, β, it holds*

$$\beta(\alpha+1) \leq \frac{1}{3}\alpha^2 + \frac{5}{3}\beta^2.$$

THEOREM 1. *For linear congestion games, the pure price of anarchy of the average social cost is at most $\frac{5}{2}$.*

PROOF. Let A be a Nash equilibrium and P be an optimal (or any other) allocation. The cost of player i at the Nash equilibrium is $c_i(A) = \sum_{e \in A_i} n_e(A)$, where $n_e(A)$ denotes the number of players that use facility e in A. We want to bound the social cost, the sum of the cost of the players: $\text{SUM}(A) = \sum_i c_i(A) = \sum_{e \in E} n_e^2(A)$, with respect to the optimal cost $\text{SUM}(P) = \sum_i c_i(P) = \sum_{e \in E} n_e^2(P)$.

At the Nash equilibrium, the cost of player i should not decrease when the player switches to strategy P_i:

$$c_i(A) = \sum_{e \in A_i} n_e(A) \leq \sum_{e \in P_i} n_e(A_{-i}, P_i) \leq \sum_{e \in P_i} (n_e(A)+1)$$

where (A_{-i}, P_i) is the usual notation in Game Theory to denote the allocation that results when we replace A_i by P_i.

If we sum over all players i, we can bound the social cost as

$$\begin{aligned} \text{SUM}(A) &= \sum_{i \in N} c_i(A) \\ &\leq \sum_{i \in N} \sum_{e \in P_i} (n_e(A)+1) \\ &= \sum_{e \in E} n_e(P)(n_e(A)+1) \end{aligned}$$

With the help of Lemma 1, the last expression is at most $\frac{1}{3}\sum_{e \in E} n_e^2(A) + \frac{5}{3}\sum_{e \in E} n_e^2(P) = \frac{1}{3}\text{SUM}(A) + \frac{5}{3}\text{SUM}(P)$ and the theorem follows. □

THEOREM 2. *There are linear congestion games with 3 or more players with pure price of anarchy for the average social cost equal to $\frac{5}{2}$.*

PROOF. We will construct a congestion game for $N \geq 3$ players and $|E| = 2N$ facilities with price of anarchy 5/2. (It is not hard to show that for $N = 2$ players, the price of anarchy is exactly 2.)

We divide the set E into two subsets $E_1 = \{h_1, \ldots, h_N\}$ and $E_2 = \{g_1, \ldots, g_N\}$, each of N facilities. Player i has two pure strategies: $\{h_i, g_i\}$ and $\{g_{i+1}, h_{i-1}, h_{i+1}\}$. The optimal allocation is for each player to select the first strategy while the worst-case Nash equilibrium is for each player to select the second strategy. It is not hard to verify that this is a Nash equilibrium in which each player has cost 5. Since at the optimal allocation the cost of each player is 2, the price of anarchy is 5/2.

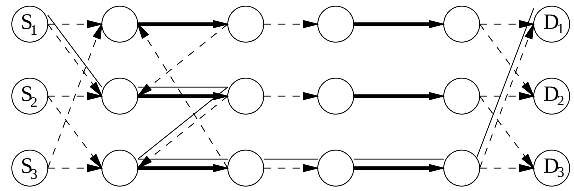

Figure 1: There are three players who want to go from S_i to D_i. The optimal strategies are for each player to move in a straight line. At the Nash equilibrium, the players use the dashed lines. The strategy of player 1 at the Nash equilibrium is shown. The bold (non-dashed) lines are long (heavy) paths.

This example is not a network congestion game, but we can turn it into a network congestion game as shown in Figure 1. □

3.2 Symmetric games - Average social cost

For symmetric congestion games and average social cost the price of anarchy is also 5/2. The upper bound follows directly from Theorem 1 because symmetric games is a special case of asymmetric games. The following theorem gives the lower bound. This would have subsumed Theorem 2 had it not had an additional term which tends to 0 as N tends to infinity. In other words, for asymmetric games the price of anarchy is exactly 5/2 for every $N \geq 3$, but for symmetric games it is somewhat less: $(5N-2)/(2N+1)$.

Another reason to include the lower bounds for both the symmetric and the asymmetric case is that in the later case the congestion game is a network congestion game, while in the former it is not. We don't know whether the bound 5/2 holds also for symmetric network games.

THEOREM 3. *For linear symmetric congestion games, the pure price of anarchy of the average social cost is at most $\frac{5N-2}{2N+1}$.*

PROOF. Let A be a Nash equilibrium and P be an optimal allocation. The cost of player i at the Nash equilibrium is greater or equal than the cost he would have if he had chosen any other strategy. Notice that since the game is symmetric, all the possible strategies are available to every player. So for every player i

$$c_i(A) \leq \sum_{e \in P_j} n_e(A) + |P_j - A_i| = \sum_{e \in P_j} n_e(A) + |P_j| - |P_j \cap A_i|$$

for all players j.

If we sum over all $j \in N$ we get a bound on player i's cost.

$$\begin{aligned} N \cdot c_i(A) &\leq \sum_{j \in N} \sum_{e \in P_j} n_e(A) + |P_j| - |P_j \cap A_i| \\ &= \sum_{e \in E} n_e(P) n_e(A) + \sum_{e \in E} n_e(P) - \sum_{j \in N} |P_j \cap A_i| \\ &= \sum_{e \in E} n_e(P) n_e(A) + \sum_{e \in E} n_e(P) - \sum_{e \in A_i} n_e(P) \end{aligned}$$

Summing over all $i \in N$ gives us a bound on the social cost of the Nash equilibrium.

$$\begin{aligned}
\textsc{Sum}(A) &= \sum_{e\in E} n_e^2(A) \\
&\leq \frac{1}{N}\sum_{i\in N}\sum_{e\in E} n_e(P)n_e(A) \\
&\quad +\frac{1}{N}\sum_{i\in N}\sum_{e\in E} n_e(P) - \frac{1}{N}\sum_{i\in N}\sum_{e\in A_i} n_e(P) \\
&= \sum_{e\in E} n_e(P)n_e(A) + \sum_{e\in E} n_e(P) \\
&\quad -\frac{1}{N}\sum_{e\in E} n_e(P)n_e(A) \\
&= \frac{N-1}{N}\sum_{e\in E} n_e(P)n_e(A) + \sum_{e\in E} n_e(P) \\
&= \frac{N-1}{N}\sum_{e\in E}(n_e(P)n_e(A) + n_e(P)) \\
&+ \frac{1}{N}\sum_{e\in E} n_e(P)
\end{aligned}$$

With the help of lemma 1 we finally get

$$\begin{aligned}
\textsc{Sum}(A) &\leq \frac{N-1}{3N}\sum_{e\in E} n_e^2(A) + \frac{5N-2}{3N}\sum_{e\in E} n_e^2(P) \\
&= \frac{N-1}{3N}\textsc{Sum}(A) + \frac{5N-2}{3N}\textsc{Sum}(P)
\end{aligned}$$

and the theorem follows.

□

THEOREM 4. *There are instances of symmetric linear congestion games for which the price of anarchy of the average social cost is* $(5N-2)/(2N+1)$.

PROOF. We construct a game as follows: We partition the facilities into sets $P_1, P_2, \ldots, P_N$ of the same cardinality and make each P_i a pure strategy. At the optimal allocation player i plays P_i.

We now define a Nash equilibrium as follows: Each P_i contains $N\alpha_1 + \binom{N}{2}\alpha_2$ facilities where α_1, α_2 are appropriate constants to be determined later. At the Nash equilibrium, each player i plays alone α_1 of the facilities of each P_j. Also, each pair of players i, k play together α_2 of the facilities of each P_j. At the Nash equilibrium, the cost for player i is $c_i(A) = N(\alpha_1 + 2(N-1)\alpha_2)$.

We select α_1, α_2 so that player i will not switch to P_j. (It is trivial that player i will not switch to the Nash strategy of some other player k.) The cost after switching is

$$\begin{aligned}
c_i(A_{-i}, P_j) &= \alpha_1 + 2(N-1)\alpha_2 \\
&+ 2(N-1)\alpha_1 + 3\binom{N-1}{2}\alpha_2 \\
&= (2N-1)\alpha_1 + (N-1)\frac{(3N-2)}{2}\alpha_2
\end{aligned}$$

We want $c_i(A) = c_i(A_{-i}, P_j)$, or equivalently $\alpha_1 = \frac{N+2}{2}\alpha_2$, which is satisfied when we select $\alpha_1 = N+2$ and $\alpha_2 = 2$.

With this, the cost of each player i at the Nash equilibrium is $c_i(A) = N(\alpha_1 + 2(N-1)\alpha_2) = N(5N-2)$ and the cost of each player at the optimal allocation is $|P_i| = N\alpha_1 + \binom{N}{2}\alpha_2 = N(2N+1)$. The theorem follows. □

3.3 Asymmetric games - Maximum social cost

THEOREM 5. *The pure price of anarchy is* $O(\sqrt{N})$ *where* N *is the number of players.*

PROOF. We will make use of Theorem 1 which bounds the average cost. Let A be a Nash equilibrium strategy profile and let P be an optimal strategy profile. Without loss of generality, the first player has maximum cost, i.e., $\textsc{Max}(A) = c_1(A)$. It suffices to bound $c_1(A)$ with respect to $\textsc{Max}(P) = \max_{j\in N} c_j(P)$.

Since A is a Nash equilibrium, we have

$$c_1(A) \leq \sum_{e\in P_1}(n_e(A)+1) \leq \sum_{e\in P_1} n_e(A) + c_1(P). \quad (1)$$

Let $I \subset N$ the subset of players in A that use facilities $f \in P_1$. The sum of their costs is

$$\sum_{i\in I} c_i(A) \geq \sum_{e\in P_1} n_e^2(A) \geq \frac{(\sum_{e\in P_1} n_e(A))^2}{|P_1|}.$$

On the other hand, by Theorem 1

$$\sum_{i\in N} c_i(A) \leq \frac{5}{2}\sum_{i\in N} c_i(P)$$

Combining the last two inequalities, we get

$$\begin{aligned}
(\sum_{e\in P_1} n_e(A))^2 &\leq |P_1|\sum_{i\in I} c_i(A) \\
&\leq |P_1|\sum_{i\in N} c_i(A) \\
&\leq \frac{5}{2}|P_1|\sum_{i\in N} c_i(P)
\end{aligned}$$

Together with (1), we get

$$c_1(A) \leq c_1(P) + \sqrt{\frac{5}{2}|P_1|\sum_{i\in N} c_i(P)}.$$

Since $|P_1| \leq c_1(P)$ and $c_j(P) \leq \textsc{Max}(P)$, we get that $c_1(A) \leq (1+\sqrt{5/2\,N})\textsc{Max}(P)$. □

The proof above may seem to employ some crude approximations, but it gives the best possible bound (up to a constant factor), as the following lower-bound lemma shows.

THEOREM 6. *There are instances of linear congestion games (even network ones) for which the pure price of anarchy of the maximum social cost is* $\Omega(\sqrt{N})$, *where* N *is the number of players.*

PROOF. For convenience, let the number of players be $N = k^2 - k + 1$ for some integer k. We will construct a game in which player 1 has the maximum cost among the players at the worst-case Nash equilibrium.

There are kN facilities in total which are partitioned into N sets $P_i = \{f_{i,\ell} : \ell = 1, \ldots, k\}$. Each P_i is a strategy for player i; the optimal allocation will be for player i to play P_i. To construct a Nash equilibrium we add for each player $i > 1$ an alternative strategy $A_i = \{f_{1,\lceil\frac{i-1}{k}\rceil}\}$. Notice that player 1 has no alternative strategy.

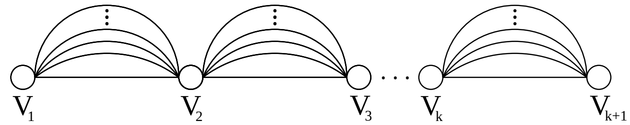

Figure 2: There is one player who goes from V_1 to V_{k+1}. For each i, there are $k-1$ players who go from V_i to V_{i+1}. In each layer $[V_i, V_{i+1}]$, there are k disjoint paths, one has length 1 and the rest have length k. The optimum allocation is for every player who goes from V_i to V_{i+1} to use separate length k paths and the player who moves from V_1 to V_{k+1}, to use the length 1 path. At the Nash equilibrium every player uses the length 1 path in every layer.

The strategy profile $A = (P_1, A_2, \ldots, A_n)$ is a Nash equilibrium in which player 1 has cost $c_1(A) = k^2$. On the other hand, the optimal strategy profile $P = (P_1, P_2, \ldots, P_n)$ has cost $c_i(P) = k$ for every player i. So the price of anarchy is $k = \sqrt{N} + O(1)$.

This is not exactly a network congestion game, but it can be turned into one as shown in Figure 2. □

3.4 Symmetric games - Maximum social cost

When we restrict the class to symmetric linear congestion games, the price of anarchy of the maximum social cost drops to 5/2, as the following Theorem shows.

THEOREM 7. *The pure price of anarchy of symmetric linear congestion games for the maximum social cost is at most* $\frac{5}{2}$.

PROOF. Let A be a Nash equilibrium and P an optimal allocation of a symmetric game. Without loss of generality, we can assume that player 1 has the maximum cost, i.e., $\text{MAX}(A) = c_1(A)$. As this game is symmetric, A is a Nash equilibrium only if player 1 has no reason to switch to P_j, for every $j \in N$:

$$c_1(A) \le c_1(A_{-1}, P_j) \le \sum_{e \in P_j} (n_e(A) + 1).$$

If we sum these inequalities for every j, we get:

$$N \cdot c_1(A) \le \sum_{e \in E} n_e(P)(n_e(A) + 1).$$

Using Lemma 1, the last expression is at most $\frac{1}{3}\sum_{e\in E} n_e^2(A) + \frac{5}{3}\sum_{e\in E} n_e^2(P)$. We can now use Theorem 1 to further bound $\sum_{e\in E} n_e^2(A) \le \frac{5}{2}\sum_{e\in E} n_e^2(P)$ and get

$$N \cdot c_1(A) \le \frac{5}{2}\sum_{e\in E} n_e^2(P) \le \frac{5}{2} N \cdot \text{MAX}(P),$$

and the proof is complete. □

This is tight in the limit as the lower bound of the following theorem shows.

THEOREM 8. *There are instances of symmetric congestion games for which the price of anarchy is* $\frac{5N+1}{2N+2}$, *for maximum social cost.*

PROOF. We construct a game possessing a Nash equilibrium A and an optimal allocation P. Player 1 has the maximum cost: $\text{MAX}(A) = c_1(A)$ and the rest of the players $i, j \neq 1$ have equal cost $c_i(A) = c_j(A)$. The optimal players have identical cost: $\text{MAX}(P) = c_i(P)$ for all i.

Let $P_1, P_2, \ldots, P_N$ be the disjoint strategies, of the same cardinality, of the optimal allocation. We define the Nash equilibrium as follows. Each P_i contains α_1 facilities that Nash player j plays alone for all $j \neq 1$. Also each P_i has α_2 facilities that Nash players j, k share for all $j, k \neq 1$. We define later parameters α_1, α_2.

So the cost for player $i \neq 1$ at the Nash equilibrium is

$$c_i(A) = \sum_{j\in N}(\alpha_1 + 2(N-2)\alpha_2) = N\alpha_1 + 2N(N-2)\alpha_2$$

We need to choose α_1, α_2 such that player i won't deviate to each of P_j. The cost of this deviation is

$$c_i(P_j, A_{-i}) = \alpha_1 + 2(N-2)\alpha_2 + 2(N-2)\alpha_1 + 3\binom{N-2}{2}\alpha_2$$

We want $c_i(A) = c_i(P_j, A_{-i})$ so

$$\frac{(N-2)(N+5)}{2}\alpha_2 = (N-3)\alpha_1 \quad (2)$$

The cost of an optimal player i is

$$c_i(P) = \sum_{e\in P_i} = (N-1)\alpha_1 + \binom{N-1}{2}\alpha_2$$

The cost of Nash player 1 is less or equal than the cost he would have if he deviated to any other strategy

$$c_1(A) = c_1(P_i, A_{-1}) = 2(N-1)\alpha_1 + 3\binom{N-1}{2}\alpha_2$$

Notice that we can construct such a strategy for player 1, disjoint to any other strategy and of suitable cardinality.

The price of anarchy using equation (2) is

$$PA = \frac{c_1(A)}{c_j(P)} = \frac{5N+1}{2N+2}.$$

□

4. POLYNOMIAL LATENCY FUNCTIONS

In this section we turn our attention to latency functions that are polynomials of bounded degree p, and in particular of the form

$$f_e(N) = \sum_{i=0}^{p} \alpha_i(e) N^i, \qquad a_i(e) \ge 0$$

The cost of a player i in a strategy profile A is

$$c_i(A) = \sum_{e\in A_i} f_e(n_e(A))$$

and the sum of all costs is

$$\text{SUM}(A) = \sum_{i\in N} c_i(A) = \sum_{e\in E} n_e(A) f_e(n_e(A))$$

The theorems and proofs about linear functions of the previous section can be extended to polynomials, in most cases with little effort. (Actually, we wrote part of the previous section with this in mind.)

4.1 Average social cost

The following lemma corresponds to Lemma 1.

LEMMA 2. *Let $f(x)$ a polynomial in x, with nonnegative coefficients, of degree p. Then for every nonnegative x and y:*

$$y \cdot f(x+1) \leq \frac{x \cdot f(x)}{2} + \frac{C_0(p) \cdot y \cdot f(y)}{2}$$

where $C_0(p) = p^{p(1-o(1))}$. The term $o(1)$ hides logarithmic terms in p.

THEOREM 9. *For polynomial latency functions of degree p, the pure price of anarchy for the average social cost is at most $p^{p(1-o(1))}$.*

PROOF. Let A be a Nash strategy profile and P an optimal strategy profile. Player i has no incentive to switch to strategy P_i when

$$c_i(A) = \sum_{e \in A_i} f_e(n_e(A)) \leq \sum_{e \in P_i} f_e(n_e(A)+1)$$

If we sum over all $i \in N$, and use Lemma 2, we get

$$\begin{aligned} \text{SUM}(A) &\leq \sum_{e \in E} n_e(P) f_e(n_e(A)+1) \\ &\leq \sum_{e \in E} \frac{n_e(A) f(n_e(A))}{2} + \\ &+ \sum_{e \in E} \frac{C_0(p) n_e(P) f(n_e(P))}{2} \end{aligned}$$

which is equal to $\frac{\text{SUM}(A)}{2} + \frac{C_0(p)\text{SUM}(P)}{2}$ and the proof is complete. □

We give below a matching lower bound. Both the upper and the lower bounds are of the form $p^{p(1-o(1))}$ but they are not exactly equal.

THEOREM 10. *There are instances of symmetric congestion games for which the price of anarchy is at least $p^{p(1-o(1))}$, for both max and sum social cost.*

PROOF. Let $P_1, P_2, \ldots, P_N$ be the disjoint strategies of the optimal allocation. We will construct a bad Nash equilibrium as follows: Each P_j has N facilities $f_{j,k}$ for $k = 1, \ldots, n$. At the Nash equilibrium $A_i = \{f_{j,k} : k \neq i\}$.

So the cost for player i at the Nash equilibrium is

$$c_i(A) = N(N-1)(N-1)^p$$

Player i has no incentive to switch to P_j when

$$c_i(A) \leq c_i(A_{-i}, P_j) = (N-1)^{p+1} + N^p$$

So, we select N to satisfy $(N-1)^{p+2} = N^p$. Since the optimum has social cost $c_i(P) = N$, the price of anarchy is $\frac{(N-1)^{p+2}}{N} = p^{p(1-o(1))}$.

The bound holds not only for this particular number of players N but for any integral multiple of it, by appropriately replicating the above construction. □

4.2 Maximum social cost

THEOREM 11. *There are instances of congestion games with polynomial latency functions for which the pure price of anarchy is $\Omega(N^{p/(p+1)})$.*

PROOF SKETCH. The proof is very similar to that of Theorem 6. In this case, the number of players is $N = k^{p+1} - k^p + 1$, and the number of facilities Nk^p. The cost of player 1 is $c_1(A) = (k^p)^2$ while every optimal player has cost k^p. The price of anarchy is $k^p = \Omega(N^{p/(p+1)})$.

Again, this can be turned into a network congestion game, similar to that of Figure 2 with k^p layers where each path inside a layer has also length k^p. □

The following upper bound is trivial:

THEOREM 12. *The pure price of anarchy for polynomial latencies is $O(N)$.*

Also, Theorem 7 can be directly extended to the case of polynomial latencies:

THEOREM 13. *The pure price of anarchy of symmetric congestion games with polynomial latencies of degree p is $O(p^{p(1-o(1))})$.*

5. THE MIXED PRICE OF ANARCHY

From Yossi Azar we learned that he and his collaborators had similar results for the case of total latency social cost and mixed Nash equilibria [1]. We then realized that some of our proofs apply directly to the mixed case as well. In particular, Lemma 1 should be relaxed to deal with reals instead of integers as follows:

LEMMA 3. *For every non negative reals x, y, it holds*

$$y(x+1) \leq \frac{\sqrt{5}-1}{4}x^2 + \frac{\sqrt{5}+5}{4}y^2$$

With this, the proof of Theorem 1 gives that the mixed price of anarchy for linear latencies is at most $\frac{3+\sqrt{5}}{2}$.

One should be careful how to define the social cost in this case. There are two ways to do it: The social cost is the average (or sum) of the expected cost of all players $\text{SUM} = \sum_{i \in N} c_i(N)$. Or, the social cost is the sum of the squares of the latencies in all facilities: $\text{SUM} = \sum_{e \in E}(n_e(A))^2$. The two are equal for pure Nash equilibria as well as for non-atomic games, but they may be different for mixed equilibria or for weighted games. From the system's designer point of view who cares about the welfare of the players, the first social cost seems to be the right choice. In any case, our proof applies to both social costs with the same price of anarchy.

THEOREM 14. *The mixed price of anarchy of linear congestion games and for the average social cost is at most $\frac{3+\sqrt{5}}{2} \approx 2.618$.*

Similarly, Theorem 9 holds also for mixed Nash equilibria.

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