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The Price of Selfish Routing[1]

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Abstract. We study the problem of *routing* traffic through a congested network. We focus on the simplest case of a network consisting of m parallel *links*. We assume a collection of n network *users*; each user employs a *mixed strategy*, which is a probability distribution over links, to control the shipping of its own assigned *traffic*. Given a *capacity* for each link specifying the rate at which the link processes traffic, the objective is to route traffic so that the maximum (over all links) *latency* is minimized. We consider both *uniform* and *arbitrary* link capacities.

How much decrease in global performance is necessary due to the absence of some central authority to regulate network traffic and implement an optimal assignment of traffic to links? We investigate this fundamental question in the context of *Nash equilibria* for such a system, where each network user selfishly routes its traffic only on those links available to it that minimize its *expected latency cost*, given the network congestion caused by the other users. We use the *Coordination Ratio*, originally defined by Koutsoupias and Papadimitriou [16], as a measure of the cost of lack of coordination among the users; roughly speaking, the Coordination Ratio is the ratio of the expectation of the maximum (over all links) latency in the *worst* possible Nash equilibrium, over the *least* possible maximum latency had global regulation been available.

Our chief instrument is a set of combinatorial *Minimum Expected Latency Cost Equations*, one per user, that characterize the Nash equilibria of this system. These are linear equations in the minimum expected latency costs, involving the user traffics, the link capacities, and the routing pattern determined by the mixed strategies. In turn, we solve these equations in the case of *fully mixed strategies*, where each user assigns its traffic with a strictly positive probability to every link, to derive the first *existence* and *uniqueness* results for *fully mixed Nash equilibria* in this setting. Through a thorough analysis and characterization of fully mixed Nash equilibria, we obtain *tight* upper bounds of no worse than $O(\ln n / \ln \ln n)$ on the Coordination Ratio for (i) the case of uniform capacities and arbitrary traffics and (ii) the case of arbitrary capacities and identical traffics.

Key Words. Nash equilibria, Coordination ratio, Selfish routing, Fully mixed Nash equilium.

1. Introduction

1.1. *Motivation-Framework.* We study a *selfish routing* problem in *noncooperative networks*; in this problem, paths from a *source* to a *destination* are to be established by

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a collection of independent entities called *users*. So, users correspond to different traffic sources, each seeking to determine the shipping of its own traffic over a shared network. However, in doing so, different users may have to optimize completely different (and even conflicting) measures of performance and demand. Such noncooperative and antagonistic scenaria apply to various modern networking environments, where the possibility of centrally optimizing given performance objectives is absent.

A natural framework for such multiobjective problems with multiple entities seeking to optimize their own payoffs in a noncooperative network is (*Noncooperative*) *Game Theory*. An appropriate, game-theoretic concept for the solution is *Nash equilibrium* [21], [22]. Roughly speaking, the operating points of a noncooperative network are the Nash equilibria of the underlying game among the users; these are points where unilateral deviation does not help any user to improve its performance. The mission of this work is to study the inherent cost incurred at Nash equilibrium due to the lack of a central authority to monitor and engineer network operation according to global objectives; this cost is often referred to as the *Price of Anarchy* [23].

We consider the simplest case of a network consisting of m parallel *links*. Here, each of n users fixes a *mixed strategy*, which is a probability distribution over links; the distribution determines the (possibly zero) probability for the user to ship its traffic through each link. We model the *latency* over each link as the ratio of the total traffic assigned to the link over its *capacity*; each user is charged, for each link it chooses, a *latency cost* equal to the latency over the link. Here, Nash equilibrium requires that for each user, its *expected latency cost* be constant across all links that are potential carriers of the user's traffic, given the congestion caused by other users; moreover, this (constant) expected latency cost should not exceed the expected latency cost of the user on any of the remaining links. So, in a Nash equilibrium no user could unilaterally decrease its expected latency cost by switching to a different strategy.

We adopt *Coordination Ratio* [16] as the measure of performance loss due to unregulated traffic in a congested network. Roughly speaking, the Coordination Ratio is the ratio of the *Social Cost* (specifically, the *expected maximum latency* in the setting we consider) in the *worst* possible Nash equlibrium, over the *Social Optimum*; the latter is the best "off-line" global cost (specifically, the *least possible* maximum latency in our setting) had all information been available to a central network authority regulating traffic.

1.2. *Results*, *Techniques*, *and Contribution*. Our departure point is a linear system of equations for the minimum expected latency costs (Proposition 4.1), which we call *Minimum Expected Latency Cost Equations*. These equations are specific to any arbitrary but fixed Nash equilibrium of the system; they are inspired by (and reminiscent of) classical equations of stochastic equilibrium, such as the *Chapman–Kolmogorov equation* which describes the steady-state equilibrium of a Markov chain (see Section 6.1 of [13]). Their coefficients and constant terms depend on the link capacities and the user traffics, as well as on the routing pattern (which user can use which links).

For the rest of our study, we focus on a natural and intuitively promising special case of routing pattern that we call *fully mixed strategies*; here, each user assigns strictly positive probability to every link. Intuitively, fully mixed strategies favor collisions of users across the links. Since collisions increase the maximum link latency, they are expected to increase Social Cost, which is the expectation of maximum link latency. This informal intuition behind fully mixed strategies suggests that they deserve study, since they provide a natural candidate for Nash equilibria maximizing Social Cost and Coordination Ratio; such equilibria have been called *worst-case equilibria* by Koutsoupias and Papadimitriou [16].

Besides their purely intuitive appeal as worst-case equilibria, fully mixed strategies are found to facilitate the solvability of the Minimum Expected Latency Cost Equations. Specifically, we are able to explicitly solve them for the case of fully mixed strategies and derive each user's probabilities in a Nash equilibrium with fully mixed strategies, henceforth called *fully mixed Nash equilibrium*; these probabilities will be referred to as *Nash probabilities*. We discover that, for the fully mixed Nash equilibrium, the Nash probabilities, cast as functions of link capacities and user traffics, enjoy a particularly insightful form (Proposition 4.4).

The requirement that the derived expressions for the fully mixed Nash probabilities represent probabilities that suffice for a Nash equilibrium yields the *first* existence and uniqueness result for fully mixed Nash equilibria in this setting (Theorem 4.7).

We use our improved understanding of the structure of fully mixed Nash equilibria to derive some new bounds on Coordination Ratio in two significant special cases of the problem.

- We first consider the case where link capacities are *uniform*, while user traffics may vary arbitrarily. Here, we observe that in a Nash equilibrium all links are *equiprobable* for each user. This allows direct use of simple results from the classical theory of *random allocations* (see, e.g., [14]), where each of *n balls* is thrown into one of *m bins*, chosen *uniformly* at random. So, in our study, we treat users and links as balls and bins, respectively.
- We also consider the case where user traffics are identical, but link capacities vary arbitrarily while sufficing for a fully mixed Nash equilibrium to exist (Corollary 4.5). In this case, links are no longer equiprobable for any particular user; nevertheless, the constraints imposed by Nash equilibrium still enable us to recall the theory of random allocations.

To prove bounds on Coordination Ratio, we develop a modular methodology that may be applicable to other instances of the problem, and even to other settings with different performance measures. The methodology consists of three major steps:

- The first step establishes a probabilistic *tail lemma*. Roughly speaking, a tail lemma assumes a *tail inequality* for the maximum number of users choosing any particular link (see Chapter 4 of [20]); that is, it is assumed that this number enjoys a sharp concentration around its expectation. Using this assumption, the tail lemma establishes an upper bound on Social Cost that may depend on parameters specifying the sharpness of the concentration. (See Propositions 5.2 and 6.2 for the two cases we consider, respectively.)
- The second step establishes a particular tail inequality for the maximum number of users choosing any particular link; this step employs the form of Nash probabilities, the constraints on link capacities and user traffics that are necessary for a Nash equilibrium (Corollary 4.5), and standard probabilistic tools such as Chernoff-like bounds [2].

The first two steps establish together a concrete upper bound on Social Cost. (See Propositions 5.3 and 5.4, and Proposition 6.3 for the two cases we consider, respectively.)

- The third step shows a *lower* bound on Social Optimum. (See Lemmas 5.5 and 6.4 for the two cases we consider, respectively.)

Overall, the proposed methodology implies concrete upper bounds on the Coordination Ratio for each specific case we consider; the bounds obtained are as follows:

- For the case of uniform capacities and arbitrary traffics:
  - Assuming that $m = n$, we prove an upper bound of $\lceil (3 \ln n)/\ln \ln n \rceil + 1$ (Theorem 5.6).
  - Assuming that $m \leq n/(16 \ln n)$, we prove an upper bound of $\frac{3}{2} + o(1)$ times the ratio of the maximum over the average user traffics (Theorem 5.7).
- For the case of arbitrary capacities and identical traffics, and assuming that $m \leq n$, we prove a strict upper bound of $(2+o(1))\ \lceil (3 \ln n)/\ln \ln n \rceil + 16/(4 - e)$ (Theorem 6.6).

This article promotes the fully mixed Nash equilibrium as a candidate equilibrium for an intuitively hard problem instance with respect to Social Cost and Coordination Ratio. We emphasize, however, that attempting to make any formal claims for this apparent hardness has remained outside the scope of the present article. We believe that the significance of fully mixed strategies, as such a central class, is due to be revealed. (See Section 7 for a discussion on some follow-up work that supports this belief.) Finally, we believe that the approach to studying the structure of Nash equilibria set forth here and the proposed analytical methodology will prove instrumental to settling other problem instances as well.

1.3. *Related Work and Comparison*. The KP model [16] initiated the algorithmic study of performance degradation caused by lack of traffic regulation in a congested network. Koutsoupias and Papadimitriou [16] obtained tight bounds on Coordination Ratio for the case where $m = 2$ and less tight ones for the general case (under both uniform and arbitrary capacities).

Our first and third bounds on the Coordination Ratio (Theorems 5.6 and 6.6) match asymptotically a corresponding lower bound of $\Omega(\ln n/\ln \ln n)$ shown by Koutsoupias and Papadimitriou [16, Theorem 6], which was conjectured by them to be the *right* bound for the model of uniform capacities. On the other hand, our second bound (Theorem 5.7) identifies the *first* conditions on (nonconstant) m and n allowing for a bound independent of m and n. Moreover, our first and second bounds surpass a *general* upper bound of $O(\sqrt{m \ln m})$ shown by Koutsoupias and Papadimitriou [16, Theorem 8] for the model of uniform capacities; that bound, however, holds for *all* possible Nash equilibria, while our bounds hold for fully mixed strategies and under particular assumptions on the relation between m and n. In the same vein, consider a corresponding (*general*) upper bound of $O\left(\sqrt{(c^m/c^1) \sum_{j \in [m]} (c^j/c^1)} \sqrt{\ln m}\right)$ shown by Koutsoupias and Papadimitriou [16, Theorem 9] for the model of arbitrary capacities; clearly, this bound can be no better in order than $\Theta(\sqrt{m \ln m})$. Hence, our third bound of $O(\ln n/\ln \ln n)$ (shown by taking $m \leq n$) surpasses [16, Theorem 9] in all cases where $\ln n/\ln \ln n \in o(\sqrt{m \ln m})$.

Inspired by the interest in Coordination Ratio, Roughgarden and Tardos [24] (and much subsequent work) investigated the degradation in network performance due to unregulated traffic for the earlier *Wardrop model* [25], which considers *splittable* traffics.

Some recent follow-up work on selfish routing in the KP model includes [3], [5], [7], [10], [15], and [18]. (See Section 7 for an expanded discussion on some of this follow-up work.) An excellent survey of research on the KP model, with some emphasis on results related to the fully mixed Nash equilibrium, appeared recently in [12].

1.4. *Road Map*. Section 2 introduces the KP model, summarizes some background material, and establishes some preliminary facts. Section 3 outlines random allocations. The structure of Nash equilibria is studied in Section 4. The case of fully mixed strategies is treated in Sections 5 and 6 under the models of uniform capacities and arbitrary capacities, respectively. We conclude, in Section 7, with a discussion of our results and suggestions for further research.

2. Definitions, Background, and Preliminaries

2.1. *Notation and Preliminary Facts*. Throughout, for any integer $m \geq 2$, denote $[m] = \{1, \ldots, m\}$. For a real interval (a, b) and a real $\delta > 0$, $(a, b) + \delta$ denotes the real interval $(a + \delta, b + \delta)$. For all integers $m \geq 2$ and $n \geq 2$, let $\mathbf{J}_{m\times n}$ denote the matrix with all entries in its m rows and n columns equal to 1; let $\mathbf{I}_{n\times n}$ denote the *identity matrix* with n rows and n columns; all of its entries vanish except for those on the main diagonal which are equal to 1. Let e denote the base of the natural logarithm. It is well known that for any sufficiently large integer n and for an integer $\vartheta \leq n$, $\binom{n}{\vartheta} \leq (ne/\vartheta)^{\vartheta}$.

For an event E in a sample space, $\mathbf{Pr}(E)$ denotes the probability of event E happening. For a random variable X, $\mathcal{E}(X)$ denotes the *expectation* of X. A *tail probability* for X is the probability for X to take values away from its expectation (see Chapter 4 of [20]).

The following simple fact will be useful for proving bounds on tail probabilities.

CLAIM 2.1. *Assume* $\rho = \lceil 3\ln n/\ln\ln n\rceil > 3$. *Then* $(e/\rho)^{\rho} \leq 1/n^2$.

PROOF. Since $\rho > 3$, it follows that $e/\rho < 1$, so that

$$\left(\frac{e}{\rho}\right)^{\rho} = \left(\frac{e}{\rho}\right)^{\lceil (3\ln n)/\ln\ln n\rceil} \leq \left(\frac{e}{\rho}\right)^{(3\ln n)/\ln\ln n} = \left(\frac{e}{\left\lceil\frac{3\ln n}{\ln\ln n}\right\rceil}\right)^{(3\ln n)/\ln\ln n}$$
$$\leq \left(\frac{e}{\frac{3\ln n}{\ln\ln n}}\right)^{(3\ln n)/\ln\ln n} = \left(\frac{e\ln\ln n}{3\ln n}\right)^{(3\ln n)/\ln\ln n}.$$

Thus, it suffices to show that

$$\left(\frac{e\ln\ln n}{3\ln n}\right)^{(3\ln n)/\ln\ln n} \leq \frac{1}{n^2},$$

or, by taking natural logarithms on both sides and using the increasing monotonicity of the logarithmic function, that

$$\frac{3\ln n}{\ln\ln n}\ \ln\left(\frac{e\ln\ln n}{3\ln n}\right)\ \leq\ -2\ln n,$$

or, since $\ln\ln n > 0$ and $\ln n > 1$, that

$$3\ \left(\ln\frac{e}{3} + \ln\ln\ln n - \ln\ln n\right)\ \leq\ -2\ \ln\ln n,$$

or

$$3\ \left(\ln\frac{e}{3} + \ln\ln\ln n\right)\ \leq\ \ln\ln n,$$

which holds as an equality for $\ln\ln n = 3$ and as a strict inequality for all $n > 3$, as needed. $\square$

2.2. *Model*. Our presentation of the KP model is patterned after the original one in Sections 1 and 2 of [16].

We consider a *network* consisting of a set of $m \geq 2$ parallel *links* $1, 2, \ldots, m$ from a *source* node to a *destination* node. Each of $n \geq 2$ *users* $1, 2, \ldots, n$ wishes to route a particular amount of traffic along a (nonfixed) link from source to destination. (Throughout, we will be using subscripts for users and superscripts for links.) Let w_i denote the *traffic* of user $i \in [n]$. Define the $n \times 1$ *traffic vector* $\mathbf{w}$ in the natural way. For a traffic vector $\mathbf{w}$, $\mathsf{max/avg}(\mathbf{w})$ denotes the ratio of the maximum over the average traffic in $\mathbf{w}$. In the model of *identical traffics*, all user traffics are equal to 1; they vary arbitrarily in the model of *arbitrary traffics*.

A *pure strategy* for user $i \in [n]$ is some specific link. A *mixed strategy* for user $i \in [n]$ is a probability distribution over pure strategies; thus, a mixed strategy is a probability distribution over the set of links. The *support* of the mixed strategy for user $i \in [n]$ is the set of pure strategies (links) to which i assigns positive probability. A *pure strategies profile* is represented by an n-tuple $\langle \ell_1, \ell_2, \ldots, \ell_n \rangle \in [m]^n$; a *mixed strategies profile* is represented by an $n \times m$ *probability matrix* $\mathbf{P}$ of nm probabilities p_i^ℓ, $i \in [n]$ and $\ell \in [m]$, where p_i^ℓ is the probability that user i chooses link ℓ. In this work, we shall be mainly concerned with mixed strategies.

For a probability matrix $\mathbf{P}$, define *indicator variables* $I_i^\ell \in \{0, 1\}$, $i \in [n]$ and $\ell \in [m]$, such that $I_i^\ell = 1$ if and only if $p_i^\ell > 0$. Thus, the support of the mixed strategy for user $i \in [n]$ is the set $\{\ell \in [m] \mid I_i^\ell = 1\}$. In the *fully mixed case*, $I_i^\ell = 1$ for all users $i \in [n]$ and links $\ell \in [m]$; here, each user assigns its traffic on each link with positive probability, and its support is $[m]$.

A *solo link* is a link $\ell \in [m]$ such that $\sum_{k=1}^{n} I_k^\ell = 1$. Clearly, there is a *single* user $s(\ell)$ such that $I_k^\ell = 1$ if $k = s(\ell)$, while $I_k^\ell = 0$ otherwise; thus, the solo link ℓ can be traversed only by user $s(\ell)$. Since $n > 1$, it follows that there are no solo links in the fully mixed case. Let $\mathcal{S} \subseteq [m]$ denote the set of solo links. A *nonsolo link* is a link that is not solo. For each link $\ell \in [m]$, the random variable θ^ℓ is the number of users choosing link ℓ.

Let c^ℓ denote the *capacity* of link $\ell \in [m]$, representing the rate at which the link processes traffic. So, the *latency* for traffic w through link ℓ equals w/c^ℓ. In the model of *uniform capacities*, all link capacities are equal to 1; they vary arbitrarily in the model of *arbitrary capacities*. Define the $m \times n$ *capacity matrix* $\mathbf{C}$ with all entries in row ℓ equal to c^ℓ; so,

$$\mathbf{C} = \begin{pmatrix} c^1 & c^1 & \cdots & c^1 \\ c^2 & c^2 & \cdots & c^2 \\ \vdots & \vdots & \ddots & \vdots \\ c^m & c^m & \cdots & c^m \end{pmatrix}.$$

For a pure strategies profile $\langle \ell_1, \ell_2, \ldots, \ell_n \rangle$, the *latency cost for user* $i \in [n]$, denoted λ_i, is $(\sum_{k|\ell_k=\ell_i} w_k)/c^{\ell_i}$; that is, the latency cost for user i is the latency of the link it chooses.

For a mixed strategies profile $\mathbf{P}$, let W^ℓ denote the *expected traffic* on link $\ell \in [m]$; clearly, $W^\ell = \sum_{i=1}^n p_i^\ell w_i$. Given $\mathbf{P}$, define the $m \times 1$ *expected traffic vector* $\mathbf{W}$ in the natural way. For a mixed strategies profile $\mathbf{P}$, the *expected latency cost* for user $i \in [n]$ on link $\ell \in [m]$, denoted λ_i^ℓ, is the expectation, over all random choices of the remaining users, of the latency cost for user i when its traffic is assigned to link ℓ; thus, clearly,

$$\begin{aligned} \lambda_i^\ell &= \frac{w_i + \sum_{k=1,k\neq i}^n p_k^\ell w_k}{c^\ell} \\ &= \frac{w_i - p_i^\ell w_i + \sum_{k=1}^n p_k^\ell w_k}{c^\ell} \\ &= \frac{(1-p_i^\ell) w_i + W^\ell}{c^\ell}. \end{aligned}$$

For each user $i \in [n]$, the *minimum expected latency cost* λ_i is the minimum, over all links $\ell \in [m]$, of the expected latency cost for user i on link ℓ; so, $\lambda_i = \min_{\ell\in[m]} \lambda_i^\ell$. For a probability matrix $\mathbf{P}$, define the $n \times 1$ *minimum expected latency cost vector* λ in the natural way.

We are interested in a special class of mixed strategies called Nash equilibria [21] that we describe below. Formally, $\mathbf{P}$ is a *Nash equilibrium* if for all users $i \in [n]$ and links $\ell \in [m]$,

$$\lambda_i^\ell \begin{cases} = \lambda_i & \text{if} \quad I_i^\ell = 1, \\ \geq \lambda_i & \text{if} \quad I_i^\ell = 0. \end{cases}$$

Thus, each user assigns its traffic with positive probability only on links (possibly more than one of them) for which its expected latency cost is minimized; this implies that there is no incentive for a user to unilaterally deviate from its mixed strategy. Call (*fully mixed*) *Nash probabilities* the probabilities in a (fully mixed) Nash equilibrium.

Associated with a traffic vector $\mathbf{w}$ and a mixed strategies profile $\mathbf{P}$ is the *Social Cost* [16, Section 2], denoted $\mathsf{SC}(\mathbf{w}, \mathbf{P})$, which is the expectation, over all random choices of the users, of the maximum (over all links) latency of traffic through a link; thus,

$$\mathsf{SC}(\mathbf{w}, \mathbf{P}) = \mathcal{E}\left(\max_{\ell\in[m]} \frac{\sum_{k:\ell_k=\ell} w_k}{c^\ell} \right) = \sum_{\langle \ell_1, \ell_2, \ldots, \ell_n \rangle \in [m]^n} \left(\prod_{k=1}^n p_k^{\ell_k} \cdot \max_{\ell\in[m]} \frac{\sum_{k:\ell_k=\ell} w_k}{c^\ell} \right).$$

On the other hand, the *Social Optimum* [16, Section 2] associated with a traffic vector $\mathbf{w}$, denoted $\mathsf{OPT}(\mathbf{w})$, is the *least possible* maximum (over all links) latency of traffic through a link; thus,

$$\mathsf{OPT}(\mathbf{w}) \;=\; \min_{\langle \ell_1, \ell_2, \ldots, \ell_n \rangle \in [m]^n} \max_{\ell \in [m]} \frac{\sum_{k:\ell_k = \ell} w_k}{c^\ell}.$$

Note that while $\mathsf{SC}(\mathbf{w}, \mathbf{P})$ is defined in relation to a mixed strategies profile $\mathbf{P}$, $\mathsf{OPT}(\mathbf{w})$ is realized by an *optimum* pure strategies profile. The *Coordination Ratio* [16], denoted CR, is the *maximum* value, over all traffic vectors $\mathbf{w}$ and Nash equilibria $\mathbf{P}$, of the ratio $\mathsf{SC}(\mathbf{w}, \mathbf{P})/\mathsf{OPT}(\mathbf{w})$.

2.3. *Properties of Nash Equilibria*. Koutsoupias and Papadimitriou [16, Section 2] provide necessary conditions for Nash equilibria.

PROPOSITION 2.2 [16]. *Take any Nash equilibrium* $\mathbf{P}$. *Then, for each user* $i \in [n]$ *and link* $\ell \in [m]$,

$$p_i^\ell \;=\; \frac{W^\ell + w_i - c^\ell \lambda_i}{w_i},$$

subject to

(1) *for each link* $\ell \in [m]$, $W^\ell = \sum_{k=1}^n I_k^\ell (W^\ell + w_k - c^\ell \lambda_k)$, *and*
(2) *for each user* $i \in [n]$, $w_i = \sum_{j=1}^m I_i^j (W^j + w_i - c^j \lambda_i)$.

We remark that the necessary conditions in Proposition 2.2 neither provide any apparent way of computing Nash probabilities nor say anything about their existence or uniqueness. It appears that existence and uniqueness are contingent upon the corresponding existence and uniqueness of solutions for $\mathbf{W}$ and λ to the conditions (1) and (2). We observe a simple rearrangement of terms in condition (1) that yields explicit expressions for the expected traffics on nonsolo links (in terms of the minimum expected latency costs). We prove:

LEMMA 2.3. *Take any Nash equilibrium* $\mathbf{P}$. *Then, for any nonsolo link* $\ell \in [m]$,

$$W^\ell \;=\; \frac{-\sum_{k=1}^n I_k^\ell w_k + c^\ell \sum_{k=1}^n I_k^\ell \lambda_k}{\sum_{k=1}^n I_k^\ell - 1}.$$

PROOF. Proposition 2.2 (condition (1)) implies that

$$\begin{aligned} W^\ell \;&=\; \sum_{k=1}^n I_k^\ell W^\ell + \sum_{k=1}^n I_k^\ell w_k - \sum_{k=1}^n I_k^\ell c^\ell \lambda_k \\ &=\; W^\ell \sum_{k=1}^n I_k^\ell + \sum_{k=1}^n I_k^\ell w_k - c^\ell \sum_{k=1}^n I_k^\ell \lambda_k, \end{aligned}$$

or

$$\left(\sum_{k=1}^{n} I_k^\ell - 1\right) W^\ell \;=\; -\sum_{k=1}^{n} I_k^\ell w_k + c^\ell \sum_{k=1}^{n} I_k^\lambda \lambda_k.$$

Since ℓ is a nonsolo link, $\sum_{k=1}^{n} I_k^\ell \neq 1$. This implies that

$$W^\ell \;=\; \frac{-\sum_{k=1}^{n} I_k^\ell w_k + c^\ell \sum_{k=1}^{n} I_k^\ell \lambda_k}{\sum_{k=1}^{n} I_k^\ell - 1},$$

as needed. □

2.4. *An Exact Lower Bound*. We conclude this section by establishing a simple lower bound on Coordination Ratio for the fully mixed case and under the model of uniform capacities; this bound is shown from first principles, and it holds for all possible values of m.

PROPOSITION 2.4. *Consider the fully mixed case under the model of uniform capacities. Then*,

$$\mathsf{CR} \;\geq\; m - \frac{1}{m^m} \cdot \sum_{\vartheta=1}^{m-1} \left(\sum_{\theta_1, \ldots, \theta_m \,\leq\, \vartheta} \binom{m}{\theta_1, \ldots, \theta_m} \right).$$

PROOF. For each user $i \in [n]$, set $w_i := 1$; set also $n := m$ and $c := 1$. Note that for these particular choices, $\mathsf{OPT}\,(\mathbf{w}) = 1$, since the *least possible* maximum (over all links) latency of traffic through a link is achieved by assigning each traffic to a different link. Hence, for any Nash equilibrium $\mathbf{P}$, $\mathsf{SC}\,(\mathbf{w}, \mathbf{P})$ is a lower bound on Coordination Ratio.

We will establish the claimed lower bound by specifying a suitable Nash equilibrium $\mathbf{P}$: for each user $i \in [n]$ and link $\ell \in [m]$, set $p_i^\ell = 1/m$. Notice that, by definition of expected latency cost, for each user $i \in [n]$ and link $\ell \in [m]$, $\lambda_i^\ell = 1 + (n-1)/m$, so that $\lambda_i = \min_{\ell\in[m]} \lambda_i^\ell = 1 + (n-1)/m = \lambda_i^\ell$ for any link $\ell \in [m]$. Thus, $\mathbf{P}$ is a Nash equilibrium. So,

$$\begin{aligned}
\mathsf{CR} \;&\geq\; \mathsf{SC}(\mathbf{w}, \mathbf{P}) \\
&=\; \mathcal{E}(\max_{\ell\in[m]} \theta^\ell) \\
&=\; \sum_{\vartheta=1}^{m} \vartheta \,\mathbf{Pr}\,(\max_{\ell\in[m]} \theta^\ell = \vartheta) \\
&=\; \sum_{\vartheta=1}^{m} \vartheta \,(\mathbf{Pr}(\max_{\ell\in[m]} \theta^\ell \leq \vartheta) - \mathbf{Pr}(\max_{\ell\in[m]} \theta^\ell \leq \vartheta - 1)) \\
&=\; m\,\mathbf{Pr}(\max_{\ell\in[m]} \theta^\ell \leq m) - \sum_{\vartheta=1}^{m-1} \mathbf{Pr}\left(\max_{\ell\in[m]} \theta^\ell \leq \vartheta\right) \qquad \text{(by sum telescoping)} \\
&=\; m - \sum_{\vartheta=1}^{m-1} \mathbf{Pr}(\max_{\ell\in[m]} \theta^\ell \leq \vartheta) \qquad (\text{since } \max_{\ell\in[m]} \theta^\ell \leq m).
\end{aligned}$$

For any fixed constant ϑ, $1 \leq \vartheta \leq m-1$, what is $\mathbf{Pr}\left(\max_{\ell\in[m]} \theta^\ell \leq \vartheta\right)$? It is the proportion of traffic assignments for which none of $\theta^1, \ldots, \theta^m$ exceeds ϑ among all possible traffic assignments. Since $\sum_{\ell\in[m]} \theta^\ell = m$, there are, clearly, $\sum_{\theta^1,\ldots,\theta^m\leq\vartheta} \binom{m}{\theta^1,\ldots,\theta^m}$ such traffic assignments, while the total number of traffic assignments is m^m. Thus,

$$\mathbf{Pr}\left(\max_{\ell\in[m]} \theta^\ell \leq \vartheta\right) = \frac{\sum_{\theta^1,\ldots,\theta^m\leq\vartheta} \binom{m}{\theta^1,\ldots,\theta^m}}{m^m},$$

so that

$$\begin{aligned}\mathsf{CR} &\geq m - \sum_{\vartheta=1}^{m-1} \sum_{\theta^1,\ldots,\theta^m\leq\vartheta} \frac{\binom{m}{\theta^1,\ldots,\theta^m}}{m^m} \\ &= m - \frac{1}{m^m} \sum_{\vartheta=1}^{m-1} \sum_{\theta^1,\ldots,\theta^m\leq\vartheta} \binom{m}{\theta^1,\ldots,\theta^m},\end{aligned}$$

as needed. □

We remark that Proposition 2.4 yields an *exact* lower bound on Coordination Ratio for *any* particular value of m, in the sense that its proof explicitly provides a traffic vector $\mathbf{w}$ and a Nash equilibrium $\mathbf{P}$ which together attain *exactly* the claimed lower bound. For example, for $m = 2$ and $m = 3$, Proposition 2.4 yields the exact lower bounds of $\frac{3}{2}$ and $\frac{51}{27} \approx 1.889$, respectively, on Coordination Ratio. (The lower bound of $\frac{3}{2}$ for $m = 2$ was shown before by Koutsoupias and Papadimitriou [16, Theorem 1].)

3. Random Allocations. In this section, we briefly outline some material on random allocations. The reader may prefer to skip this section for now, returning to it later when its results are required.

Recall that a discrete random variable X follows the *binomial distribution with parameters n and p* if for each integer ϑ, $0 \leq \vartheta \leq n$, $\mathbf{Pr}(X = \vartheta) = \binom{n}{\vartheta} p^\vartheta (1-p)^{n-\vartheta}$; then, $\mathcal{E}(X) = np$. In this case, X may be cast as a sum $\sum_{i=1}^n X_i$ of n independent and identically distributed Bernoulli trials X_i, where $\mathbf{Pr}(X_i = 1) = p$ and $\mathbf{Pr}(X_i = 0) = 1 - p$ for each index $i \in [n]$.

We shall use concepts and tools from the theory of *random allocations* (see, e.g., [14]), studying the size of the fullest *bin* when each of *n balls* is independently thrown into one of m bins, chosen according to some specific probability distribution. We shall exploit arising analogies between selfish routing and random allocation problems, and we shall interchangeably use the terms balls and users, and bins and links, respectively; thus, for example, for any bin $\ell \in [m]$, θ^ℓ denotes the random variable for the number of balls thrown into it.

In the special case where all balls choose bin $\ell \in [m]$ with the same probability $p(\ell)$, each random variable θ^ℓ, where $\ell \in [m]$, may be cast as a sum of n independent and identically distributed Bernoulli trials; each trial represents the choice made by each specific ball to drop onto bin ℓ (with probability $p(\ell)$) or onto a different bin (with probability $1 - p(\ell)$). Thus, in this case, θ^ℓ follows the binomial distribution with parameters n and $p(\ell)$.

The Case $p(\ell) = 1/m$. This is the case where each ball chooses a bin *uniformly* at random. Then, $\mathcal{E}(\theta^\ell) = n/m$, for all bins $\ell \in [m]$. A classical tool from the theory of random allocations provides then an upper bound on an appropriate tail probability involving the size of the fullest bin (see, e.g., Theorem 3.1 of [20]) and assuming that $m = n$.

LEMMA 3.1. *Assume each of n balls is thrown uniformly at random into one of n bins. Then,*

$$\mathbf{Pr}\left(\max_{\ell\in[n]} \theta^\ell > \left\lceil \frac{3\ln n}{\ln\ln n} \right\rceil\right) \le \frac{1}{n}.$$

The Case of General $p(\ell)$. For the general case, arbitrary probabilities $p(\ell)$ are allowed for each bin $\ell \in [m]$; then, $\mathcal{E}(\theta^\ell) = np(\ell)$. In this case, classical Chernoff-like results [2] apply to provide bounds for tail probabilities; we will use a particular such result derived by Angluin and Valiant [1].

LEMMA 3.2 [1]. *Consider any bin* $\ell \in [m]$. *Then, for any parameter* $\beta \in (0, 1)$,

$$\mathbf{Pr}(\theta^\ell > (1+\beta)\,\mathcal{E}(\theta^\ell)) \le \exp\left(-\frac{\beta^2}{3} \cdot \mathcal{E}(\theta^\ell)\right).$$

We conclude this section with a technical fact that will be used later; it derives a bound on an appropriate tail probability involving the size of any particular bin.

LEMMA 3.3. *Consider any bin* $\ell \in [m]$. *Then, for any integer* $n \ge 3$,

$$\mathbf{Pr}\left(\theta^\ell \ge \left\lceil \frac{3\ln n}{\ln\ln n} \right\rceil \max\left\{1, \mathcal{E}\left(\theta^\ell\right)\right\}\right) < \frac{4\max\{1, np(\ell)\}}{(4-e)n^2}.$$

The proof of Lemma 3.3 appears in the Appendix. We remark that Lemma 3.3 generalizes Lemma 3.1 to the case of arbitrary probabilities $p(\ell)$, $\ell \in [m]$. Moreover, the proof of Lemma 3.3 generalizes the one given for Theorem 3.1 of [20].

4. The Structure of Nash Equilibria. Section 4.1 derives the Minimum Expected Latency Cost Equations. These are used in Section 4.2 for showing existence and uniqueness results for fully mixed Nash equilibria.

4.1. *Minimum Expected Latency Cost Equations*. We remind the reader that $\mathcal{S} \subseteq [m]$ denotes the set of solo links. We show:

PROPOSITION 4.1 (Minimum Expected Latency Cost Equations). *Take any Nash equilibrium* $\mathbf{P}$. *Then, for any user* $i \in [n]$,

$$\lambda_i \left(\sum_{j=1}^{m} I_i^j c^j - \sum_{j=1, j \notin \mathcal{S}}^{m} \frac{I_i^j}{\sum_{k=1}^n I_k^j - 1} c^j \right) - \sum_{k=1, k \neq i}^{n} \lambda_k \left(\sum_{j=1, j \notin \mathcal{S}}^{m} \frac{I_i^j I_k^j}{\sum_{k=1}^n I_k^j - 1} c^j \right)$$
$$= w_i \left(\sum_{j=1}^{m} I_i^j - 1 - \sum_{j=1, j \notin \mathcal{S}}^{m} \frac{I_i^j}{\sum_{k=1}^n I_k^j - 1} \right)$$
$$- \sum_{k=1, k \neq i}^{n} w_k \left(\sum_{j=1, j \notin \mathcal{S}}^{m} \frac{I_i^j I_k^j}{\sum_{k=1}^n I_k^j - 1} \right) + \sum_{j=1, j \in \mathcal{S}}^{m} I_i^j W^j.$$

PROOF. We start with an informal outline of our proof. For any user $i \in [n]$, we derive two alternative expressions for $\sum_{j=1}^m I_i^j W^j$. The first one

$$\sum_{j=1}^{m} I_i^j W^j = w_i \left(1 - \sum_{j=1}^{m} I_i^j \right) + \lambda_i \sum_{j=1}^{m} I_i^j c^j$$

follows directly from Proposition 2.2 (condition (2)). The second expression

$$\sum_{j=1}^{m} I_i^j W^j = - \sum_{j=1, j \notin \mathcal{S}}^{m} I_i^j \frac{\sum_{k=1}^n I_k^j w_k}{\sum_{k=1}^n I_k^j - 1} + \sum_{j=1, j \notin \mathcal{S}}^{m} I_i^j c^j \frac{\sum_{k=1}^n I_k^j \lambda_k}{\sum_{k=1}^n I_k^j - 1} + w_i \sum_{j=1, j \in \mathcal{S}}^{m} I_i^j$$

will follow by using the expressions for the expected traffics on nonsolo links that were derived in Lemma 2.3. Equating the two derived expressions will yield the Minimum Expected Latency Cost Equations. We now continue with the details of the formal proof.

Fix any user $i \in [n]$. By Proposition 2.2 (condition (2)), it follows that

$$\sum_{j=1}^{m} I_i^j W^j = w_i \left(1 - \sum_{j=1}^{m} I_i^j \right) + \lambda_i \sum_{j=1}^{m} I_i^j c^j.$$

We now derive an alternative expression for the sum $\sum_{j=1}^m I_i^j W^j$ by using the expressions for the expected traffics on nonsolo links that were derived in Lemma 2.3. Clearly,

$$\sum_{j=1}^{m} I_i^j W^j = \sum_{j=1, j \notin \mathcal{S}}^{m} I_i^j W^j + \sum_{j=1, j \in \mathcal{S}}^{m} I_i^j W^j$$
$$= \sum_{j=1, j \notin \mathcal{S}}^{m} I_i^j \left(\frac{- \sum_{k=1}^n I_k^j w_k + c^j \sum_{k=1}^n I_k^j \lambda_k}{\sum_{k=1}^n I_k^j - 1} \right) + \sum_{j=1, j \in \mathcal{S}}^{m} I_i^j W^j$$
$$= - \sum_{j=1, j \notin \mathcal{S}}^{m} I_i^j \frac{\sum_{k=1}^n I_k^j w_k}{\sum_{k=1}^n I_k^j - 1} + \sum_{j=1, j \notin \mathcal{S}}^{m} I_i^j c^j \frac{\sum_{k=1}^n I_k^j \lambda_k}{\sum_{k=1}^n I_k^j - 1} + \sum_{j=1, j \in \mathcal{S}}^{m} I_i^j W^j.$$

Equating the two derived expressions for $\sum_{j=1}^{m} I_i^j W^j$ yields that

$$w_i \left(1 - \sum_{j=1}^{m} I_i^j\right) + \lambda_i \sum_{j=1}^{m} I_i^j c^j = -\sum_{j=1,\, j\notin\mathcal{S}}^{m} I_i^j \, \frac{\sum_{k=1}^{n} I_k^j w_k}{\sum_{k=1}^{n} I_k^j - 1} + \sum_{j=1,\, j\notin\mathcal{S}}^{m} I_i^j c^j \, \frac{\sum_{k=1}^{n} I_k^j \lambda_k}{\sum_{k=1}^{n} I_k^j - 1} + \sum_{j=1,\, j\in\mathcal{S}}^{m} I_i^j W^j,$$

or

$$\lambda_i \sum_{j=1}^{m} I_i^j c^j - \sum_{j=1,\, j\notin\mathcal{S}}^{m} I_i^j c^j \, \frac{\sum_{k=1}^{n} I_k^j \lambda_k}{\sum_{k=1}^{n} I_k^j - 1} = -w_i \left(1 - \sum_{j=1}^{m} I_i^j\right) - \sum_{j=1,\, j\notin\mathcal{S}}^{m} I_i^j \, \frac{\sum_{k=1}^{n} I_k^j w_k}{\sum_{k=1}^{n} I_k^j - 1} + \sum_{j=1,\, j\in\mathcal{S}}^{m} I_i^j W^j.$$

It follows that

$$\lambda_i \sum_{j=1}^{m} I_i^j c^j - \sum_{j=1,\, j\notin\mathcal{S}}^{m} I_i^j c^j \, \frac{I_i^j \lambda_i + \sum_{k=1, k\neq i}^{n} I_k^j \lambda_k}{\sum_{k=1}^{n} I_k^j - 1} = w_i \left(\sum_{j=1}^{m} I_i^j - 1\right) - \sum_{j=1,\, j\notin\mathcal{S}}^{m} I_i^j \, \frac{\sum_{k=1}^{n} I_k^j w_k}{\sum_{k=1}^{n} I_k^j - 1} + \sum_{j=1,\, j\in\mathcal{S}}^{m} I_i^j W^j.$$

Thus,

$$\lambda_i \sum_{j=1}^{m} I_i^j c^j - \lambda_i \sum_{j=1,\, j\notin\mathcal{S}}^{m} \frac{I_i^j}{\sum_{k=1}^{n} I_k^j - 1} \, c^j - \sum_{j=1,\, j\notin\mathcal{S}}^{m} I_i^j c^j \, \frac{\sum_{k=1, k\neq i}^{n} I_k^j \lambda_k}{\sum_{k=1}^{n} I_k^j - 1} = w_i \left(\sum_{j=1}^{m} I_i^j - 1\right) - \sum_{j=1,\, j\notin\mathcal{S}}^{m} I_i^j \, \frac{\sum_{k=1}^{n} I_k^j w_k}{\sum_{k=1}^{n} I_k^j - 1} + \sum_{j=1,\, j\in\mathcal{S}}^{m} I_i^j W^j,$$

or

$$\lambda_i \left(\sum_{j=1}^{m} I_i^j c^j - \sum_{j=1,\, j\notin\mathcal{S}}^{m} \frac{I_i^j}{\sum_{k=1}^{n} I_k^j - 1} \, c^j\right) - \sum_{j=1,\, j\notin\mathcal{S}}^{m} I_i^j c^j \, \frac{\sum_{k=1, k\neq i}^{n} I_k^j \lambda_k}{\sum_{k=1}^{n} I_k^j - 1} = w_i \left(\sum_{j=1}^{m} I_i^j - 1\right) - \sum_{j=1,\, j\notin\mathcal{S}}^{m} I_i^j \, \frac{\sum_{k=1}^{n} I_k^j w_k}{\sum_{k=1}^{n} I_k^j - 1} + \sum_{j=1,\, j\in\mathcal{S}}^{m} I_i^j W^j.$$

Thus, interchanging the order of summation,

$$\lambda_i \left(\sum_{j=1}^{m} I_i^j c^j - \sum_{j=1,\, j\notin\mathcal{S}}^{m} \frac{I_i^j}{\sum_{k=1}^{n} I_k^j - 1} \, c^j\right) - \sum_{k=1, k\neq i}^{n} \lambda_k \left(\sum_{j=1,\, j\notin\mathcal{S}}^{m} \frac{I_i^j I_k^j}{\sum_{k=1}^{n} I_k^j - 1} \, c^j\right) = w_i \left(\sum_{j=1}^{m} I_i^j - 1\right) - \sum_{k=1}^{n} w_k \left(\sum_{j=1,\, j\notin\mathcal{S}}^{m} \frac{I_i^j I_k^j}{\sum_{k=1}^{n} I_k^j - 1}\right) + \sum_{j=1,\, j\in\mathcal{S}}^{m} I_i^j W^j$$

$$= w_i \left(\sum_{j=1}^{m} I_i^j - 1 - \sum_{j=1,\, j \notin \mathcal{S}}^{m} \frac{I_i^j}{\sum_{k=1}^{n} I_k^j - 1} \right)$$
$$- \sum_{k=1,k\neq i}^{n} w_k \left(\sum_{j=1,\, j \notin \mathcal{S}}^{m} \frac{I_i^j I_k^j}{\sum_{k=1}^{n} I_k^j - 1} \right) + \sum_{j=1,\, j \in \mathcal{S}} I_i^j W^j,$$

as needed. □

4.2. *Fully Mixed Strategies*. We now focus on the fully mixed case. (Recall that there are no solo links in the fully mixed case. This implies, in particular, that the last term $\sum_{j=1,\, j\in\mathcal{S}} I_i^j W^j$ in the left-hand side of the Minimum Expected Latency Cost Equations is eliminated.) We set $I_i^\ell = 1$ for all users $i \in [n]$ and links $\ell \in [m]$ in the Minimum Expected Latency Cost Equations (Proposition 4.1) and solve the resulting linear system to obtain that λ is a linear transformation of $\mathbf{w}$.

LEMMA 4.2. *Consider any Nash equilibrium* $\mathbf{P}$, *in the fully mixed case. Then*

$$\lambda = \frac{1}{\sum_{j=1}^{m} c^j} \begin{pmatrix} m & 1 & \cdots & 1 \\ 1 & m & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & m \end{pmatrix} \cdot \mathbf{w}.$$

PROOF. By Proposition 4.1, for each user $i \in [n]$,

$$\lambda_i \left(\sum_{j=1}^{m} c^j - \sum_{j=1}^{m} \frac{1}{\sum_{k=1}^{n} 1 - 1} c^j \right) - \sum_{k=1,k\neq i}^{n} \lambda_k \left(\sum_{j=1}^{m} \frac{1}{\sum_{k=1}^{n} 1 - 1} c^j \right)$$
$$= w_i \left(\sum_{j=1}^{m} 1 - 1 - \sum_{j=1}^{m} \frac{1}{\sum_{k=1}^{n} 1 - 1} \right) - \sum_{k=1,k\neq i}^{n} w_k \left(\sum_{j=1}^{m} \frac{1}{\sum_{k=1}^{n} 1 - 1} \right),$$

or

$$\lambda_i \left(\sum_{j=1}^{m} c^j - \frac{1}{n-1} \sum_{j=1}^{m} c^j \right) - \sum_{k=1,k\neq i}^{n} \lambda_k \left(\frac{1}{n-1} \sum_{j=1}^{m} c^j \right)$$
$$= w_i \left(m - 1 - \frac{m}{n-1} \right) - \sum_{k=1,k\neq i}^{n} w_k \left(\frac{m}{n-1} \right),$$

or

$$\lambda_i (n-2) \left(\sum_{j=1}^{m} c^j \right) - \sum_{k=1,k\neq i}^{n} \lambda_k \left(\sum_{j=1}^{m} c^j \right) = w_i ((m-1)(n-1) - m) - m \sum_{k=1,k\neq i}^{n} w_k,$$

or

$$\left(\sum_{j=1}^{m} c^j \right) \left((n-1)\lambda_i - \sum_{k=1}^{n} \lambda_k \right) = (m-1)(n-1) w_i - m \sum_{k=1}^{n} w_k.$$

Thus, in matrix form,

$$\left(\sum_{j=1}^{m} c^j\right)\begin{pmatrix} n-2 & -1 & \cdots & -1 \\ -1 & n-2 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & n-2 \end{pmatrix}\cdot\lambda$$

$$=\begin{pmatrix} (m-1)(n-1)-m & -m & \cdots & -m \\ -m & (m-1)(n-1)-m & \cdots & -m \\ \vdots & \vdots & \ddots & \vdots \\ -m & -m & \cdots & (m-1)(n-1)-m \end{pmatrix}\cdot\mathbf{w}.$$

We observe that

$$\begin{pmatrix} n-2 & -1 & \cdots & -1 \\ -1 & n-2 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & n-2 \end{pmatrix}\cdot\begin{pmatrix} 0 & -\frac{1}{n-1} & \cdots & -\frac{1}{n-1} \\ -\frac{1}{n-1} & 0 & \cdots & -\frac{1}{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{n-1} & -\frac{1}{n-1} & \cdots & 0 \end{pmatrix} = \mathbf{I}_{n\times n},$$

so that inverting yields that

$$\lambda = \frac{1}{\sum_{j=1}^{m} c^j}\begin{pmatrix} 0 & -\frac{1}{n-1} & \cdots & -\frac{1}{n-1} \\ -\frac{1}{n-1} & 0 & \cdots & -\frac{1}{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{n-1} & -\frac{1}{n-1} & \cdots & 0 \end{pmatrix}$$

$$\cdot\begin{pmatrix} (m-1)(n-1)-m & -m & \cdots & -m \\ -m & (m-1)(n-1)-m & \cdots & -m \\ \vdots & \vdots & \ddots & \vdots \\ -m & -m & \cdots & (m-1)(n-1)-m \end{pmatrix}\cdot\mathbf{w}$$

$$= \frac{1}{\sum_{j=1}^{m} c^j}\begin{pmatrix} m & 1 & \cdots & 1 \\ 1 & m & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & m \end{pmatrix}\cdot\mathbf{w},$$

as needed. □

We now substitute the expressions for the minimum expected latency costs (Lemma 4.2) into the expressions for the expected traffics (Lemma 2.3) to obtain that $\mathbf{W}$ is a linear transformation of $\mathbf{w}$.

LEMMA 4.3. *Consider any Nash equilibrium* $\mathbf{P}$, *in the fully mixed case. Then,*

$$\mathbf{W} = \frac{1}{n-1}\left(-\mathbf{J}_{m\times n} + \frac{m+n-1}{\sum_{j=1}^{m} c^j}\mathbf{C}\right)\cdot\mathbf{w}.$$

PROOF. Since, in the fully mixed case, there are no solo links and $I_i^j = 1$ for all users $i \in [n]$ and links $j \in [m]$, Lemma 2.3 implies that for all links $\ell \in [m]$,

$$W^\ell = \frac{-\sum_{k=1}^n w_k + c^\ell \sum_{k=1}^n \lambda_k}{n-1}.$$

Thus, in matrix form, Lemma 4.2 implies that

$$\begin{aligned}
\mathbf{W} &= \frac{1}{n-1}\left(-\mathbf{J}_{m\times n}\cdot \mathbf{w} + \mathbf{C}\cdot \lambda\right)\\
&= \frac{1}{n-1}\left(-\mathbf{J}_{m\times n}\cdot \mathbf{w} + \mathbf{C}\cdot \frac{1}{\sum_{j=1}^m c^j}\begin{pmatrix} m & 1 & \cdots & 1\\ 1 & m & \cdots & 1\\ \vdots & \vdots & \ddots & \vdots\\ 1 & 1 & \cdots & m\end{pmatrix}\cdot \mathbf{w}\right)\\
&= \frac{1}{n-1}\left(-\mathbf{J}_{m\times n} + \frac{m+n-1}{\sum_{j=1}^m c^j}\mathbf{C}\right)\cdot \mathbf{w},
\end{aligned}$$

as needed. □

We now derive expressions for the fully mixed Nash probabilities.

PROPOSITION 4.4. *Consider any Nash equilibrium* $\mathbf{P}$, *in the fully mixed case. Then, for all users* $i \in [n]$ *and links* $\ell \in [m]$,

$$p_i^\ell = \left(1 - \frac{mc^\ell}{\sum_{j=1}^m c^j}\right)\left(1 - \frac{\sum_{k=1}^n w_k}{(n-1)w_i}\right) + \frac{c^\ell}{\sum_{j=1}^m c^j}.$$

PROOF. The proof amounts to substituting the expressions for λ_i, $i \in [n]$, and W^ℓ, $\ell \in [m]$, derived in Lemmas 4.2 and 4.3, respectively, into the expressions for the Nash probabilities p_i^ℓ from Proposition 2.2. So, for a user $i \in [n]$ and link $\ell \in [m]$, Proposition 2.2 implies that

$$\begin{aligned}
p_i^\ell &= 1 + \frac{W^\ell}{w_i} - \frac{c^\ell}{w_i}\lambda_i\\
&= 1 + \frac{1}{w_i}\frac{1}{n-1}\left(-1 + \frac{(m+n-1)c^\ell}{\sum_{j=1}^m c^j}\right)\sum_{k=1}^n w_k\\
&\quad -\frac{c^\ell}{w_i}\frac{1}{\sum_{j=1}^m c^j}\left((m-1)w_i + \sum_{k=1}^n w_k\right) \qquad \text{(by Lemmas 4.2 and 4.3)}\\
&= 1 - \frac{(m-1)c^\ell}{\sum_{j=1}^m c^j} + \frac{\sum_{k=1}^n w_k}{w_i}\left(-\frac{1}{n-1} + \frac{(m+n-1)c^\ell}{(n-1)\sum_{j=1}^m c^j} - \frac{c^\ell}{\sum_{j=1}^m c^j}\right)\\
&= 1 - \frac{(m-1)c^\ell}{\sum_{j=1}^m c^j} + \frac{\sum_{k=1}^n w_k}{w_i}\left(-\frac{1}{n-1} + \frac{mc^\ell}{(n-1)\sum_{j=1}^m c^j}\right)
\end{aligned}$$

$$= 1 - \frac{(m-1)c^\ell}{\sum_{j=1}^m c^j} - \frac{1}{n-1}\,\frac{\sum_{k=1}^n w_k}{w_i}\left(1 - \frac{mc^\ell}{\sum_{j=1}^m c^j}\right)$$
$$= \left(1 - \frac{mc^\ell}{\sum_{j=1}^m c^j}\right)\left(1 - \frac{\sum_{k=1}^n w_k}{(n-1)w_i}\right) + \frac{c^\ell}{\sum_{j=1}^m c^j},$$

as needed. □

Do the quantities p_i^ℓ determined in Proposition 4.4 indeed represent probabilities? For them to do so, it must be that for each user $i \in [n]$, (1) $\sum_{j=1}^m p_i^j = 1$, and (2) for each link $\ell \in [m]$, $0 \le p_i^\ell \le 1$. Also, since these quantities were specifically derived for the case of fully mixed strategies, condition (2) should more accurately be stated as (2′) for each link $\ell \in [m]$, $0 < p_i^\ell < 1$. A straightforward calculation verifies that conditions (1) and (2′) may or may not hold, depending on the particular values of the user traffics and link capacities. Hence, we obtain an inexistence result for fully mixed Nash equilibria.

COROLLARY 4.5. *Assume that there exist a user $i \in [n]$ and a link $\ell \in [m]$ such that*

$$\left(1 - \frac{mc^\ell}{\sum_{j=1}^m c^j}\right)\left(1 - \frac{\sum_{k=1}^n w_k}{(n-1)w_i}\right) + \frac{c^\ell}{\sum_{j=1}^m c^j} \notin (0,1).$$

Then, in the fully mixed case, there exists no Nash equilibrium.

We continue to show that the necessary condition for a Nash equilibrium (in the fully mixed case) determined in Corollary 4.5 is also sufficient.

PROPOSITION 4.6. *Assume that for all users $i \in [n]$ and links $\ell \in [m]$,*

$$\left(1 - \frac{mc^\ell}{\sum_{j=1}^m c^j}\right)\left(1 - \frac{\sum_{k=1}^n w_k}{(n-1)w_i}\right) + \frac{c^\ell}{\sum_{j=1}^m c^j} \in (0,1).$$

Then, in the fully mixed case, the probabilities

$$p_i^\ell = \left(1 - \frac{mc^\ell}{\sum_{j=1}^m c^j}\right)\left(1 - \frac{\sum_{k=1}^n w_k}{(n-1)\,w_i}\right) + \frac{c^\ell}{\sum_{j=1}^m c^j},$$

for each user $i \in [n]$ and link $\ell \in [m]$, are Nash probabilities.

PROOF. The assumption implies that for any user $i \in [n]$ and link $\ell \in [m]$, $0 < p_i^\ell < 1$. Thus, by definition of Nash equilibrium, we need to show that for any user i and link ℓ, $\lambda_i^\ell = \lambda_i$. So fix any user i and link ℓ. Clearly, by definition of expected latency cost,

$$\lambda_i^\ell = \frac{w_i + \sum_{k=1,k\neq i}^n p_k^\ell w_k}{c^\ell}$$
$$= \frac{w_i}{c^\ell} + \frac{1}{c^\ell}\sum_{k=1,k\neq i}^n p_k^\ell w_k$$

$$
\begin{aligned}
&= \frac{w_i}{c^\ell} + \frac{1}{c^\ell} \sum_{k=1, k \neq i}^{n} \left(\left(1 - \frac{m c^\ell}{\sum_{j=1}^{m} c^j} \right) \left(1 - \frac{\sum_{k'=1}^{n} w_{k'}}{(n-1) w_k} \right) + \frac{c^\ell}{\sum_{j=1}^{m} c^j} \right) w_k \\
&= \frac{w_i}{c^\ell} + \frac{1}{c^\ell} \left(1 - \frac{m c^\ell}{\sum_{j=1}^{m} c^j} \right) \sum_{k=1, k \neq i}^{n} \left(1 - \frac{\sum_{k'=1}^{n} w_{k'}}{(n-1) w_k} \right) w_k \\
&\quad + \frac{1}{c^\ell} \frac{c^\ell}{\sum_{j=1}^{m} c^j} \sum_{k=1, k \neq i}^{n} w_k \\
&= \frac{w_i}{c^\ell} + \frac{1}{c^\ell} \left(1 - \frac{m c^\ell}{\sum_{j=1}^{m} c^j} \right) \cdot \sum_{k=1, k \neq i}^{n} w_k \\
&\quad - \frac{1}{c^\ell} \left(1 - \frac{m c^\ell}{\sum_{j=1}^{m} c^j} \right) \frac{1}{n-1} \sum_{k'=1}^{n} w_{k'} \sum_{k=1, k \neq i}^{n} \frac{w_k}{w_k} + \frac{1}{\sum_{j=1}^{m} c^j} \sum_{k=1, k \neq i}^{n} w_k \\
&= \frac{w_i}{c^\ell} + \frac{1}{c^\ell} \left(1 - \frac{m c^\ell}{\sum_{j=1}^{m} c^j} \right) \sum_{k=1, k \neq i}^{n} w_k - \frac{1}{c^\ell} \left(1 - \frac{m c^\ell}{\sum_{j=1}^{m} c^j} \right) \sum_{k'=1}^{n} w_{k'} \\
&\quad + \frac{1}{\sum_{j=1}^{m} c^j} \sum_{k=1, k \neq i}^{n} w_k \\
&= \frac{w_i}{c^\ell} - \frac{1}{c^\ell} \left(1 - \frac{m c^\ell}{\sum_{j=1}^{m} c^j} \right) w_i + \frac{1}{\sum_{j=1}^{m} c^j} \sum_{k=1, k \neq i}^{n} w_k \\
&= \frac{m}{\sum_{j=1}^{m} c^j} w_i + \frac{1}{\sum_{j=1}^{m} c^j} \sum_{k=1, k \neq i}^{n} w_k \\
&= \frac{1}{\sum_{j=1}^{m} c^j} \left(m w_i + \sum_{k=1, k \neq i}^{n} w_k \right).
\end{aligned}
$$

Lemma 4.2, implies now that $\lambda_i^\ell = \lambda_i$, so that by definition of Nash equilibrium and fully mixed strategies, the probabilities p_i^ℓ are Nash probabilities, as needed. $\square$

Propositions 4.4 and 4.6 together establish:

THEOREM 4.7 (Existence and Uniqueness of Nash Equilibria). *Consider the fully mixed case. Then, for all users $i \in [n]$ and links $\ell \in [m]$,*

$$
\left(1 - \frac{m c^\ell}{\sum_{j=1}^{m} c^j} \right) \cdot \left(1 - \frac{\sum_{k=1}^{n} w_k}{(n-1)\, w_i} \right) + \frac{c^\ell}{\sum_{j=1}^{m} c^j} \;\in\; (0, 1)
$$

if and only if there exists a Nash equilibrium, which must be unique and has associated Nash probabilities

$$p_i^\ell = \left(1 - \frac{mc^\ell}{\sum_{j=1}^m c^j}\right)\left(1 - \frac{\sum_{k=1}^n w_k}{(n-1)\,w_i}\right) + \frac{c^\ell}{\sum_{j=1}^m c^j},$$

for each user $i \in [n]$ and link $\ell \in [m]$.

The expressions for the Nash probabilities p_i^ℓ in Theorem 4.7 enjoy, as functions of the link capacities and user traffics, a particularly insightful form. Their first term is the product of two factors: the first one $1 - mc^\ell/\sum_{j=1}^m c^j$ depends solely on link capacities, while the second one $1 - \sum_{k=1}^n w_k/[(n-1)\,w_i]$ depends solely on user traffics. Their second term $c^\ell/\sum_{j=1}^m c^j$ depends only on link capacities. The first factor in the first term vanishes for the model of uniform capacities; thus, it is responsible for eliminating the dependence of Nash probabilities on user traffics in this model. A corresponding elimination is lacking for the case of identical traffics, since the second factor in the first term of the Nash probabilities does *not* vanish when user traffics are identical; thus, Nash probabilities do depend on link capacities in the model of identical traffics. This subtle difference manifests an inherent asymmetry between link capacities and user traffics, as parameters determining the fully mixed Nash probabilities.

We finally remark that Theorem 4.7 implies that for the fully mixed case, Nash equilibrium can be checked for existence and evaluated (if existing) in time $\Theta(mn)$.

5. Uniform Capacities and Arbitrary Traffics. In this section we derive upper bounds on the Coordination Ratio for the case of fully mixed strategies and under the model of uniform capacities, where $c^\ell = 1$ for each link $\ell \in [m]$.

5.1. *Preliminaries*. We start with a characterization of fully mixed Nash probabilities.

LEMMA 5.1. *Consider the fully mixed case under the model of uniform capacities. Then there exists a unique Nash equilibrium with associated Nash probabilities $p_i^\ell = 1/m$, for each user $i \in [n]$ and link $\ell \in [m]$.*

PROOF. Since $1/m \in (0, 1)$, the claim follows immediately since, by Theorem 4.7, for any user $i \in [n]$ and link $\ell \in [m]$,

$$\begin{aligned}\left(1 - \frac{mc^\ell}{\sum_{j=1}^m c^j}\right)&\left(1 - \frac{\sum_{k=1}^n w_k}{(n-1)\,w_i}\right) + \frac{c^\ell}{\sum_{j=1}^m c^j}\\ &= \left(1 - \frac{m}{m}\right)\left(1 - \frac{\sum_{k=1}^n w_k}{(n-1)\,w_i}\right) + \frac{1}{m} = \frac{1}{m}.\end{aligned}$$ □

Lemma 5.1 determines the Nash probabilities associated with the unique Nash equilibrium. So, the discussion in Section 3 applies to yield that for each link $\ell \in [m]$,

$$\mathbf{Pr}(\theta^\ell = \vartheta) = \binom{n}{\vartheta}\left(\frac{1}{m}\right)^{\vartheta}\left(1-\frac{1}{m}\right)^{n-\vartheta},$$

where $0 \le \vartheta \le n$, and that $\mathcal{E}(\theta^\ell) = n/m$.

5.2. *First Tail Lemma*. We prove an upper bound on Social Cost under a certain assumption on the tail of the probability distribution of $\max_{\ell\in[m]} \theta^\ell$.

PROPOSITION 5.2 (First Tail Lemma). *Consider the fully mixed case under the model of uniform capacities. Assume that, for a Nash equilibrium* $\mathbf{P}$, *there exists a function* $\rho(m, n)$ *such that for every link* $\ell \in [m]$,

$$\mathbf{Pr}\left(\max_{\ell\in[m]} \theta^\ell > \rho(m,n)\,\mathcal{E}(\theta^\ell)\right) \le \frac{1}{n}.$$

Then

$$\mathsf{SC}\,(\mathbf{w},\mathbf{P}) \le \max_{k\in[n]} w_k \cdot \left(\rho(m,n)\,\frac{n}{m} + 1\right).$$

PROOF. We start with an informal outline of our proof. We use the definition for Social Cost to observe that

$$\mathsf{SC}\,(\mathbf{w},\mathbf{P}) \le \max_{k\in[n]} w_k \cdot \sum_{\vartheta=0}^{n} \vartheta\,\mathbf{Pr}\left(\max_{\ell\in[m]} \theta^\ell = \vartheta\right).$$

We then use the assumption on $\mathbf{Pr}(\max_{\ell\in[m]} \theta^\ell > \rho(m,n)\,\mathcal{E}(\theta^\ell))$ and split the summation across $\vartheta = \rho(m,n)\mathcal{E}(\theta^{\ell_0})$, for any particular link $\ell_0 \in [m]$, to derive the claimed upper bound on $\mathsf{SC}\,(\mathbf{w},\mathbf{P})$. We now continue with the details of the formal proof.

We analyze the Social Cost for the Nash equilibrium $\mathbf{P}$. Clearly,

$$\begin{aligned}
\mathsf{SC}\,(\mathbf{w},\mathbf{P}) &= \mathcal{E}\left(\max_{\ell\in[m]} \frac{\sum_{k:\ell_k=\ell} w_k}{c^\ell}\right)\\
&= \mathcal{E}\left(\max_{\ell\in[m]}\left(\sum_{k:\ell_k=\ell} w_k\right)\right)\\
&\le \mathcal{E}\left(\max_{\ell\in[m]}\left(\theta^\ell \max_{k:\ell_k=\ell} w_k\right)\right)\\
&\le \mathcal{E}\left(\max_{\ell\in[m]}\left(\theta^\ell \max_{1\le k\le n} w_k\right)\right)\\
&= \max_{k\in[n]} w_k \cdot \mathcal{E}\left(\max_{\ell\in[m]} \theta^\ell\right)\\
&= \max_{k\in[n]} w_k \cdot c \sum_{\vartheta=0}^{n} \vartheta\,\mathbf{Pr}\left(\max_{\ell\in[m]} \theta^\ell = \vartheta\right).
\end{aligned}$$

We break the sum in the left-hand side according to the event that for some arbitrary (but fixed) link $\ell_0 \in [m]$, $\max_{\ell \in [m]} \theta^\ell < \rho(m,n)\mathcal{E}(\theta^{\ell_0})$, to obtain that

$$
\begin{aligned}
\mathsf{SC}(\mathbf{w}, \mathbf{P}) &\leq \max_{k \in [n]} w_k \\
&\quad \cdot \left(\sum_{0 \leq \vartheta \leq \rho(m,n)\mathcal{E}(\theta^{\ell_0})} \vartheta \, \mathbf{Pr}\left(\max_{\ell \in [m]} \theta^\ell = \vartheta \right) \right. \\
&\qquad \left. + \sum_{\rho(m,n)\mathcal{E}(\theta^{\ell_0}) < \vartheta \leq n} \vartheta \, \mathbf{Pr}\left(\max_{\ell \in [m]} \theta^\ell = \vartheta \right) \right) \\
&\leq \max_{k \in [n]} w_k \\
&\quad \cdot \left(\sum_{0 \leq \vartheta \leq \rho(m,n)\mathcal{E}(\theta^{\ell_0})} \rho(m,n) \, \mathcal{E}(\theta^{\ell_0}) \, \mathbf{Pr}\left(\max_{\ell \in [m]} \theta^\ell = \vartheta \right) \right. \\
&\qquad \left. + \sum_{\rho(m,n)\mathcal{E}(\theta^{\ell_0}) < \vartheta \leq n} n \, \mathbf{Pr}\left(\max_{\ell \in [m]} \theta^\ell = \vartheta \right) \right) \\
&= \max_{k \in [n]} w_k \\
&\quad \cdot \left(\rho(m,n) \, \mathcal{E}(\theta^{\ell_0}) \sum_{0 \leq \vartheta \leq \rho(m,n)\mathcal{E}(\theta^{\ell_0})} \mathbf{Pr}\left(\max_{\ell \in [m]} \theta^\ell = \vartheta \right) \right. \\
&\qquad \left. + \, n \sum_{\rho(m,n)\mathcal{E}(\theta^{\ell_0}) < \vartheta \leq n} \mathbf{Pr}\left(\max_{\ell \in [m]} \theta^\ell = \vartheta \right) \right) \\
&= \max_{k \in [n]} w_k \\
&\quad \cdot \left(\rho(m,n) \, \mathcal{E}(\theta^{\ell_0}) \, \mathbf{Pr}\left(\max_{\ell \in [m]} \theta^\ell \leq \rho(m,n) \, \mathcal{E}(\theta^{\ell_0}) \right) \right. \\
&\qquad \left. + \, n \, \mathbf{Pr}\left(\max_{\ell \in [m]} \theta^\ell > \rho(m,n) \, \mathcal{E}(\theta^{\ell_0}) \right) \right) \\
&\leq \max_{k \in [n]} w_k \cdot \left(\rho(m,n) \, \mathcal{E}\left(\theta^{\ell_0}\right) \cdot 1 + n \cdot \frac{1}{n} \right) \\
&\quad \text{(by assumption on } \mathbf{Pr}(\max_{\ell \in [m]} \theta^\ell > \rho(m,n) \, \mathcal{E}(\theta^{\ell_0}))) \\
&= \max_{k \in [n]} w_k \cdot \left(\rho(m,n) \, \frac{n}{m} + 1 \right),
\end{aligned}
$$

as needed. □

The function $\rho(m, n)$ in Proposition 5.2, called a *tail function*, provides a multiplicative *tail threshold* for the random variable $\max_{\ell \in [m]} \theta^\ell$ such that the measure of the tail distribution of it that lies above $\rho(m, n)\, \mathcal{E}(\theta^\ell)$, for any link $\ell \in [m]$, is sufficiently small.

Proposition 5.2 implies that to show an upper bound on Social Cost, it suffices to determine a suitable tail function $\rho(m, n)$. We now do so for two particular instances of the problem.

5.3. *Upper Bounds on Social Cost*. We determine a suitable tail function under two particular assumptions on how m and n compare to each other.

The Case $m = n$. We prove:

PROPOSITION 5.3. *Consider the fully mixed case under the model of uniform capacities. Assume that $m = n$. Then, for a Nash equilibrium* $\mathbf{P}$,

$$\mathsf{SC}\,(\mathbf{w}, \mathbf{P}) \;\leq\; \max_{k \in [n]} w_k \cdot \left(\left\lceil \frac{3 \ln n}{\ln \ln n} \right\rceil + 1 \right).$$

PROOF. Take any link $\ell \in [n]$. Since $\mathcal{E}(\theta^\ell) = 1$, Lemma 3.1 implies that for each link $\ell \in [n]$,

$$\mathbf{Pr}\left(\max_{\ell \in [n]} \theta^\ell > \left\lceil \frac{3 \ln n}{\ln \ln n} \right\rceil \cdot \mathcal{E}\left(\theta^\ell\right) \right) \;\leq\; \frac{1}{n}.$$

Thus, Proposition 5.2 applies with $\rho(m, n) = \lceil (3 \ln n)/(\ln \ln n) \rceil$ and yields

$$\mathsf{SC}\,(\mathbf{w}, \mathbf{P}) \;\leq\; \max_{k \in [n]} w_k \cdot \left(\left\lceil \frac{3 \ln n}{\ln \ln n} \right\rceil + 1 \right),$$

as needed. □

The Case $m \leq n/16 \ln n$. We prove:

PROPOSITION 5.4. *Consider the fully mixed case under the model of uniform capacities. Assume that $m \leq n/(16 \ln n)$. Then, for a Nash equilibrium* $\mathbf{P}$,

$$\mathsf{SC}\,(\mathbf{w}, \mathbf{P}) \;\leq\; \max_{k \in [n]} w_k \cdot \left(\frac{3}{2} \frac{n}{m} + 1 \right).$$

PROOF. Take any link $\ell \in [m]$. Since $\mathcal{E}(\theta^\ell) = n/m$, Lemma 3.2 implies that for any parameter $\beta \in (0, 1)$,

$$\begin{aligned}
\mathbf{Pr}(\theta^\ell > (1 + \beta)\, \mathcal{E}(\theta^\ell)) \;&\leq\; \exp\left(-\frac{\beta^2}{2} \frac{n}{m} \right) \\
&\leq\; \exp\left(-\frac{\beta^2}{2}\, 16 \ln n \right) \qquad \left(\text{since } m \leq \frac{n}{16 \ln n} \right).
\end{aligned}$$

Now fix $\beta = \frac{1}{2}$, so that

$$\mathbf{Pr}(\theta^\ell > \tfrac{3}{2}\mathcal{E}(\theta^\ell)) \ \le\ \exp\left(-2\ln n\right) = \frac{1}{n^2}.$$

Thus,

$$\begin{aligned}
\mathbf{Pr}\left(\max_{\ell\in[m]} \theta^\ell > \tfrac{3}{2}\mathcal{E}\left(\theta^\ell\right)\right) \ &=\ \mathbf{Pr}\left(\bigvee_{\ell\in[m]} (\theta^\ell > \tfrac{3}{2}\mathcal{E}(\theta^\ell))\right)\\
&\le\ \sum_{\ell\in[m]} \mathbf{Pr}(\theta^\ell > \tfrac{3}{2}\mathcal{E}(\theta^\ell))\\
&\le\ \sum_{\ell\in[m]} \frac{1}{n^2}\\
&=\ m\cdot\frac{1}{n^2}\\
&\le\ \frac{n}{24\ln n}\cdot\frac{1}{n^2}\\
&<\ \frac{1}{n}.
\end{aligned}$$

Thus, Proposition 5.2 applies with $\rho(m, n) = \frac{3}{2}$ to yield the claim. □

5.4. *Lower Bound on Social Optimum*. In this section we prove a lower bound on Social Optimum.

LEMMA 5.5. *Consider the model of uniform capacities. Then*

$$\mathsf{OPT}(\mathbf{w}) \ \ge\ \max\left\{\frac{\sum_{1\le k\le n} w_k}{m},\ \max_{1\le k\le n} w_k\right\}.$$

PROOF. Clearly, in the optimal assignment of traffics to links, some link must receive traffic no less than the average (over all links) traffic, and some link must receive the maximum (over all users) traffic. Thus,

$$\mathsf{OPT}(\mathbf{w}) \ \ge\ \max\left\{\frac{\sum_{1\le k\le n} w_k}{m},\ \max_{k\in[n]} w_k\right\} = \max\left\{\frac{\sum_{1\le k\le n} w_k}{m},\ \max_{k\in[n]} w_k\right\},$$

as needed. □

5.5. *Combining the Bounds*. Assuming first that $m = n$ and appealing to Proposition 5.3 and Lemma 5.5, we obtain:

THEOREM 5.6. *Consider the fully mixed case under the model of uniform capacities. Assume that $m = n$. Then*

$$\mathsf{CR} \ \le\ \left\lceil\frac{3\ln n}{\ln\ln n}\right\rceil + 1.$$

PROOF. By the definition of Coordination Ratio,

$$\begin{aligned}
\mathsf{CR} &= \max_{\mathbf{w},\mathbf{P}} \frac{\mathsf{SC}\,(\mathbf{w},\mathbf{P})}{\mathsf{OPT}\,(\mathbf{w})} \qquad \text{(where } \mathbf{P} \text{ is a Nash equilibrium)} \\
&\leq \max_{k\in[n]} w_k \cdot \left(\left\lceil \frac{3\ln n}{\ln\ln n} \right\rceil + 1 \right) \frac{1}{\max_{k\in[n]} w_k} \qquad \text{(by Proposition 5.3 and Lemma 5.5)} \\
&= \left\lceil \frac{3\ln n}{\ln\ln n} \right\rceil + 1,
\end{aligned}$$

as needed. □

With $m \leq n/(16\ln n)$ appealing to Proposition 5.4 and Lemma 5.5 yields:

THEOREM 5.7. *Consider the fully mixed case under the model of uniform capacities. Assume that $m \leq n/(16\ln n)$. Then,*

$$\mathsf{CR} \;\leq\; (\tfrac{3}{2} + o(1))\,\mathsf{max/avg}(\mathbf{w}).$$

PROOF. By the definition of Coordination Ratio,

$$\begin{aligned}
\mathsf{CR} &= \max_{\mathbf{w},\mathbf{P}} \frac{\mathsf{SC}\,(\mathbf{w},\mathbf{P})}{\mathsf{OPT}\,(\mathbf{w})} \qquad \text{(where } \mathbf{P} \text{ is a Nash equilibrium)} \\
&\leq \max_{k\in[n]} w_k \left(\frac{3}{2}\,\frac{n}{m} + 1 \right) \frac{m}{\sum_{k\in[n]} w_k} \qquad \text{(by Proposition 5.4 and Lemma 5.5)} \\
&\leq \left(\frac{3}{2} + \frac{1}{16\ln n} \right) \mathsf{max/avg}(\mathbf{w}) \qquad \left(\text{since } m \leq \frac{n}{16\ln n} \right) \\
&= \left(\frac{3}{2} + o(1) \right) \mathsf{max/avg}(\mathbf{w}),
\end{aligned}$$

as needed. □

6. Arbitrary Capacities and Identical Traffics. In this section, we derive upper bounds on the Coordination Ratio for the case of fully mixed strategies under the model of arbitrary capacities and identical traffics.

For each link $\ell \in [m]$, let $\widetilde{c^\ell} = c^\ell / (\sum_{j=1}^{n} c^j)$ denote the *reduced capacity* of link ℓ. (Clearly, $\sum_{\ell\in[m]} \widetilde{c^\ell} = 1$.)

6.1. *Preliminaries*. We start with a simple characterization of existence and uniqueness of fully mixed Nash equilibria for this case.

LEMMA 6.1. *Consider the fully mixed case under the model of arbitrary capacities. Assume that all traffics are identical. Then, for all links $\ell \in [m]$,*

$$\widetilde{c^\ell} \;\in\; \left(\frac{1}{m+n-1}\;,\; \frac{n}{m+n-1} \right),$$

if and only if there exists a Nash equilibrium, which must be unique and has associated Nash probabilities

$$p_i^\ell \;=\; \frac{(m+n-1)\,\widetilde{c^\ell}-1}{n-1},$$

for any user $i \in [n]$ *and link* $\ell \in [m]$.

PROOF. By Theorem 4.7, there exists a Nash equilibrium, which must be unique, if and only if for all users $i \in [n]$ and links $\ell \in [m]$,

$$\left(1 - m\widetilde{c^\ell}\right)\left(1 - \frac{nw}{(n-1)w}\right) + \widetilde{c^\ell} \;\in\; (0,1),$$

or

$$\left(\frac{m}{n-1}+1\right)\widetilde{c^\ell} \;\in\; (0,1) + \frac{1}{n-1} = \left(\frac{1}{n-1}\,,\,\frac{n}{n-1}\right).$$

This is equivalent to

$$\frac{m+n-1}{n-1}\cdot\widetilde{c^\ell} \in \left(\frac{1}{n-1}\,,\,\frac{n}{n-1}\right) \quad \text{or} \quad \widetilde{c^\ell} \in \left(\frac{1}{m+n-1}\,,\,\frac{n}{m+n-1}\right),$$

as needed. Also, by Theorem 4.7, the associated Nash probabilities are

$$\begin{aligned} p_i^\ell &= \left(1 - m\widetilde{c^\ell}\right)\left(1 - \frac{nw}{(n-1)w}\right) + \widetilde{c^\ell} \\ &= \left(1 - m\widetilde{c^\ell}\right)\left(-\frac{1}{n-1}\right) + \widetilde{c^\ell} = \frac{(m+n-1)\,\widetilde{c^\ell}-1}{n-1}, \end{aligned}$$

for all users $i \in [n]$ and links $\ell \in [m]$, as needed. □

Lemma 6.1 describes the Nash probabilities for the case of identical traffics and under the model of arbitrary capacities; thus, it is the analog of Lemma 5.1 that holds for the case of arbitrary traffics and under the model of uniform capacities. We remark that these two lemmas stand in contrast to each other, since Lemma 5.1 establishes the unconditional existence of a (unique) Nash equilibrium, while Lemma 6.1 provides conditions on link capacities under which a (then unique) Nash equilibrium exists. Thus, Lemmas 5.1 and 6.1 reveal an essential difference with respect to existence of Nash equilibria between the case of uniform capacities and arbitrary traffics, and the case of arbitrary capacities and identical traffics, respectively.

Lemma 6.1 shows that each Nash probability is now *independent* of the particular user and depends only on the link; to emphasize this independence, we write $p(\ell)$ to denote p_i^ℓ for a user $i \in [n]$ and link $\ell \in [m]$. So, the discussion in Section 3 applies to yield that for each link $\ell \in [m]$, $\mathbf{Pr}(\theta^\ell = \vartheta) = \binom{n}{\vartheta} p^\vartheta(\ell)(1-p(\ell))^{n-\vartheta}$, where $0 \le \vartheta \le n$, and that $\mathcal{E}(\theta^\ell) = np(\ell)$.

6.2. *Second Tail Lemma*. We prove an upper bound on Social Cost under a certain assumption on the tails of the probability distributions of the random variables θ^{ℓ}, where $\ell \in [m]$.

PROPOSITION 6.2 (Second Tail Lemma). *Consider the fully mixed case under the model of arbitrary capacities. Assume that all traffics are equal to w. Assume that, for a Nash equilibrium* $\mathbf{P}$, *there exists, for each link $\ell \in [m]$, a function $\rho_{\ell}(m, n)$ such that*

$$\mathbf{Pr}\left(\bigwedge_{\ell \in [m]} (\theta^{\ell} \leq \rho_{\ell}(m, n)\ \max\{1, \mathcal{E}(\theta^{\ell})\})\right) \ > \ 1 - \frac{\delta}{n},$$

for some constant $\delta > 0$. Then, in a Nash equilibrium $\mathbf{P}$,

$$\mathsf{SC}\,(\mathbf{w}, \mathbf{P}) \ < \ \frac{m + n - 1}{\sum_{\ell \in [m]} c^{\ell}} \cdot \left(\max_{\ell \in [m]} \rho_{\ell}(m, n)\, \frac{n}{n-1} + \delta\right).$$

PROOF. We start with an informal outline of our proof. We use the definition for Social Cost to observe that $\mathsf{SC}(\mathbf{w}, \mathbf{P}) = w\, \mathcal{E}(\max_{\ell \in [m]} \theta^{\ell}/c^{\ell})$. To bound from above the expectation of the (discrete) random variable $\max_{\ell \in [m]} \theta^{\ell}/c^{\ell}$, we partition its sample space according to the event that for *all* links $\ell \in [m]$, $\theta^{\ell} \leq \rho_{\ell}(m, n)\ \max\{1, \mathcal{E}(\theta^{\ell})\}$; we obtain that

$$\mathsf{SC}\,(\mathbf{w}, \mathbf{P}) \ \leq \ \max_{\ell \in [m]} \rho_{\ell}(m, n) \ \cdot \max\left\{\max_{\ell \in [m]} \frac{1}{c^{\ell}},\ \max_{\ell \in [m]} \frac{\mathcal{E}(\theta^{\ell})}{c^{\ell}}\right\} + \frac{\delta}{\min_{\ell \in [m]} c^{\ell}}.$$

We then use the necessary conditions on reduced capacities derived in Lemma 6.1 as well as the expressions for the Nash probabilities $p(\ell)$, where $\ell \in [m]$, derived there to establish the claim. We now continue with the details of the formal proof.

We analyze the Social Cost for the Nash equilibrium $\mathbf{P}$. By definition of Social Cost,

$$\begin{aligned}
\mathsf{SC}\,(\mathbf{w}, \mathbf{P}) \ &= \ \mathcal{E}\left(\max_{\ell \in [m]} \frac{\sum_{k : \ell_k = \ell} w_k}{c^{\ell}}\right) \\
&= \ \mathcal{E}\left(\max_{\ell \in [m]} \frac{\theta^{\ell}}{c^{\ell}}\right) \qquad \text{(since all traffics are identical)} \\
&= \ \mathcal{E}\left(\max_{\ell \in [m]} \frac{\theta^{\ell}}{c^{\ell}}\right).
\end{aligned}$$

We continue to analyze (and bound from above) the expectation of the random variable $\max_{\ell \in [m]} \theta^{\ell}/c^{\ell}$. Since each variable θ^{ℓ} may only take on values from $\{0, 1, \ldots, n\}$, it follows that, given the link capacities, the random variable $\max_{\ell \in [m]} \theta^{\ell}/c^{\ell}$ is a *discrete* random variable, taking on only finitely many values; thus, its expectation is a finite sum, so that

$$\mathsf{SC}\,(\mathbf{w}, \mathbf{P}) \ = \sum_{\vartheta : \max_{\ell \in [m]} \theta^{\ell}/c^{\ell} = \vartheta} \vartheta\ \mathbf{Pr}\left(\max_{\ell \in [m]} \frac{\theta^{\ell}}{c^{\ell}} = \vartheta\right).$$

We break the sum in the left-hand side according to the event that for *all* links $\ell \in [m]$, $\theta^\ell \leq \rho_\ell(m,n)\ \max\{1, \mathcal{E}(\theta^\ell)\}$, to obtain that

$$\begin{aligned}\mathsf{SC}\,(\mathbf{w},\mathbf{P}) \;=\; & \sum_{\substack{\vartheta:\ \vartheta=\max_{\ell\in[m]}\theta^\ell/c^\ell\ \&\\ \forall \ell\in[m]:\\ \theta^\ell\leq\rho_\ell(m,n)\max\{1,\mathcal{E}(\theta^\ell)\}}} \vartheta\, \mathbf{Pr}\left(\max_{\ell\in[m]}\theta^\ell/c^\ell=\vartheta\right)\\ &+ \sum_{\substack{\vartheta:\ \vartheta=\max_{\ell\in[m]}\frac{\theta^\ell}{c^\ell}\ \&\\ \exists \ell\in[m]:\\ \theta^\ell>\rho_\ell(m,n)\max\{1,\mathcal{E}(\theta^\ell)\}}} \vartheta\, \mathbf{Pr}\left(\max_{\ell\in[m]}\frac{\theta^\ell}{c^\ell}=\vartheta\right).\end{aligned}$$

We proceed to calculate upper bounds on the dummy variable ϑ involved in each of the two sums in the right-hand side of the last expression.

- Consider the first sum, taking that for all links $\ell \in [m]$, $\theta^\ell \leq \rho_\ell(m,n)\max\{1,\mathcal{E}(\theta^\ell)\}$; it follows that
$$\max_{\ell\in[m]}\theta^\ell/c^\ell \;\leq\; \max_{\ell\in[m]}\frac{\rho_\ell(m,n)\max\{1,\mathcal{E}(\theta^\ell)\}}{c^\ell}.$$
- Now consider the second sum. The condition that there exists a link $\ell \in [m]$ such that $\theta^\ell > \rho_\ell(m,n)\ \max\{1,\mathcal{E}(\theta^\ell)\}$ is not particularly helpful; so, we use the (trivial) upper bound $\theta^\ell \leq n$ for all links $\ell \in [m]$. Since ϑ represents all possible values taken on by the random variable $\max_{\ell\in[m]}\theta^\ell/c^\ell$, it follows that for the second sum, $\vartheta \leq \max_{\ell\in[m]} n/c^\ell$.

It follows that

$$\begin{aligned}\mathsf{SC}\,(\mathbf{w},\mathbf{P}) \;\leq\; & \sum_{\substack{\vartheta:\vartheta=\max_{\ell\in[m]}\theta^\ell/c^\ell\ \&\\ \forall \ell\in[m]:\\ \theta^\ell\leq\rho_\ell(m,n)\ \max\{1,\mathcal{E}(\theta^\ell)\}}} \max_{\ell\in[m]}\frac{\rho_\ell(m,n)\ \max\{1,\mathcal{E}(\theta^\ell)\}}{c^\ell}\ \mathbf{Pr}\left(\max_{\ell\in[m]}\theta^\ell/c^\ell=\vartheta\right)\\ &+ \sum_{\substack{\vartheta:\ \vartheta=\max_{\ell\in[m]}\theta^\ell/c^\ell\ \&\\ \exists \ell\in[m]:\\ \theta^\ell>\rho_\ell(m,n)\ \max\{1,\mathcal{E}(\theta^\ell)\}}} \max_{\ell\in[m]}\frac{n}{c^\ell}\ \mathbf{Pr}\left(\max_{\ell\in[m]}\theta^\ell/c^\ell=\vartheta\right)\\ =\; & \max_{\ell\in[m]}\frac{\rho_\ell(m,n)\ \max\{1,\mathcal{E}(\theta^\ell)\}}{c^\ell}\cdot \sum_{\substack{\vartheta:\ \vartheta=\max_{\ell\in[m]}\theta^\ell/c^\ell\ \&\\ \forall \ell\in[m]:\\ \theta^\ell\leq\rho_\ell(m,n)\ \max\{1,\mathcal{E}(\theta^\ell)\}}} \mathbf{Pr}\left(\max_{\ell\in[m]}\theta^\ell/c^\ell=\vartheta\right)\\ &+\max_{\ell\in[m]}\frac{n}{c^\ell}\cdot \sum_{\substack{\vartheta:\ \vartheta=\max_{\ell\in[m]}\theta^\ell/c^\ell\ \&\\ \exists \ell\in[m]:\\ \theta^\ell>\rho_\ell(m,n)\ \max\{1,\mathcal{E}(\theta^\ell)\}}} \mathbf{Pr}\left(\max_{\ell\in[m]}\theta^\ell/c^\ell=\vartheta\right).\end{aligned}$$

Clearly,

$$\sum_{\substack{\vartheta:\ \vartheta=\max_{\ell\in[m]}\theta^\ell/c^\ell\ \&\\ \forall \ell\in[m]:\\ \theta^\ell\leq\rho_\ell(m,n)\ \max\{1,\mathcal{E}(\theta^\ell)\}}} \mathbf{Pr}\left(\max_{\ell\in[m]}\theta^\ell/c^\ell=\vartheta\right)$$

$$
\begin{aligned}
&= \mathbf{Pr}\left(\bigwedge_{\ell \in [m]} (\theta^{\ell} \leq \rho_{\ell}(m,n)\ \max\{1, \mathcal{E}(\theta^{\ell})\})\right) \\
&\leq\ 1,
\end{aligned}
$$

while

$$
\begin{aligned}
\sum_{\substack{\vartheta:\ \vartheta = \max_{\ell \in [m]} \theta^{\ell}/c^{\ell}\ \& \\ \exists \ell \in [m]: \\ \theta^{\ell} > \rho_{\ell}(m,n)\ \max\{1, \mathcal{E}(\theta^{\ell})\}}} \mathbf{Pr}\left(\max_{\ell \in [m]} \frac{\theta^{\ell}}{c^{\ell}} = \vartheta\right) &= \mathbf{Pr}\left(\bigvee_{\ell \in [m]} (\theta^{\ell} > \rho_{\ell}(m,n)\ \max\{1, \mathcal{E}(\theta^{\ell})\})\right) \\
&< \frac{\delta}{n}.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\mathsf{SC}(\mathbf{w}, \mathbf{P}) &< \max_{\ell \in [m]} \frac{\rho_{\ell}(m,n)\ \max\{1, \mathcal{E}(\theta^{\ell})\}}{c^{\ell}} \cdot 1 + \max_{\ell \in [m]} \frac{n}{c^{\ell}} \cdot \frac{\delta}{n} \\
&\leq \max_{\ell \in [m]} \rho_{\ell}(m,n)\ \cdot \max_{\ell \in [m]} \frac{\max\{1, \mathcal{E}(\theta^{\ell})\}}{c^{\ell}} + \frac{\delta}{\min_{\ell \in [m]} c^{\ell}} \\
&= \max_{\ell \in [m]} \rho_{\ell}(m,n)\ \cdot \max_{\ell \in [m]} \max\left\{\frac{1}{c^{\ell}}, \frac{\mathcal{E}(\theta^{\ell})}{c^{\ell}}\right\} + \frac{\delta}{\min_{\ell \in [m]} c^{\ell}} \\
&= \max_{\ell \in [m]} \rho_{\ell}(m,n)\ \cdot \max\left\{\max_{\ell \in [m]} \frac{1}{c^{\ell}}, \max_{\ell \in [m]} \frac{\mathcal{E}(\theta^{\ell})}{c^{\ell}}\right\} + \frac{\delta}{\min_{\ell \in [m]} c^{\ell}} \\
&= \max_{\ell \in [m]} \rho_{\ell}(m,n)\ \cdot \max\left\{\frac{1}{\min_{\ell \in [m]} c^{\ell}}, \max_{\ell \in [m]} \frac{\mathcal{E}(\theta^{\ell})}{c^{\ell}}\right\} + \frac{\delta}{\min_{\ell \in [m]} c^{\ell}}.
\end{aligned}
$$

By Lemma 6.1,

$$
\frac{1}{\min_{\ell \in [m]} c^{\ell}}\ <\ \frac{m+n-1}{\sum_{\ell \in [m]} c^{\ell}}.
$$

On the other hand,

$$
\begin{aligned}
\max_{\ell \in [m]} \frac{\mathcal{E}\left(\theta^{\ell}\right)}{c^{\ell}} &= \max_{\ell \in [m]} \frac{n\ p(\ell)}{c^{\ell}} \qquad \left(\text{since } \mathcal{E}\left(\theta^{\ell}\right) = np(\ell)\right) \\
&= n\ \max_{\ell \in [m]} \frac{p(\ell)}{c^{\ell}} \\
&= n\ \max_{\ell \in [m]} \frac{(m+n-1)\,\widetilde{c^{\ell}} - 1}{(n-1)\,c^{\ell}} \qquad \text{(by Lemma 6.1)} \\
&< n\ \max_{\ell \in [m]} \frac{(m+n-1)\,\widetilde{c^{\ell}}}{(n-1)\,c^{\ell}} \\
&= \frac{n(m+n-1)}{n-1} \max_{\ell \in [m]} \frac{1}{\sum_{\ell \in [m]} c^{\ell}} \qquad \text{(by definition of reduced capacities)} \\
&= \frac{n(m+n-1)}{(n-1)\ \sum_{\ell \in [m]} c^{\ell}}.
\end{aligned}
$$

Thus,

$$\max\left\{\frac{1}{\min_{\ell\in[m]} c^\ell}\ ,\max_{\ell\in[m]}\frac{\mathcal{E}\left(\theta^\ell\right)}{c^\ell}\right\} \ \le\ \max\left\{\frac{m+n-1}{\sum_{\ell\in[m]} c^\ell},\frac{n\,(m+n-1)}{(n-1)\ \sum_{\ell\in[m]} c^\ell}\right\}$$
$$= \frac{m+n-1}{\sum_{\ell\in[m]} c^\ell}\ \max\left\{1,\frac{n}{n-1}\right\}$$
$$= \frac{m+n-1}{\sum_{\ell\in[m]} c^\ell}\ \frac{n}{n-1}.$$

It follows that

$$\mathsf{SC}\,(\mathbf{w},\mathbf{P})\ <\ \max_{\ell\in[m]}\rho_\ell(m,n)\,\frac{m+n-1}{\sum_{\ell\in[m]} c^\ell}\cdot\frac{n}{n-1}+\delta\,\frac{m+n-1}{\sum_{\ell\in[m]} c^\ell}$$
$$= \frac{m+n-1}{\sum_{\ell\in[m]} c^\ell}\ \left(\max_{\ell\in[m]}\rho_\ell(m,n)\,\frac{n}{n-1}+\delta\right),$$

as needed. □

We remark that the assumption that **P** is a Nash equilibrium has been crucial for proving Proposition 6.2, since it allowed Lemma 6.1 to be used, which provides conditions on reduced capacities that are necessary for a Nash equilibrium but may fail to hold in general.

6.3. *Upper Bound on Social Cost.* In this section we determine a suitable tail function for each link $\ell \in [m]$, under the additional assumption that $m \le n$. Taking the maximum of these tail functions will yield, via Proposition 6.2, an upper bound on Social Cost. We show:

PROPOSITION 6.3. *Consider the fully mixed case under the model of arbitrary capacities. Assume that all traffics are equal to* w, *and that* $m \le n$. *Then, in a Nash equilibrium* **P**,

$$\mathsf{SC}\,(\mathbf{w},\mathbf{P})\ <\ \frac{2n-1}{\sum_{\ell\in[m]} c^\ell}\ \left(\left\lceil\frac{3\ln n}{\ln\ln n}\right\rceil\frac{n}{n-1}+\frac{8}{4-e}\right).$$

PROOF. We start with an informal outline of our proof. We set $\rho_\ell(m,n) = \lceil(3\ln n)/(\ln\ln n)\rceil$ for each link $\ell \in [m]$; by appealing to Lemma 3.3, we prove that

$$\mathbf{Pr}\left(\bigwedge_{\ell\in[m]}\left(\theta^\ell \le \left\lceil\frac{3\ln n}{\ln\ln n}\right\rceil \max\{1,\mathcal{E}(\theta^\ell)\}\right)\right)\ >\ 1-\frac{8}{(4-e)n}.$$

Then Proposition 6.2 applies to establish the claim. We now present the details of the formal proof.

For each link $\ell \in [m]$, set $\rho_\ell(m,n) = \lceil (3 \ln n)/(\ln \ln n) \rceil$. Then,

$$
\begin{aligned}
&\mathbf{Pr}\left(\bigwedge_{\ell \in [m]} (\theta^\ell \leq \rho_\ell(m,n)\ \max\{1, \mathcal{E}(\theta^\ell)\}) \right) \\
&\quad = 1 - \mathbf{Pr}\left(\bigvee_{\ell \in [m]} \left(\theta^\ell > \left\lceil \frac{3 \ln n}{\ln \ln n} \right\rceil \max\{1, \mathcal{E}(\theta^\ell)\} \right) \right) \\
&\quad \geq 1 - \mathbf{Pr}\left(\bigvee_{\ell \in [m]} \left(\theta^\ell \geq \left\lceil \frac{3 \ln n}{\ln \ln n} \right\rceil \max\{1, \mathcal{E}(\theta^\ell)\} \right) \right) \\
&\quad \geq 1 - \sum_{\ell \in [m]} \mathbf{Pr}\left(\theta^\ell \geq \left\lceil \frac{3 \ln n}{\ln \ln n} \right\rceil \max\left\{1, \mathcal{E}\left(\theta^\ell\right)\right\} \right) \\
&\quad > 1 - \sum_{\ell \in [m]} \frac{4 \max\{1, np(\ell)\}}{(4-e)n^2} \qquad \text{(by Lemma 3.3)} \\
&\quad = 1 - \frac{4}{(4-e)n^2} \sum_{\ell \in [m]} \max\{1, np(\ell)\} \\
&\quad = 1 - \frac{4}{(4-e)n^2} \left(\sum_{\substack{\ell \in [m]: \\ np(\ell) \leq 1}} \max\{1, np(\ell)\} + \sum_{\substack{\ell \in [m]: \\ np(\ell) > 1}} \max\{1, np(\ell)\} \right) \\
&\quad = 1 - \frac{4}{(4-e)n^2} \left(\sum_{\substack{\ell \in [m]: \\ np(\ell) \leq 1}} 1 + \sum_{\substack{\ell \in [m]: \\ np(\ell) > 1}} np(\ell) \right) \\
&\quad \geq 1 - \frac{4}{(4-e)n^2} \left(\sum_{\ell \in [m]} 1 + \sum_{\ell \in [m]} np(\ell) \right) \\
&\quad = 1 - \frac{4}{(4-e)n^2} \left(m + n \sum_{\ell \in [m]} p(\ell) \right) \\
&\quad = 1 - \frac{4}{(4-e)n^2}\ (m+n) \\
&\quad \geq 1 - \frac{4}{(4-e)n^2}\ (2n) \qquad \text{(since } m \leq n\text{)} \\
&\quad = 1 - \frac{8}{(4-e)n}.
\end{aligned}
$$

Thus, Proposition 6.2 implies that

$$
\begin{aligned}
\mathsf{SC}(\mathbf{w}, \mathbf{P}) &< \frac{m+n-1}{\sum_{\ell \in [m]} c^\ell} \cdot \left(\max_{\ell \in [m]} \left\lceil \frac{3 \ln n}{\ln \ln n} \right\rceil \cdot \frac{n}{n-1} + \frac{8}{4-e} \right) \\
&\leq \frac{m+n-1}{\sum_{\ell \in [m]} c^\ell} \cdot \left(\left\lceil \frac{3 \ln n}{\ln \ln n} \right\rceil \cdot \frac{n}{n-1} + \frac{8}{4-e} \right)
\end{aligned}
$$

$$\leq \frac{2n-1}{\sum_{\ell \in [m]} c^{\ell}} \left(\left\lceil \frac{3 \ln n}{\ln \ln n} \right\rceil \frac{n}{n-1} + \frac{8}{4-e} \right) \qquad \text{(since } m \leq n\text{)},$$

as needed. □

6.4. *Lower Bound on Social Optimum.* In this section, we prove a lower bound on Social Optimum.

LEMMA 6.4. *Consider the fully mixed case under the model of uniform capacities. Assume that all traffics are equal to w. Then*

$$\mathsf{OPT}(\mathbf{w}) \geq \frac{n}{\sum_{\ell \in [m]} c^{\ell}}.$$

PROOF. We start with an informal outline of our proof. The proof considers an alternative model where each individual traffic may be "split" among more than one link; clearly, the Social Optimum is no less than the value it attains in this model. This splitting assumption is modeled by introducing a *split fraction* $\varphi_\ell \in [0, 1]$ for each link $\ell \in [m]$, representing the fraction of the total traffic nw received by link ℓ. Thus, the Social Optimum for this model is the minimum, over all possible choices of split fractions φ_ℓ for links, of the maximum, over all links, of $\varphi_\ell n / c^\ell$. The claim follows easily then from the observation that there must exist a link $\ell \in [m]$ such that $\varphi_\ell \geq \widetilde{c^\ell}$. We now continue with the details of the formal proof.

Clearly, the Social Optimum is no less than the value it attains in a model where each individual traffic may be "split" among more than one link. So, define a *split fraction* $\varphi_\ell \in [0, 1]$ for each link $\ell \in [m]$, representing the fraction of the total traffic $\sum_{k=1}^{n} w_k = n$ received by link ℓ. Then, the Social Optimum for this model is given by

$$\begin{aligned} \mathsf{OPT}(\mathbf{w}) &\geq \min_{\{\varphi_\ell \in [0,1]\}_{\ell \in [m]} \mid \Sigma_{\ell \in [m]} \varphi_\ell = 1} \max_{\ell \in [m]} \frac{\varphi_\ell n w}{c^\ell} \\ &= n \cdot \min_{\{\varphi_\ell \in [0,1]\}_{\ell \in [m]} \mid \Sigma_{\ell \in [m]} \varphi_\ell = 1} \max_{\ell \in [m]} \frac{\varphi_\ell}{c^\ell} \\ &= \frac{n}{\sum_{\ell \in [m]} c^\ell} \cdot \min_{\{\varphi_\ell \in [0,1]\}_{\ell \in [m]} \mid \Sigma_{\ell \in [m]} \varphi_\ell = 1} \max_{\ell \in [m]} \frac{\varphi_\ell}{\widetilde{c^\ell}}. \end{aligned}$$

We continue to prove a simple fact.

CLAIM 6.5. *There exists a link $\ell \in [m]$ such that $\varphi_\ell \geq \widetilde{c^\ell}$.*

PROOF. Assume, by way of contradiction, that for each link $\ell \in [m]$, $\varphi_\ell < \widetilde{c^\ell}$, so that $\sum_{\ell \in [m]} \varphi_\ell < \sum_{\ell \in [m]} \widetilde{c^\ell} = 1$. By definition of split fractions, $\sum_{\ell \in [m]} \varphi_\ell = 1$. A contradiction. □

Claim 6.5 implies now that

$$\mathsf{OPT}(\mathbf{w}) \geq \frac{n}{\sum_{\ell \in [m]} c^\ell} \cdot \min_{\{\varphi_\ell \in [0,1]\}_{\ell \in [m]} \mid \Sigma_{\ell \in [m]} \varphi_\ell = 1} 1 = \frac{n}{\sum_{\ell \in [m]} c^\ell},$$

as needed. □

6.5. *Combining the Bounds*. Assuming that $m \leq n$ and appealing to Proposition 6.3 and Lemma 6.4, we obtain:

THEOREM 6.6. *Consider the fully mixed case under the model of arbitrary capacities. Assume that all traffics are identical. Assume also that $m \leq n$. Then*

$$\mathsf{CR} \ < \ (2 + o(1)) \left\lceil \frac{3 \ln n}{\ln \ln n} \right\rceil + \frac{16}{4 - e}.$$

PROOF. By the definition of Coordination Ratio,

$$\begin{aligned}
\mathsf{CR} \ &\leq \ \max_{\mathbf{w},\mathbf{P}} \frac{\mathsf{SC}\,(\mathbf{w}, \mathbf{P})}{\mathsf{OPT}\,(\mathbf{w})} \qquad \text{(where } \mathbf{P} \text{ is a Nash equilibrium)} \\
&< \ \frac{2n-1}{\sum_{\ell \in [m]} c^{\ell}} \left(\left\lceil \frac{3 \ln n}{\ln \ln n} \right\rceil \frac{n}{n-1} + \frac{8}{4-e} \right) \frac{\sum_{\ell \in [m]} c^{\ell}}{n} \\
&\quad \text{(by Proposition 6.3 and Lemma 6.4)} \\
&= \ \frac{2n-1}{n-1} \left\lceil \frac{3 \ln n}{\ln \ln n} \right\rceil + \frac{2n-1}{n} \frac{8}{4-e} \\
&< \ (2 + o(1)) \left\lceil \frac{3 \ln n}{\ln \ln n} \right\rceil + \frac{16}{4-e},
\end{aligned}$$

as needed. □

7. Epilogue. Our work has been the *first* attempt to understand and analyze a new genre of algorithmic problems arising from the KP model and the Coordination Ratio [16]. We introduced the fully mixed Nash equilibrium and proved for it asymptotically tight (within small constants), and sometimes even constant, upper bounds of no worse than $\Theta(\lg n / \lg \lg n)$ on the Coordination Ratio for two interesting instances of the problem: all links have the same capacity while user traffics may vary, or all users carry the same traffic while link capacities vary.

Although the fully mixed Nash equilibrium is one out of the exponentially many possible Nash equilibria, we believe that it encapsulates the difficulty of the whole problem; that is, we believe that the fully mixed Nash equilibrium is an abstraction of a hard problem instance with respect to Social Cost and Coordination Ratio. A substantial body of recent research that followed the original conference publication of our work in 2001 has provided some concrete evidence to this belief. To be more specific, there has been a substantial body of recent research work addressing the so-called *Fully Mixed Nash Equilibrium Conjecture* [7], [10], [18], which asserts that the fully mixed Nash equilibrium maximizes, when it exists, Social Cost.

This natural conjecture was initially motivated by some preliminary results by Fotakis et al. [7], explicitly formulated by Gairing et al. [10], and first studied in a systematic way by Lücking et al. [18]. The conjecture could be proved for several special cases of the problem [7], [10], [11], [18]. For the special case of arbitrary users and identical links, Gairing et al. [11, Theorem 5.1] prove that for any arbitrary vector $\mathbf{w}$, the Social Cost of any Nash equilibrium is within $2\mathsf{max}/\mathsf{avg}(\mathbf{w})(1+\varepsilon)$ of that of the fully mixed Nash equilibrium, for any constant $\varepsilon > 0$. Combined with our upper bounds on the Coordination Ratio restricted to the fully mixed Nash equilibrium (Theorems 5.6, 5.6, and 6.6) this implies corresponding upper bounds (multiplied by $2\mathsf{max}/\mathsf{avg}(\mathbf{w})(1+\varepsilon)$) on Coordination Ratio for the general case.

The fully mixed Nash equilibrium conjecture was recently disproved in [6] for the case of arbitrary users and uniform capacities. It is an interesting open problem whether the conjecture holds for the case of identical users and arbitrary capacities. Finally, conjectures motivated by and similar to the Fully Mixed Nash Equilibrium Conjecture were recently formulated and studied in an intensive way for several variants of the KP model [4], [8], [9], [17], [19]. For an advocate of conjectures related to the fully mixed Nash equilibrium, we refer the reader to the recent survey [12].

Two independent research teams, one of Koutsoupias et al. [15] and another of Czumaj and Vöcking [3], have subsequently bounded (in a nonconstructive way) the Coordination Ratio to be $\Theta(\lg m/\lg\lg m)$ for the model of uniform capacities and arbitrary traffics. Their corresponding, nonconstructive proofs have not identified the worst-case Nash equilibrium [16] for this model—they have only provided a tight upper bound for *any* Nash equilibrium (and, therefore, for the worst-case one). Theorem 5.7 implies that this bound is *not* tight for the restriction of the model of uniform capacities and arbitrary traffics to the fully mixed Nash equilibrium. A corresponding, nonconstructive tight upper bound of $\Theta(\lg m/\lg\lg\lg m)$ on the Coordination Ratio has been shown by Czumaj and Vöcking [3] for the model of arbitrary traffics and arbitrary capacities. Theorem 6.6 implies that this bound is *not* tight for the restriction of the model of arbitrary capacities and identical traffics to the fully mixed Nash equilibrium.

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Appendix. Proof of Lemma 3.3. Since θ^ℓ follows the binomial distribution with parameters n and $p(\ell)$, it holds, for any integer ϑ such that $1 \le \vartheta \le n$, that

$$\begin{aligned}\mathbf{Pr}\left(\theta^\ell = \vartheta\right) &= \binom{n}{\vartheta} p^\vartheta(\ell)\,(1-p(\ell))^{n-\vartheta} \le \binom{n}{\vartheta} p^\vartheta(\ell)\\ &\le \left(\frac{en}{\vartheta}\right)^\vartheta \cdot p^\vartheta(\ell) = \left(\frac{enp(\ell)}{\vartheta}\right)^\vartheta\\ &\le \left(\frac{e\max\{1, np(\ell)\}}{\vartheta}\right)^\vartheta.\end{aligned}$$

Consider now some integer parameter $\rho > 3$ that will be determined later. Clearly,

$$\begin{aligned}&\mathbf{Pr}(\theta^\ell \ge \rho\max\{1, \mathcal{E}(\theta^\ell)\})\\ &\qquad = \mathbf{Pr}\left(\theta^\ell \ge \left\lceil \rho\max\left\{1, \mathcal{E}\left(\theta^\ell\right)\right\}\right\rceil\right) \qquad \text{(since } \theta^\ell \text{ is integer)}\end{aligned}$$

$$
\begin{aligned}
&= \sum_{\vartheta=\lceil \rho \max\{1,\mathcal{E}(\theta^\ell)\}\rceil}^{n} \mathbf{Pr}(\theta^\ell = \vartheta) \\
&= \sum_{\vartheta=\lceil \rho \max\{1,np(\ell)\}\rceil}^{n} \mathbf{Pr}(\theta^\ell = \vartheta) \\
&\leq \sum_{\vartheta=\lceil \rho \max\{1,np(\ell)\}\rceil}^{n} \left(\frac{e\ \max\{1, np(\ell)\}}{\vartheta}\right)^{\vartheta} \\
&\leq \sum_{\vartheta=\lceil \rho \max\{1,np(\ell)\}\rceil}^{n} \left(\frac{e \max\{1, np(\ell)\}}{\lceil \rho \max\{1, np(\ell)\}\rceil}\right)^{\vartheta} \\
&\leq \sum_{\vartheta=\lceil \rho \max\{1,np(\ell)\}\rceil}^{n} \left(\frac{e}{\rho}\right)^{\vartheta} \\
&= \left(\frac{e}{\rho}\right)^{\lceil \rho \max\{1,np(\ell)\}\rceil} \sum_{\vartheta=0}^{n-\lceil \rho \max\{1,np(\ell)\}\rceil} \left(\frac{e}{\rho}\right)^{\vartheta} \\
&< \left(\frac{e}{\rho}\right)^{\lceil \rho \max\{1,np(\ell)\}\rceil} \sum_{\vartheta=0}^{\infty} \left(\frac{e}{\rho}\right)^{\vartheta} \\
&\leq \left(\frac{e}{\rho}\right)^{\lceil \rho \max\{1,np(\ell)\}\rceil} \sum_{\vartheta=0}^{\infty} \left(\frac{e}{4}\right)^{\vartheta} \qquad \text{(since } \rho \geq 4\text{)} \\
&= \left(\frac{e}{\rho}\right)^{\lceil \rho \max\{1,np(\ell)\}\rceil} \cdot \frac{1}{1-\frac{e}{4}} \\
&\leq \frac{4}{4-e} \left(\frac{e}{\rho}\right)^{\rho \max\{1,np(\ell)\}} \qquad \text{(since } \tfrac{e}{\rho} < 1 \text{ and } \rho \max\{1, np(\ell)\} > 3\text{)} \\
&= \frac{4}{4-e} \left(\left(\frac{e}{\rho}\right)^{\rho}\right)^{\max\{1,np(\ell)\}} \\
&\leq \frac{4}{4-e} \left(\frac{e}{\rho}\right)^{\rho} \cdot \max\{1, np(\ell)\} \\
&\qquad \text{(since } (e/\rho)^{\rho} < 1 \text{ and } \max\{1, np(\ell)\} \geq 1\text{).}
\end{aligned}
$$

Now fix $\rho = \lceil (3 \ln n)/\ln\ln n \rceil$. Since $n \geq 3$, $\ln n > 1$ and $\ln\ln n > 0$ so that $(3 \ln n)/\ln\ln n > 3$ and $\rho = \lceil (3 \ln n)/\ln\ln n \rceil > 3$, as presumed. It follows by Claim 2.1 that

$$
\begin{aligned}
\mathbf{Pr}\left(\theta^\ell \geq \left\lceil \frac{3 \ln n}{\ln\ln n} \right\rceil \max\{1, \mathcal{E}(\theta^\ell)\}\right) &< \frac{4}{4-e} \cdot \frac{1}{n^2} \cdot \max\{1, np(\ell)\} \\
&= \frac{4 \max\{1, np(\ell)\}}{(4-e)n^2},
\end{aligned}
$$

as needed.

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