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ABSTRACT

The elastostatic problem for an infinite orthotropic strip containing a crack is considered. It is assumed that the orthogonal axes of material orthotropy may have an arbitrary angular orientation with respect to the orthogonal axes of geometric symmetry of the uncracked strip. The crack is located along an axis of orthotropy, hence at an arbitrary angle with respect to the sides of the strip. The general problem is formulated in terms of a system of singular integral equations for arbitrary crack surface tractions. As examples Modes I and II stress intensity factors are calculated for the strip having an internal or an edge crack with various lengths and angular orientations. In most calculations uniform tension or uniform bending away from the crack region is used as the external load. Limited results are also given for uniform normal or shear tractions on the crack surface.

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1. INTRODUCTION

Because of the ever increasing use of fiber-reinforced composites in a great variety of engineering structures, in recent years the problems regarding their structural integrity and failure have been studied quite extensively. In these studies the material is generally assumed to be homogeneous and orthotropic if either the structure is free from flaws which may be the cause of an eventual failure initiation, or the structure may have a flaw but its size is large in comparison with the local microstructural length parameters such as the fiber diameter and the distance between the neighboring fibers. Otherwise, in failure initiation studies the material has to be treated as a nonhomogeneous continuum containing local flaws with certain geometries. In composites, as well as in wood and certain metallic materials, from the viewpoint of structural failure, a distinguishing feature of material orthotropy is that the material is generally not isotropic with respect to its fracture resistance. Furthermore, in most cases the planes of orthotropy are generally also the planes of weak fracture resistance. Thus, in orthotropic materials regardless of the overall geometry and loading conditions, the fracture propagation would be either along a plane of orthotropy or would have a zig-zag path.

Partly because of the fact that some of the most important structural applications of composites have been in sheet form, and partly for analytical reasons, the crack problems in orthotropic materials have been studied mostly for the cases of plane stress or plane strain. In plane problems, if the medium is infinite containing a line crack or a series of collinear cracks, it was shown that the stress intensity factor is identical to that found for an isotropic plane with the same crack geometry [1-4]. However, it was also shown that if the medium is bounded the material orthotropy would have an influence on the stress intensity factors, and depending on the nature of the orthotropy, the stress intensity factors may be greater or smaller than the corresponding isotropic values [5]. In [5] a uniformly loaded orthotropic strip having cracks perpendicular to the sides was considered and the plane of the crack was assumed to be one of the planes of material orthotropy. This and similar solutions would be adequate to study the fracture problems in sheet structures in which the stress-free boundary is parallel to one of the planes of material orthotropy. On the other hand if the stress-free boundary of the sheet does not coincide with a plane of orthotropy and yet, as expected, if the crack lies on a plane of orthotropy, then the solution of the so-called inclined crack would be necessary to study the related fracture problem. Such a problem is considered in this paper for an infinite strip. The crack is assumed to have an arbitrary location and orientation in the strip (Figure 1), the only restriction being that the plane of the crack is a plane of material orthotropy. The problem is formulated for arbitrary normal and shear tractions on the crack surface and the cases of both internal and edge cracks are considered. The corresponding internal crack problem for an isotropic strip was considered in [6].

2. FORMULATION

The plane elastostatic problem under consideration is described in Figure 1 where x_1 and x_2 refer to the axos of orthotropy and the crack is located on the line $x_1 = 0$, $a<x_2<b$. The solution of the problem is expressed as the sum of two states of stress derived from the Airy stress functions $F_1(x_1,x_2)$ and $F_2(x,y)$ where the coordinates (x_1,x_2) and (x,y) are defined in Figure 1. Referred to (x_1,x_2) axes, in terms of the stress function F_1 the stress components are given by

$$\sigma_{11}^{(1)} = \frac{\partial^2 F_1}{\partial x_2^2} , \quad \sigma_{22}^{(1)} = \frac{\partial^2 F_1}{\partial x_1^2} , \quad \sigma_{12}^{(1)} = - \frac{\partial^2 F_1}{\partial x_1 \partial x_2} . \tag{1}$$

The stress function F_1 must satisfy the following differential equation [7]:

$$\frac{\partial^4 F_1}{\partial x_1^4} + \beta_2 \frac{\partial^4 F_1}{\partial x_1^2 \partial x_2^2} + \beta_1 \frac{\partial^4 F_1}{\partial x_2^4} = 0 \quad , \tag{2}$$

where

$$\beta_1 = \frac{a_{11}}{a_{22}} \quad , \quad \beta_2 = \frac{2a_{12} + a_{66}}{a_{22}} \quad . \tag{3}$$

The elastic constants a_{ij} are defined through the stress-strain relations as follows

$$\varepsilon_{11} = a_{11}\sigma_{11} + a_{12}\sigma_{22} \quad , \quad \varepsilon_{22} = a_{21}\sigma_{11} + a_{22}\sigma_{22} \quad , \quad 2\varepsilon_{12} = a_{66}\sigma_{12} \ . \tag{4}$$

In terms of the engineering constants they are given by

$$a_{11} = 1/E_{11} \ , \ a_{22} = 1/E_{22} \ , \ a_{12} = -\nu_{12}/E_{11} = a_{21} \ , \ a_{66} = 1/G_{12} \ . \tag{5}$$

By using the Fourier transform in the variable x_2, (2) gives the following characteristic equation:

$$m^4 - \beta_2 m^2 + \beta_1 = 0 \quad . \tag{6}$$

Let the roots of the characteristic equation (6) be

$$m_1 = \omega_1 = -m_3 \quad , \quad m_2 = \omega_2 = -m_4 \quad . \tag{7}$$

The known constants ω_1 and ω_2 are real if $\beta_2^2 > 4\beta_1$ and are complex conjugates if $\beta_2^2 < 4\beta_1$. The solution of (2) may then be expressed in terms of the following Fourier integrals:

$$F_1(x_1,x_2) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[A(s)e^{-\omega_1|s|x_1} + B(s)e^{-\omega_2|s|x_1}\right]e^{-isx_2}ds \quad , \ x_1>0 \ ,$$

$$F_1(x_1,x_2) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[A_1(s)e^{\omega_1|s|x_1} + B_1(s)e^{\omega_2|s|x_1}\right]e^{-isx_2}ds \quad , \quad x_1<0 \quad , \tag{8}$$

where ω_1 and ω_2 are selected in such a way that they have positive real parts. Observing that

$$F_1(+0,x_2) = F_1(-0,x_2) \quad , \quad \frac{\partial}{\partial x_1}F_1(+0,x_2) = \frac{\partial}{\partial x_1}F_1(-0,x_2) \quad , \tag{9}$$

equations (8) may be written as

$$F_1(x_1,x_2) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[A(s)e^{-\omega_1|s|x_1} + B(s)e^{-\omega_2|s|x_1}\right]e^{-isx_2}ds \quad , \quad x_1>0 \quad ,$$

$$F_1(x_1,x_2) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\left[c_1A(s) + c_2B(s)\right]e^{\omega_1|s|x_1} + [c_3A(s) - c_1B(s)]e^{\omega_2|s|x_1}\right\}e^{-isx_2}ds \quad , \quad x_1<0 \quad , \tag{10}$$

where

$$c_1 = -\frac{\omega_1+\omega_2}{\omega_1-\omega_2} \quad , \quad c_2 = -\frac{2\omega_2}{\omega_1-\omega_2} \quad , \quad c_3 = \frac{2\omega_1}{\omega_1-\omega_2} \quad . \tag{11}$$

If we now define the discontinuity in the displacement derivatives by

$$f_1(x_2) = \frac{\partial}{\partial x_2}[u_1^{(1)}(+0,x_2) - u_1^{(1)}(-0,x_2)] \quad ,$$

$$f_2(x) = \frac{\partial}{\partial x_2}[u_2^{(1)}(+0,x_2) - u_2^{(1)}(-0,x_2)] \quad , \tag{12}$$

and assume that

$$f_1(x_2) = 0 \quad , \quad f_2(x_2) = 0 \quad , \quad -\infty<x_2<a \quad , \quad b<x_2<\infty \quad , \tag{13}$$

after some manipulations the unknown functions A and B can be obtained in terms of f_1 and f_2 and the stress components may be expressed as

$$\sigma_{11}^{(1)}(x_1,x_2) = \frac{1}{2\pi a_{22}(\omega_1^2-\omega_2^2)}\int_a^b \left[-\frac{(t-x_2)f_1(t)/\omega_1+\omega_1 x_1 f_2(t)}{(t-x_2)^2+\omega_1^2 x_1^2}\right.$$

$$\left.+\frac{(t-x_2)f_1/\omega_2+\omega_2 x_1 f_2(t)}{(t-x_2)^2+\omega_2^2 x_1^2}\right] dt \quad , \tag{14}$$

$$\sigma_{12}^{(1)}(x_1,x_2) = \frac{1}{2\pi a_{22}(\omega_1^2-\omega_2^2)}\int_a^b \left[\frac{\omega_1(t-x_2)f_2(t)-\omega_1 x_1 f_1(t)}{(t-x_2)^2+\omega_1^2 x_1^2}\right.$$

$$\left.+\frac{\omega_2 x_1 f_1(t)-\omega_2(t-x_2)f_2(t)}{(t-x_2)^2+\omega_2^2 x_1^2}\right] dt \quad , \tag{15}$$

$$\sigma_{22}^{(1)}(x_1,x_2) = \frac{1}{2\pi a_{22}(\omega_1^2-\omega_2^2)}\int_a^b \left[\frac{\omega_1(t-x_2)f_1(t)+\omega_1^3 x_1 f_2(t)}{(t-x_2)^2+\omega_1^2 x_1^2}\right.$$

$$\left.-\frac{\omega_2(t-x_2)f_1(t)+\omega_2^3 x_1 f_2(t)}{(t-x_2)^2+\omega_2^2 x_1^2}\right] dt \quad . \tag{16}$$

Referring now to the second solution in which the stress function F_2 is expressed in coordinates x,y (see Figure 1), it can be shown that the compatibility condition reduces the following differential equation:

$$\frac{\partial^4 F_2}{\partial x^4} + \gamma_1 \frac{\partial^4 F_2}{\partial x^3 \partial y} + \gamma_2 \frac{\partial^4 F_2}{\partial x^2 \partial y^2} + \gamma_3 \frac{\partial^4 F_2}{\partial x \partial y^3} + \gamma_4 \frac{\partial^4 F_2}{\partial y^4} = 0 \quad , \tag{17}$$

where

$$\gamma_1 = -\frac{2H_6}{H_2} \quad , \quad \gamma_2 = \frac{2H_4+H_3}{H_2} \quad , \quad \gamma_3 = -\frac{2H_5}{H_2} \quad , \quad \gamma_4 = \frac{H_1}{H_2} \quad , \tag{18}$$

$$H_1 = a_{11}\cos^4\theta + a_{22}\sin^4\theta + (2a_{12}+a_{66})\sin^2\theta\cos^2\theta \quad ,$$

$$H_2 = a_{11}\sin^4\theta + a_{22}\cos^4\theta + (2a_{12}+a_{66})\sin^2\theta\cos^2\theta \quad ,$$

$$H_3 = a_{66}(\cos^2\theta-\sin^2\theta)^2 + 4(a_{11}+a_{22}-2a_{12})\sin^2\theta\cos^2\theta \quad ,$$

$$H_4 = a_{12}(\cos^4\theta+\sin^4\theta) + (a_{11}+a_{22}-a_{66})\sin^2\theta\cos^2\theta \quad ,$$

$$H_5 = [(a_{66}+2a_{12})(\cos^2\theta-\sin^2\theta) - 2(a_{11}\cos^2\theta-a_{22}\sin^2\theta)]\sin\theta\cos\theta \quad ,$$

$$H_6 = [-(a_{66}+2a_{12})(\cos^2\theta-\sin^2\theta) - 2(a_{11}\sin^2\theta-a_{22}\cos^2\theta)]\sin\theta\cos\theta \quad . \tag{19}$$

Let the solution of (17) be expressed by the following Fourier integral

$$F_2(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\sum_{1}^{4} C_k(s)e^{r_k ys}e^{-ixs}ds \quad . \tag{20}$$

Then, substituting from (20) into (17), after some analysis the characteristic equation giving $r_1,\dots,r_4$ is obtained as

$$\gamma_r r^4 - i\gamma_3 r^3 - \gamma_2 r^2 + i\gamma_1 r + 1 = 0 \quad . \tag{21}$$

The roots $r_1,\dots,r_4$ of (21) are complex and satisfy(*)

$$r_3 = -\bar{r}_2 \quad , \quad r_4 = -\bar{r}_1 \quad . \tag{22}$$

For the second solution the stress components are found to be

$$\sigma_{xx}^{(2)}(x,y) = \frac{\partial^2 F_2}{\partial y^2} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\sum_{1}^{4} s^2 r_k^2 C_k e^{r_k sy}e^{-ixs}ds \quad , \tag{23}$$

(*)One may also note that for $\theta=0$ the roots are real, and if $r_1,\dots,r_4$ are the roots corresponding to the angle θ_1, then for the angle $\theta=\theta_1\mp\pi/2$ the roots are $\bar{r}_1,\dots,\bar{r}_4$.

$$\sigma_{xy}^{(2)}(x,y) = -\frac{\partial^2 F_2}{\partial x \partial y} = \frac{i}{2\pi}\int_{-\infty}^{\infty}\sum_{1}^{4} s^2 r_k C_k e^{r_k s y} e^{-ixs} ds \quad , \tag{24}$$

$$\sigma_{yy}^{(2)}(x,y) = \frac{\partial^2 F_2}{\partial x^2} = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\sum_{1}^{4} s^2 C_k e^{r_k s y} e^{-ixs} ds \quad . \tag{25}$$

It will now be assumed that at a given point in the cracked orthotropic strip shown in Figure 1 the stress state can be expressed by the sum of the stresses given by equations (14-16) and (23-25), namely

$$\sigma_{ij}(x_1,x_2) = \sigma_{ij}^{(1)}(x_1,x_2) + \sigma_{ij}^{(2)}(x_1,x_2) \quad , \quad (i,j=1,2) \quad , \tag{26}$$

or

$$\sigma_{\alpha\beta}(x,y) = \sigma_{\alpha\beta}^{(1)}(x,y) + \sigma_{\alpha\beta}^{(2)}(x,y) \quad , \quad (\alpha,\beta) = (x,y) \quad . \tag{27}$$

In applying to the boundary conditions, (26) and (27) should be used with the following transformations:

$$\sigma_{11}^{(2)}(x_1,x_2) = n_1^2\sigma_{xx}^{(2)} + n_2^2\sigma_{yy}^{(2)} - 2n_1n_2\sigma_{xy}^{(2)} \quad ,$$

$$\sigma_{12}^{(2)}(x_1,x_2) = (n_1^2-n_2^2)\sigma_{xy}^{(2)} + n_1n_2(\sigma_{xx}^{(2)}-\sigma_{yy}^{(2)}) \quad , \tag{28a,b}$$

and

$$\sigma_{xx}^{(1)}(x,y) = n_1^2\sigma_{11}^{(1)} + n_2^2\sigma_{22}^{(1)} + 2n_1n_2\sigma_{12}^{(1)} \quad ,$$

$$\sigma_{xy}^{(1)}(x,y) = (n_1^2-n_2^2)\sigma_{12}^{(1)} - n_1n_2(\sigma_{11}^{(1)}-\sigma_{22}^{(1)}) \quad , \tag{29a,b}$$

where the direction cosines are given by

$$n_1 = \cos\theta \quad , \quad n_2 = \sin\theta \quad . \tag{30}$$

3. THE INTEGRAL EQUATIONS

The formulation of the problem given in the previous section contains six unknown functions, $C_k(s)$, $(k=1,..,4)$ and $f_j(t)$, $(j=1,2)$. Referring to Figure 1, these unknowns can be determined by using the following boundary conditions:

$$\sigma_{yy}(x,0) = \sigma_{xy}(x,0) = \sigma_{yy}(x,h) = \sigma_{xy}(x,h) = 0 \quad , \quad -\infty<x<\infty \quad , \tag{31}$$

$$\sigma_{11}(0,x_2) = p_1(x_2) \quad , \quad \sigma_{12}(0,x_2) = p_2(x_2) \quad , \quad a<x_2<b \tag{32}$$

where the crack surface tractions p_1 and p_2 are known functions and are assumed to be the only external loads applied to the strip. Solutions to other types of loading may be obtained by using the standard superposition technique. Substituting from equations (14-16, 24, 25, 27, 29) into (31), we obtain the following system of algebraic equations expressing $C_k(x)$, $k=1,..4$, in terms of f_1 and f_2:

$$\sum_1^4 C_k(s) = R_1(s) \quad , \quad \sum_1^4 r_k C_k(s) = R_2(s) \quad ,$$
$$\sum_1^4 C_k(s)e^{r_k sh} = R_3(s) \quad , \quad \sum_1^4 r_k C_k(s)e^{r_k sh} = R_4(s) \quad , \tag{33a-d}$$

where the functions $R_j(s)$, $j=1,..,4$, as well as the solution of the algebraic system (33) are given in the Appendix A.

Substituting now from equations (14, 15, 23-26, 28) into (32) and using the appropriate expressions for $C_k(s)$ found in Appendix A, the following system of singular integral equations are obtained for the functions f_1 and f_2:

$$\sigma_{1i}(0,x_2) = \frac{1}{\pi D_i}\int_a^b \sum_1^2 \left[\frac{\delta_{ij}}{t-x_2} + k_{ij}(x_2,t)\right] f_j(t)dt = p_i(x_2) \quad , \quad i=1,2 \quad ,$$
$$a<x_2<b \quad , \tag{34}$$

where

$$D_1 = 2a_{22}\omega_1\omega_2(\omega_1+\omega_2) \quad , \quad D_2 = 2a_{22}(\omega_1+\omega_2) \quad . \tag{35}$$

and the expressions of the kernels k_{ij}, $(i,j=1,2)$ are given in the Appendix B. Referring to the definition of f_1 and f_2 given by equation (12) and the assumptions (13), it is clear that, in addition to (34) f_1 and f_2 must satisfy the following single-valuedness conditions:

$$\int_a^b f_j(t)dt = 0 \quad , \quad j=1,2 \quad . \tag{36}$$

From the results given in Appendix B, the kernels $k_{ij}(x_2,t)$, $(i,j=1,2)$ appear to be complex valued functions. However, by using the properties of the roots ω_j, $(j=1,2)$ and r_k, $(k=1,..,4)$ of the characteristic equations, it can be shown that, as expected, k_{ij} are indeed real functions.

Note that the index of the singular integral equations (34) is +1. Therefore, the solution is of the following form:

$$f_i(t) = g_i(t)[(t-a)(b-t)]^{-\frac{1}{2}} \quad , \quad a<t<b \quad , \quad i=1,2 \quad , \tag{37}$$

where the functions g_1 and g_2 are bounded and continuous in $[a,b]$. It may also be noted that equations (34) give the stress components $\sigma_{11}(0,x_2)$ and $\sigma_{12}(0,x_2)$ outside as well as inside the region $(x_1 = 0,\ a<x_2<b)$. Therefore, from (34) one may easily obtain the stress intensity factors in terms of the unknown functions g_1 and g_2. The stress intensity factors are defined by

$$k_1(a) = \lim_{x_2\to a} \sqrt{2(a-x_2)}\ \sigma_{11}(0,x_2) \quad ,$$

$$k_2(a) = \lim_{x_2 \to a} \sqrt{2(a-x_2)}\, \sigma_{12}(0,x_2) \quad ,$$

$$k_1(b) = \lim_{x_2 \to b} \sqrt{2(x_2-b)}\, \sigma_{11}(0,x_2) \quad ,$$

$$k_2(b) = \lim_{x_2 \to b} \sqrt{2(x_2-b)}\, \sigma_{12}(0,x_2) \quad . \qquad (38a\text{-}d)$$

Since the kernels $k_{ij}(x_2,t)$, $(i,j=1,2)$ are bounded in the closed interval $[a,b]$, from (37) it follows that the functions

$$k_i(x_2) = \frac{1}{\pi}\int_a^b \sum_1^2 k_{ij}(x_2,t)f_j(t)dt \quad , \quad (i=1,2) \quad , \quad (0 \le x_2 \le h/\cos\theta) \qquad (39)$$

are also bounded. Thus, defining the fundamental function

$$X(z) = \sqrt{(z-b)(z-a)} \quad , \quad (z=x_2+ix_2') \qquad (40)$$

from (34) and (37) we obtain

$$D_j\sigma_{1j}(0,x_2) = \frac{i}{\pi}\int_a^b \frac{g_j(t)dt}{(t-x_2)X^+(t)} + k_j(x_2) \quad , \quad (j=1,2) \quad . \qquad (41)$$

Defining now the sectionally holomorphic functions

$$\Phi_j(z) = \frac{1}{\pi i}\int_a^b \frac{g_j(t)dt}{(t-z)X^+(t)} \quad , \quad (j=1,2) \qquad (42)$$

and observing that Φ_1 and Φ_2 are holomorphic outside the cut $(a<x_2<b, x_2'=0)$, we find

$$D_j\sigma_{1j}(x_2,0) = -\Phi_j(x_2) + k_j(x_2) \quad , \quad (j=1,2,\ x_2<a,\ x_2>b) \quad . \qquad (43)$$

On the other hand, following [8] from (42) it can be shown that

$$\Phi_j(z) = \frac{g_j(z)}{X(z)} - P_j(z) \quad , \quad j=1,2 \quad , \tag{44}$$

where $P_j(z)$ is the principal part of g_j/X at infinity. Thus, it is seen that

$$D_j\sigma_{1j}(x_2,0) = -\frac{g_j(x_2)}{X(x_2)} + P_j(x_2) + k_j(x_2) \quad , \quad (x_2<a,\ x_2>b) \quad . \tag{45}$$

Finally, from (38), (45), and

$$X(x_2) = \sqrt{(x_2-b)(x_2-a)} = -\sqrt{(b-x_2)(a-x_2)} \quad , \tag{46}$$

we find

$$k_1(a) = \frac{1}{D_1} g_1(a)/\sqrt{(b-a)/2} \quad , \quad k_2(a) = \frac{1}{D_2} g_2(a)/\sqrt{(b-a)/2} \quad ,$$

$$k_1(b) = -\frac{1}{D_1} g_1(b)/\sqrt{(b-a/2} \quad , \quad k_2(b) = -\frac{1}{D_2} g_2(b)/\sqrt{(b-a)/2} \quad . \tag{47a-d}$$

4. NUMERICAL SOLUTION AND RESULTS

The system of singular integral equations (34) is solved numerically by first normalizing the interval (a,b) to (-1,1) and then using the Gauss-Chebyshev integration formulas [9]. The important problem in the numerical analysis is the evaluation of the kernels $k_{ij}(i,j=1,2)$. To do this a highly accurate and relatively simple technique for the calculation of the roots r_i, $(i=1,..,4)$ of the characteristic equation (21) was needed. An outline of such a technique may be found in [10]. Even though complex algebra had to be used throughout the numerical calculations, values of the kernels were, of course, always real. First, changing the material constants or the geometry, the isotropic results given in [6] and the results of the symmetric crack geometry for the orthotropic strip found in [5] were verified. The numerical results are then obtained for the following two basic loading conditions (see Figure 1):

$$p_1(x_2) = -\sigma_m \cos^2\theta \quad , \quad p_2(x_2) = -\sigma_m \sin\theta\cos\theta \quad , \qquad (48a,b)$$

which correspond to uniform (membrane) loading $\sigma_{xx}(y, \mp\infty) = \sigma_m$, and

$$p_1(x_2) = \sigma_b(\frac{2x_2}{h}\cos\theta - 1)\cos^2\theta \quad ,$$

$$p_2(x_2) = \sigma_b(\frac{2x_2}{h}\cos\theta - 1)\sin\theta\cos\theta \qquad (49a,b)$$

which correspond to "pure bending." Here σ_b is the surface stress in the strip under bending away from the crack region. Some results are also obtained for uniform normal or shear tractions on the crack surface in order to explain certain anomalies arising from the inclined crack solution. As an example a boron-epoxy composite sheet with the following material constants is considered (see equations 4 and 5):

$$E_{11} = 24.75 \times 10^6 \text{psi } (170.65 \times 10^9 \text{ N/m}^2) \quad ,$$

$$E_{22} = 8 \times 10^6 \text{ psi } (55.6 \times 10^9 \text{ N/m}^2) \quad ,$$

$$G_{12} = 0.7 \times 10^6 \text{ psi } (4.83 \times 10^9 \text{ N/m}^2) \quad ,$$

$$\nu_{12} = 0.1114 \quad .$$

For this material the roots m_j or ω_j, $(j=1,2)$ of the characteristic equation (6) turn out to be real.

The results for the strip containing an internal crack are given in Tables 1-4. The stress intensity factors given in the tables are defined by equations (38a-d) and are normalized with respect to

$\sigma_m\sqrt{c}$ or $\sigma_b\sqrt{c}$, $c=(b-a)/2$. Table 1 shows the results for a symmetrically located internal crack (i.e., for $a=(h/\cos\theta)-b$) and for various values of the angle θ. Table 2 shows the results for an excentrically located internal crack. In this case the crack tip $x_2=a$ and the crack angle θ are fixed ($a=0.2h/\cos\theta$, $\theta=\pi/4$) and the crack length b-a is varied. The stress intensity ratios k_1' and k_2' shown in this table are defined in Table 1. The general rule for an excentric crack perpendicular to the sides of the strip is that $k_1(a)$ is always greater than $k(b)$ if $a<h-b$. This result is also expected for an inclined crack provided the external load is either uniform pressure or uniform shear traction on the crack surface. However, in the inclined crack case under more general loading conditions this rule may not always be valid. For example, from Table 2 it is seen that for $b=0.4h/\cos\theta$, $k_1(b)>k_1(a)$. Even though this result appears to be somewhat unexpected, it can easily be explained by the coupling effect between the shear and normal crack surface loadings arising from the inclined crack geometry. The stress intensity factors due to only normal or shear traction on the crack surface are shown in Table 3. Note that for the primary stress intensity factors (i.e., k_1 for normal loading and k_2 for shear loading) the general rule mentioned above remains to be valid. However, since the coupling effects (i.e., k_1 for shear loading and k_2 for normal loading) can be positive or negative, the type of anomalous results observed in Table 2 should not be entirely unexpected.

In reference [5] it was shown that in an infinite orthotropic strip containing cracks perpendicular to the sides the stress state in the plane of the crack in general and the stress intensity factors at the crack tips in particular are not affected by a 90° rotation of the axes of material orthotropy. From the proof given in [5] it can be seen that this rather general result will not remain valid for an inclined crack. Table 4 shows the result of an example regarding the rotation of material axes. In the strip labeled by 30° the stiffer material axis E_{11} makes 30° with the x-axis, and in that labeled

Table 1. Stress intensity factors for a symmetrically located internal crack in an orthotropic strip under tension or bending, $c = (b-a)/2$

$\frac{2c}{h/\cos\theta}$	$\theta = 0$		$\theta = \pi/6$		$\theta = \pi/4$		$\theta = \pi/3$	
	Tension: $\sigma_{xx}(\mp\infty, y) = \sigma_m$; $k_1' = k_1/\sigma_m\sqrt{c}$, $k_2' = k_2/\sigma_m\sqrt{c}$							
	k_1'	k_2'	k_1'	k_2'	k_1'	k_2'	k_1'	k_2'
0.2	1.018	0	0.841	0.444	0.629	0.521	0.380	0.465
0.4	1.081	0	1.061	0.479	0.865	0.578	0.566	0.533
0.6	1.226	0	1.420	0.553	1.211	0.686	0.807	0.647
0.8	1.624	0	2.155	0.739	1.877	0.939	1.682	0.924
0.9	2.249	0	3.151	1.022	2.602	1.324		
	Bending: $\sigma_{xx}(\mp\infty, y) = \sigma_b(1-2y/h)$, $k_1' = k_1/\sigma_b\sqrt{c}$, $k_2' = k_2/\sigma_b\sqrt{c}$.							
0.2	0.100	0	0.076	0.044	0.052	0.050	0.028	0.044
0.4	0.202	0	0.164	0.088	0.121	0.103	0.073	0.092
0.6	0.315	0	0.288	0.141	0.229	0.169	0.143	0.155
0.8	0.493	0	0.535	0.233	0.440	0.288	0.314	0.279
0.9	0.709	0	0.868	0.355	0.684	0.450		

Table 2. Stress intensity factors for an excentrically located internal crack in an orthotropic strip under uniform tension or bending away from the crack region; $\theta = \pi/4$, $a/(h/\cos\theta) = 0.2$, $c = (b-a)/2$

$\frac{b}{h/\cos\theta}$	Tension: $\sigma_{xx}^{\infty} = \sigma_m$				Bending: $\sigma_{xx}^{\infty} = \sigma_b(1-2y/h)$			
	$k_1'(a)$	$k_2'(a)$	$k_1'(b)$	$k_2'(b)$	$k_1'(a)$	$k_2'(a)$	$k_1'(b)$	$k_2'(b)$
0.4	0.675	0.552	0.685	0.513	0.327	0.271	0.223	0.155
0.6	0.903	0.628	0.893	0.556	0.315	0.231	0.065	0.008
0.8	1.211	0.686	1.211	0.686	0.229	0.169	-0.229	-0.169

Table 3. Stress intensity factors for an excentrically located internal crack under uniform crack surface tractions; $\theta = \pi/4$, $a/(h/\cos\theta) = 0.2$, $c = (b-a)/2$

$\frac{b}{h/\cos\theta}$	$p_1(x_2) = -1,\ p_2(x_2) = 0$				$p_1(x_2) = 0,\ p_2(x_2) = -1$			
	$k_1(a)/\sqrt{c}$	$k_2(a)/\sqrt{c}$	$k_1(b)/\sqrt{c}$	$k_2(b)/\sqrt{c}$	$k_1(a)/\sqrt{c}$	$k_2(a)/\sqrt{c}$	$k_1(b)/\sqrt{c}$	$k_2(b)/\sqrt{c}$
0.4	1.402	0.048	1.330	-0.019	-0.052	1.056	0.040	1.045
0.6	1.910	0.109	1.769	-0.006	-0.103	1.147	0.018	1.118
0.7	2.218	0.122	2.094	0.026	-0.136	1.200	-0.049	1.172
0.8	2.611	0.107	2.611	0.107	-0.189	1.265	-0.189	1.265

Table 4. Comparison of the stress intensity factors for isotropic and orthotropic strips with a symmetrically located internal crack. Tension: $\sigma_m = \sigma_{xx}(\mp\infty, y)$, bending: $\sigma_{xx}^{\infty} = \sigma_b(1-2y/h)$, $(b-a)/(h/\cos\theta) = 0.6$, $c = (b-a)/2$, $a = (h/\cos\theta) - b$

	$\theta = 0$	$\theta = \pi/6$			
	Tension	Tension		Bending	
	$k_1/\sigma_m\sqrt{c}$	$k_1/\sigma_m\sqrt{c}$	$k_2/\sigma_m\sqrt{c}$	$k_1/\sigma_b\sqrt{c}$	$k_2/\sigma_b\sqrt{c}$
Isotropic	1.303	1.080	0.504	0.248	0.137
Ortho. (30°)	1.226	1.420	0.553	0.288	0.141
Ortho. (120°)	1.226	1.172	0.518	0.258	0.138

by 120° E_{11} axis makes 120° with the x-axis, i.e., in the latter case the material has been rotated by 90° (see Figure 1). The isotropic results are also given in the table. The table shows that in the inclined crack problem not only the material orthotropy but also the orientation of the axes of orthotropy may have a significant effect on the stress intensity factors.

In the case of an edge crack, i.e., for $a = 0$, $b < h/\cos\theta$, the integral equations (34) remain unchanged. However, the unknown functions $f_1(t)$ and $f_2(t)$ are bounded at $t = 0$ and the conditions (36) are no longer valid. In this case the integral equations can be solved numerically by first normalizing the interval (0,b) to (-1,1) through the change in variables

$$t = \frac{b}{2}(r+1) \quad , \quad x_2 = \frac{b}{2}(s+1) \quad , \; -1<(s,r)<1 \quad , \tag{50}$$

and then using again a Gauss-Chebyshev integration formula. A convenient technique in this problem is defining the unknown functions by

$$f_i(t) = G_i(r)/\sqrt{1-r^2} \quad , \quad i=1,2 \tag{51}$$

and using the collocation points s_j obtained from $U_{n-1}(s_j) = 0$, $(j=1,..,n-1)$ and the condition $G_i(-1) = 0$ (to account for boundedness of $f_i(t)$ at $t=0$) to calculate $G_i(r_k)$, $(k=1,..,n)$ $T_n(r_k) = 0$, where T_n and U_n are Chebyshev polynomials. Table 5 shows the calculated results for the edge crack. In this problem too the external load is either a uniform tension or a uniform bending applied to the strip away from the crack region.

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Table 5. Stress intensity factors for an edge crack (a=0) in an orthotropic strip under tension or bending away from the crack region

$\frac{b}{h/\cos\theta}$	Tension: $\sigma_{xx}^{\infty} = \sigma_m$, $k_1' = k_1(b)/\sigma_m\sqrt{b}$, $k_2' = k_2(b)\sigma_m\sqrt{b}$							
	$\theta = 0$		$\theta = \pi/6$		$\theta = \pi/4$		$\theta = \pi/3$	
	k_1'	k_2'	k_1'	k_2'	k_1'	k_2'	k_1'	k_2'
0.1	1.13	0	1.19	0.32	0.99	0.38	0.66	0.35
0.2	1.32	0	1.38	0.34	1.16	0.41	0.79	0.57
0.3	1.61	0	1.70	0.39	1.42	0.47	0.98	0.44
0.4	2.04	0	2.34	0.47				
0.5	2.72	0	3.57	0.63				
0.6	3.86	0	6.09	0.97				
	Bending: $\sigma_{xx}(\mp\infty, y) = \sigma_b(1-2y/h)$, $k_1 = k_1(b)/\sigma_b\sqrt{b}$, $k_2 = k_2(b)/\sigma_b\sqrt{b}$							
0.1	0.99	0	1.06	0.28	0.89	0.33	0.59	0.30
0.2	1.01	0	1.09	0.24	0.92	0.29	0.64	0.27
0.3	1.08	0	1.20	0.23	1.01	0.28	0.71	0.26
0.4	1.21	0	1.51	0.23				
0.5	1.43	0	2.13	0.27				
0.6	1.81	0	3.33	0.41				

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APPENDIX A

Expressions of the functions $R_j(s)$ and the solution of equations (33):

$$R_1(s) = \Delta_1\{(\omega_1 n_1^2 - \frac{n_2^2}{\omega_1})I_1^1 + (\frac{n_2^2}{\omega_2} - n_1^2\omega_1)I_2^1 + (n_1^2\omega_1^3 - n_2^2\omega_1)J_1^1$$

$$+ (n_2^2\omega_2 - n_1^2\omega_2^3)J_2^1 + 2n_1 n_2[\omega_1 K_1^1 - \omega_2 K_2^1 - \omega_1 L_1^1 + \omega_2 L_2^1]\} \quad ,$$

$$R_2(s) = \frac{\Delta_1}{i}\{n_1 n_2[(-\frac{1}{\omega_1} - \omega_1)I_1^1 + (\frac{1}{\omega_2} + \omega_2)I_2^1 - (\omega_1 + \omega_1^3)J_1^1$$

$$+ (\omega_2 + \omega_2^3)J_2^1] + (n_1^2 - n_2^2)[\omega_1 K_1^1 - \omega_2 K_2^1 - \omega_1 L_1^1 + \omega_2 L_2^1]\} \quad ,$$

$$R_3(s) = \Delta_1\{(n_1^2\omega_1 - \frac{n_2^2}{\omega_1})I_1^2 + (\frac{n_2^2}{\omega_2} - n_1^2\omega_1)I_2^2 + (n_1^2\omega_1^3 - n_2^2\omega_1)J_1^2$$

$$+ (n_2^2\omega_2 - n_1^2\omega_2^3)J_2^2 + 2n_1 n_2[\omega_1 K_1^2 - \omega_2 K_2^2 - \omega_1 L_1^2 + \omega_2 L_2^2]\} \quad ,$$

$$R_4(s) = \frac{\Delta_1}{i}\{n_1 n_2[-(\frac{1}{\omega_1} + \omega_1)I_1^2 + (\frac{1}{\omega_2} + \omega_2)I_2^2 - (\omega_1 + \omega_1^3)J_1^2$$

$$+ (\omega_2 + \omega_2^3)J_2^2] + (n_1^2 - n_2^2)[\omega_1 K_1^2 - \omega_2 K_2^2 - \omega_1 L_1^2 + \omega_2 L_2^2]\} \quad ; \qquad \text{(A1-A4)}$$

$$\Delta_1 = \frac{1}{2\pi a_{22}(\omega_1^2 - \omega_2^2)s^2} \quad ; \qquad \text{(A5)}$$

$$I_j^k(s) = \int_a^b E_j^k(s,t) f_1(t)dt \quad ,$$

$$J_j^k(s) = \int_a^b F_j^k(s,t) f_2(t)dt \quad ,$$

$$K_j^k(s) = \int_a^b F_j^k(s,t) f_1(t)dt \quad ,$$

$$L_j^k(s) = \int_a^b E_j^k(s,t) f_2(t)dt \quad , \quad (j,k) = (1,2) \quad ; \qquad \text{(A6-A9)}$$

$$E_j^1(s,t) = \pi e^{-|s|tn_1\lambda_j}[n_1\lambda_j\cos c_jt + c_j\sin|\alpha|c_jt$$

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$$+ i \frac{s}{|s|} c_j\cos c_jst - in_1\lambda_j\sin c_jst] \quad ,$$

$$E_j^2(s,t) = \pi e^{-|s|\lambda_j(h-n_1t)}\{-n_1\lambda_j\cos[c_js(t-n_1h+n_1\omega_j^2h)]$$

$$+ c_j\sin[|s|c_j(t-n_1h+n_1\omega_j^2h)]$$

$$+ ic_j \frac{s}{|s|} \cos [sc_j(t-n_1h+\omega_j^2n_1h)]$$

$$+ in_1\lambda_j\sin [sc_j(t-n_1h+\omega_j^2n_1h)]\} \quad , \quad j=1,2 \quad ; \qquad (A10,A11)$$

$$F_j^1(s,t) = e^{-|s|\lambda_jn_1t}[-\frac{c_j}{\omega_j}\cos c_jst + n_1b_j\sin c_j|s|t$$

$$+ i \frac{s}{|s|} n_1b_j\cos c_jst + i \frac{c_j}{\omega_j} \sin c_jst] \quad ,$$

$$F_j^2(s,t) = \pi e^{-|s|\lambda_j(h-n_1t)} \{\frac{c_j}{\omega_j} \cos[sc_j(t-n_1h+\omega_j^2n_1h)]$$

$$+ n_1b_j\sin[|s|c_j(t-n_1h+\omega_j^2n_1h)]$$

$$+ i \frac{s}{|s|} n_1b_j\cos[sc_j(t-n_1h+\omega_j^2n_1h)]$$

$$- i \frac{c_j}{\omega_j} \sin[sc_j(t-n_1h+\omega_j^2n_1h)] \quad , \quad j-1,2 \quad ; \qquad (A12,A13)$$

$$\lambda_j = \omega_j/(n_1^2\omega_j^2+n_2^2) \ , \ b_j = 1/(n_1^2\omega_j^2+n_2^2) \ , \ c_j = -n_2/(n_1^2\omega_j^2+n_2^2) \quad ,$$

$$j=1,2 \quad ; \qquad (A14-A15)$$

Solution of equations (33):

$$C_k(s) = \frac{1}{\Delta(s)} \sum_{1}^{4} m_{kj}(s) R_j(s) \quad , \quad k=1,\ldots,4 \quad ; \tag{A17}$$

$$\begin{aligned}
\Delta(s) = {} & (r_1-r_3)(r_1-r_4)(e^{r_1 sh} - e^{r_2 sh})(e^{r_4 sh} - e^{r_3 sh}) \\
& - (r_1-r_3)(r_1-r_2)(e^{r_2 sh} - e^{r_3 sh})(e^{r_1 sh} - e^{r_4 sh}) \\
& + (r_1-r_2)(r_1-r_4)(e^{r_2 sh} - e^{r_4 sh})(e^{r_1 sh} - e^{r_3 sh}) \quad ;
\end{aligned} \tag{A18}$$

$$\begin{aligned}
m_{11}(s) = {} & r_4(r_3-r_2)e^{(r_2+r_3)sh} + r_3(r_2-r_4)e^{(r_2+r_4)sh} \\
& + r_2(r_4-r_3)e^{(r_3+r_4)sh} \quad ,
\end{aligned}$$

$$m_{12}(s) = (r_2-r_3)e^{(r_2+r_3)sh} - (r_2-r_4)e^{(r_2+r_4)sh} - (r_4-r_3)e^{(r_3+r_4)sh} \quad ,$$

$$m_{13}(s) = r_2(r_4-r_3)e^{r_2 sh} + r_3(r_2-r_4)e^{r_3 sh} + r_4(r_3-r_2)e^{r_4 sh} \quad ,$$

$$m_{14}(s) = (r_3-r_4)e^{r_2 sh} + (r_4-r_2)e^{r_3 sh} + (r_2-r_3)e^{r_4 sh} \quad ,$$

$$\begin{aligned}
m_{21}(s) = {} & r_4(r_1-r_3)e^{(r_1+r_3)sh} - r_3(r_1-r_4)e^{(r_1+r_4)sh} \\
& + r_1(r_3-r_4)e^{(r_3+r_4)sh} \quad ,
\end{aligned}$$

$$\begin{aligned}
m_{22}(s) = {} & (r_1-r_3)e^{(r_1+r_3)sh} + (r_1-r_4)e^{(r_1+r_4)sh} \\
& + (r_4-r_3)e^{(r_3+r_4)sh} \quad ,
\end{aligned}$$

$$m_{23}(s) = r_1(r_3-r_4)e^{r_1 sh} - r_3(r_1-r_4)e^{r_3 sh} + r_4(r_1-r_3)e^{r_4 sh} \quad ,$$

$$m_{24}(s) = (r_4-r_3)e^{r_1 sh} + (r_1-r_4)e^{r_3 sh} - (r_1-r_3)e^{r_4 sh} \quad ,$$

$$m_{31}(s) = r_2(r_1-r_4)e^{(r_1+r_4)sh} + r_1(r_4-r_2)e^{(r_2+r_4)sh}$$

$$- r_4(r_1-r_2)e^{(r_1+r_2)sh} \quad ,$$

$$m_{32}(s) = (r_1-r_4)e^{(r_1+r_4)sh} + (r_1-r_2)e^{(r_1+r_2)sh}$$

$$+ (r_4-r_1)e^{(r_1+r_4)sh} \quad ,$$

$$m_{33}(s) = r_1(r_4-r_2)e^{r_1 sh} - r_2(r_1-r_4)e^{r_2 sh} - r_4(r_1-r_2)e^{r_4 sh} \quad ,$$

$$m_{34}(s) = (r_2-r_4)e^{r_1 sh} - (r_1-r_4)e^{r_2 sh} + (r_1-r_2)e^{r_4 sh} \quad ,$$

$$m_{41}(s) = r_3(r_1-r_2)e^{(r_1+r_2)sh} - r_2(r_1-r_3)e^{(r_1+r_3)sh}$$

$$+ r_1(r_2-r_3)e^{(r_2+r_3)sh} \quad ,$$

$$m_{42}(s) = -(r_1-r_2)e^{(r_1+r_2)sh} + (r_1-r_3)e^{(r_1+r_3)sh}$$

$$+ (r_3-r_2)e^{(r_3+r_2)sh} \quad ,$$

$$m_{43}(s) = r_1(r_2-r_3)e^{r_1 sh} - r_2(r_1-r_3)e^{r_2 sh} + r_3(r_1-r_2)e^{r_3 sh} \quad ,$$

$$m_{44}(s) = (r_3-r_2)e^{r_1 sh} + (r_1-r_3)e^{r_2 sh} - (r_1-r_2)e^{r_3 sh} \quad . \qquad \text{(A19-A34)}$$

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APPENDIX B

Expressions of the kernels $k_{ij}(x_2,t)$, $(i,j=1,2)$:

$$k_{ij}(x_2,t) = d_i \int_0^\infty [G_{ij}(x_2,t,s) + G_{ij}(x_2,t,-s)]ds \quad , \quad (i,j) = (1,2) \; ; \tag{B1}$$

$$d_1 = \frac{\omega_1\omega_2}{2\pi(\omega_1-\omega_2)} \quad , \quad d_2 = \frac{1}{2\pi(\omega_1-\omega_2)} \; ; \tag{B2,B3}$$

$$G_{11}(x_2,t,s) = \frac{e^{-in_2x_2s}}{\Delta(s)} \Big[h_1E_1^1+h_2E_2^1+h_3E_1^2+h_4E_2^2+h_5\omega_1F_1^1 - h_5\omega_2F_2^1+h_6\omega_1F_1^2-h_6\omega_2F_2^2\Big] \; ,$$

$$G_{12}(x_2,t,s) = \frac{e^{-in_2x_2s}}{\Delta(s)} \Big[-\omega_1h_5E_1^1+\omega_2h_5E_2^1-\omega_1h_6E_1^2+\omega_2h_6E_2^2 + \omega_1^2h_1F_1^1+\omega_2^2h_2F_2^1+\omega_1^2h_3F_1^2+\omega_2^2h_4F_2^2\Big] \; ,$$

$$G_{21}(x_2,t,s) = \frac{e^{-in_2x_2s}}{\Delta(s)} \Big[v_1E_1^1+v_2E_2^1+v_3E_1^2+v_4E_2^2+v_5\omega_1F_1^1 - v_5\omega_2F_2^1+v_6\omega_1F_1^2-v_6\omega_2F_2^2\Big] \; ,$$

$$G_{22}(x_2,t,s) = \frac{e^{-in_2x_2s}}{\Delta(s)} \Big[-\omega_1v_5E_1^1+\omega_2v_5E_2^1-\omega_1v_6E_1^2+\omega_2v_6E_2^2 + \omega_1^2v_1F_1^1+\omega_2^2v_2F_2^1+\omega_1^2v_3F_1^2+\omega_2^2v_4F_2^2\Big] \; ; \tag{B4-B7}$$

where the functions $E_j^k(x,t)$ and $F_j^k(s,t)$, $(j,k=1,2)$ are given by equations (A10-A13), $\Delta(s)$ is given by (A18), and

$$h_1(x_2,s) = \sum_1^4 \alpha_k(x_2,s)[a_1m_{k1}+ia_2m_{k2}] \quad ,$$

$$h_2(x_2,s) = \sum_1^4 \alpha_k(a_3m_{k1}+ia_4m_{k2}) \quad ,$$

$$h_3(x_2,s) = \sum_1^4 \alpha_k(a_1m_{k3}+ia_2m_{k4}) \quad ,$$

$$h_4(x_2,s) = \sum_1^4 \alpha_k(a_3m_{k3}+ia_4m_{k4}) \quad ,$$

$$h_5(x_2,s) = \sum_1^4 \alpha_k[2n_1n_2m_{k1}-i(n_1^2-n_2^2)m_{k2}] \quad ,$$

$$h_6(x_2,s) = \sum_1^4 \alpha_k[2n_1n_2m_{k3}-i(n_1^2-n_2^2)m_{k4}] \quad ; \qquad \text{(B8-B13)}$$

$$\alpha_k(x_2,s) = (n_1^2r_k^2-n_2^2-2in_1n_2r_k)e^{r_kn_1x_2s} \quad , \quad (k=1,\ldots,4) \quad ; \qquad \text{(B14)}$$

$$v_1(x_2,s) = \sum_1^4 \beta_j(a_1m_{j1}+ia_2m_{j2}) \quad ,$$

$$v_2(x_2,s) = \sum_1^4 \beta_j(a_3m_{j1}+ia_4m_{j2}) \quad ,$$

$$v_3(x_2,s) = \sum_1^4 \beta_j(a_1m_{j3}+ia_2m_{j4}) \quad ,$$

$$v_4(x_2,s) = \sum_1^4 \beta_j(a_3m_{j3}+ia_4m_{j4}) \quad ,$$

$$v_5(x_2,s) = \sum_1^4 \beta_j[2n_1n_2m_{j1}-i(n_1^2-n_2^2)m_{j2}] \quad , \qquad \text{(B15-B20)}$$

$$v_6(x_2,s) = \sum_1^4 \beta_j[2n_1n_2m_{j3}-i(n_1^2-n_2^2)m_{j4}] \quad ;$$

$$\beta_j(x_2,s) = [n_1n_2r_j^2+n_1n_2+i(n_1^2-n_2^2)r_j]e^{r_jn_1x_2s} \quad , \quad (j=1,\ldots,4) \qquad \text{(B21)}$$

$$a_1 = n_1^2\omega_1 - \frac{n_2^2}{\omega_1} \quad , \quad a_2 = \frac{n_1 n_2}{\omega_1} + n_1 n_2 \omega_1 \quad ,$$

$$a_3 = -n_1^2\omega_2 + \frac{n_2^2}{\omega_2} \quad , \quad a_4 = -\frac{n_1 n_2}{\omega_2} - n_1 n_2 \omega_2 \quad ; \qquad \text{(B22-B25)}$$

and the functions $m_{kj}(s)$, $(k,j=1,..,4)$ are given by equations (A19-A34).

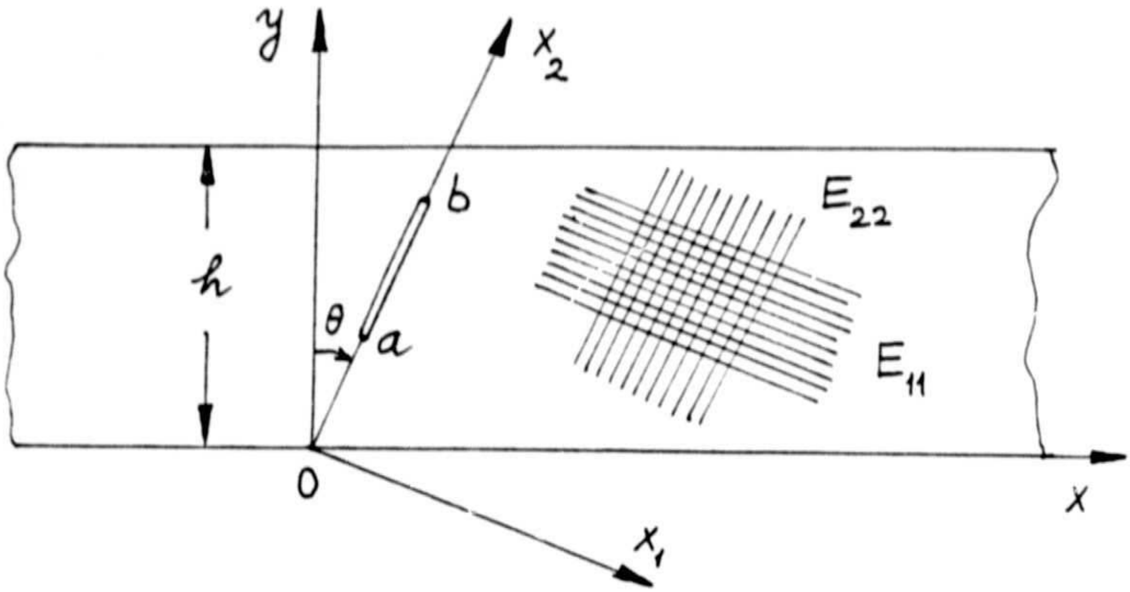

Figure 1.