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THE PROBLEM OF ESTIMATION OF UNKNOWN MEAN VALUE FOR SOME CORRELATION MODELS OF HOMOGENEOUS AND ISOTROPIC RANDOM FIELDS

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M. I. YADRENKO AND N. SEMENOVS'KA

Abstract. We obtain explicit formulas for the variance of a linear unbiased estimator of unknown mean value for some correlation models of homogeneous and isotropic random fields.

1. Introduction

Explicit formulas for the linear extrapolation to the center of a ball are constructed in the paper [2] for some correlation models of homogeneous and isotropic random fields observed on a sphere. We considered in [2] the so-called Buell list of "empirical formulas of correlation functions" (see Table 1), which is nowadays a standard source of correlation models in meteorology. We also considered in [2] the Whittle–Mattern class of correlation functions given by

$$\varphi_\nu(r) = \frac{2^{1-\nu}}{\Gamma(\nu)} r^\nu K_\nu(r), \tag{1}$$

where K is the modified Bessel function and $\nu > 0$.

In the present paper, we study the problem of estimation of the unknown mean value of homogeneous and isotropic random fields for these two families of the correlation functions.

2. Homogeneous and isotropic random fields on $\mathbf{R}^n$

We recall some notions of the spectral theory of random fields.

A random function $\xi(x)$, $x \in \mathbf{R}^n$, is called a homogeneous and isotropic random field if $\mathsf{E}\,\xi(x) = \text{const}$, $\mathsf{E}\,\xi^2(x) < \infty$, and $\mathsf{E}\,\xi(x)\xi(y) = \varphi(r)$ is a function of the distance $r = |x-y|$ between points x and y (we assume in what follows that $\mathsf{E}\,\xi(x) = 0$).

The correlation function $\varphi(r)$ of a homogeneous and isotropic random field admits the following representation:

$$\varphi(r) = 2^{(n-2)/2}\Gamma\left(\frac{n}{2}\right)\int_0^\infty \frac{J_{\frac{n-2}{2}}(\lambda r)}{(\lambda r)^{(n-2)/2}}\,dF(\lambda) \tag{2}$$

(see, for example, [1, 3]), where F is a nondecreasing bounded function on $[0,+\infty]$ and J is the Bessel function of a real argument. Relation (2) is called the *spectral expansion* of a homogeneous and isotropic random field, while $F(\lambda)$ is called its *spectral function.* If $F(\lambda)$ is absolutely continuous, then there exists a function $f(\lambda)$ such that

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TABLE 1. Buell's list of correlation functions

Index	Model $\varphi(r)$
1	e^{-r^2}
2	$(1+r^2)^{-\nu}$
3	$(1+r+\kappa r^2)e^{-r}$
3a	$(1+r)e^{-r}$
3b	$(1+r+r^2/3)e^{-r}$
3c	$(1+r-r^2/2)e^{-r}$
4	$\frac{ae^{-br}-be^{-ar}}{a-b}$
5	$\frac{\sin r}{r}$
6	$\frac{2^{1/3}}{\Gamma(2/3)}r^{2/3}K_{2/3}(r)$

$F(\lambda) = \int_0^\lambda f(u)\,du$; $f(\lambda)$ is called the *spectral density* of a homogeneous and isotropic random field. Relation (2) reduces to the well-known expansion of the correlation function of a real stationary process

$$\varphi(r) = \int_0^\infty \cos(\lambda r)\,dF(\lambda)$$

if $n=1$.

One can express the spectral density $f(\lambda)$ in terms of the correlation function $\varphi(r)$; namely,

$$f(\lambda) = \int_0^\infty \sqrt{\lambda r}\,J_{\frac{n-2}{2}}(\lambda r)\frac{(\lambda r)^{(n-1)/2}}{2^{(n-2)/2}\Gamma\left(\frac{n}{2}\right)}\varphi(r)\,dr. \tag{3}$$

The spectral function $F(\lambda)$ can also be expressed via the correlation function $\varphi(r)$ as follows:

$$F(\lambda) = \frac{1}{2^{\frac{n-2}{2}}\Gamma\left(\frac{n}{2}\right)}\int_0^\infty \sqrt{\lambda r}\,J_{\frac{n-2}{2}}(\lambda r)(\lambda r)^{n/2}\frac{\varphi(r)}{r}\,dr \tag{4}$$

(see [1]). Note that $\varphi(0) = F(+\infty)$ (this follows from (2)).

3. The problem of estimation of unknown mean value

Assume that a random field $\xi(x) = a + \eta(x)$ is observed on a sphere of radius r, where $\eta(x)$ is a homogeneous and isotropic random field with zero mean and known correlation function $\varphi(r)$. The problem is to obtain a linear unbiased estimator for a that has the minimal variance.

Theorem 3.1. *The estimator*

$$\widehat{a} = \frac{1}{\omega_n} \int_{S_n} \xi(x)\, dm_n \tag{5}$$

has the minimal variance in the class of linear unbiased estimators of the unknown mean value of a homogeneous and isotropic random field observed on a sphere where $m_n(\cdot)$ is the Lebesgue measure on the sphere in $\mathbf{R}^n$ and ω_n is the area of the surface of the unit sphere in $\mathbf{R}^n$. Note that the right-hand side of (5) *is the average of the field $\xi(x)$ over the sphere S_n. The variance of estimator* (5) *is given by*

$$D\widehat{a} = \frac{2^{n-2}\Gamma^2\left(\frac{n}{2}\right)}{r^{n-2}} \int_0^\infty \lambda^{2-n} J^2_{\frac{n-2}{2}}(\lambda r)\, dF(\lambda). \tag{6}$$

This result can be found in [1].

The evaluation of the variance of $\widehat{a}$ by using relation (6) meets some technical problems if the correlation function is general, since one needs to evaluate the spectral function $F(\lambda)$ and the integral

$$\int_0^\infty \lambda^{2-n} J^2_{\frac{n-2}{2}}(\lambda r)\, dF(\lambda). \tag{7}$$

4. Explicit formulas for some correlation models

Consider the Whittle–Mattern family (1). We noticed in [2] that models (1) for $\nu = 2/3$, $3/2$, and $5/2$ become the Buell models of indices 6, 3a, and 3b, respectively. Note also that

$$\left(1 + r + \kappa r^2\right) e^{-r} = (1 - 3\kappa)\varphi_{3/2}(r) + 3\kappa\varphi_{5/2}(r) \tag{8}$$

and the Buell model of index 3 can be treated as a linear combination of the Whittle–Mattern models with $\nu = 3/2$ and $\nu = 5/2$.

Prior to considering the problem of estimation of the unknown mean value for the above correlation models we recall the following result (see [2]).

Lemma 4.1. *We have*

$$\begin{aligned} &\int_0^\infty J^2_{\frac{n-2}{2}}(\lambda r) \frac{\lambda}{(\lambda^2 + a^2)^{n/2+\nu}}\, d\lambda \\ &\quad = \frac{r^{n-2} B(1/2, (n-1)/2 + \nu)}{2\pi} \\ &\qquad \times \left(\frac{B((n-1)/2, \nu)}{a^{2\nu}\Gamma(n-1)} \cdot {}_1F_2\big((n-1)/2; 1-\nu, n-1; a^2 r^2\big) \right. \\ &\qquad\qquad \left. + \frac{r^{2\nu}\Gamma(-\nu)}{\Gamma(n+\nu-1)} \cdot {}_1F_2((n-1)/2+\nu; n+\nu-1, \nu+1; a^2 r^2) \right) \end{aligned} \tag{9}$$

for $n > \nu > 0$, where ${}_pF_q$ is the generalized hypergeometric series defined by

$${}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; z) = \sum_{k=0}^\infty \frac{(a_1)_k \cdots (a_p)}{(b_1)_k \cdots (b_q)} \cdot \frac{z^k}{k!}.$$

Model 1. Consider a homogeneous and isotropic random field whose correlation function is given by (1). The spectral density for this model is of the form

$$f_\nu(\lambda) = \frac{2\lambda^{n-1}}{B(\nu, n/2)(\lambda^2+1)^{n/2+\nu}}, \qquad n > \nu \tag{10}$$

(see [2]). Considering (9) for $a = 1$ we get

$$
\begin{aligned}
&\int_0^\infty \lambda^{2-n} J^2_{\frac{n-2}{2}}(\lambda r)\, dF(\lambda) \\
&\quad = \frac{B(1/2, (n-1)/2+\nu) r^{n-2}}{\pi B(\nu, n/2)} \\
&\qquad \times \Bigg(\frac{B((n-1)/2, \nu)}{\Gamma(n-1)} \cdot {}_1F_2\big((n-1)/2; 1-\nu, n-1; r^2\big) \\
&\qquad\quad + \frac{r^{2\nu}\Gamma(-\nu)}{\Gamma(n+\nu-1)} \cdot {}_1F_2\big((n-1)/2+\nu; n+\nu-1, \nu+1; r^2\big)\Bigg).
\end{aligned} \tag{11}
$$

Put

$$
G_1(r) = \frac{B((n-1)/2, \nu)}{\Gamma(n-1)} \cdot {}_1F_2\big((n-1)/2; 1-\nu, n-1; r^2\big),
$$

$$
G_2(r) = \frac{\Gamma(-\nu)}{\Gamma(n+\nu-1)} \cdot {}_1F_2\big((n-1)/2+\nu; n+\nu-1, \nu+1; r^2\big).
$$

Then the variance of the estimator (5) is given by

$$
\operatorname{Var}\widehat{a} = \frac{2^{n-2}\Gamma^2\left(\frac{n}{2}\right) B(1/2, (n-1)/2+\nu)}{\pi B(\nu, n/2)} \big(G_1(r) + r^{2\nu} G_2(r)\big). \tag{12}
$$

Model 2. Let $\varphi(r) = e^{-r^2}$. This is a particular case of a more general model

$$
\varphi(r) = e^{-cr^2}
$$

considered in [1].

The spectral density for this model is known:

$$
f(\lambda) = \frac{\lambda^{n-1} e^{-\lambda^2/4}}{2^{n-1}\Gamma\left(\frac{n}{2}\right)}
$$

(see [2]). It follows from the relation

$$
\int_0^\infty x^{1/2} e^{-\beta x^2} J_\nu(\alpha x) J_\nu(xy)(xy)^{1/2}\, dx = \frac{y^{1/2}}{2\beta} e^{-\frac{\alpha^2+y^2}{4\beta}} I_\nu\left(\frac{\alpha y}{2\beta}\right)
$$

([9, 8.11, (23)]) that

$$
\int_0^\infty \lambda^{2-n} J^2_{\frac{n-2}{2}}(\lambda r) f(\lambda)\, d\lambda = \frac{e^{-2r^2} I_{\frac{n-2}{2}}\left(2r^2\right)}{2^{n-2}\Gamma\left(\frac{n}{2}\right)}.
$$

After simple algebra and using (6) we obtain an explicit form of the variance of the linear unbiased estimator $\widehat{a}$:

$$
\operatorname{Var}\widehat{a} = \Gamma\left(\frac{n}{2}\right) e^{-2r^2} r^{2-n} I_{\frac{n-2}{2}}\left(2r^2\right).
$$

Model 3. Let $\varphi(r) = (1+r^2)^{-\nu}$ for $\nu > 0$. The spectral density for this model of the correlation function is given by

$$
f(\lambda) = \frac{\lambda^{\frac{n-2}{2}+\nu} K_{n/2-\nu}(\lambda)}{2^{\frac{n-2}{2}+\nu-1}\Gamma\left(\frac{n}{2}\right)\Gamma(\nu)}
$$

(see [2]). According to [9, 10.3, (25)],

$$
\int_0^\infty \lambda^{2-n} J^2_{\frac{n-2}{2}}(\lambda r) f(\lambda)\, d\lambda = \frac{2^{2-n} r^{n-2}}{\Gamma^2(n/2)} F\left(\frac{n-1}{2}, \nu; n-1; -4r^2\right),
$$

where F is the Gauss hypergeometric series defined by

$$F(a,b;c;z) = {}_2F_1(a,b;c;z).$$

After simple algebra and using (6) we obtain an explicit form of the variance of the linear unbiased estimator $\widehat{a}$:

$$\operatorname{Var}\widehat{a} = F\left(\frac{n-1}{2}, \nu; n-1; -4r^2\right).$$

Model 4. Let $\varphi(r) = (1+r)e^{-r}$. As we noticed above, this model of the correlation function corresponds to the Whittle–Mattern model $\varphi_{3/2}(r)$. Thus relation (10) with $\nu = 3/2$ implies that

$$f_{3/2}(\lambda) = \frac{2\lambda^{n-1}}{B(3/2, n/2)(\lambda^2+1)^{(n+3)/2}}, \qquad n > 3/2.$$

Similarly to (11) we get

$$\int_0^\infty \lambda^{2-n} J^2_{\frac{n-2}{2}}(\lambda r)\, dF(\lambda) = \frac{nr^{n-2}}{\pi}\big(G_1(r) + r^3 G_2(r)\big),$$

where

$$G_1(r) = \frac{B((n-1)/2, 3/2)}{\Gamma(n-1)} \cdot {}_1F_2\big((n-1)/2; -1/2, n-1; r^2\big),$$

$$G_2(r) = \frac{4\sqrt{\pi}}{3\Gamma(n+1/2)} \cdot {}_1F_2\big(n/2+1; n+1/2, 5/2; r^2\big).$$

Using (6) we obtain the variance of the linear unbiased estimator $\widehat{a}$:

$$\operatorname{Var}\widehat{a} = \frac{n2^{n-2}\Gamma^2\left(\frac{n}{2}\right)}{\pi}\big(G_1(r) + r^3 G_2(r)\big).$$

Model 5. Let $\varphi(r) = (1+r+r^2/3)e^{-r}$. This model of the correlation function corresponds to the Whittle–Mattern model $\varphi_{5/2}(r)$. It follows from equation (10) with $\nu = 5/2$ that

$$f_{5/2}(\lambda) = \frac{2\lambda^{n-1}}{B(5/2, n/2)(\lambda^2+1)^{(n+5)/2}}, \qquad n > 5/2.$$

Similarly to (11) we get

$$\int_0^\infty \lambda^{2-n} J^2_{\frac{n-2}{2}}(\lambda r)\, dF(\lambda) = \frac{n(n+2)r^{n-2}}{3\pi}\big(g_1(r) - r^5 g_2(r)\big),$$

where

$$g_1(r) = \frac{B((n-1)/2, 5/2)}{\Gamma(n-1)} \cdot {}_1F_2\big((n-1)/2; -3/2, n-1; r^2\big),$$

$$g_2(r) = \frac{8\sqrt{\pi}}{15\Gamma(n+3/2)} \cdot {}_1F_2\big(n/2+2; n+3/2, 7/2; r^2\big).$$

Using (12) we obtain the variance of the linear unbiased estimator $\widehat{a}$:

$$D\widehat{a} = \frac{2^{n-2}\Gamma^2\left(\frac{n}{2}\right) n(n+2)}{3\pi}\big(g_1(r) - r^5 g_2(r)\big).$$

Model 6. Let $\varphi(r) = (1+r+\kappa r^2)e^{-r}$. This model of the correlation function corresponds to a linear combination of models $\varphi_{3/2}(r)$ and $\varphi_{5/2}(r)$, namely

$$\big(1+r+\kappa r^2\big)e^{-r} = (1-3\kappa)\varphi_{3/2}(r) + 3\kappa\varphi_{5/2}(r),$$

according to equality (8). Thus the spectral density $f(\lambda)$ for this correlation model also is the linear combination of $f_{3/2}(r)$ and $f_{5/2}(r)$ with the same coefficients. The results obtained for the two latter models show that

$$f(\lambda) = \frac{2(n+1)\lambda^{n-1}}{B}(1/2, n/2)(1+\lambda^2)^{(n+5)/2}\big((1-3\kappa)\lambda^2 + n\kappa + 1\big).$$

Similarly

$$(13)\quad \begin{aligned} &\int_0^\infty \lambda^{2-n} J^2_{\frac{n-2}{2}}(\lambda r)\, dF(\lambda) \\ &= \frac{n}{\pi} r^{n-2} \left((1-3\kappa)\big(G_1(r) + r^3 G_2(r)\big) + \kappa(n+2)\big(g_1(r) - r^5 g_2(r)\big)\right), \end{aligned}$$

where

$$\begin{aligned} G_1(r) &= \frac{B((n-1)/2, 3/2)}{\Gamma(n-1)} \cdot {}_1F_2\big((n-1)/2; -1/2, n-1; r^2\big), \\ G_2(r) &= \frac{4\sqrt{\pi}}{3\Gamma(n+1/2)} \cdot {}_1F_2\big(n/2+1; n+1/2, 5/2; r^2\big), \\ g_1(r) &= \frac{B((n-1)/2, 5/2)}{\Gamma(n-1)} \cdot {}_1F_2\big((n-1)/2; -3/2, n-1; r^2\big), \\ g_2(r) &= \frac{8\sqrt{\pi}}{15\Gamma(n+3/2)} \cdot {}_1F_2\big(n/2+2; n+3/2, 7/2; r^2\big). \end{aligned}$$

Using (6) we obtain the variance of the linear unbiased estimator $\widehat{a}$:

$$(14)\quad \operatorname{Var}\widehat{a} = \frac{n2^{n-2}\Gamma^2(n/2)}{\pi}\left((1-3\kappa)\big(G_1(r) + r^3 G_2(r)\big) + \kappa(n+2)\big(g_1(r) - r^5 g_2(r)\big)\right).$$

Model 7. Consider the case of $\kappa = -1/2$ in the latter model; that is, we deal with the case of the correlation function $\varphi(r) = (1 + r - r^2/2)e^{-r}$, $n = 2$. Applying the above general result to the particular case of $k = -1/2$ we obtain

$$f(\lambda) = \frac{15}{2}\lambda^3\big(1+\lambda^2\big)^{-7/2}.$$

Furthermore, using (13) we get

$$\int_0^\infty J_0^2(\lambda r)\, dF(\lambda) = \frac{2}{\pi}\left(5/2\big(G_1(r) + r^3 G_2(r)\big) + 2\big(r^5 g_2(r) - g_1(r)\big)\right),$$

where

$$\begin{aligned} G_1(r) &= \frac{\pi}{2} \cdot {}_1F_2\big(1/2; -1/2, 1; r^2\big), \\ G_2(r) &= \frac{2^5}{5\cdot 3^2} \cdot {}_1F_2\big(2; 5/2, 5/2; r^2\big), \\ g_1(r) &= \frac{3\pi}{4} \cdot {}_1F_2\big(1/2; -3/2, 1; r^2\big), \\ g_2(r) &= \frac{2^6}{5^2\cdot 3^2} \cdot {}_1F_2\big(3; 7/2, 7/2; r^2\big). \end{aligned}$$

Now the variance for the linear unbiased estimator $\widehat{a}$ follows from (14):

$$\operatorname{Var}\widehat{a} = \frac{2}{\pi}\left(5/2\big(G_1(r) + r^3 G_2(r)\big) + 2\big(r^5 g_2(r) - g_1(r)\big)\right).$$

Model 8. Consider the correlation model

$$\varphi(r) = \frac{ae^{-br} - be^{-ar}}{a-b}.$$

Let $\varphi_a(r) = e^{-ar}$ and $\varphi_b(r) = e^{-br}$. The corresponding spectral densities are $f_a(\lambda)$ and $f_b(\lambda)$, respectively. The correlation function $\varphi(r)$ is a linear combination of $\varphi_a(r)$ and $\varphi_b(r)$. Then the spectral density $f(\lambda)$ corresponding to the correlation function $\varphi(r)$ is the linear combination of $f_a(\lambda)$ and $f_b(\lambda)$ with the same coefficients. Using representation (3) and the following relation

$$\int_0^\infty x^{\nu+1/2} e^{-\alpha x} J_\nu(xy)(xy)^{1/2}\,dx = \pi^{-1/2} 2^{\nu+1}\Gamma(\nu+3/2)\alpha y^{\nu+1/2}\left(\alpha^2+y^2\right)^{-\nu-3/2}$$

([9, 8.6, (4)]) we evaluate the spectral density $f_a(\lambda)$:

$$f_a(\lambda) = \frac{2a\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{n}{2}\right)} \cdot \frac{\lambda^{n-1}}{(\lambda^2+a^2)^{(n+1)/2}}.$$

Now we apply (9) to evaluate the integral

$$\int_0^\infty \lambda^{2-n} J^2_{\frac{n-2}{2}}(\lambda r) f_a(\lambda)\,d\lambda.$$

We have

$$\int_0^\infty \lambda^{2-n} J^2_{\frac{n-2}{2}}(\lambda r) \cdot \frac{2a\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{n}{2}\right)} \cdot \frac{\lambda^{n-1}}{(a^2+\lambda^2)^{\frac{n+1}{2}}}\,d\lambda$$
$$= \frac{2ar^{n-2}}{\pi}\left(\frac{B\left(\frac{n-1}{2},\frac{1}{2}\right)}{a\Gamma(n-1)} \cdot {}_1F_2\left(\frac{n-1}{2};\frac{1}{2},n-1;a^2r^2\right)\right.$$
$$\left. - \frac{2\sqrt{\pi}}{\Gamma\left(n-\frac{1}{2}\right)} r \cdot {}_1F_2\left(\frac{n}{2};\frac{3}{2},n-\frac{1}{2};a^2r^2\right)\right).$$

Put

$$L(a,r) = \left(\frac{B\left(\frac{n-1}{2},\frac{1}{2}\right)}{a\Gamma(n-1)} \cdot {}_1F_2\left(\frac{n-1}{2};\frac{1}{2},n-1;a^2r^2\right)\right.$$
$$\left. - \frac{2\sqrt{\pi}}{\Gamma\left(n-\frac{1}{2}\right)} r \cdot {}_1F_2\left(\frac{n}{2};\frac{3}{2},n-\frac{1}{2};a^2r^2\right)\right).$$

Then

$$\int_0^\infty \lambda^{2-n} J^2_{\frac{n-2}{2}}(\lambda r) f_a(\lambda)\,d\lambda = \frac{2ar^{n-2}}{\pi} L(a,r).$$

Similarly

$$f_b(\lambda) = \frac{2b\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{n}{2}\right)} \cdot \frac{\lambda^{n-1}}{(\lambda^2+b^2)^{(n+1)/2}}$$

and

$$\int_0^\infty \lambda^{2-n} J^2_{\frac{n-2}{2}}(\lambda r) f_b(\lambda)\,d\lambda = \frac{2br^{n-2}}{\pi} L(b,r),$$

where

$$L(b,r) = \left(\frac{B\left(\frac{n-1}{2},\frac{1}{2}\right)}{b\Gamma(n-1)} \cdot {}_1F_2\left(\frac{n-1}{2};\frac{1}{2},n-1;b^2r^2\right)\right.$$
$$\left. - \frac{2\sqrt{\pi}}{\Gamma\left(n-\frac{1}{2}\right)} r \cdot {}_1F_2\left(\frac{n}{2};\frac{3}{2},n-\frac{1}{2};b^2r^2\right)\right).$$

Thus the spectral density for this model of the correlation function is given by

$$f(\lambda) = \frac{2ab}{a-b}\frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{n}{2}\right)}\left(\frac{\lambda^{n-1}}{(\lambda^2+b^2)^{(n+1)/2}} - \frac{\lambda^{n-1}}{(\lambda^2+a^2)^{(n+1)/2}}\right),$$

and the integral on the right-hand side of (6) can be evaluated explicitly as

$$\int_0^\infty \lambda^{2-n} J^2_{\frac{n-2}{2}}(\lambda r) f(\lambda)\, d\lambda = \frac{2abr^{n-2}(L(b,r) - L(a,r))}{\pi(a-b)}.$$

Then the variance of the linear unbiased estimator $\widehat{a}$ can be rewritten in a simpler form as follows:

$$D\widehat{a} = \frac{ab2^{n-1}\Gamma^2(n/2)}{\pi(a-b)}(L(b,r) - L(a,r)).$$

Model 9. Let $\varphi(r) = \sin(r)/r$, $n = 3$, be the correlation function of the so-called random field of Markov type. The problem of estimation of the unknown mean value for a field of Markov type in $\mathbf{R}^n$ is considered in [1]. For example, the variance of the linear unbiased estimator (6) is given by

$$\operatorname{Var} \widehat{a} = 4r^{-2} \sin^2 r.$$

Model 10. Let

$$\varphi(r) = \frac{2^{1/3}}{\Gamma(2/3)} r^{2/3} K_{2/3}(r).$$

This model of the correlation function corresponds to the Whittle–Mattern model $\varphi_{2/3}(r)$. According to equality (10) with $\nu = 1/3$ we have

$$f_{2/3}(\lambda) = \frac{2\lambda^{n-1}}{B(2/3, n/2)(\lambda^2+1)^{n/2+2/3}}, \qquad n > 2/3.$$

Put

$$G_1(r) = \frac{B((n-1)/2, 2/3)}{\Gamma(n-1)} \cdot {}_1F_2\big((n-1)/2; 1/3, n-1; r^2\big),$$

$$G_2(r) = \frac{\Gamma(-2/3)}{\Gamma(n-1/3)} \cdot {}_1F_2\big(n/2 + 1/6; n - 1/3, 5/3; r^2\big).$$

Using (12) we obtain the variance of the linear unbiased estimator $\widehat{a}$ in an explicit form, namely,

$$\operatorname{Var} \widehat{a} = \frac{2^{n-2}\Gamma^2(n/2) B\left(\frac{1}{2}, \frac{2}{3} + \frac{n-1}{2}\right)}{\pi B\left(\frac{2}{3}, \frac{n}{2}\right)} \big(G_1(r) + r^{4/3} G_2(r)\big).$$

5. Concluding remarks

The problem of estimation of the unknown mean value meets many applications in practice. Isotropic random fields play an important role when modelling spatial phenomena for the natural sciences and engineering. The results obtained in this paper can be used to construct estimators of the unknown mean value in geodesy, geology, geomorphology, hydrology, meteorology, agronomy, and geophysics.

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Department of Probability Theory and Mathematical Statistics, Faculty for Mathematics and Mechanics, National Taras Shevchenko University, Academician Glushkov Avenue 6, Kyiv 03127, Ukraine

Department of Probability Theory and Mathematical Statistics, Faculty for Mathematics and Mechanics, National Taras Shevchenko University, Academician Glushkov Avenue 6, Kyiv 03127, Ukraine
E-mail address: semenovsky@hotmail.ru

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