The Problem of Conservation Laws and the Poincaré Quasigroup in General Relativity

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1. Introduction

The role played by conservation laws in physics is well known. One can hardly imagine a physics text that makes no use, one way or the other, of the concepts of energy, momentum, and angular momentum engendered by the respective conservation laws. (I will also use the term *moment* for the total 4-moment of momentum, i.e., the angular momentum and the velocity of the center of inertia.)

The physico-mathematical essence of conservation laws, as consequences of a theory's symmetries, was clarified in the second decade of the twentieth century by Felix Klein, David Hilbert, and Emmy Noether on the basis of the special theory of relativity (SR). This approach was intended to be used in the general theory of relativity (GR). Ironically, it so happened that the status of conservation laws was soon questioned in GR itself (see Vizgin 1972, 1981).

In creating his general theory of relativity, Einstein used the law of conservation of energy-momentum as one of his main tools. In the final theory, however, conservation laws became a problem, rather than a consequence of the theory. The problem arose from the idea, fundamental to GR, of geometrizing physical interactions, the idea that led to the notion of space-time with variable curvature. After short but heated debates, Einstein found a solution to this problem in a very important, albeit special case of an "island configuration," an isolated system where the geometry is noticeably curved only in a finite region, whereas at infinity, it is asymptotically flat (Einstein 1918). Einstein's solution is based on the so-called pseudotensor of the gravitational field's energy-momentum. Most

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specialists in gravitation still believe that the pseudotensor approach is the best possible one (see, e.g., Faddeev 1982).

Such a limited solution has, however, been unable to satisfy everyone. Some theorists believe that the situation with conservation laws in GR is so unsatisfactory that they reject Einstein's theory of gravitation as a whole (Logunov 1998). The criticism of such views has so far avoided the problem of conservation laws in the general formulation, that is, in the case of arbitrary geometries admitted by GR. Yet the existence of the problem is clearly revealed when one attempts to establish a correspondence between SR and GR using conserved quantities as a "bridge." Those specialists who do not question GR have also evinced dissatisfaction with this situation. Penrose (1982), for one, put the energy-momentum-moment problem at the very top of his list of unsolved problems in classical GR.

Below we suggest a novel analysis of the problem of conservation laws in GR.

We take the field theory in SR as our point of departure. The situation with conservation laws in this case is well known and quite clear. But the special relativistic case is "degenerate" from the point of view of GR. Curved space-time "splits" various properties that are equivalent in SR, making some of them independent and rendering some others meaningless. Therefore, not just any formulation of conservation laws that is "natural" and customary in SR can be taken as basic if one's target is GR.

The Noether theorem (the so-called *first* theorem), which completed the theory of conservation laws, affords the deepest explanation of them in SR, and this theorem will be used as a basis for the analysis of the problem in GR. (The so-called *second* Noether theorem is irrelevant to conservation laws in SR and will not be discussed in this paper.)

The key question of this paper is whether there is a connection, in the sense of the correspondence principle, between the inferior (pseudo-) conservation laws in GR and the ten absolutely clear conservation laws in SR. More specifically, is there a formal construction, definable on a generic space-time of GR, that corresponds to the ten-parameter Poincaré group in SR?

After a brief account of the history of the pseudotensor approach we undertake a Noether-type approach to the general problem of conservation laws in GR, based on the concept of the Poincaré quasigroup (Gorelik 1988).

2. The Noether Theorem and Conservation Laws in Special Relativity

The soundest foundation for discussing conservation laws in a sufficiently developed theory is provided by the Noether theorem (see Bogolyubov and Shirkov 1980, Vizgin 1972). This theorem establishes a link between the theory's symmetries (the invariance of its action) and its conservation laws:

> *Each one-parameter symmetry (a one-parameter set of transformations of the theory's variables keeping the action invariant) corresponds to a separate conservation law.*

Let us briefly recall the Noether-type formulation leading to conservation laws for the field $u_a(x^i)$, where a is an index or a set of indices, x^i are coordinates in space-time, and $i = 1, \ldots, n$ (for generality, let us allow an arbitrary number n of dimensions of space-time). The field dynamics is described by the Lagrangian $\mathfrak{L}[u]$ and the corresponding action, the integral of the Lagrangian over the space-time region Ω:

$$A = \int_{\Omega} \mathfrak{L}[u]\, d^n x. \tag{1}$$

The principle of stationary action, $\delta A = 0$, then gives the field equations

$$\delta\mathfrak{L}/\delta u_a = 0. \tag{2}$$

Let the theory have an s-parameter symmetry Σ^s; that is, transformations of coordinates $x^i \to x'^i$ and of the field variables $u_a \to u'_a$, corresponding to this symmetry, do not change the action:

$$A(x', u') = A(x, u). \tag{3}$$

For infinitesimal symmetry parameters Δ^A ($A = 1, \ldots, s$), the action is invariant with regard to the transformations $x^i \to x^i + \delta x^i$, $u_a \to u_a + \delta u_a$, where

$$\delta x^i = X^i_A \Delta^A, \tag{3a}$$

$$\delta u_a = \Psi_{aA} \Delta^A, \tag{3b}$$

and $X_A^{\ i}$ and Ψ_{aA} are definite quantities reflecting the structure of the symmetry Σ^s and the properties of the field variables u_a. Then, according to the Noether theorem, $\mathfrak{L}$, u_a, X, and Ψ give rise to the quantities

$$\Theta_A^i = \Theta_A^i(\mathfrak{L}, u, X, \Psi), \tag{4}$$

which yield, via the field laws (2), s equations of the form

$$\partial_i \Theta_A^i = 0, \quad A = 1, \ldots, s; \quad \partial_i \equiv \partial/\partial x^i. \tag{5}$$

Equations (5) are the conservation laws *in differential form*. The name is appropriate because integrating (5) over some space-time region Ω limited by the hypersurface $\partial\Omega$ leads, by Gauss's theorem, to the relations

$$\oint_{\partial\Omega} \Theta_A^i \, d\sigma_i = 0, \tag{6}$$

which are sometimes called *integrated balance equations*. And these, in turn, entail conservation laws in the conventional sense. To see this, consider Minkowski (or Newtonian) space-time, and, as the hypersurface $\partial\Omega$, choose a four-dimensional cylinder with the lower and upper bases corresponding to the three-dimensional volume V at the times t_1 and t_2. Then Equations (6) take the form of the balance relations

$$C_A(V, t_2) - C_A(V, t_1) = \int_{t_1}^{t_2} Q_A \, dt, \tag{7}$$

where

$$C_A(V, t) \equiv \int_V \Theta_A^0 \, dV, \tag{7a}$$

$$Q_A(\partial V, t) \equiv \oint_{\partial V} \Theta_A^\alpha \, ds_\alpha, \quad \alpha = 1, 2, 3, \tag{7b}$$

that is, the change of the quantity Θ_A in the volume V is equal to the flow of that quantity through the surface ∂V bounding the volume.

If we move the surface ∂V to spatial infinity and assume that the fields $u_a(x)$ (as well as Θ_A) vanish on the surface (the assumption of a closed system), we obtain a conservation law in its pure form:

$$C_A(t_2) = C_A(t_1), \tag{8}$$

with the integration in (7a) performed over the entire three-dimensional space. It is Equations (7) and (8) that justify the name, "conservation laws," for Equations (5), from which they follow.

The energy-momentum-moment conservation laws in classical mechanics and in SR can be obtained in the manner just described, and they turn out to correspond to the space-time symmetries—the invariance of the action with respect to translations and rotations of the reference system (that is, to the space-time homogeneity and isotropy).

The Noether theorem indicates that these laws are grounded precisely in the properties of space-time itself, rather than in the specific properties of any physical systems (though, of course, the particular expressions of the conserved quantities Θ_A are also determined by the Lagrangian, $\mathfrak{L}$, that is, by the properties of the system in question). Hence the general character of conservation laws in classical mechanics and SR. This is also the reason for regarding the energy-momentum-moment conservation laws as *space-time conservation laws*. We will talk, henceforth, only about such conservation laws, and therefore the adjective will be omitted. (The Noether theorem also gives rise to other conservation laws (e.g., of electric charge), corresponding to transformations that do not involve the space-time coordinates and are characterized by $X = 0$, $\Psi \neq 0$.)

The central symmetry of SR is the invariance with respect to transformations from one inertial reference system to another. This symmetry is described by the ten-parameter Poincaré group, $\mathcal{P}^{10}$, and, therefore, generates ten "conserved" quantities $\Theta_A{}^i$:

$$\partial_i \Theta_A^i = 0, \quad A = 1, \ldots, 10. \tag{9}$$

The quotation marks here are meant to stress a simple, albeit important point that conservation laws depend on the assumption about the behavior of the field at infinity.

The number of conservation laws in SR is equal to the dimensionality of the Poincaré group and is determined by the number of space-time dimensions, *n*:

$$s = n(n+1)/2 = 10 \quad \text{for} \quad n = 4. \tag{10}$$

At first glance, the quantities $\Theta_A{}^i$ (Equation (9)) look somewhat unusual, since the two indices have quite different characters: index i ranges over four space-time coordinates, whereas index A ranges over ten independent symmetries. Usually, however, one deals with the energy-momentum tensor T^{ik}, symmetric in the indices ik, and—not that usually—the tensor of moment $M^i_{l\,m}$, which is anti-symmetric in the indices lm. Nevertheless, the set of all such quantities T and M constitutes precisely the $\Theta_a{}^i$:

$$\Theta^i_A = \{T^{ik}, M^i_{lm}\}.$$

The possibility of such a division is grounded in the specific symmetry properties of Minkowski space-time, but for the Noether scheme itself, these are just additional conditions.

3. Conservation Laws at the Time of the Creation of General Relativity

The history of conservation laws in GR began even before the creation of this theory (Vizgin 1972, 1981). While thinking about the construction of a relativistic theory of gravitation in 1907, Einstein identified, in the principle of equivalence, the seed from which the theory grew over the next eight years. It was not until 1913, however, that an adequate mathematical expression of this principle emerged: a Riemannian geometrization of gravitational interaction. In Einstein's strenuous and tormenting efforts from 1913 to 1915 to build a theory of gravitation on this basis, a major role was played by the energy-momentum conservation law borrowed from electrodynamics and SR, in the form

$$\partial_i \Theta^{ik} = 0. \tag{11}$$

Speaking about the postulates that should figure in the foundations of a relativistic theory of gravitation, Einstein mentioned, first and foremost, the "satisfaction of the laws of conservation of momentum and energy" (Einstein 1913, p. 1250). It was just then that Einstein discovered the crux of the problem of conservation laws in GR: the apparent incompatibility of general covariance with the energy-momentum conservation law (or, more exactly, with the equation that Einstein referred to as a conservation law). At first, because of his inadequate knowledge of Riemannian geometry, Einstein decided to sacrifice general covariance in favor of the conservation law. But after two years of painful search for noncovariant field equations, Einstein decided to place his trust in the power of Riemannian geometry and to jettison the universal validity of conservation laws.

4. Einstein's Attitude toward Conservation Laws

Einstein's attitude toward conservation laws was shaped by his primarily physical (rather than mathematical) approach to formulating GR. And the law of energy conservation was undoubtedly something to be respected by him, due to its historical merits in physics and its general validity. Moreover, Einstein had, in fact, no other tool for constructing a theory that could match the power of the law of energy conservation.

Guided by the idea that the relativistic gravitation theory should be a generalization of the special principle of relativity (asserting the admissibility of any, not only inertial, reference systems), Einstein arrived, fairly early in the process, at the conclusion that space-time coordinates in GR could not be assigned an operational or objective metrical significance. Hence the motto of general covariance.

At the start of the path towards GR, energy conservation was for Einstein one of the main arguments against the general covariance of the field equations. But even after completing GR, when Einstein returned to generally covariant field equations, he was not inclined to abandon the conservation law. He sought to import the relationship (11) from electrodynamics and SR into gravitation theory, generalizing this relationship or deducing it from the field equations. This effort led to the introduction of the so-called energy-momentum pseudotensor of the gravitational field t^{ik}. A relation somewhat similar to (11) was introduced for the sum of this pseudotensor with the energy-momentum tensor of matter

$$\partial_i[(-g)(T^{ik} + t^{ik})] = 0. \qquad (12)$$

This equation was derived from the fundamental property of covariant field equations

$$D_i T^{ik} = 0 \qquad (13)$$

or (which is equivalent, given Einstein's equations $R^{ik} - (½)g^{ik}R = \kappa T^{ik}$) from the general property of Riemannian geometry:

$$D_i(R^{ik} - \frac{1}{2}g^{ik}R) = 0. \tag{14}$$

The covariant derivative $D_i = \partial_i + \Gamma_i$, being a characteristic feature of a generally covariant theory, differs from the ordinary one by the component Γ_i, inevitable in a curved geometry. Yet because of this addition, Gauss's theorem cannot be used to turn Equation (13), which seems to be a natural covariant generalization of the differential conservation law of SR (11), into a balance equation similar to (6) and a conservation law similar to (8).

Einstein worked out his compromise pseudotensor solution (12) under the influence of two essentially methodological principles: confidence in the law of energy conservation and confidence in general covariance. The starting point was the covariantly formulated equation (13). By a formal transformation, it was turned into Equation (12) expressing the fact of vanishing of the ordinary divergence, yet the energy-momentum of the gravitational field was to be described by the quantity t^{ik} made up of the gravitational potentials (the metric) g^{ik}. But this quantity depends on the choice of the coordinate system in a noncovariant and nontensorial manner: hence the name, the t^{ik} *pseudotensor*.

The noncovariant character of the t^{ik} was immediately demonstrated in the examples adduced by Schrödinger and Bauer, where, in certain coordinates, the t^{ik} become zero for a definitely non-zero field but do not vanish in empty Minkowski space-time. In combination with the motto of GR (all coordinate systems are permissible), these examples appeared to be fatal to the pseudotensor approach (see Cattani and De Maria 1993).

The way out of this difficult situation suggested by Einstein in 1918 was as follows: The ambiguity in the energy-momentum of the gravitational field should be resolved for the integrated values $\int t^{ik}dV_i$ by limiting the domain of applicability of the pseudotensor approach. Attention was restricted to island situations, where all matter is concentrated in a certain finite spatial volume, and, outside this volume, Cartesian coordinates (Galilean, in Einstein's words) were employed. Since Einstein's approach to conservation laws in GR (worked out by him in 1918) continues to be accepted, let us have a closer look at this work.

Einstein begins his 1918 paper with this dramatic statement: "Although the general theory of relativity has found acceptance among the majority of theoretical physicists and mathematicians, almost all of my colleagues object to my formulation of the law of conservation of energy-momentum" (Einstein 1918, p. 448). But this circumstance did not undermine Einstein's confidence in the viability and importance of the conservation law. In defending his point of view, Einstein begins thus:

> Just like the law of conservation of momentum, which is formed out of three similar conservation equations, the law [of conservation of energy] was, in its original formulation, an integral law. The special theory of relativity blended all four conservation laws into a unified differential law, which asserts the vanishing of the divergence of the "energy tensor." This differential law is equivalent to the integral laws abstracted from experience; it is here, alone, that its significance lies. (Einstein 1918, p. 448)

And somewhat later he says: "*Experience* clearly compels us to seek a differential law that is equivalent to the integral laws of conservation of momentum and energy" (Einstein 1918, p. 449).

Einstein points out that his pseudotensor formulation runs into his colleagues' "objections because . . . they expect all physically significant quantities to be capable of being conceived as scalars or components of tensors" (Einstein 1918, p. 449). In refuting this objection, Einstein wanted to show that, with the aid of the pseudotensor equation (12),

> the concepts of energy and momentum are established just as strictly as we are accustomed to demand in classical mechanics. The energy and momentum of a closed system are completely determined, independently of the choice of the coordinate system, if only the state of motion of the system (considered as a whole) is given relative to the coordinate system. (Einstein 1918, pp. 449–450)

In support of the limitation to the island situations (closed systems), Einstein says: "In order for us to be able to speak of the energy and momentum of a system, the density of energy and momentum must vanish outside a certain region" (Einstein 1918, p. 450). And he goes on to show that if, outside this region, only Cartesian coordinates are used, then the integral values of energy and momentum (throughout the occupied region) do not depend on the choice of the coordinate system inside the region: "Thus, contrary to what is now our customary way of thinking, we come to ascribe more reality to an integral than to its differentials" (Einstein 1918, p. 452).

Yet restriction to an island situation could not satisfy Einstein himself (who, by 1918, had already established relativistic cosmology), and more than half of his paper is devoted to integral conservation laws for a closed universe.

There is no indication in Einstein's works that he was aware of the connection between symmetry and conservation principles. (He discussed the importation into GR, not of all ten conservation laws of SR, but only of those pertaining to energy and momentum). This is not surprising. It was only in 1911 that Gustav Herglotz established, for the first time, a link between ten conservation laws and ten symmetries of the Poincaré group in the context of the mechanics of continuous media in SR (Vizgin 1972), which was rather remote from Einstein's domain of interest. And Noether's work, in which the symmetry-conservation link was elaborated in a general form, appeared only in 1918, when Einstein had already worked out his pseudotensor solution.

Had Einstein realized the Noether nature of any conservation law when he was thinking about the problem of conservation laws in GR, he would have had to look for some ten-parameter symmetry in general Riemannian space. According to contemporary experts (Trautman 1962; Schmutzer 1970, 1979), such a strategy is doomed to failure. This pessimistic view presupposes, however, that in GR the symmetries suitable for the Noether theorem can only be the *symmetries of the movement* of space itself (those associated with the Killing vectors). And such symmetries indeed single out only very special geometries and are nonexistent in the generic Riemannian space.

Another way of understanding symmetries in GR, the approach applicable to the generic Riemannian case—the ten-parameter *symmetry of description*—will be discussed in §9.

5. Conservation Laws in General Relativity after 1918

After Einstein's work of 1918, discussion of the problem of conservation laws in GR faded for nearly forty years. It looked as though the absence of a general solution to the problem and the absence of a Noether interpretation did not worry theorists. Moreover, when conservation laws were going through hard times in the domain of relativistic quantum mechanics in the 1920s and 1930s, "theoretical" aid was proffered from GR (Gorelik and Frenkel 1994).

Interest in the problem of conservation laws in GR flared up again in the late 1950s, giving rise to an abundant literature and diverse physico-mathematical treatments, with the Noether theorem now taking the central place in the ensuing discussions (Trautman 1962, Schmutzer 1970). A solution that could satisfy specialists did not come forth, however. Over the past seventy years, the pseudotensor approach has not undergone any fundamental changes. The ambiguity in the expression for Einstein's pseudotensor was identified and other versions were suggested; the law of conservation of angular momentum was introduced on the same pseudotensor basis (Landau and Lifshitz 1973, Trautman 1962, Schmutzer 1970); conditions at "Galilean" infinity were specified for the island system (Faddeev 1982), and some general verbal arguments for the non-localizability of gravitational energy were set forth.

Yet many questions remain unanswered. There is no explanation for why, in a generally covariant theory, the asymptotically flat space outside the island system must necessarily be described in Cartesian coordinates (and not, say, in spherical coordinates). No explanation has been given yet why conservation laws can be applied only to two extreme situations in GR: the island system in empty (flat) space and the universe as a whole. After all, the differential conservation law in SR (5) produces an integral conservation law (8) only in a degenerate, idealized situation, and, in the generic case, it gives a balance equation (7): The change of the quantity Θ_A in the volume V equals the flow of that quantity through the boundary of V.

And finally, there are two other unanswered questions related to the concepts that are central to this essay. First, how can one establish a connection, in terms of the correspondence principle, between the inferior pseudotensor conservation laws of GR and the ten absolutely clear conservation laws of SR? And, second, what does the Noether theorem, the most general and fundamental basis for conservation laws, have to do with the pseudotensor laws?

6. The Pseudotensor Approach from the Noether Point of View

Instead of repeating the arguments pro and con, let us have a look at the pseudotensor proposal from the Noether point of view. One would think that Equation (11), $\partial_i\Theta^{ik} = 0$, is beyond doubt as a point of departure. Both the quantities Θ^{ik} and Equation (11) itself can be obtained in SR by the Noether procedure. But matters stand quite differently with regard to Equation (13), $D_iT^{ik} = 0$. The tensor T^{ik} is obtained in GR, not from a Noether formulation, but from the variation of the Lagrangian of matter for the metric tensor:

$$T^{ik} = \delta\mathfrak{L}/\delta g_{ik}.$$

This variation is, in fact, part of the derivation of the equations of motion for the system consisting of the gravitational field and matter, that is, Einstein's equations (Landau and Lifshitz 1973). Tensor T^{ik} plays the role of the source in these equations. The reason for calling T^{ik} the energy-momentum tensor for matter consists in the fact that, in a flat geometry, the expression for T^{ik} turns into a (spatially symmetrized) energy-momentum tensor Θ^{ik} (obtained by the Noether procedure applied to the group of translations of Minkowski space), whereas Equation (13) for GR turns into a differential form of the conservation law of SR.

But even though Equation (13) seems to be a general relativistic generalization of (11), it cannot be called a conservation law, since it has no Noether interpretation. Besides, from the Noether point of view, the two indices in the special relativistic tensor Θ^{ik} are not quite equivalent. Their seeming equivalence is a consequence of the "accidental" fact that the Poincaré group contains a subgroup of translations, which can be parameterized by a 4-vector. Strictly speaking, the quantity Θ^{ik} should be designated Θ^{iA}, where $A = 1, \ldots, 4$ (see Equations (9), with only a subset of them forming (11)). Rotations, which the Noether approach associates with the 4-moment of momentum, cannot be parameterized by a 4-vector, but they can be parameterized by the antisymmetric tensor ω^{kl} (the Lorentz rotations in the *xt*-planes, i.e., relative motions of reference systems, are also included among the rotations). This is why the moment is described, in SR, by the quantities Θ^{iA}, (where $A = (kl)$, $1 \leq k < l \leq 4$), or by the third-rank tensor $M_{k\,l}^{\,i}$, which is anti-symmetrical in the indices *kl*. All of the quantities conserved in SR can be expressed in the form Θ^{iA}, where $A = 1, \ldots, 10$. Translations and rotations are independent symmetries of the space-time description in SR, therefore energy-momentum and the moment of momentum are independent (and equal in their status) quantities.

This makes it hard to understand the desire to solve the energy-momentum problem in GR separately from and independently of the problem of angular momentum. In fact, there is a unified problem of *energy-momentum-moment* in GR, and therefore, it is necessary, in this theory, to look for the quantities Θ^{iA}, where $A = 1, \ldots, 10$. However, it is impossible to identify any asymmetry of the indices (in order to turn one index into $A = 1, \ldots, 10$) in the tensor T^{ik}, the right part of the Einstein equations. Due to their nature, the indices in T^{ik} are absolutely symmetrical.

Division of the quantities Θ^{iA} into two sets (the energy-momentum tensor and the moment tensor), covariant with respect to the Poincaré group, reflects precisely the composition of that group. The impossibility of such a division in GR in the generic situation should not be surprising.

The transition from Newtonian mechanics to SR led to the unification of the concepts of energy and momentum into that of energy-momentum; the concepts of angular momentum and the velocity of the center of inertia have also been united. The transition from SR to GR leads to the unification of all of these quantities (see §§9–11).

The desire to interpret the equation $D_i T^{ik} = 0$ as a conservation law in GR must surprise one no less than, say, the desire to attach some special meaning in general Riemannian space to the lines $x^i = \kappa^i \tau$, $\kappa^i = const.$ (and to call these lines "straight") in an arbitrary system of coordinates, only because it holds true for Cartesian coordinates in Minkowski space.

7. The Problem of Conservation Laws in General Relativity and the Correspondence Principle

The concepts of energy, momentum and angular momentum, and the appropriate conservation laws are indispensable to the rest of "nongravitational" physics, providing the most general, basic means of description. At the same time, there exist quite different opinions about the importance of conservation laws in GR.

The most common view simply denies that there is any problem here, the argument being that, because of the special character of the gravitational field, energy-type concepts and the appropriate conservation laws lose their meaning in most situations admitted by GR. At the same time, the Einstein equations are held to give a complete description of any situation.

In the opinion of other specialists, the problem of conservation laws in GR remains unresolved and presents a challenge for the theory (Penrose 1982, Trautman 1962, Schmutzer 1970).

One of the simple arguments used by the advocates of the first position consists in the following: The equivalence principle underlying GR allegedly dooms to failure any attempt to introduce a concept of the gravitational field energy. Any gravitational field is meant to disappear for the observer who steps into Einstein's elevator, a freely-falling reference system; in that case, the field energy must disappear, too.

Does this argument point to the impossibility of introducing the notion of gravitational energy in GR? No more so than in classical mechanics, where a transition to the reference system moving with a given body, thus "eliminating" the motion of this body, does not prevent the introduction of the concept of kinetic energy. Such an argument, however, indicates quite clearly that, in GR, the question of conservation laws is intimately linked to the concept of reference systems; see §8.

The role played by the equivalence principle in GR has been extensively debated (see, e.g., Synge 1960, Fock 1979, Ginzburg 1979). The differing attitudes to the principle are prompted mainly by the different frameworks in which it is considered. Within the framework of Newtonian mechanics, the equivalence principle has a quite definite meaning, and, from the viewpoint that "reveals" GR's links with classical physics and experiment, the equivalence principle is really a fundamental property of gravitation. If it is viewed from within GR, however, it appears meaningless, at least if the concept of an accelerated reference system has not yet been introduced. Within the theoretical framework of GR the equivalence principle "dissolves" into the very notion of geometrizing gravity.

This situation can be compared with the discussion of the principle of spatial isotropy between a supporter of Aristotle's doctrine of space and an advocate of the Newtonian conception of space, in which the principle of isotropy is similarly "dissolved."

Those who say that energy-type concepts lose all meaning in GR sometimes point to the tendency for long-standing notions to die off in the course of scientific progress. Thus, the transition from classical mechanics to quantum mechanics, for example, led to the extinction of the notion of (observable) particle trajectories. In quantum mechanics, however, it is shown *how and when* (in terms of the correspondence principle) the concept of a trajectory may become justified.

In the case of conservation laws in GR the situation is different. It is not known how to effect a transition (in the sense of correspondence) to conservation laws in the case of small deviations of geometry from the flat case; isolated distributions of matter do not exhaust all possibilities, for the curvature may be globally small but without Euclidean properties at infinity as, for example, in the case of constant large-radius curvature.

What are the general concepts and structures of GR that transform into conservation laws in SR? An answer to this question is necessary irrespective of one's attitude to the problem of conservation laws in GR.

Since the situation with conservation laws in SR is quite clear, and since SR is (logically and historically) a natural starting point and a limiting case for GR, it is useful to resort to the correspondence principle in analyzing the issue of conservation laws. One ought to clarify in this context *what* can or should be called a conservation law in GR; that is to say, what quantities, properties, and facts known in SR can be generalized in GR.

The connection between a theory's symmetries and its conservation laws established by the Noether theorem greatly facilitates understanding the essence of the problem. In classical mechanics and in SR, there are ten space-time conservation laws: of energy (1), of momentum (3), of angular momentum (3), and of the velocity of the center of inertia (3). In accordance with the Noether theorem, the number of conservation laws is directly determined by the fact that classical mechanics is based on the ten-parameter Galilean group, while SR rests on the ten-parameter Poincaré group.

In classical mechanics, the laws of nature or, more to the point, the action is invariant with regard to the following transformations of the Cartesian system of spatial coordinates and time $(t, x, y, z) \rightarrow (t', x', y', z')$: displacement of the origin of time (one parameter, Δt), rectilinear displacement of spatial coordinates (three parameters, $\Delta \mathbf{x}$), rotations of the system of coordinates (three parameters; e.g., three Euler angles), and transformation to another inertial system (three parameters, the components of relative velocity).

In SR, a similar role is played by the Poincaré group, which includes four space-time translations and six rotations (three purely spatial and three Lorentz "rotations"). In each case, the breakdown of the number of conserved quantities, 10 = 1+3+3+3 and 10 = 4+6, reflects the structure of the corresponding group, and their total number, 10, is determined by the number of space-time dimensions (see §§9, 10).

In terms of the correspondence principle and the Noether theorem, the question may now be posed as follows: What kind of structure in GR corresponds to the ten-parameter Poincaré group describing the geometry of Minkowski space? Sure enough, this is not the group of all continuous coordinate transformations: not only because it is an infinite-parameter group (though this, by itself, is sufficient to dismiss such a proposal), but also because such a group can be easily introduced even in Minkowski space-time.

The locally flat character of the geometry of space-time in GR is usually not connected with the understanding of the Poincaré group as the limit of a certain structure defined in general Riemannian space (see, for example, Misner, Thorne, and Wheeler 1973). At the same time, the description of the geometry of Minkowski space-time based on the properties of the Poincaré group proved very fruitful in physics. It is believed that in a generic Riemannian space (lacking movement symmetries), there is no natural way of defining a "natural" finite-parameter family of transformations of coordinates (Trautman 1962) and that it makes sense to speak only of the infinite-parameter group of all smooth transformations of coordinates. Yet, as the subsequent analysis will show, the space-time symmetries can be linked not only to motions in space, but also to the observer's displacements, that is, to the displacements of an appropriately defined reference system.

In the case of a homogenous space, and in particular in SR, we know of two equivalent interpretations of spatial transformations: *active* and *passive*. According to the active interpretation, the system of coordinates does not change, while the geometrical or physical system under consideration, for example, the region in which the field variables do not vanish, is transformed. The active viewpoint may alternatively be characterized as follows: The observer (reference system) stands still, whereas the entire space moves "dragging" all objects along with it. According to the passive viewpoint, space and the objects stand motionless while the observer (reference system) moves. As a result of each of these procedures, the coordinates in the spatial region under consideration undergo change, but the difference between the active and passive approaches cannot even be expressed in "internal" terms, that is, not supposing some embedding space.

In a generic Riemannian (arbitrarily curved) space, the active approach is impossible: In the general situation, space cannot be shifted along itself, and a complex enough system (for example, of n+2 points in n-dimensional space) cannot be appropriately displaced without internal changes (in the above example, without changing the distances between the points). However, the passive approach—the "movement" of an observer, that is, of a properly defined reference system—leads, as we shall see, for n-dimensional Riemann space, to an $n(n+1)/2$-parameter set of coordinate transformations, the set possessing a quasigroup structure generalizing the structure of the Poincaré group (§9).

But before considering this suggestion in detail, let us discuss the connection between conservation laws and the concept of reference system.

8. Conservation Laws and the Concept of Reference System

The problem of conservation laws in GR is inseparable from the issue of space-time reference systems. The fairly common neglect of the concept of reference system in GR could be justified if physicists were only interested in scalar quantities, which do not depend on the reference system; but energy, at any rate, does not belong among such quantities. There is a clear connection between conservation laws and reference systems in SR, where the conserved quantities are defined by the invariance of the action with regard to Poincaré transformations connecting various inertial reference systems. From a mathematical point of view, one selects a set of standard coordinate systems in Minkowski space (those that allow a physical interpretation in terms of reference systems), and the coordinates are generated by a quite rigorous procedure.

The mixing of the concepts of reference system and coordinate system in general relativity—which is justly criticized—has nevertheless a reasonable foundation. After all, a reference system in physics is, first and foremost, a concrete way of establishing coordinates for space-time points. Such a physical incarnation of a coordinate system is called below a *coordinate reference system*.

It is true that such clarifications may seem pointless in GR, since there is an opinion, traceable to Einstein, that, in the Riemannian geometry of GR, it is not possible to impart a metrical (measurable, physical) significance to coordinates. Yet, for all the heuristic importance of this idea for Einstein, it is wrong. Suffice it to recall that the very first coordinate system introduced into Riemannian geometry by Riemann himself (1854) had a quite definite metrical significance and was applicable to arbitrary Riemannian spaces. Other similar ways of introducing coordinates are also known. One of these—perhaps logically the simplest—will be used in the next section to characterize the symmetries of general Riemannian space.

In SR, inertial reference systems are typically described in Cartesian coordinate systems in Minkowski space. Space-time in SR can surely be described by means of any coordinates (spherical, cylindrical, etc.), but as far as conservation laws are concerned, only Cartesian coordinates are usually considered.

Is such a restriction necessary? A restriction to the class of *standard* coordinate systems is certainly necessary; this much is required to bring out the ten-parameter ($10 = n(n+1)/2$, with $n = 4$) transformations describing the space-time symmetry in SR (homogeneity and isotropy). But restriction just to Cartesian coordinates is not necessary. For example, one may employ spherical coordinates or any other class of *standard*, that is, equivalently defined, coordinates. Each of these classes of coordinates is fit for describing the space-time symmetry in SR.

In SR, the inertial coordinate reference systems are called “privileged,” and not without reason. The notion of a privileged class of coordinate systems in GR will be linked here only to the possibility of a general, standard, and constructively described (physically realizable) way of introducing coordinates. In this sense, there may be many classes of “privileged” coordinates.

The need for a preliminary restriction and standardization in the way of describing coordinates is quite natural. To identify the invariant properties of some object, various observers have to study it with the possibly strictest limitations in their techniques of investigation. All observers must follow standard methods so that the results of their investigations disclose the object's properties, rather than the properties of the observers.

Therefore, in GR it is essential first to determine the class of standard coordinate reference systems. After all, conservation laws emerge, according to the Noether theorem, for any action, including the general relativistic action, only if a finite-parameter set of coordinate transformations is identified in a physically sensible way. This is why, in pursuing the Noether approach to conservation laws, one has to identify a natural, finite-parameter family of coordinates in general Riemannian space.

9. The Poincaré Quasigroup

Let us start with a simple question: Why are there exactly *ten* conservation laws in SR? The answer is: Because space-time has *four* dimensions.

Indeed, a Noether-type connection between conservation laws and space-time symmetries provides a relationship between the number of symmetries in the n-dimensional Minkowski space-time and n itself: there are n independent translations along n axes and $C_n^2 = n(n-1)/2$ independent rotations; altogether this yields $s = n(n+1)/2$ independent symmetry transformations. If $n = 4$, then $s = 10$.

The origin of the ten-parameter Poincaré group describing space-time symmetries in SR is similar. This group, like any other, may be represented in a purely algebraic fashion, but physicists do not always take care to distinguish between the Poincaré group and one of its representations: the linear representation in a 3+1-dimensional Minkowski M^{3+1} space. This representation is formed by linear transformations from one Cartesian system of coordinates in M^{3+1} to another, which conserve the metrical structure—the expression for the metrical interval

$$I = \Delta s^2 = -c^2(\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2.$$

It should be emphasized that the linear representation is just one of many possible representations of the Poincaré group, and that it cannot be extended to Riemannian space, for this representation relies upon the linear structure of M^{3+1}, which is not available in curved space.

So the conservation laws in SR are generated by the symmetries of the metrical space-time structure, the number of the symmetries being determined by the number of space-time dimensions. The space-time in question is a linear space. The concept of linear space is also sufficient for constructing a finite-parameter (in the n-dimensional case, an $n(n+1)/2$-parameter) set of coordinate systems, or, in physical terms, a set of inertial reference systems in SR. It is through the Noether theorem that this ten-parameter set of coordinate reference systems leads to ten conservation laws.

Let us now turn to the Riemannian geometry of GR, where we also want to identify a natural $n(n+1)/2$-parameter set of coordinate reference systems in an arbitrary n-dimensional Riemannian space.

The obvious and seemingly crucial objection to such an undertaking hinges on the fact that the presence of a natural, finite-parameter set of coordinate transformations must reflect some sort of space-time symmetry, a property that does not change from point to point. And such a property, it would seem, must be nonexistent in a space with variable curvature. And yet, even in an arbitrarily curved space-time of GR, there is a property that does not change from point to point: the *space-time dimensionality*. The task now is to turn this trivial observation into a productive strategy.

First of all, we cannot use the concept of linear dimensionality, which ensures the connection between conservation laws and the dimensionality in SR. We also cannot use the topological concept of dimensionality, since it ignores the metrical space-time structure, which is basic to GR.

Yet a metrical concept of dimensionality is available. Consider a generalization of the following geometrical fact:

> *In a Euclidean (pseudo-Euclidean) n-dimensional space, the position of an arbitrary point can be determined solely in terms of its distances (intervals) to n fixed points.* (Gorelik 1978, 1979)

In a Riemannian space, a suitable measure of "distance" can be provided by the interval (or the world function; see Synge 1960) $I(p,p')$ between two arbitrary space-time points p and p'. The description of space-time in terms of I is equivalent to the conventional description employing the metric tensor g_{ik} or the intervals ds^2 between infinitely close points:

$$I(p,p') = \left(\int_p^{p'} ds\right)^2, \quad ds^2 = g_{ik}dx^i dx^k. \tag{15}$$

Here integration from p and p' is done along a *geodesic*.

Let us now fix a certain point b in space. This defines the function $x(p) \equiv I(p,b)$ on this space. Fixing n such points $\{b^i\}$, $i = 1, \ldots n$, in an n-

dimensional space provides a full-blown *basis* and, hence, generates a definite coordinate system: each point p is assigned n numbers $x^i(p) = I(p,b^i)$. These coordinates, introduced with the aid of the spatial metric, can be called *metric coordinates*. In the four-dimensional Minkowski space, any four points in a generic position (that is, not lying on the same plane) provide a basis.

Consider the set of all such metric coordinate systems generated by various bases. Since each coordinate system is completely determined by the position of its basis points, the transformation from one such system to another can be accomplished by indicating the positions of new basis points $\{b'^i\}$ in the old basis $\{b^k\}$, that is, by supplying n^2 numbers $I(b'^i,b^k)$, $i,k = 1, \ldots, n$, which can be represented in the form of a matrix.

Not all of these n^2 parameters are equally essential. To exclude "coordinate" effects, one should use standard, normalized bases, with the intervals between basis points fixed, for example, in such a way that they coincide with the intervals between the following points in Minkowski space: (0000), (0a00), (00a0), (000a). Each of these bases is characterized by the basis diameter a, which can be made infinitesimal thus effecting a local correspondence of GR with SR.

The use of normalized bases reduces the number of parameters by $n(n–1)/2$. So the set of bases (or of metric coordinate systems) is characterized by $n^2 – n(n–1)/2 = n(n+1)/2$ parameters. Thus in four-dimensional space, there are ten such parameters, just as in a Poincaré group.

Each coordinate reference system thus constructed is obtained in the same way and can be introduced in an arbitrarily curved space.

The $n(n+1)/2$-parameter set of bases (or of coordinate reference systems) introduced via the above procedure generates a *quasigroup structure*. Indeed, fix a particular basis b_0 (the lower index will now range over bases). Then any other basis, as mentioned, corresponds to a certain $n \times n$ matrix. The sequence of transformations from basis b_0 to basis b_1 and then from b_1 to b_2 is equivalent to a transformation from b_0 directly to b_2. This defines the group *product* of two matrices (not in the sense of matrix algebra, of course). This product is determined by the geometry of a given space-time $\mathbb{R}^n$, specifically, by a function of $2n$ variables, the function defining the interval between two points p and p' in terms of the intervals between these points and the points of both bases:

$$I(p,p') = \Gamma(\ldots (p,b^i)\ldots; \ldots I(p',b^k)\ldots). \qquad (16)$$

The function Γ provides a global characterization of the geometry of $\mathbb{R}^n$.

If $B^{ik} = I(b_1{}^i, b_2{}^k)$ is the matrix effecting the transformation from basis b_1 to b_2, the *inverse* matrix (in the group sense) should naturally be the transposed matrix B^T responsible for the transformation from b_2 to b_1. The product and inverse operations define an $n(n+1)/2$-parameter quasigroup in space $\mathbb{R}^n$: the *Poincaré quasigroup* $\mathcal{P}\mathbb{R}^n$ of this space (Gorelik 1981, 1988).

The mathematical notion of group has long been customary in physics. A quasigroup, on the other hand, is a relatively new object. What makes it different is the *nonassociative* character of the law of multiplication; that is, for three quasigroup elements A, B, C, in general, $(AB)C \neq A(BC)$ (Bruck 1971). As algebraic objects, quasigroups have been known in mathematics for about sixty years. Yet the significant role played by them in differential geometry (Sabinin 1981) has only recently been recognized. Nonassociativity is the price to pay for the shift from highly symmetrical spaces (e.g., those of constant curvature) to spaces with variable curvature.

In Minkowski space, the quasigroup $\mathcal{P}\mathbb{R}^n$ becomes isomorphic to the ordinary Poincaré group: $\mathcal{P}M^n = \mathcal{P}^{(n)}$. Indeed, every transformation from the Poincaré group in the standard linear representation is in one-to-one correspondence with a certain transformation of spatial unit vectors (0100), (0010), and (0001). The basis formed by the initial point of the unit vectors and by their ends, corresponds, one-to-one, to a set of spatial unit vectors. And every transformation of unit vectors corresponds, one-to-one, to a transformation of the basis.

The set of transformations of one basis into another in Minkowski space—those expressed in terms of mutual distances (intervals) of the points constituting the bases—forms a *nonlinear representation* of the Poincaré group. This representation is more complex than the conventional linear representation, but it *allows* a generalization in the case of curved space.

10. What Kind of Symmetry is Described by the Poincaré Quasigroup?

It is widely believed that the notion of symmetry can be expressed mathematically by an associated group (Weyl 1952, Vizgin 1972), yet the concept of symmetry is consistent with a more general mathematical structure. There are no apparent reasons, for example, for precluding a set of transformations from being associated with a symmetry, if the set is closed under composition, contains a unit (identity) transformation, and, for each transformation, contains the inverse one. But this is precisely what a quasigroup is. The demand for associativity (converting a quasigroup into a group) seems no more imperative for the description of symmetry than the demand for commutativity (valid only for the simplest symmetries).

Although the requirement that the set of coordinate transformations must form a group is frequently mentioned in connection with the Noether theorem, it plays no role there. The structure of a set of transformations remains, of course, essential for their physical interpretation and for the specification of the corresponding conserved quantities.

What kind of symmetry of the arbitrarily curved space $\mathbb{R}^n$ is described by the quasigroup $\mathcal{P}\mathbb{R}^n$? A generic curved space-time $\mathbb{R}^n$ retains a property that does not change from point to point: the number of its dimensions. It is an aspect of the space-time homogeneity that is not affected by the transition from flat to curved space-time. The quasigroup $\mathcal{P}\mathbb{R}^n$ represents a symmetry of a *description* of space $\mathbb{R}^n$ by an observer (in the four-dimensional case—a coordinate reference system produced by four space-time points with fixed intervals between them). This symmetry can be looked upon as a relation among space-time descriptions of one and the same physical system by different, albeit standard, observers (coordinate reference systems).

Does the use of special, "privileged" coordinates contradict the principle of general covariance underlying GR? Not at all. What really matters in GR is the possibility to have a construction realizable in an *arbitrary geometry* rather than the possibility of using *arbitrary coordinates*. After all, the latter possibility does not hinder conservation laws in Minkowski space.

When the invariance of a certain object is at issue, one should be clear about instrumental procedures—their standardization and the "calibration" of the instruments themselves—employed in describing that object. In our case, the object in question is curved space-time.

In speaking of an arbitrary reference system in SR, one usually means (explicitly or not) "any inertial Cartesian reference system." This is why in GR, too, it is necessary to seek a constraint on reference systems, the constraint "equal in power" to the restriction to inertial and Cartesian systems in SR. In a privileged class of coordinate systems in GR, there must be "as many elements" as there are in the class of inertial reference systems in SR. In the same way, reference systems in GR must be capable of being constructed "everywhere." If a reference system is related to some space-time structure, then it must be possible to relate the reference system with a similar structure located at any other position in space-time.

To describe a curved (as well as flat) space, one is free to use any coordinates. Yet the task of describing a physical situation with the aid of conserved quantities implies a restriction on coordinate systems. The demand for such a restriction is not unique to GR. In SR, the very expressions for the conserved quantities contain information about the type of coordinate reference systems employed. This information is packed into the quantities X and Ψ describing coordinate transformation; see Equations (3a, b).

Likewise, in GR, the conserved quantities, which are engendered by the ten-parameter Poincaré quasigroup and the general relativistic action, contain information about the type of coordinates employed, and also about the geometry of a given space-time $\mathbb{R}^n$. This information is packed into the function Γ; see Equation (16). Of course, the conserved quantities of SR also encapsulate information about the geometry of Minkowski space-time, but only in a trivial sort of way, because this geometry is fixed and global.

11. The Ten Noether Laws of Conservation of Energy-Momentum-Moment in GR

Thus the desired $n(n+1)/2$-parameter set of transformations of coordinate systems (or, remembering the need for a physical interpretation, of coordinate *reference* systems) is built into an arbitrarily curved Riemannian space-time and has the structure of a quasigroup.

The general relativistic action

$$A = \int (Rg^{1/2} + \mathfrak{L}_m)\, d^4x, \tag{17}$$

which is invariant with respect to arbitrary transformations of coordinates, is invariant, in particular, with regard to transformations from $\mathcal{P}\mathbb{R}^n$. This is why, thanks to the Noether theorem for the action (17), the $n(n+1)/2$-parameter set of coordinate transformations $\mathcal{P}\mathbb{R}^n$ generates $n(n+1)/2$ conservation laws of the type $\partial_i \Theta_A{}^i = 0$, $A = 1, \ldots, n(n+1)/2$. It seems natural to refer to the quantity $\Theta_A{}^i$ as the *density of energy-momentum-moment*.

In Minkowski geometry, $n(n+1)/2$ quantities Θ_A can be easily divided into n and $n(n-1)/2$ values—energy-momentum and moment—since the Poincaré group contains translation and rotation subgroups. In the case of a general Riemannian space, whose symmetries are described by a *quasigroup*, the division of the conserved quantities Θ_A into similar sets is a special task that cannot be accomplished uniquely.

The ten conserved components of energy-momentum-moment, $\Theta_A{}^i$, $A = 1, \ldots, 4(4+1)/2$, generated by the Poincaré quasigroup $\mathcal{P}\mathbb{R}^4$, satisfy the Noether-type conservation laws

$$\partial_i \Theta_A{}^i = 0,$$

but in the generic Riemannian space $\mathbb{R}^4$, these differential laws can take on an integral form only as balance equations

$$\oint \Theta^i_A d^3\sigma_i = 0.$$

In general, however, it is not possible to turn these balance equations into the conventional conservation laws, as in SR (see §2, Equations (6–8)), since in GR, the supposition that "the field at infinity must vanish" radically restricts the space-time geometry. Recall that this problem was actually discussed in 1918 by Einstein, who supported the restriction to island situations (see §4).

The density of energy-momentum-moment $\Theta_A{}^i$ is a *nonlocal* quantity, since it is determined by the entire geometry (via the function Γ); this is why $\Theta_A{}^i$ cannot, in general, have simple transformation properties (similar to tensor properties in SR). And the ten energy-momentum-moment quantities do not, in general, decompose into smaller sets like energy-momentum and angular momentum.

However, the quantities $\Theta_A{}^i$ have resulted from the same general Noether approach as in the rest of physics.

12. Conclusion

The problem of conservation laws in GR has a dramatic history. The gravitational energy-momentum pseudotensor emerged as early as 1913, two years before the real birth of GR. And yet, many decades after the emergence of GR, experts do not regard the pseudotensor solution to be completely satisfactory.

The drawbacks of the pseudotensor approach are especially obvious from the viewpoint of the correspondence principle and in light of the Noether theorem.

The theory's symmetry, required by the Noether theorem, can be identified, in the case of GR, with the dimensional homogeneity of space-time: the number of dimensions of Riemannian space is the same at each point. The description of the dimensionality of Riemannian space $\mathbb{R}^n$ in metric terms makes it possible to introduce the Poincaré quasigroup $\mathcal{P}\mathbb{R}^n$, with the number of parameters, $n(n+1)/2$, connected with the number of dimensions in the same way as in the case of the Poincaré group in SR. And in the limiting case of Minkowski space, this quasigroup becomes isomorphic with the Poincaré group.

By the standard Noether procedure, this symmetry of the generic Riemannian space-time $\mathbb{R}^n$ turns up ten laws of conservation of energy-momentum-moment, or, more precisely, ten balance equations.

The conservation laws, which are based on the Poincaré quasigroup, have an advantage over the pseudotensor laws, not only because "pseudo" smacks of "sham" and "quasi" suggests "having resemblance to." Philology aside, the quasigroup approach also has methodological advantages.

Abandonment of associativity makes the quasigroup structure less definite and more flexible than the group structure. Groups are distinguished in a discrete way. Their differences cannot be made infinitely small. For example, the group of circular motions does not collapse with the group of rectilinear motions with the increase of the circle radius. But quasigroups can change continuously, turning into each other. The Poincaré group is sharply separated from the de Sitter group and other groups realizable in the homogeneous spaces of GR. But the Poincaré quasigroup connects and unites all these cases. This suggests yet another way of looking at the transition from SR to GR.

And finally, the Noether-type approach to conservation laws in GR provides a novel basis for the study of concrete situations arising in this theory, where the conserved quantities may be useful.

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