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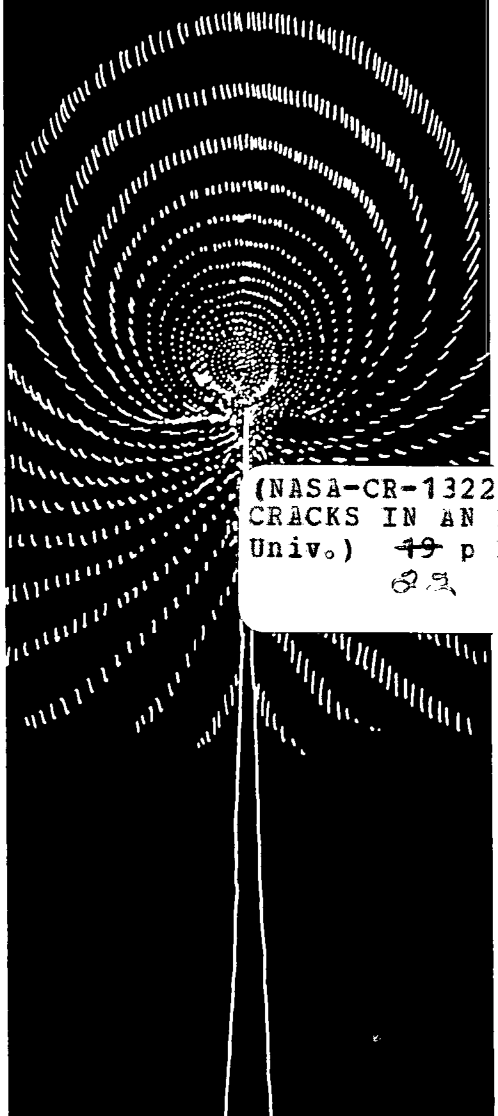

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THE PROBLEM OF EDGE CRACKS IN AN INFINITE STRIP*

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Abstract

The elastostatic plane problem of an infinite strip containing two symmetrically located internal cracks perpendicular to the boundary is formulated in terms of a singular integral equation with the derivative of the crack surface displacement as the density function. The solution of the problem is obtained for various crack geometries and for uniaxial tension applied to the strip away from the crack region. The limiting case of the edge cracks is then considered in some detail. The fundamental function of the integral equation is obtained and a numerical technique for solving the singular integral equations with this particular type of fundamental function which is characteristic of the edge cracks is described. The stress intensity factor for the complete range of net ligament-to-width ratio $0 \leq a/h \leq 1$ is calculated. The results also include the solution of the edge crack problem in an elastic half plane.

1. INTRODUCTION

Within the past decade the plane elastostatic problem of an "edge crack" has been considered by many investigators using a variety of techniques. Among the notable contributions, we may mention [1,2] where the Wiener-Hopf technique is used to solve the problem, [3,4] where a variety of crack problems are formulated as integral equations in terms of some weight functions, [5] in which the problem is treated by using the mapping

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technique, and [6,7] where the finite element methods are used. In this paper the problem is reduced to a singular integral equation with the derivative of the crack surface displacement as the density function. The method offers certain flexibilities which become very useful in dealing with the cracks in nonhomogeneous materials. One of the primary aims of the paper is to show that the numerical techniques developed for solving singular integral equations with weight functions $w(x) = (1-x)^{\alpha}(1+x)^{\beta}$, $(-1 < Re(\alpha,\beta) < 0)$ [8] may be extended to treat the problems in which formally α or β is zero, which invariably is the case for the edge cracks.

First the problem of an infinite strip with two collinear cracks perpendicular to the sides will be considered. The limiting cases of this problem are a single crack when the distance between the inner crack tips becomes zero [9,10], and the edge crack when the distance from the outer crack tips to the sides becomes zero. The results for the edge crack in a half plane [2] will also be given largely for the purpose of verification of the technique.

2. THE INTEGRAL EQUATIONS

Consider the strip problem shown in Figure 1. Let the geometry of the medium and the external loads be symmetric with respect to x=0 and y=0 planes. Thus, using symmetry consideration and the superposition technique, the singular part of the solution may be obtained by solving the problem under the

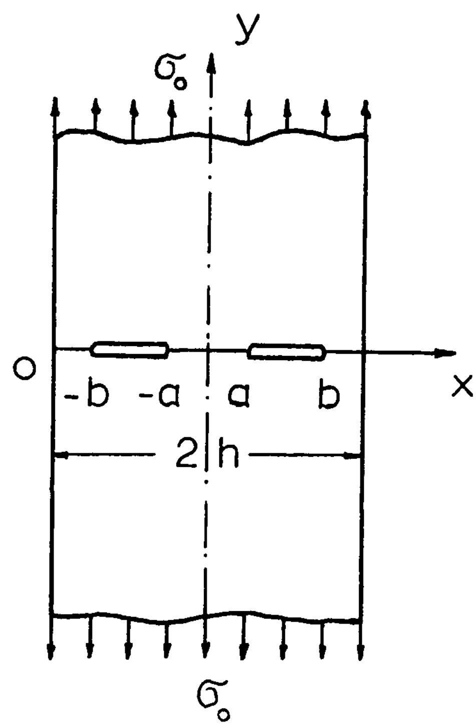

Figure 1. Infinite strip with two internal cracks.

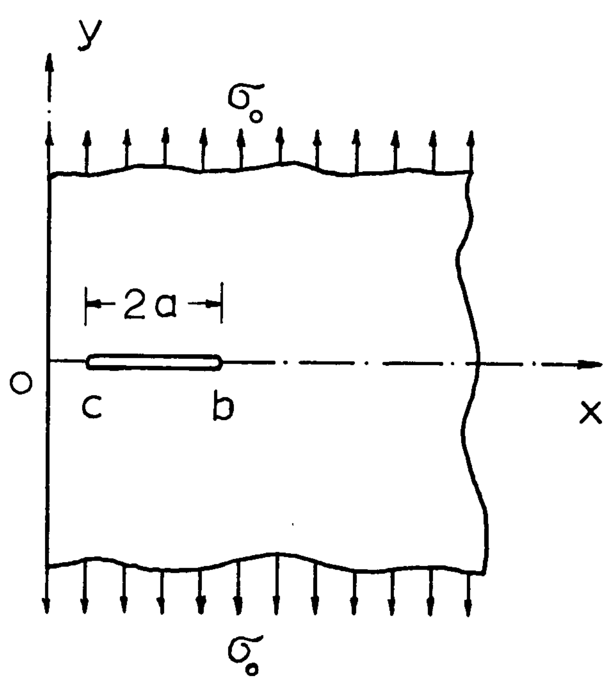

Figure 2. Half plane with an internal crack.

following boundary conditions*:

$$u(0,y) = 0\ , \qquad \sigma_{xy}(0,y) = 0\ , \qquad (0\le y<\infty), \tag{1.a,b}$$

$$\sigma_{xx}(h,y) = 0\ , \qquad \sigma_{xy}(h,y) = 0\ , \qquad (0\le y\le\infty), \tag{2.a,b}$$

$$\sigma_{yy}(x,\infty) = 0\ , \qquad \sigma_{xy}(x,\infty) = 0\ , \qquad (0\le x\le h), \tag{3.a,b}$$

$$\sigma_{xy}(x,0) = 0\ , \qquad (0\le x\le h), \tag{4}$$

$$\sigma_{yy}(x,0) = -\sigma(x)\ , \qquad (a<x<b),$$

$$\frac{\partial}{\partial x}\, v(x,0) = 0\ , \qquad (0\le x<a,\ b<x\le h), \tag{5.a,b}$$

$$\int_a^b \frac{\partial}{\partial x}\, v(x,0)dx = 0\ , \tag{6}$$

where u and v are the x and y-components of the displacement vector. The conditions (5.b) and (6) are clearly equivalent to $v(x,0) = 0$, $(0\le x<a,\ b<x\le h)$. The solution of the problem satisfying the necessary field equations and the conditions (1), (3) and (4) may be expressed as [8]

$$u(x,y) = -\frac{2}{\pi}\int_0^\infty \frac{m(p)}{p}\left(\frac{\kappa-1}{2} - py\right)e^{-py}\sin(px)dp$$

$$-\frac{2}{\pi}\int_0^\infty \left\{\frac{1}{s}\left[f(s) - \frac{\kappa-1}{2}\, g(s)\right]\sinh(sx)\right.$$

$$\left. + xg(s)\cosh(sx)\right\}\cos(sy)ds\ ,$$

---

*If the external load is the uniform tension σ_0 as shown in Figures 1 and 2, then in (5a) $\sigma(x) = \sigma_0$. However, in formulating both problems it is assumed that the crack surface traction may be a function of x. This may arise from symmetric but irregular loading of the strip and the half plane.

$$v(x,y) = \frac{2}{\pi}\int_0^\infty \frac{m(p)}{p}\left(\frac{\kappa+1}{2} + py\right)e^{-py}\cos(px)dp$$

$$+ \frac{2}{\pi}\int_0^\infty \left\{\frac{1}{s}\left[f(s) + \frac{\kappa+1}{2}\, g(s)\right]\cosh(sx)\right.$$

$$\left. + xg(s)\sinh(sx)\right\}\sin(sy)ds \; , \qquad (7.a,b)$$

where $\kappa = 3-4\nu$ for plane strain and $\kappa = (3-\nu)/(1+\nu)$ for generalized plane stress, ν being the Poisson's ratio. In (7) m, f, and g are unknown functions and may, in principle, be determined from the remaining boundary conditions (2) and (5). From (7) the stresses may be obtained as

$$\frac{1}{2\mu}\,\sigma_{xx} = -\frac{2}{\pi}\int_0^\infty m(p)(1-py)e^{-py}\cos(px)dp$$

$$-\frac{2}{\pi}\int_0^\infty [f(s)\cosh(sx) + sxg(s)\sinh(sx)]\cos(sy)ds \; ,$$

$$\frac{1}{2\mu}\,\sigma_{yy} = -\frac{2}{\pi}\int_0^\infty m(p)(1+py)e^{-py}\cos(px)dp$$

$$+ \frac{2}{\pi}\int_0^\infty \{[f(s) + 2g(s)]\cosh(sx)$$

$$+ sxg(s)\sinh(sx)\}\cos(sy)ds \; ,$$

$$\frac{1}{2\mu}\,\sigma_{xy} = -\frac{2}{\pi}\int_0^\infty ypm(p)e^{-py}\sin(px)dp$$

$$+ \frac{2}{\pi}\int_0^\infty \{[f(s) + g(s)]\sinh(sx)$$

$$+ sxg(s)\cosh(sx)\}\sin(sy)ds \; , \qquad (8.a\text{-}c)$$

where μ is the shear modulus. Substituting from (8) into (2) it is found that

$$f(s)\cosh(sh) + shg(s)\sinh(sh) = -\frac{4s^2}{\pi}\int_0^\infty \frac{pm(p)}{(p^2+s^2)^2}\cosh(ph)dp ,$$

$$[f(s) + g(s)]\sinh(sh) + shg(s)\cosh(sh)$$

$$= \frac{4s}{\pi}\int_0^\infty \frac{p^2m(p)}{(p^2+s^2)^2}\sin(ph)dp . \qquad (9.a,b)$$

The only unknown function is now m(p) which is determined from the mixed boundary conditions (5). (7.b), (8.b) and (9), substituted into (5), give a set of dual integral equations for m(p). However, a more direct approach to solve the problem would be defining a new unknown function G(x) as

$$G(x) = \frac{\partial}{\partial x} v(x,0) , \qquad (0\le x\le h) \qquad (10)$$

and from (5.b) and (7.b) observing that

$$-\frac{\kappa+1}{2} m(p) = \int_a^b G(t)\sin(pt)dt . \qquad (11)$$

Thus, substituting from (8.b), (9) and (11) into (5.a), we obtain an integral equation to determine G(x) of the following form:

$$\int_a^b H(x,t)G(t)dt = -\frac{1+\kappa}{4\mu}\pi\sigma_0 , \qquad (a<x<b). \qquad (12)$$

It may easily be shown that at x=t the kernel H(x,t) has a Cauchy-type singularity. Separating this singularity, it is found that

$$\int_a^b [\frac{1}{t-x} + \frac{1}{t+x} + k(x,t) - k(x,-t)]G(t)dt = -\frac{1+\kappa}{4\mu}\pi\sigma_0 ,$$

$$(a<x<b), \qquad (13)$$

where

$$k(x,t) = \int_0^\infty K(x,t,s)e^{-(h-t)s} ds ,$$

$$K(x,t,s) = e^{-sh}\{-[1 + (3+2sh)e^{-2sh}]\cosh(sx) - 2sx\sinh(sx)e^{-2sh} - [2sx\sinh(sx) + (3 - 2sh + e^{-2sh})\cosh(sx)][1 - 2s(h-t)]\}/(1 + 4she^{-2sh} - e^{-4sh}) . \qquad (14.a,b)$$

The index of the singular integral equation is +1; hence, its solution is determinate within an arbitrary constant which is determined from (6), or by (10), from

$$\int_a^b G(x)dx = 0 . \qquad (15)$$

Similarly, for a semi-infinite plane shown in Figure 2 we find [11]

$$\int_c^b \frac{G(t)dt}{t-x} + \int_c^b N(x,t)G(t)dt = -\frac{1+\kappa}{4\mu}\pi\sigma_0 , \qquad (c<x<b), \qquad (16)$$

$$N(x,t) = -\frac{1}{t+x} + \frac{6x}{(t+x)^2} - \frac{4x^2}{(t+x)^3} , \qquad \int_c^b G(t)dt = 0 . \qquad (17.a,b)$$

For $0<a<b<h$ and $0<c<b<\infty$ the kernels $k(x,t)$, $k(x,-t)$, and $N(x,t)$ are bounded, and hence (13) and (17) are ordinary singular integral equations with fundamental functions

$$w(x) = [(x-a)(b-x)]^{-1/2}, \qquad w(x) = [(x-c)(b-x)]^{-1/2} \qquad (18.a,b)$$

and may be solved in a straightforward manner (see, e.g., [8]).

3. LIMITING CASE: THE EDGE CRACKS

In the limiting case of $a=0$, $b<h$, the problem reduces to that of a single crack $(-b,b)$ in an infinite strip which was considered in [9]. For this case, noting that $G(x) = -G(-x)$,

(13) becomes

$$\int_{-b}^{b} \left[\frac{1}{t-x} + k(x,t)\right]G(t)dt = -\frac{1+\kappa}{4\mu}\pi\sigma_0 , \qquad (-b<x<b), \tag{19}$$

where the bounded function $k(x,t)$ is given by (14). The fundamental function of (19) is $w(x) = (b^2-x^2)^{-1/2}$ and the solution subject to

$$\int_{-b}^{b} G(x)dx = 0 \tag{20}$$

may be obtained by following any one of the standard techniques.

In the other limiting case, i.e., for $a>0$, $b=h$, the problem becomes one of two edge cracks in a strip. In this case, through an asymptotic analysis, it can be shown that as both x and t approach the end point h the kernel $k(x,t)$ in (13) becomes unbounded and hence influences the singular nature of the solution. Since the integrand $K(x,t,s)$ in (14) is bounded and continuous everywhere in $0\le s<\infty$, the unbounded terms in $k(x,t)$ will be the consequence of the asymptotic behavior of $K(x,t,s)$ for $s\to\infty$. Thus, adding and subtracting the asymptotic value of $K(x,t,s)$ to and from the integrand in (14.a) and using the relation [12]

$$\int_0^\infty s^m e^{-s(2h-t)}\begin{Bmatrix}\sinh(sx)\\ \cosh(sx)\end{Bmatrix}ds = \frac{d^m}{dt^m}\int_0^\infty e^{-s(2h-t)}\begin{Bmatrix}\sinh(sx)\\ \cosh(sx)\end{Bmatrix}ds$$

$$= \frac{d^m}{dt^m}\left[\frac{1}{(2h-t)^2 - x^2}\begin{Bmatrix}x\\ 2h-t\end{Bmatrix}\right] , \tag{21}$$

we obtain

$$\int_a^h \left[\frac{1}{t-x} + k_s(x,t) + k_f(x,t)\right]G(t)dt = -\frac{1+\kappa}{4\mu}\sigma_0\pi ,$$

$$(a<x<h), \tag{22}$$

$$k_s(x,t) = [-1 + 6(h-x)\frac{d}{dx} - 2(h-x)^2\frac{d^2}{dx^2}]\,\frac{1}{t-(2h-x)} ,$$

$$k_f(x,t) = (t+x)^{-1} - k(x,-t) + \int_0^\infty [K(x,t,s) - K_\infty(x,t,s)]e^{-(h-t)s}ds ,$$

$$K_\infty(x,t,s) = e^{-sh}\{-\cosh(sx) - [1 - 2s(h-t)][2sx\sinh(sx) + (3-2sh)\cosh(sx)]\} . \qquad (23.a\text{-}c)$$

The first two terms of the kernel of (22) give a generalized Cauchy kernel. To obtain the fundamental function $w(t)$ of the integral equation, let us first proceed in a routine manner [13, Chapter 4] and investigate the possibility of power singularities by defining

$$G(t) = r(t)w(t) , \qquad w(t) = (h-t)^{-\alpha}(t-a)^{-\beta} ,$$

$$F(z) = \frac{1}{\pi}\int_a^h \frac{G(t)}{t-z}\,dt = \frac{1}{\pi}\int_a^h \frac{r(t)e^{\pi i\alpha}dt}{(t-h)^\alpha(t-a)^\beta(t-z)} , \qquad (24.a\text{-}c)$$

where $r(t)$ is Hölder-continuous in $a\le t\le h$ and $(t-h)^\alpha(t-a)^\beta$ is any definite branch which varies continuously in $a<t<h$. In (24) it is assumed that

$$r(a) \neq 0 , \qquad r(h) \neq 0 , \qquad 0 < Re(\alpha,\beta) < 1 . \qquad (25)$$

Noting that the point $(2h-x)$ is outside the cut (a,h) on the real axis, (24) gives the following asymptotic relations [13]:

$$F(z) = \frac{r(a)e^{\pi i\beta}}{(h-a)^\alpha\sin\pi\beta}\,\frac{1}{(z-a)^\beta} - \frac{r(h)}{(h-a)^\beta\sin\pi\alpha}\,\frac{1}{(z-h)^\alpha} + F_0(z) ,$$

$$\frac{1}{\pi}\int_a^h \frac{G(t)dt}{t-x} = \frac{r(a)}{(h-a)^\alpha}\,\frac{\cot\pi\beta}{(x-a)^\beta} - \frac{r(h)}{(h-a)^\beta}\,\frac{\cot\pi\alpha}{(h-x)^\alpha} + F_1(x) ,$$

$$\frac{1}{\pi}\int_a^h \frac{G(t)dt}{t-(2h-x)} = F(2h-x) = -\frac{r(h)}{(h-a)^\beta\sin\pi\alpha}\,\frac{1}{(h-x)^\alpha} + F_2(x) ,$$

$$(26.a\text{-}c)$$

where the functions $F_j(z)$, $(j=0,1,2)$, are bounded everywhere with a possible exception of the end points c_k, $(c_1=a,\ c_2=h)$ near which [13]

$$|F_j(z)| < \frac{C_k}{|z-c_k|^{p_k}}\ , \quad p_k < \mathrm{Re}(\alpha,\beta), \qquad (j=0,1,2;\ k=1,2), \tag{27}$$

C_k and p_k being real constants. Substituting now from (26) into (22), multiplying both sides first by $(x-a)^{\beta}$ and letting $x=a$, and then by $(h-x)^{\alpha}$ and letting $x=h$, it is found that

$$r(a)\cot\pi\beta = 0\ ,$$

$$\frac{r(h)}{\sin\pi\alpha}\,[2(1-\alpha)^2 - 1 - \cos\pi\alpha] = 0\ . \tag{28.a,b}$$

(25) and (28.a) give $\beta = 0.5$ which is the well-known result. For $r(h) \neq 0$ the only possible root of (28.b) is $\alpha = 1$ which is unacceptable. It is, therefore, clear that the function $G(x)$ does not have a power singularity at the end point $x = h$. Before proceeding with the determination of the fundamental function $w(t)$, it should be pointed out that (28.b) is identical to the characteristic equation resulting from the asymptotic behavior of the solution for a 90-degree wedge. For example, if the wedge $(0<r<\infty,\ 0<\theta<\pi/2)$ is subjected to boundary tractions

$$\sigma_{\theta\theta} + i\sigma_{r\theta} = 0\ , \qquad (\theta = \tfrac{\pi}{2}\ ,\ 0<r<\infty),$$

$$\sigma_{\theta\theta} + i\sigma_{r\theta} = \sigma(r)\ , \qquad (\theta=0,\ r_1<r<r_2), \tag{29.a,b}$$

it can be shown that

$$\frac{4\mu}{1+\kappa}\,\frac{\partial}{\partial r}\,u_\theta(r,0) = -\int_{r_1}^{r_2}\sigma(t)dt\,\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{t^{\lambda-1}}{r^{\lambda}}\,\frac{d\lambda}{D(\lambda)}\ ,$$

$$D(\lambda) = [2(1-\lambda)^2 - 1 - \cos\pi\lambda]/\sin\pi\lambda\ , \tag{30.a,b}$$

where $D(\lambda)$ is the characteristic function.

Since the function $G(x)$ must have an integrable singularity at $x=a$, and since the foregoing analysis indicates that it cannot have a power singularity at $x=h$, the next step is to look for a solution of the following form:

$$G(x) = r_0(x)(t-a)^{-\beta} , \qquad (0 < Re(\beta) < 1) \tag{31}$$

where $r_0(a)$ and $r_0(h)$ are bounded and nonzero. Defining again the following sectionally holomorphic function

$$F(z) = \frac{1}{\pi} \int_a^h \frac{G(t)}{t-z} dt = \frac{1}{\pi} \int_a^h \frac{r_0(t)dt}{(t-z)(t-a)^{\beta}} , \tag{32}$$

we obtain

$$F(z) = \frac{r_0(a)e^{\pi i\beta}}{\sin\pi\beta} \frac{1}{(z-a)^{\beta}} + \frac{r_0(h)}{\pi(h-a)^{\beta}} \log(z-h) + F_3(z) ,$$

$$\frac{1}{\pi} \int_a^h \frac{G(t)dt}{t-x} = \frac{r_0(a)\cot\pi\beta}{(x-a)^{\beta}} + \frac{r_0(h)}{\pi(h-a)^{\beta}} \log(h-x) + F_4(x) ,$$

$$\frac{1}{\pi} \int_a^h \frac{G(t)dt}{t-(2h-x)} = F(2h-x) = \frac{r_0(h)}{\pi(h-a)^{\beta}} \log(h-x) + F_5(x)$$

$$(h-x) \frac{1}{\pi} \frac{d}{dx} \int_a^h \frac{G(t)dt}{t-(2h-x)} = F_6(x) ,$$

$$(h-x)^2 \frac{1}{\pi} \frac{d^2}{dx^2} \int_a^h \frac{G(t)dt}{t-(2h-x)} = F_7(x) , \tag{33.a-e}$$

where the functions F_3 and F_4 are bounded everywhere except possibly at the end point $z=a$ where they have a behavior similar to (27); in particular they tend to definite limits as $z \to h$ and $x \to h$. The functions F_5, F_6 and F_7 are bounded and tend to definite limits as $x \to a$ and $x \to h$. Now substituting from (33) into

(22) it is seen that the logarithmic singularities are cancelled and the power singularity $(x-a)^{-\beta}$ can be eliminated by requiring that $\cot\pi\beta = 0$, which again gives $\beta = 1/2$. We may thus conclude that the solution of (22) is of the form (31). Formally then the weight function of the integral equation is $(h-t)^0(t-a)^{-1/2}$ with the related polynomials $P_n^{(0,-1/2)}(t)$. Similarly in (16) for $c = 0$ by letting $G(t) = r_1(t)(b-t)^{-\beta}$ (where r_1 is a bounded function), it may easily be shown that $\beta = 0.5$. It should be pointed out that for $c = 0$ and $b = h$ the general solution of (16) and (22) no longer contains an arbitrary constant; therefore the conditions (15) and (17.b) are not necessary for a unique solution which is also suggested by the physics of the problem.

4. ON THE NUMERICAL SOLUTION OF THE INTEGRAL EQUATION

Consider the singular integral equation (16) which, for $c = 0$, has a generalized Cauchy kernel. To solve the integral equation numerically by techniques similar to that described in [8], we first normalize the interval (0,b) by defining

$$\tau = t/b \ , \qquad \xi = x/b \ ,$$

$$G(t) = \phi(\tau) = r_1(t)(b-t)^{-1/2} = R(\tau)(1-\tau)^{-1/2} \ . \qquad (34)$$

Using (34) (16) may be expressed as

$$\int_0^1 M(\xi,\tau)\phi(\tau)d\tau = -\frac{1+\kappa}{4\mu}\pi\sigma(b\xi) \ , \qquad (0<\xi<1) \ ,$$

$$M(\xi,\tau) = (\tau-\xi)^{-1} + bN(b\xi,b\tau) \ . \qquad (35.a,b)$$

To solve (35) numerically the most practical technique appears to be to extend the definition of $\phi(\tau)$ in an appropriate manner into the interval (-1,0) and to use the corresponding Jacobian integration formula with the related orthogonal polynomials $P_n^{(-1/2,-1/2)}(\tau)$. Noting that $\phi(0) \neq 0$, an appropriate extension of ϕ may be an even continuation as follows:

$$\phi(\tau) = p(\tau)(1-\tau^2)^{-1/2} , \quad p(\tau) = p(-\tau) , \quad (-1<\tau<1),$$

$$p(\tau)(1+\tau)^{-1/2} = R(\tau) , \quad (0<\tau<1). \qquad (36)$$

With (36) (35) becomes

$$\int_0^1 M(\xi,\tau)\phi(\tau)d\tau = \frac{1}{2} \int_{-1}^{1} M(\xi,|\tau|) \frac{p(\tau)d\tau}{(1-\tau^2)^{1/2}} = - \frac{1+\kappa}{4\mu} \pi\sigma(b\xi) ,$$

$$(0<\xi<1). \qquad (37)$$

Using now the integration formula corresponding to the weight function $(1-\tau^2)^{-1/2}$ [8] and from (13), (16) and (17) observing that $\tau=0$ is a zero of $M(\xi,\tau)$, we obtain

$$- \frac{1+\kappa}{4\mu} \pi\sigma(b\xi_j) = \frac{1}{2} \sum_{i=1}^{2n+1} A_i p(\tau_i) M(\xi_j,|\tau_i|)$$

$$= \sum_{i=1}^{n} A_i p(\tau_i) M(\xi_j,\tau_i) , \quad (j=1,...,n), \qquad (38)$$

$$T_{2n+1}(\tau_i) = 0 , \quad \tau_i = \cos(\frac{2i-1}{4n+2} \pi) , \quad A_i = \frac{\pi}{2n+1}$$

$$U_{2n}(\xi_j) = 0 , \quad \xi_j = \cos(\frac{\pi j}{2n+1}) . \qquad (39)$$

(38) provides a system of n linear algebraic equations to determine the unknowns $p(\tau_i)$, $(0<\tau_i<1, i=1,...,n)$. Once the density function G(t) is obtained the stress intensity factors and the crack surface displacement may be obtained from (see Figure 1)

$$k(a) = \lim_{t\to a} \sqrt{2(a-t)}\, \sigma_{yy}(x,0) = \lim_{t\to a} \frac{4\mu}{1+\kappa} \sqrt{2(t-a)}\, G(t) ,$$

$$k(b) = \lim_{t\to b} \sqrt{2(t-b)}\, \sigma_{yy}(x,0) = -\lim_{t\to b} \frac{4\mu}{1+\kappa} \sqrt{2(b-t)}\, G(t) ,$$

$$v(x,0) = \int_a^x G(t)dt . \qquad (40.a-c)$$

5. NUMERICAL RESULTS

As a first example consider the half plane problem shown in Figure 2. The numerical results given in this section are obtained for a uniform pressure $\sigma(x) = \sigma_0$ applied to the crack surface which corresponds to the uniform tension applied to the strip (Figure 1) or the half plane (Figure 2) perpendicular to and away from the plane of the crack.

Table 1. Stress intensity factor at $x = b$ for the half plane ($a_0 = (b-c)/2$, $c_0 = (b+c)/2$).

c_0/a_0	1.05	1.01	1.001	1.0	1.0 (Ref.2)
$\frac{k(b)}{\sigma_0\sqrt{a_0}}$	1.2540	1.3303	1.3987	1.5869	1.5861

The results for the half plane are given in Table 1 which also shows the result given in [2]. The agreement appears to be quite good. Further results for larger values of c_0/a_0 and the values of $k(c)$ may be found in [11]. Figure 3 shows the crack surface displacement for the edge crack, $c_0/a_0 = 1$ and for an internal crack, $c_0/a_0 = 1.01$ obtained from

$$v(x,0) = -\int_x^d G(t)dt . \qquad (41)$$

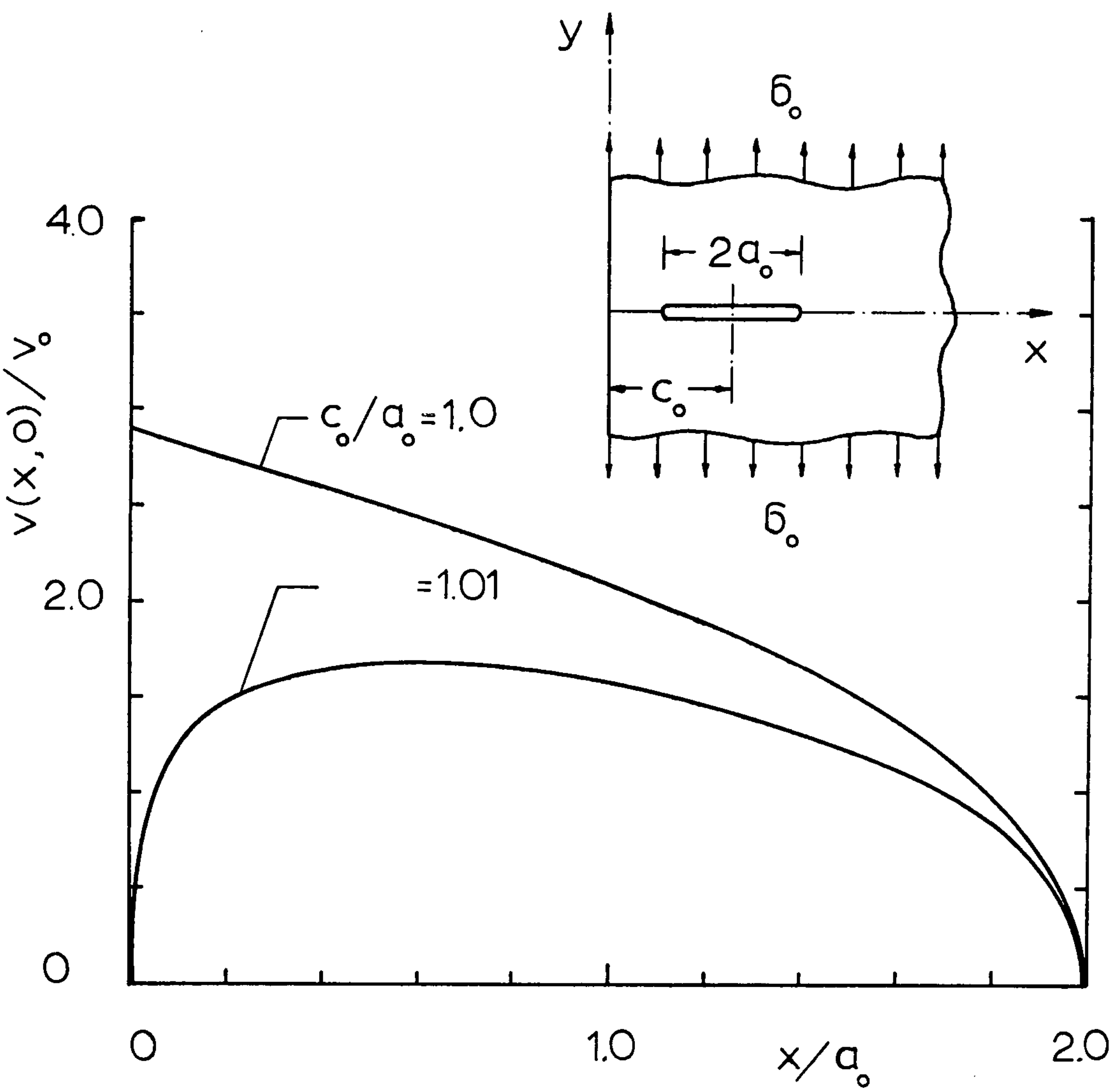

Figure 3. The crack surface displacement for an internal and for the edge crack in a half plane($v_o = \sigma_o a_o(1+x)/4\mu$).

The normalizing constant v_0 shown in the figure is

$$v_0 = \sigma_0 a_0 (1+\kappa)/4\mu \ . \tag{42}$$

The results for the strip are given in Tables 2 and 3, and in Figures 4 and 5. Table 2 gives the results for two symmetrically located collinear internal cracks shown in Figure 1. In limit when a=0 the results reduce to that found in [10] for a single (central) crack. As the size of the net ligaments 2a and (h-b) approach zero, the corresponding stress intensity factors go to infinity. These limiting values are indicated in Table 2 by an arrow. From the Table it is seen that the rupture of one of these ligaments causes a sharp increase in the stress intensity factor at the other crack tip (for example, compare the lines 1,2 and 5,6).

Table 2. Stress intensity factors for collinear internal cracks in a strip (Figure 1, $a_0 = (b-a)/2$).

a/h	b/h	$\frac{k(a)}{\sigma_0\sqrt{a_0}}$	$\frac{k(b)}{\sigma_0\sqrt{a_0}}$
0	0.4	$(\rightarrow \infty)$	1.5690
0.1	0.5	1.1746	1.1169
0.2	0.6	1.1102	1.0961
0.4	0.8	1.0984	1.1250
0.5	0.9	1.1290	1.2278
0.6	1.0	1.6080	$(\rightarrow \infty)$
0	0.8	$(\rightarrow \infty)$	2.5680
0.1	0.9	1.6730	1.7451
0.2	1.0	2.1769	$(\rightarrow \infty)$
0.5	0.95	1.1960	1.4711
0.5	0.98	1.2713	1.9008
0.5	1.0	1.6228	$(\rightarrow \infty)$

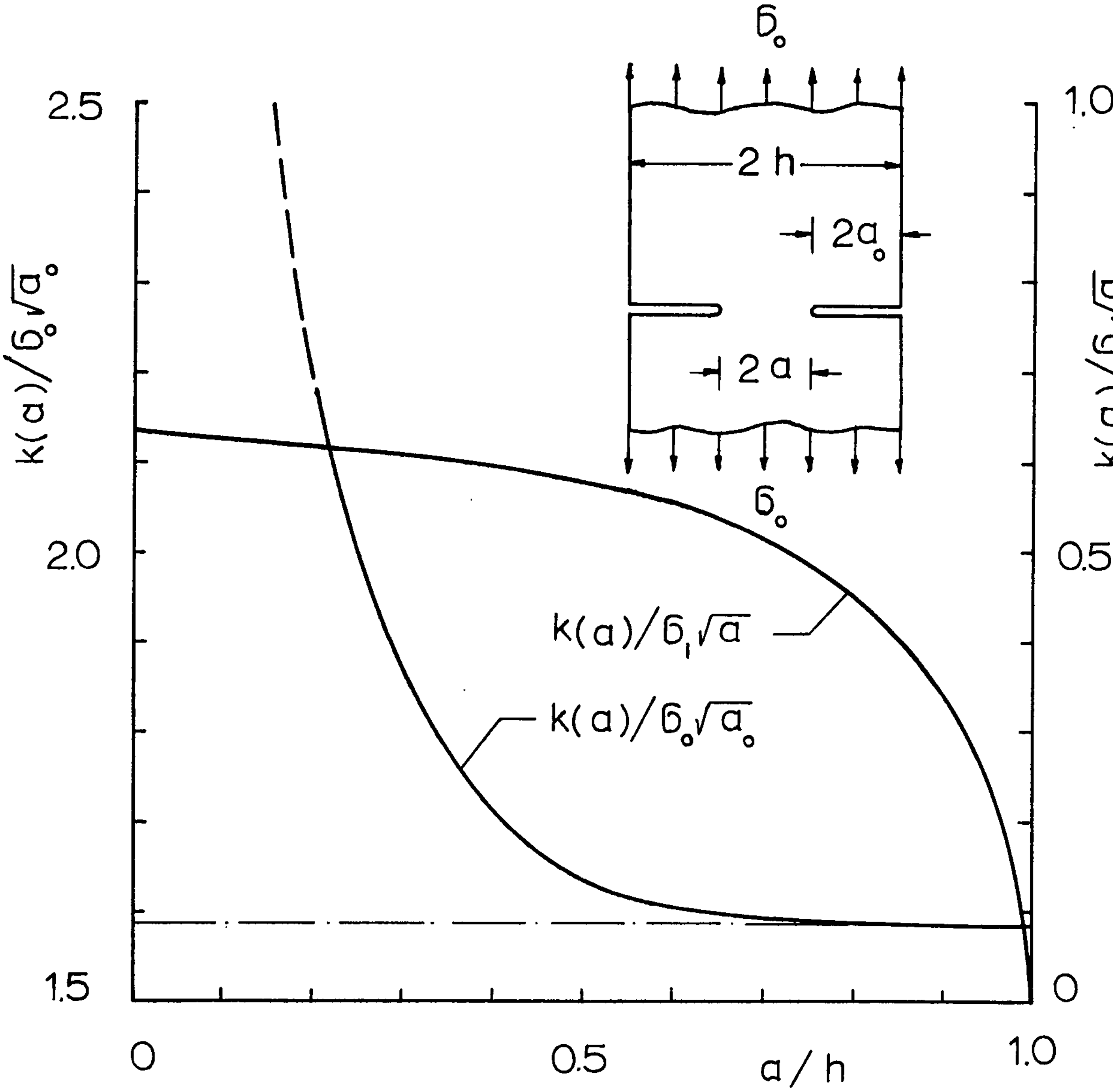

Figure 4. The stress intensity factor for the edge cracks in an infinite strip ($\sigma_1 = \sigma_0 h/a$).

Table 3. Stress intensity factor for the edge cracks in a strip of finite width ($b = h$, $a_0 = (h-a)/2$, net section stress: $\sigma_1 = \sigma_0 h/a$).

a/h	$\frac{k(a)}{\sigma_0\sqrt{a_0}}$	$\frac{k(a)}{\sigma_1\sqrt{a}}$
0	$\to\infty$	$\frac{2}{\pi} = 0.63662$
0.1	2.9467	0.62510
0.2	2.1769	0.61572
0.3	1.8744	0.60738
0.4	1.7136	0.59361
0.5	1.6328	0.57728
0.6	1.6080	0.55703
0.7	1.5970	0.51749
0.8	1.5915	0.45014
0.9	1.5883	0.33690
1.0	1.5869	$\to 0$

Table 3 gives the results for the edge cracks which are also shown in Figure 4. The limiting values of these results are that of the edge crack in a half plane for $a/h \to 1$ (see Table 1) and infinity for $a/h \to 0$. For the case of $(h/a) = \infty$ the closed form solution is given by (e.g., [14])

$$\sigma_{yy}(x,0) = \frac{P}{\pi\sqrt{a^2-x^2}}\,, \qquad (-a<x<a),$$

$$k(a) = \lim_{x\to a} \sqrt{2(a-x)}\;\sigma_{yy}(x,0) = \frac{P}{\pi\sqrt{a}}\,, \tag{43.a,b}$$

where P is the resultant load (per unit thickness) acting along the y-axis. Defining now a net section stress σ_1 by

$$\sigma_1 = \frac{P}{2a} = \sigma_0\,\frac{h}{a} \tag{44}$$

for $a/h \to 0$, the stress intensity factor becomes

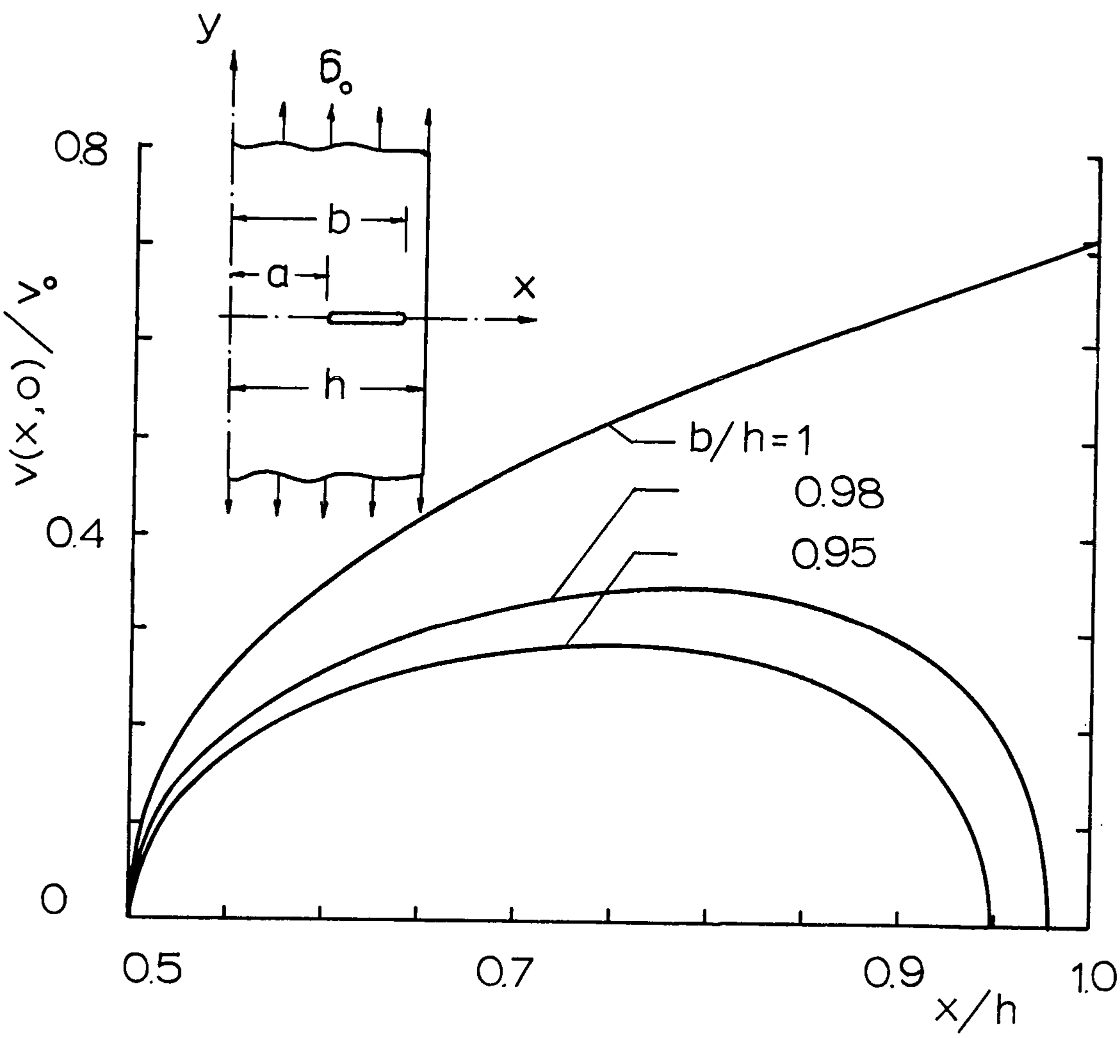

Figure 5. The crack surface displacement for internal and for edge cracks in an infinite strip($v_o = \sigma_o a_o (1+\kappa)/4\mu$, $a_o = (b-a)/2$).

$$\frac{k(a)}{\sigma_1\sqrt{a}} = \frac{2}{\pi} = 0.63662 \ . \qquad (45)$$

The two stress intensity factor ratios shown in Table 3 and Figure 4 are related by

$$\frac{k(a)}{\sigma_1\sqrt{a}} = \frac{k(a)}{\sigma_0\sqrt{a_0}} \left[\frac{a}{2h} \left(1 - \frac{a}{h}\right)\right]^{1/2} . \qquad (46)$$

Note that for $h/a \to \infty$, for a fixed a $k(a)$ is proportional to $\sigma_0 h$ or $\sigma_0 a_0$. Hence the ratio $k(a)/(\sigma_0\sqrt{a_0}) \to \infty$ as $h/a \to \infty$ or $a_0/a \to \infty$.

Examples of the crack surface displacement $v(x,0)$, $(a<x<b)$ for the internal and edge cracks in the strip are shown in Figure 5. The normalization constant v_0 used in Figure 5 is also given by (42).

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