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THE PROJECTIVITY OF THE MODULI SPACE OF STABLE CURVES I: PRELIMINARIES ON "det" AND "Div"

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Introduction.

This paper is the first in a sequence of three. In the last paper Mumford will prove that the coarse moduli space of "stable" curves is a projective variety. The proof is a direct application of the very powerful Grothendieck relative Riemann-Roch Theorem.

The notion of a stable curve was introduced by Deligne and Mumford [1]. A stable curve is a reduced, connected curve with at most ordinary double points such that every non-singular rational component meets the other components in at least 3 points.

In this first paper we deal with some essential preliminary constructions which may also have other applications.

In the first paragraph we give the details of a construction whose existence was asserted by Grothendieck and described in the unpublished expose of Ferrand in SGA "Theorie des Intersections —". The construction is to assign to every perfect complex $\mathscr{F}^{\cdot}$ an invertible sheaf det $\mathscr{F}^{\cdot}$ in such a way that det becomes a functor from the category of perfect complexes and isomorphisms (in the derived categorical sense) to the category of invertible sheaves and isomorphisms. Roughly det $\mathscr{F}^{\cdot}$ is the alternating tensor product of the top exterior products of a locally free resolution of $\mathscr{F}^{\cdot}$. However in making this precise a certain very nasty problem of sign arises. The authors' first solution to these sign problems was described by Grothendieck in a letter as very alambicated* and he suggested to use the "Koszul rule of signs" which we follow in this paper.

The second paragraph deals with a generalization of Chow's construction assigning a "chow form" to every subvariety of $\mathbf{P}^n$. We functorialize this and analyse the invertible sheaves involved, following some ideas in an unpublished letter of Grothendieck to Mumford (1962) and in

*) This apparently means similar to an alchemical apparatus.

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[3, p. 109]. Finally we must mention that we have several overlaps with J. Fogarty "Truncated Hilbert functors" [4]. He analyses the relation between Div and Chow in the case $\mathscr{F}^{\cdot}$ is an 0-dimensional perfect complex, i.e. a coherent sheaf of finite Tor. dimension. In his notation Div and Chow correspond to ∇ and ω respectively.

Chapter I: det.

Let X be a scheme. We denote by $\mathscr{P}_X$ the category of graded invertible $\mathcal{O}_X$-modules. An object of $\mathscr{P}_X$ is a pair (L,α) where L is an invertible $\mathcal{O}_X$-module and α is a continouus function:

$$\alpha : X \to \mathbf{Z} .$$

A homomorphism $h:(L,\alpha) \to (M,\beta)$ is a homomorphism of $\mathcal{O}_X$-modules such that for each $x \in X$ we have:

$$\alpha(x) \neq \beta(x) \Rightarrow h_x = 0 .$$

We denote by $\mathscr{P}is_X$ the subcategory of $\mathscr{P}_X$ whose morphisms are isomorphisms only.

The tensor product of two objects in $\mathscr{P}_X$ is given by:

$$(L,\alpha) \otimes (M,\beta) = (L \otimes M, \alpha+\beta) .$$

For each pair of objects $(L,\alpha),(M,\beta)$ in $\mathscr{P}_X$ we have an isomorphism:

$$\psi_{(L,\alpha),(M,\beta)}\colon (L,\alpha) \otimes (M,\beta) \xrightarrow{\sim} (M,\beta) \otimes (L,\alpha)$$

defined as follows: If $l \in L_x$ and $m \in M_x$ then

$$\psi(l \otimes m) = (-1)^{\alpha(x)+\beta(x)} \cdot m \otimes l .$$

Clearly:

$$\psi_{(M,\beta),(L,\alpha)} \cdot \psi_{(L,\alpha),(M,\beta)} = 1_{(L,\alpha)\otimes(M,\beta)} .$$

We denote by 1 the object $(\mathcal{O}_X, 0)$. A right inverse of an object (L,α) in $\mathscr{P}_X$ will be an object (L',α') together with an isomorphism

$$\delta : (L,\alpha) \otimes (L',\alpha') \xrightarrow{\sim} 1 .$$

Of course $\alpha' = -\alpha$.

A right inverse will be considered as a left inverse via:

$$(L',\alpha') \otimes (L,\alpha) \xrightarrow[\sim]{\psi} (L,\alpha) \otimes (L',\alpha') \xrightarrow[\sim]{\delta} 1 .$$

We denote by $\mathscr{C}_X$ the category of finite locally free $\mathcal{O}_X$-modules, and by $\mathscr{C}is_X$ the subcategory whose morphisms are isomorphisms only.

If $F \in \mathrm{ob}(\mathscr{C}_X)$ we define:

$$\det{}^*(F) = (\Lambda^{\max} F, \operatorname{rank} F)$$

(where$(\Lambda^{\max} F)_x = \Lambda^{\operatorname{rank} F_x} F_x$) .

It is well known that $\det^*$ is a functor from $\mathscr{C}is_X$ to $\mathscr{P}is_X$.

For every short-exact sequence of objects in $\mathscr{C}_X$

$$0 \to F' \xrightarrow{\alpha} F \xrightarrow{\beta} F'' \to 0$$

we have an isomorphism:

$$\mathrm{i}^*(\alpha,\beta)\colon \det^*F' \otimes \det^*F'' \xrightarrow{\sim} \det^*F$$

such that locally,

$$\mathrm{i}^*(\alpha,\beta)\big((e_1 \wedge \ldots \wedge e_t) \otimes (\beta f_1 \wedge \ldots \wedge \beta f_s)\big) = \alpha e_1 \wedge \ldots \wedge \alpha e_t \wedge f_1 \wedge \ldots \wedge f_s$$

for $e_i \in \Gamma(U,F')$ and $f_j \in \Gamma(U,F)$.

The following proposition is well known:

Proposition 1. i) i^* *is functorial*, i.e., *given a diagram*:

$$\begin{tikzcd}
0 \arrow[r] & F' \arrow[r, "\alpha"] \arrow[d, "\lambda'", "\wr"'] & F \arrow[r, "\beta"] \arrow[d, "\lambda", "\wr"'] & F'' \arrow[r] \arrow[d, "\lambda''", "\wr"'] & 0 \\
0 \arrow[r] & G' \arrow[r, "\gamma"] & G \arrow[r, "\delta"] & G'' \arrow[r] & 0
\end{tikzcd}$$

where the rows are short-exact sequences of objects in $\mathscr{C}_X$, *and the columns are isomorphisms, the diagram*:

$$\begin{tikzcd}
\det^*F' \otimes \det^*F'' \arrow[r, "\mathrm{i}^*(\alpha,\beta)", "\sim"'] \arrow[d, "\det^*\lambda' \otimes \det^*\lambda''", "\wr"'] & \det^*F \arrow[d, "\det^*\lambda", "\wr"'] \\
\det^*G' \otimes \det^*G'' \arrow[r, "\sim", "\mathrm{i}^*(\gamma,\delta)"'] & \det^*G
\end{tikzcd}$$

commutes.

ii) *Given a commutative diagram of objects in* $\mathscr{C}_X$

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & F' \arrow[r, "\alpha'"] \arrow[d, "\gamma'"] & G' \arrow[r, "\beta'"] \arrow[d, "\gamma"] & H' \arrow[r] \arrow[d, "\gamma''"] & 0 \\
0 \arrow[r] & F \arrow[r, "\alpha"] \arrow[d, "\delta'"] & G \arrow[r, "\beta"] \arrow[d, "\delta"] & H \arrow[r] \arrow[d, "\delta''"] & 0 \\
0 \arrow[r] & F'' \arrow[r, "\alpha''"] \arrow[d] & G'' \arrow[r, "\beta''"] \arrow[d] & H'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

where each row and each column is a short-exact sequence, the diagram

$$\begin{tikzcd}
\det^* F' \otimes \det^* F'' \otimes \det^* H' \otimes \det^* H'' \arrow[r, "\mathrm{i}^*(\gamma', \delta') \otimes \mathrm{i}^*(\gamma'', \delta'')", "\sim"'] \arrow[d, "\mathrm{i}^*(\alpha', \beta') \otimes \mathrm{i}^*(\alpha'', \beta'') \cdot (1 \otimes \psi_{\det^* F'', \det^* H'} \otimes 1)", "\wr"'] & \det^* F \otimes \det^* H \arrow[d, "\mathrm{i}^*(\alpha, \beta)", "\wr"'] \\
\det^* G' \otimes \det^* G'' \arrow[r, "\sim", "\mathrm{i}^*(\gamma, \delta)"'] & \det^* G
\end{tikzcd}$$

commutes.

iii) $\det^*$ *and* i^* *commute with base change.*

The isomorphism i^* is a special case of a more general canonical isomorphism: suppose E is a locally free $\mathcal{O}_X$-module and:

$$(0) = F^0E \subset F^1E \subset \ldots \subset F^rE = E$$

is a filtration such that $F^iE/F^{i-1}E$ are all locally free. Then there is a canonical isomorphism:

$$\mathrm{i}^*(\{FE\}) \colon \otimes_{i=1}^r \det^*(F^iE/F^{i-1}E) \xrightarrow{\approx} \det^*(E)\,.$$

Moreover these isomorphisms satisfy the following basic compatibility generalizing (ii) above: suppose $\{F^0E\}$ and $\{G^0E\}$ are 2 filtrations on E such that for all i,j

$$G^{i,j} = F^iE \cap G^jE/(F^{i-1}E \cap G^jE) + (F^iE \cap G^{j-1}E)$$

is locally free. For each fixed i, the $G^{i,j}$ are the graded objects associated to a filtration on $F^iE/F^{i-1}E$, and for each fixed j, they are the graded objects associated to a filtration on $G^jE/G^{j-1}E$. Thus the i's give us a diagram:

$$\begin{tikzcd}
\otimes_{i,j} \det^*(G^{i,j}) \arrow[r, "\sim"] \arrow[d, "\wr"'] & \otimes_i \det^*(F^iE/F^{i-1}E) \arrow[d, "\wr"'] \\
\otimes_j \det^*(G^jE/G^{j-1}E) \arrow[r, "\sim"] & \det^* E
\end{tikzcd}$$

This then commutes. We will not enter into the details here however, because the general isomorphism i can be defined inductively as a composition of the special isomorphisms i associated to short filtrations:

$$(0) = F^0E \subset F^1E \subset F^2E = E\,,$$

which is then just the i associated to the exact sequence:

$$0 \to F^1E \to E \to E/F^1E \to 0\,.$$

Moreover, the general compatibility property is just a formal consequence of the special one – (ii) above.

Next we consider the category $\mathscr{C}^{\cdot}{}_X$ of bounded complexes of objects in $\mathscr{C}_X$, morphisms being all maps of complexes. A map of complexes which induces an isomorphism in cohomology will be called a *quasi-isomorphism*. The subcategory of $\mathscr{C}^{\cdot}{}_X$ whose maps are quasi-isomorphisms will be called $\mathscr{C}^{\cdot}is_X$.

Definition 1. A determinant functor from $\mathscr{C}^{\cdot}is$ to $\mathscr{P}is$ consists of the following data:

I) For each scheme X a functor f_X from $\mathscr{C}^{\cdot}is_X$ to $\mathscr{P}is_X$.

II) For each scheme X and for each short-exact sequence:

$$0 \to F^{\cdot\prime} \xrightarrow{\alpha} F^{\cdot} \xrightarrow{\beta} F^{\cdot\prime\prime} \to 0$$

in $\mathscr{C}^{\cdot}{}_X$ an isomorphism:

$$\mathrm{i}_X(\alpha,\beta): \mathrm{f}(F^{\cdot\prime}) \otimes \mathrm{f}(F^{\cdot\prime\prime}) \xrightarrow{\sim} \mathrm{f}(F^{\cdot}) .$$

This data is to satisfy the following requirements:

i) Given a commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & F^{\cdot\prime} \arrow[r, "\alpha"] \arrow[d, "\lambda'"] & F^{\cdot} \arrow[r, "\beta"] \arrow[d, "\lambda"] & F^{\cdot\prime\prime} \arrow[r] \arrow[d, "\lambda''"] & 0 \\
0 \arrow[r] & G^{\cdot\prime} \arrow[r, "\gamma"] & G^{\cdot} \arrow[r, "\delta"] & G^{\cdot\prime\prime} \arrow[r] & 0
\end{tikzcd}$$

where the rows are short-exact sequences of objects in $\mathscr{C}^{\cdot}{}_X$ and λ',λ and λ'' are quasi-isomorphisms, the diagram:

$$\begin{tikzcd}
\mathrm{f}(F^{\cdot\prime}) \otimes \mathrm{f}(F^{\cdot\prime\prime}) \arrow[r, "\mathrm{i}_X(\alpha,\beta)", "\sim"'] \arrow[d, "\mathrm{f}(\lambda')\otimes \mathrm{f}(\lambda'')", "\wr"'] & \mathrm{f}(F^{\cdot}) \arrow[d, "\mathrm{f}(\lambda)", "\wr"'] \\
\mathrm{f}(G^{\cdot\prime}) \otimes \mathrm{f}(G^{\cdot\prime\prime}) \arrow[r, "\mathrm{i}_X(\gamma,\delta)", "\sim"'] & \mathrm{f}(G^{\cdot})
\end{tikzcd}$$

commutes.

ii) Given a commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & F^{\cdot\prime} \arrow[r, "\alpha'"] \arrow[d, "\gamma'"] & G^{\cdot\prime} \arrow[r, "\beta'"] \arrow[d, "\gamma"] & H^{\cdot\prime} \arrow[r] \arrow[d, "\gamma''"] & 0 \\
0 \arrow[r] & F^{\cdot} \arrow[r, "\alpha"] \arrow[d, "\delta'"] & G^{\cdot} \arrow[r, "\beta"] \arrow[d, "\delta"] & H^{\cdot} \arrow[r] \arrow[d, "\delta''"] & 0 \\
0 \arrow[r] & F^{\cdot\prime\prime} \arrow[r, "\alpha''"] \arrow[d] & G^{\cdot\prime\prime} \arrow[r, "\beta''"] \arrow[d] & H^{\cdot\prime\prime} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

where each row and each column is a short-exact sequence, the diagram:

$$\begin{tikzcd}
\mathrm{f}(F^{\cdot\prime}) \otimes \mathrm{f}(F^{\cdot\prime\prime}) \otimes \mathrm{f}(H^{\cdot\prime}) \otimes \mathrm{f}(H^{\cdot\prime\prime}) \arrow[r, "\mathrm{i}_X(\gamma',\delta')\,\mathrm{i}_X(\gamma''\ \delta'')", "\sim"'] \arrow[d, "\mathrm{i}_X(\alpha',\beta')\otimes\mathrm{i}_X(\alpha'',\beta'')\otimes 1\otimes\psi_{\mathrm{f}(F^{\cdot\prime\prime}),\mathrm{f}(H^{\cdot\prime})}\otimes 1", "\wr"'] & \mathrm{f}(F^{\cdot}) \otimes \mathrm{f}(H^{\cdot}) \arrow[d, "\mathrm{i}_X(\alpha,\beta)", "\wr"'] \\
\mathrm{f}(G^{\cdot\prime}) \otimes \mathrm{f}(G^{\cdot\prime\prime}) \arrow[r, "\sim", "\mathrm{i}_X(\gamma,\delta)"'] & \mathrm{f}(G^{\cdot})
\end{tikzcd}$$

commutes.

iii) f and i both commute with base change.

iv) f and i are normalized as follows:

a) $\mathrm{f}(0^{\cdot}) = 1$

b) For the exact sequence:

$$0 \to F^{\cdot} \xrightarrow{1_{F^{\cdot}}} F^{\cdot} \xrightarrow{0} 0^{\cdot} \to 0$$

the map

$$\mathrm{f}(F^{\cdot})\otimes 1 \xrightarrow[\mathrm{i}_X(1,\,0)]{\sim} \mathrm{f}(F^{\cdot})$$

is the canonical one,

b′) For the exact sequence:

$$0 \to 0^{\cdot} \xrightarrow{0} F^{\cdot} \xrightarrow{1_{F^{\cdot}}} F^{\cdot} \to 0$$

the map

$$\mathrm{f}(F^{\cdot}) \otimes 1 \xrightarrow[\mathrm{i}_X(1,\,0)]{\sim} \mathrm{f}(F^{\cdot})$$

is the canonical one.

v) We consider $\mathscr{C}is$ as a full subcategory of $\mathscr{C}^{\cdot}is$ by viewing objects of $\mathscr{C}is$ as complexes with only one nonvanishing term, this term being placed in degree zero. Then for such objects:

$$\mathrm{f}(F) = \det{}^*F$$
$$\mathrm{i}_X(\alpha,\beta) = \mathrm{i}^*(\alpha,\beta)\ .$$

The main theorem of this chapter is

THEOREM 1. *There is one and, up to canonical isomorphism, only one determinant functor* (f,i), *which we will write* (det,i).

Let X be a scheme, $H^\cdot$ an acyclic object in $\mathscr{C}^\cdot_X$. If (f,i) is a determinant functor, we have an isomorphism:

$$\mathrm{f}(0)\colon \mathrm{f}(H^\cdot) \to 1 .$$

If

$$0 \to H^{\cdot\prime} \xrightarrow{\alpha} H^\cdot \xrightarrow{\beta} H^{\cdot\prime\prime} \to 0$$

is an exact sequence of acyclic objects it follows from Definition 1, i) and iv a) that the diagram

$$\begin{tikzcd} \mathrm{f}(H^{\cdot\prime}) \otimes \mathrm{f}(H^{\cdot\prime\prime}) \arrow[r, "\mathrm{i}_X(\alpha,\beta)", "\sim"'] \arrow[d, "\wr"'] & \mathrm{f}(H^\cdot) \arrow[d, "\wr"'] \\ 1 \otimes 1 \arrow[r, "\mathrm{mult.}", "\sim"'] & 1 \end{tikzcd}$$

commutes.

Let $\alpha\colon F^\cdot \to G^\cdot$ be an injective quasi-isomorphism such that the cokernel is again an object of $\mathscr{C}^\cdot_X$, i.e., we have a short-exact sequence:

$$0 \to F^\cdot \xrightarrow{\alpha} G^\cdot \xrightarrow{\beta} H^\cdot \to 0$$

such that $H^\cdot$ is acyclic.

From the diagram:

$$\begin{tikzcd} 0 \arrow[r] & F^\cdot \arrow[r] \arrow[d, "1_{F^\cdot}"] & F^\cdot \arrow[r] \arrow[d, "\alpha"] & 0^\cdot \arrow[r] \arrow[d, "0"] & 0 \\ 0 \arrow[r] & F^\cdot \arrow[r, "\alpha"] & G^\cdot \arrow[r, "\beta"] & H^\cdot \arrow[r] & 0 \end{tikzcd}$$

we get a commutative diagram:

$$\begin{tikzcd} \mathrm{f}(F^\cdot) \otimes 1 \arrow[r, "\mathrm{mult.}", "\sim"'] \arrow[d, "\wr"'] & \mathrm{f}(F^\cdot) \arrow[d, "\wr"'] \\ \mathrm{f}(F^\cdot) \otimes \mathrm{f}(H^\cdot) \arrow[r, "\mathrm{i}_X(\alpha,\beta)", "\sim"'] & \mathrm{f}(G^\cdot) \end{tikzcd}$$

hence we see that $\mathrm{f}(\alpha)$ is determined by the maps $\mathrm{i}_X(\alpha,\beta)$ and $\mathrm{f}(0)\colon \mathrm{f}(H^\cdot) \to 1$.

Let $\lambda\colon F^\cdot \to G^\cdot$ be an arbitrary quasi-isomorphism. We denote by $Z_\lambda^\cdot$ the following complex:

$$Z_\lambda^{\ i} = F^i \oplus G^i \oplus F^{i+1}$$

$$d^i{}_{Z_\lambda} = \begin{pmatrix} d^i & 0 & -1 \\ 0 & d^i & \lambda^{i+1} \\ 0 & 0 & -d^{i+1} \end{pmatrix}$$

Consider the diagram:

$$F^{\cdot} \xrightarrow{\alpha} Z_\lambda^{\cdot} \underset{\beta}{\overset{\beta'}{\rightleftarrows}} G^{\cdot}$$

where

$$\alpha = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \beta' = (\lambda, 1, 0)\,.$$

We leave to the reader to check that these are all quasi-isomorphisms and furthermore,

$$\beta' \circ \alpha = \lambda, \quad \beta' \circ \beta = 1_G\,.$$

Hence we have:

$$\begin{aligned} \mathrm{f}(\lambda) &= \mathrm{f}(\beta') \circ \mathrm{f}(\alpha) = \mathrm{f}(\beta') \circ \mathrm{f}(\beta) \circ \mathrm{f}(\beta)^{-1} \circ \mathrm{f}(\alpha) \\ &= \mathrm{f}(\beta' \circ \beta) \circ \mathrm{f}(\beta)^{-1} \circ \mathrm{f}(\alpha) = \mathrm{f}(\beta)^{-1} \circ \mathrm{f}(\alpha)\,. \end{aligned}$$

Hence, since both α and β are injective quasi-isomorphisms, the map $\mathrm{f}(\lambda)$ is determined by the maps i and $\mathrm{f}(0)$ from $\mathrm{f}(H^{\cdot}) \to 1$ for acyclic $H^{\cdot}$. We summarize this in the following:

LEMMA 1. *Let* (f,i) *and* (g,j) *be two determinant functors from* $\mathscr{C}^{\cdot}$*is to* $\mathscr{P}$*is. Suppose we are given* θ *as follows:*

i) *For each scheme* X *and each object* $F^{\cdot}$ *in* $\mathscr{C}^{\cdot}_X$ *we have an isomorphism:*

$$\theta_{X,F^{\cdot}} : \mathrm{f}(F^{\cdot}) \xrightarrow{\sim} \mathrm{g}(F^{\cdot})\,.$$

ii) *For all acyclic* $H^{\cdot}$ *the diagram:*

$$\begin{array}{ccc} \mathrm{f}(H^{\cdot}) & \xrightarrow{\theta_{X,H^{\cdot}}} & \mathrm{g}(H^{\cdot}) \\ \wr\Big\downarrow \mathrm{f}(0) & & \wr\Big\downarrow \mathrm{g}(0) \\ 1 & = & 1 \end{array}$$

commutes.

iii) *For all short-exact sequences:*

$$0 \to F^{\cdot} \xrightarrow{\alpha} G^{\cdot} \xrightarrow{\beta} H^{\cdot} \to 0$$

with $H^{\cdot}$ *acyclic, the diagram:*

$$\begin{CD} \mathrm{f}(F^\cdot) \otimes \mathrm{f}(H^\cdot) @>{\mathrm{i}_X(\alpha,\beta)}>{\sim}> \mathrm{f}(G^\cdot) \\ @V{\wr}V{\theta_{X,F^\cdot}\otimes\theta_{X,H^\cdot}}V @V{\wr}V{\theta_{X,G^\cdot}}V \\ \mathrm{g}(F^\cdot) \otimes \mathrm{g}(H^\cdot) @>{\mathrm{j}_X(\alpha,\beta)}>{\sim}> \mathrm{g}(G^\cdot) \end{CD}$$

commutes.

iv) *θ commutes with base change.*

Then for all quasi-isomorphisms λ: $F^\cdot \to G^\cdot$ the diagram:

$$\begin{CD} \mathrm{f}(F^\cdot) @>{\mathrm{f}(\lambda)}>{\sim}> \mathrm{f}(G^\cdot) \\ @V{\wr}V{\theta_{X,F^\cdot}}V @V{\wr}V{\theta_{X,G^\cdot}}V \\ \mathrm{g}(F^\cdot) @>{\mathrm{g}(\lambda)}>{\sim}> \mathrm{g}(G^\cdot) \end{CD}$$

commutes.

As a side remark, notice that these methods prove:

PROPOSITION 2. *Let* (f, i) *be a determinant functor from $\mathscr{C}^\cdot$is to $\mathscr{P}$is, and let*

$$\lambda, \mu : F^\cdot \rightrightarrows G^\cdot$$

be two quasi-isomorphism such that locally on X, λ is homotopic to μ, then

$$f(\lambda) = \mathrm{f}(\mu)\,.$$

PROOF. Two maps being equal is a local property, and since f commutes with base change we may assume that X is affine. However in the affine case locally homotopic maps are homotopic so let H be such a homotopy, i.e.,

$$\lambda - \mu = dH + Hd\,.$$

We leave to the reader to check that we have an isomorphism of complexes:

$$Z_\lambda^\cdot \xrightarrow{\sim} Z_\mu^\cdot$$

given by the matrix:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & H \\ 0 & 0 & 1 \end{pmatrix}$$

such that the diagram

$$\begin{tikzcd} & Z_\lambda^\cdot \arrow[dd, "\wr"'] & \\ F^\cdot \arrow[ur, hook, "\alpha"] \arrow[dr, hook, "\gamma"'] & & G^\cdot \arrow[ul, hook', "\beta"'] \arrow[dl, hook', "\delta"] \\ & Z_\mu^\cdot & \end{tikzcd}$$

commutes. But we have already seen that

$$\mathrm{f}(\lambda) = \mathrm{f}(\beta)^{-1} \circ \mathrm{f}(\alpha) \quad \text{and} \quad \mathrm{f}(\mu) = \mathrm{f}(\delta)^{-1} \circ \mathrm{f}(\gamma)$$

hence the proposition.

Lemma 2. *Suppose we are given a pair* (f, i) *satisfying all the axioms of definition* 1 *except*:

I) *is replaced by*:

I′) *For each scheme X we have a map*

$$\mathrm{f}_X : \mathrm{ob}(\mathscr{C}_X^\cdot) \to \mathrm{ob}(\mathscr{P}_X)$$

such that $\mathrm{f}_X(0^\cdot) = 1$ *and for each acyclic complex $H^\cdot$ on X an isomorphism*:

$$\mathrm{f}_X(0) : \mathrm{f}_X(H^\cdot) \xrightarrow{\sim} 1\,.$$

i) *is replaced by*

i′) *For each scheme X and for each short-exact sequence of acyclic objects*:

$$0 \to H'^\cdot \xrightarrow{\alpha} H^\cdot \xrightarrow{\beta} H''^\cdot \to 0$$

the diagram

$$\begin{tikzcd} \mathrm{f}(H'^\cdot) \otimes \mathrm{f}(H''^\cdot) \arrow[r, "\mathrm{i}_X(\alpha,\beta)", "\sim"'] \arrow[d, "\wr"', "\mathrm{f}(0)\otimes\mathrm{f}(0)"] & \mathrm{f}(H^\cdot) \arrow[d, "\wr"', "\mathrm{f}(0)"] \\ 1 \otimes 1 \arrow[r, "\text{mult.}", "\sim"'] & 1 \end{tikzcd}$$

commutes.

(The rest is left unaltered.)

Then there exists up to caninical isomorphism a unique determinant functor $(\tilde{\mathrm{f}}, \mathrm{i})$ *such that for all $F^\cdot$ we have*

$$\tilde{\mathrm{f}}(F^\cdot) = \mathrm{f}(F^\cdot)$$

and for each quasi-isomorphism

$$H^\cdot \xrightarrow{0} 0\,.$$

we have:

$$\tilde{\mathrm{f}}(0) = \mathrm{f}(0)\,.$$

Proof. Uniqueness follows immediately from Lemma 1. Suppose we have defined $\tilde{\mathrm{f}}$ for all affine schemes, then since $\tilde{\mathrm{f}}$ commutes with base change, the maps patch together to give $\tilde{\mathrm{f}}$ on all schemes, hence we may assume that X is affine.

Let $F^{\cdot} \xrightarrow{\alpha} G^{\cdot}$ be an injective quasi-isomorphism. We will say that α is good if the cokernel of α is again in $\mathscr{C}^{\cdot}_X$. Let $H^{\cdot} =$ cokernel of α. Then we get a short-exact sequence of complexes

$$0 \to F^{\cdot} \xrightarrow{\alpha} G^{\cdot} \xrightarrow{\beta} H^{\cdot} \to 0$$

such that $H^{\cdot}$ is acyclic. We define $\mathrm{f}'(\alpha)$ via:

$$\begin{tikzcd} \mathrm{f}(F^{\cdot}) & \mathrm{f}(F^{\cdot}) \otimes 1 \arrow[l, "\text{mult.}"', "\sim"] & \mathrm{f}(F^{\cdot}) \otimes \mathrm{f}(H^{\cdot}) \arrow[l, "1\otimes \mathrm{f}(0)"', "\sim"] & \mathrm{f}(G^{\cdot}) \arrow[l, "i_X(\alpha,\beta)"', "\sim"] \arrow[from=1-1, to=1-4, bend right, "\mathrm{f}'(\alpha)", "\sim"'] \end{tikzcd}$$

Let $\alpha: E^{\cdot} \to F^{\cdot}$ and $\beta: F^{\cdot} \to G^{\cdot}$. be two good injective quasi-isomorphisms. We have a commutative diagram:

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & E \arrow[r, "\alpha"] \arrow[d, equal] & F \arrow[r, "\gamma"] \arrow[d, "\beta"] & H \arrow[r] \arrow[d, "\xi"] & 0 \\ 0 \arrow[r] & E \arrow[r, "\beta\circ\alpha"] \arrow[d] & G \arrow[r, "\delta"] \arrow[d, "\varepsilon"] & K \arrow[r] \arrow[d, "\zeta"] & 0 \\ 0 \arrow[r] & 0 \arrow[r] \arrow[d] & L \arrow[r, equal] \arrow[d] & L \arrow[r] \arrow[d] & 0 \\ & 0 & 0 & 0 & \end{tikzcd}$$

so by axiom ii), iv) and i′) we have:

$$(**) \qquad \mathrm{f}'(\beta)\circ\mathrm{f}'(\alpha) = \mathrm{f}'(\beta\circ\alpha)\,.$$

If $\lambda: F^{\cdot} \to G^{\cdot}$ is an arbitrary quasi-isomorphism, we have a diagram:

$$F^{\cdot} \xrightarrow{\alpha} Z_{\lambda}^{\cdot} \xrightarrow{\beta} G^{\cdot}$$

where

$$\alpha = \begin{pmatrix}1\\0\\0\end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix}0\\1\\0\end{pmatrix}.$$

Clearly α and β are both good injective quasi-isomorphisms, and we define

$$\tilde{\mathrm{f}}(\lambda) = \mathrm{f}'(\beta)^{-1}\cdot\mathrm{f}'(\alpha)\,.$$

To see that $\tilde{\mathrm{f}}$ is functorial, let

$$\lambda : E^\cdot \to F^\cdot \quad \text{and} \quad \mu : F^\cdot \to G^\cdot$$

be quasi-isomorphisms: we define a complex $W^\cdot$ as follows:

$$W^i = E^i \oplus F^i \oplus G^i \oplus E^{i+1} \oplus F^{i+1}$$

$$d_W = \begin{pmatrix} d & 0 & 0 & -1 & 0 \\ 0 & d & 0 & \lambda & -1 \\ 0 & 0 & d & 0 & \mu \\ 0 & 0 & 0 & -d & 0 \\ 0 & 0 & 0 & 0 & -d \end{pmatrix}$$

We then have a commutative diagram:

$$\begin{tikzcd}
 & Z_\lambda^\cdot \arrow[dr, "p"] & & F^\cdot \arrow[ll] \arrow[dr] & \\
E^\cdot \arrow[ur] \arrow[dr] & & W^\cdot & & Z_\mu^\cdot \arrow[ll, "q"'] \\
 & Z^\cdot_{\mu\circ\lambda} \arrow[ur, "r"] & & G^\cdot \arrow[ll] \arrow[ur] &
\end{tikzcd}$$

where

$$p = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad q = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad r = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & \lambda \end{pmatrix}.$$

The fact that $\tilde{\mathrm{f}}(\mu\circ\lambda) = \tilde{\mathrm{f}}(\mu)\circ\tilde{\mathrm{f}}(\lambda)$ now follows from this diagram and the functoriality of f'. We leave to the reader to check axiom i): this is not hard. It is also easy to check that $\tilde{\mathrm{f}} = \mathrm{f}'$ where f' is defined, but this is not needed.

For each scheme X and each object L in $\mathscr{P}_X$ we fix a right inverse L^{-1} of L, i.e., an isomorphism

$$\delta_L : L \otimes L^{-1} \xrightarrow{\sim} 1 .$$

If $\alpha : L \xrightarrow{\sim} M$ is an isomorphism in $\mathscr{P}_X$ we denote by α^{-1} the unique isomorphism making the diagram:

$$\begin{tikzcd}
L \otimes L^{-1} \arrow[r, "\sim"] \arrow[d, "\wr"', "\alpha\otimes\alpha^{-1}"] & 1 \arrow[d, equal] \\
M \otimes M^{-1} \arrow[r, "\sim"] & 1
\end{tikzcd}$$

commutative.

For every pair of objects L, M we denote by $\theta_{L,M}$ the unique isomorphism making the diagram

$$\begin{array}{ccc}
(M\otimes L)\otimes(M\otimes L)^{-1} & \xrightarrow[\sim]{\delta} & 1 \\
\wr\downarrow 1\otimes\theta_{M,L} & & \wr\downarrow (\delta\cdot\delta)^{-1} \\
(M\otimes L)\otimes(M^{-1}\otimes L^{-1}) & \xrightarrow[\sim]{1\otimes\psi_{L,M}\otimes 1} & M\otimes M^{-1}\otimes L\otimes L^{-1}
\end{array}$$

Then $^{-1}$ is a functor from $\mathscr{P}is$ to $\mathscr{P}is$ which commutes with base change, and for each pair L, M the diagram:

$$\begin{array}{ccc}
(M\otimes L)^{-1} & \xrightarrow[\sim]{\theta_{M,L}} & M^{-1}\otimes L^{-1} \\
\wr\downarrow (\psi_{M,L})^{-1} & & \wr\downarrow \psi_{M^{-1},L^{-1}} \\
(L\otimes M)^{-1} & \xrightarrow[\sim]{\theta_{L,M}} & L^{-1}\otimes M^{-1}
\end{array}$$

commutes.

If F^i is an indexed object of $\mathscr{C}_X$ we define:

$$\det(F^i) = \begin{Bmatrix} \det{}^*(F^i) & \text{for } i \text{ even} \\ \det{}^*(F^i)^{-1} & \text{for } i \text{ odd} \end{Bmatrix}$$

If

$$0 \to F^{i\prime} \xrightarrow{\alpha^i} F^i \xrightarrow{\beta^i} F^{i\prime\prime} \to 0$$

is an indexed short-exact sequence of objects in $\mathscr{C}_X$, we define

$$\mathrm{i}(\alpha^i,\beta^i) = \begin{Bmatrix} \mathrm{i}^*(\alpha^i,\beta^i) & \text{for } i \text{ even} \\ \mathrm{i}^*(\alpha^i,\beta^i)^{-1} & \text{for } i \text{ odd} \end{Bmatrix}$$

If $F^{\cdot}$ is an object of $\mathscr{C}^{\cdot}_X$ we define

$$\det(F^{\cdot}) = \ldots\otimes\det(F^{i+1})\otimes\det(F^i)\otimes\det(F^{i-1})\otimes\ldots$$

Finally if

$$0 \to F^{\cdot\prime} \xrightarrow{\alpha} F^{\cdot} \xrightarrow{\beta} F^{\cdot\prime\prime} \to 0$$

is a short exact sequence of objects in $\mathscr{C}_X{}^{\cdot}$ we define

$$\mathrm{i}(\alpha,\beta) : \det(F^{\cdot\prime})\otimes\det(F^{\cdot\prime\prime}) \xrightarrow{\sim} \det(F^{\cdot})$$

to be the composite:

$$\begin{aligned}
&\det(F^{\cdot\prime})\otimes\det(F^{\cdot\prime\prime}) = \ldots\otimes\det(F^{i\prime})\otimes\det(F^{i-1\prime})\otimes\ldots \\
&\otimes\det(F^{i\prime\prime})\otimes\det(F^{i-1\prime\prime})\otimes\ldots \xrightarrow{\sim} \ldots\otimes\det(F^{i\prime})\otimes\det(F^{i\prime\prime}) \\
&\otimes\det(F^{i-1\prime})\otimes\det(F^{i-1\prime\prime})\otimes\ldots \xrightarrow[\sim]{\otimes_i \mathrm{i}(\alpha^i,\beta^i)} \ldots\otimes\det(F^i) \\
&\otimes\det(F^{i-1})\otimes\ldots = \det(F^{\cdot}).
\end{aligned}$$

the most amazing thing is that we can construct for each acyclic object $H^\cdot$ in $\mathscr{C}^\cdot_X$ an isomorphism:

$$\det_X(0) : \det_X(H^\cdot) \xrightarrow{\sim} 1$$

such that all the axioms of lemma 2 holds.

These axioms are all trivially verified execpt for I′) and i′). We will verify these simultaneously and we use induction with respect to the length of the complexes.

STEP 1. Complexes of length 2.

Consider first an acyclic complex

$$H^\cdot = \ldots \to 0 \to H^i \xrightarrow{d} H^{i+1} \to 0 \to \ldots$$

with i an odd integer. Since d is an isomorphism we get an isomorphism:

$$\begin{aligned}\det(H^\cdot) = \det{}^*H^{i+1})\otimes\det{}^*(H^i)^{-1} &\xrightarrow[\sim]{1\otimes\det^*(d)^{-1}} \\ \det{}^*(H^{i+1})\otimes\det{}^*(H^{i+1})^{-1} &\xrightarrow{\sim} 1 .\end{aligned}$$

We define this isomorphism to be det(0).

Given a short exact sequence of acyclic length 2-complexes:

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F^i \arrow[r, "\alpha^i"] \arrow[d, "d"] & G^i \arrow[r, "\beta^i"] \arrow[d, "d"] & H^i \arrow[r] \arrow[d, "d"] & 0 \\
0 \arrow[r] & F^{i+1} \arrow[r] \arrow[d] & G^{i+1} \arrow[r, "\beta^i"] \arrow[d] & H^{i+1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

we get a diagram:

$$\begin{tikzcd}
\det(F^\cdot)\otimes\det(H^\cdot) \arrow[rr, "i(\alpha,\beta)", "\sim"'] \arrow[d, "\wr"'] \arrow[ddd, bend right=60, "\det(0)\otimes\det(0)"', "\wr"] & & \det(G^\cdot) \arrow[d, equal] \arrow[ddd, bend left=60, "\det(0)"] \\
\det{}^*(F^{i+1})\otimes\det{}^*(H^{i+1})\otimes\det{}^*(F^i)^{-1}\otimes\det{}^*(H^i)^{-1} \arrow[rr, "\sim"] \arrow[d, "1\otimes 1\otimes\det^*(d)^{-1}\otimes\det^*(d)^{-1}", "\wr"'] & \mathrm{I} & \det{}^*(G^{i+1})\otimes\det{}^*(G^i)^{-1} \arrow[d, "1\otimes\det^*(d)^{-1}", "\wr"'] \\
\det{}^*(F^{i+1})\otimes\det{}^*(H^{i+1})\otimes\det{}^*(F^{i+1})^{-1}\otimes\det{}^*(H^{i+1})^{-1} \arrow[rr, "\sim"] \arrow[d, "\delta\otimes\delta\circ(1\otimes\psi\otimes 1)", "\wr"'] & \mathrm{II} & \det{}^*(G^{i+1})\otimes\det{}^*(G^{i+1})^{-1} \arrow[d, "\delta", "\wr"'] \\
1\otimes 1 \arrow[rr, "\sim"] & \mathrm{III} & 1
\end{tikzcd}$$

IV V

The square I is commutative by the definition of i, and the squares IV and V are commutative by the definition of det(0). The square III is commutative by the definition of i^{-1} and finally II is commutative by axioms iii) of definition 1. Hence the whole diagram commutes. If

$$H^\cdot = \ldots \to 0 \to H^i \xrightarrow{d} H^{i+1} \to 0 \to \ldots$$

is an acyclic complex with i even we define det(0) to be the composite

$$\det(H^\cdot) = \det{}^*(H^{i+1})^{-1}\otimes\det{}^*(H^i) \xrightarrow[\sim]{1\otimes\det^*(d)} \det{}^*(H^{i+1})^{-1}\otimes\det{}^*(H^{i+1})$$
$$\xrightarrow[\sim]{\psi} \det{}^*(H^{i+1})\otimes\det{}^*(H^{i+1})^{-1} \xrightarrow[\sim]{\delta} 1\ \ldots .$$

Given a short-exact sequence of acyclic length 2-complexes

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F^i \arrow[r, "\alpha^i"] \arrow[d, "d"] & G^i \arrow[r, "\beta^i"] \arrow[d, "d"] & H^i \arrow[r] \arrow[d, "d"] & 0 \\
0 \arrow[r] & F^{i+1} \arrow[r, "\alpha^{i+1}"] \arrow[d] & G^{i+1} \arrow[r, "\beta^{i+1}"] \arrow[d] & H^{i+1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

with i even we get just as before a commutative diagram.

$$\begin{array}{ccc} \det(F^\cdot)\otimes\det(H^\cdot) & \xrightarrow{\mathrm{i}(\alpha,\beta)} & \det(G^\cdot) \\ \Big\downarrow{\scriptstyle \det(0)\otimes\det(0)} & & \Big\downarrow{\scriptstyle \det(0)} \\ 1\otimes 1 & \xrightarrow{\sim} & 1 \end{array}$$

Hence I′) and i′) holds for all acyclic complexes of length 2.

STEP 2. Suppose I′) and i′) hold for all acyclic complexes of length $\leq n$, and let

$$H^\cdot = \ldots \to 0 \to H^i \to H^{i+1} \to H^{i+2} \to \ldots \to H^{i+n} \to 0 \to \ldots$$

be an acyclic complex of length $n+1$. We then get a short-exact sequence of complexes:

$$\begin{array}{ccccccccc} & & \vdots & & \vdots & & \vdots & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & 0 & \to & 0 & \longrightarrow & 0 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & H^i & = & H^i & \longrightarrow & 0 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & H^i & \to & H^{i+1} & \to & H^{i+1}/H^i & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & 0 & \to & H^{i+2} & \to & H^{i+2} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & \vdots & & \vdots & & \vdots & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & 0 & \to & H^{i+n} & \to & H^{i+n} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & 0 & \to & 0 & \longrightarrow & 0 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & \vdots & & \vdots & & \vdots & & \\ & & \| & & \| & & \| & & \\ 0 & \to & H^\cdot_{\mathrm{I}} & \xrightarrow{\alpha} & H^\cdot & \xrightarrow{\beta} & H^\cdot_{\mathrm{II}} & \longrightarrow & 0 \end{array}$$

Since $H^\cdot_{\mathrm{I}}$ and $H^\cdot_{\mathrm{II}}$ are of length $\leq n$ we define det(0) so as to make the diagram

$$\begin{array}{ccc} \det(H^\cdot_{\mathrm{I}})\otimes\det(H^\cdot_{\mathrm{II}}) & \xrightarrow[\sim]{\mathrm{i}(\alpha,\beta)} & \det(H^\cdot) \\ \Big|{\scriptstyle \det(0)\otimes\det(0)} & & \Big|{\scriptstyle \det(0)} \\ 1\otimes 1 & \xrightarrow[\sim]{\text{mult.}} & 1 \end{array}$$

commutative. It is then easy to check that i′) follows from axiom ii) of definition 2. Now by Lemma 2, the pair (det, i) is a determinant functor $\mathscr{C}is$ to $\mathscr{P}is$. Now say (f,j) is any determinant functor. If $E^{\cdot}$ is any complex we define an isomorphism

$$\theta_1 : \mathrm{f}(E^{\cdot}) \xrightarrow{\sim} \mathrm{f}(TE^{\cdot})^{-1}$$

in such a way that the diagram:

$$\begin{tikzcd}
\mathrm{f}(E^{\cdot})^{-1}\otimes\mathrm{f}(E^{\cdot}) \arrow[r, "\psi", "\sim"'] \arrow[d, "1\otimes\theta_1"] & \mathrm{f}(E^{\cdot})\otimes\mathrm{f}(E^{\cdot})^{-1} \arrow[r, "\delta", "\sim"'] & 1 \arrow[d, equal] \\
\mathrm{f}(E^{\cdot})^{-1}\otimes\mathrm{f}(TE^{\cdot})^{-1} \arrow[r, "\sim"] & \mathrm{f}(C^{\cdot}_{1_{E^{\cdot}}})^{-1} \arrow[r, "\mathrm{f}'(0)"] & 1
\end{tikzcd}$$

commutes. Here T stands for the shift operator defined by

$$(TE^{\cdot})^n = E^{n+1} \quad \text{and} \quad Td = -d\,.$$

$C^{\cdot}$ is the mapping cylinder complex. Inductively we define

$$\theta_n(E^{\cdot}) = \theta_{n-1}(TE^{\cdot})^{-1}\cdot\theta_1(E^{\cdot}) \quad \text{and} \quad \theta_{-n}(E^{\cdot}) = \theta_n(T^{-n}E^{\cdot})^{-1}$$

(note this $^{-1}$ is the functor mentioned on p. 31.)

It is straightforward to check that given a short-exact sequence

$$0 \to E^{\cdot} \xrightarrow{\alpha} F^{\cdot} \xrightarrow{\beta} G^{\cdot} \to 0$$

the diagram

$$\begin{tikzcd}
\mathrm{f}(E^{\cdot}) \otimes \mathrm{f}(G^{\cdot}) \arrow[r, "\mathrm{i}(\alpha,\beta)", "\sim"'] \arrow[d, "\theta_1\otimes\theta_1", "\wr"'] & \mathrm{f}(F^{\cdot}) \arrow[d, "\theta_1", "\wr"'] \\
\mathrm{f}(TE^{\cdot})^{-1}\otimes\mathrm{f}(TG^{\cdot})^{-1} \arrow[r, "\mathrm{i}(T\alpha,T\beta)^{-1}", "\sim"'] & \mathrm{f}(TF^{\cdot})^{-1}
\end{tikzcd}$$

commutes.

And for any quasi-isomorphism

$$\lambda : E^{\cdot} \to F^{\cdot}$$

the diagram

$$\begin{tikzcd}
\mathrm{f}(E^{\cdot}) \arrow[r, "\mathrm{f}(\lambda)", "\sim"'] \arrow[d, "\theta_1", "\wr"'] & \mathrm{f}(F^{\cdot}) \arrow[d, "\theta_1", "\wr"'] \\
\mathrm{f}(TE^{\cdot})^{-1} \arrow[r, "\mathrm{f}(T\lambda)^{-1}", "\sim"'] & \mathrm{f}(TF^{\cdot})^{-1}
\end{tikzcd}$$

commutes.

We proceed to define an isomorphism of functors:

$$\eta : (\mathrm{f},\mathrm{j}) \xrightarrow{\sim} (\det,\mathrm{i})\,.$$

First consider a complex $E^{\cdot}$ concentrated in degree i.

We define η as follows:

$$\begin{tikzcd}
\mathfrak{f}(E^\cdot) \arrow[r, "\theta_i", "\sim"'] \arrow[rr, bend right, "\eta", "\sim"'] & \mathfrak{f}(T^i E^\cdot)^{(-1)^i} \arrow[r, equal] & \det(E^\cdot)
\end{tikzcd}$$

It is then obvious that restricted to all complexes which are concentrated in a single degree, η is an isomorphism of functors. If

$$E^\cdot = \ldots \to 0 \to E^i \to E^{i+1} \to \ldots \to E^{i+n} \to 0$$

is a complex of length $n+1$, we get a short-exact sequence of complexes

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & 0 & \longrightarrow & E^i & \longrightarrow & E^i & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & E^{i+1} & \to & E^{i+1} & \to & 0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & E^{i+2} & \to & E^{i+2} & \to & 0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & E^{i+n} & \to & E^{i+n} & \to & 0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & & \\
 & & \| & & \| & & \| & & \\
0 & \to & E^\cdot_{\mathrm{I}} & \xrightarrow{\alpha} & E^\cdot & \xrightarrow{\beta} & E^\cdot_{\mathrm{II}} & \to & 0
\end{array}$$

Inductively we can define η such that the diagram

$$\begin{tikzcd}
\mathfrak{f}(E^\cdot_{\mathrm{I}}) \otimes \mathfrak{f}(E^\cdot_{\mathrm{II}}) \arrow[r, "\mathfrak{j}(\alpha, \beta)", "\sim"'] \arrow[d, "\wr"', "\eta\otimes\eta"] & \mathfrak{f}(E^\cdot) \arrow[d, "\wr"', "\eta"] \\
\det(E^\cdot_{\mathrm{I}}) \otimes \det(E^\cdot_{\mathrm{II}}) \arrow[r, "\mathfrak{i}(\alpha, \beta)", "\sim"'] & \det(E^\cdot)
\end{tikzcd}$$

commutes.

Using axiom ii) it is easy to check that for all short-exact sequences of complexes

$$0 \to E^\cdot \xrightarrow{\alpha} F^\cdot \xrightarrow{\beta} G^\cdot \to 0$$

the diagram

$$\begin{array}{ccc} \mathrm{f}(E^\cdot)\otimes\mathrm{f}(G^\cdot) & \xrightarrow[\sim]{\mathrm{i}(\alpha,\beta)} & \mathrm{f}(F^\cdot) \\ \wr\downarrow{\scriptstyle \eta\otimes\eta} & & \downarrow{\scriptstyle \eta} \\ \det(E^\cdot)\otimes\det(G^\cdot) & \xrightarrow[\sim]{i(\alpha,\beta)} & \det(F^\cdot) \end{array}$$

commutes.

Finally we want to show that for each acyclic complex $H^\cdot$ the diagram

$$\begin{tikzcd} \mathrm{f}(H^\cdot) \arrow[r, "\eta", "\sim"'] \arrow[dr, "\mathrm{f}(0)"', "\wr"] & \det(H^\cdot) \arrow[d, "\det(0)", "\wr"'] \\ & 1 \end{tikzcd}$$

commutes.

By induction we only have to prove this in case of a length 2 acyclic complex. Note that any such complex is the mapping cylinder of an isomorphism of pointed complexes, say:

$$H^\cdot = C_\lambda^\cdot \quad \text{where} \quad \lambda : A^\cdot \xrightarrow{\sim} B^\cdot .$$

We have then a short-exact sequence

$$0 \to B^\cdot \xrightarrow{\alpha} H^\cdot \xrightarrow{\beta} TA^\cdot \to 0$$

and f(0) is given as the composite:

$$\mathrm{f}(H^\cdot) \xrightarrow[\sim]{\mathrm{i}(\alpha,\beta)} \mathrm{f}(B^\cdot)\otimes\mathrm{f}(TA^\cdot) \xrightarrow{1\otimes\theta_{-1}} \mathrm{f}(B)\otimes\mathrm{f}(A)^{-1} \xrightarrow{1\otimes\mathrm{f}(\lambda)^{-1}}$$
$$\mathrm{f}(B)\otimes\mathrm{f}(B)^{-1} \xrightarrow{\delta} 1$$

The same formula holds for det, and so by Lemma 1 η is an isomorphism of functors.

We can in fact extend det even further. We need some preliminaries concerning derived categories for this.

Let $\mathscr{A}$ be an abelian category; we denote by $\mathscr{A}_3$ the following category.

i) The objects of $\mathscr{A}_3$ are sequences of the form

$$E'' \xrightarrow{\alpha} E \xrightarrow{\beta} E'$$

such that $\beta\cdot\alpha=0$.

ii) The morphisms in $\mathscr{A}_3$ are triples of maps in $\mathscr{A}$ making the resulting diagram commute.

DEFINITION 2. The subcategory of $\mathrm{D}(\mathscr{A}_3)$ whose objects are short-exact sequences of complexes will be denoted by $\mathrm{VT}(\mathscr{A})$ and, we will call it the category of true triangles of $\mathrm{D}(\mathscr{A})$.

REMARK. Let

$$X = 0 \to E^{\cdot\prime\prime} \xrightarrow{\alpha} E^{\cdot} \xrightarrow{\beta} E^{\cdot\prime} \to 0$$

be a true triangle. Taking the mapping cylinder of the first map we get an ordinary triangle

$$\ldots \to E^{\cdot\prime\prime} \to E^{\cdot} \to C_\alpha^{\cdot} \to TE^{\cdot\prime\prime} \to TE^{\cdot} \to TC_\alpha^{\cdot} \to \ldots .$$

If $1_{E^{\cdot\prime\prime}}$ is the identity map on $E^{\cdot\prime\prime}$ we have a short-exact sequence

$$0 \to C^{\cdot}_{1_{E^{\cdot\prime\prime}}} \to C_\alpha^{\cdot} \xrightarrow{u} E^{\cdot\prime} \to 0 .$$

But $C_{1_{E^{\cdot\prime\prime}}}$ is acyclic so u is a quasi-isomorphism, and hence the composition

$$E^{\cdot\prime} \xrightarrow{u^{-1}} C_\alpha^{\cdot} \to TE^{\cdot\prime\prime}$$

gives us a triangle which we call

$$\delta(X) = \to E^{\cdot\prime\prime} \xrightarrow{\alpha} E^{\cdot} \xrightarrow{\beta} E^{\cdot\prime} \to TE^{\cdot\prime\prime} \to \ldots .$$

In fact δ is a functor from true triangles to the category $\mathrm{TD}(\mathscr{A})$ of triangles in $\mathrm{D}(\mathscr{A})$. Note that the homomorphism

$$\delta : \mathrm{Hom}_{\mathrm{VT}(\mathscr{A})}(XY) \to \mathrm{Hom}_{\mathrm{TD}(\mathscr{A})}(\delta(X),\delta(Y))$$

is in general neither injective nor surjective.

PROPOSITION 3. *Let* $f : X \to Y$ *be a morphism of schemes and let* $\mathrm{Mod}(X)$, $\mathrm{Mod}(Y)$ *be the category of* $\mathcal{O}_X$- *and* $\mathcal{O}_Y$-*modules. Then left and right derived functors*

$$\mathrm{Lf}^* : \mathrm{VT}(\mathrm{Mod}(Y))^- \to \mathrm{VT}(\mathrm{Mod}(X))^-$$
$$\mathrm{Rf}_* : \mathrm{VT}(\mathrm{Mod}(X))^+ \to \mathrm{VT}(\mathrm{Mod}(Y))^+$$

exists.

PROOF. According to Hartshorne: Residues and Duality, Chapter I, Theorem 5.1, the proposition follows if each true triangle bounded below allows a quasi-isomorphism into a true triangle consisting of injective $\mathcal{O}_Y$-modules, respectively each true triangle bounded above is quasi-isomorphic to a true triangle consisting of flat $\mathcal{O}_X$-modules. The fact that such quasi-isomorphisms exist follows from the following:

i) A short-exact sequence of injective $\mathcal{O}_X$-modules is an injective object in the category $\mathrm{Mod}(X)_3$, and every object of $\mathrm{Mod}(X)_3$ with the first map injective admits an embedding into a short-exact sequence of injectives.
ii) Every object of $\mathrm{Mod}(X)_3$ with the last map surjective is the quotient of a short-exact sequence of flat $\mathcal{O}_X$-modules.

This proves the proposition.

Recall the definition of a perfect complex $\mathscr{F}^{\cdot}$ on a scheme X [2]. This means that $\mathscr{F}^{\cdot}$ is a complex of $\mathcal{O}_X$-modules (not necessary quasi coherent) such that locally on X there exists a bounded complex $\mathscr{G}^{\cdot}$ of finite free $\mathcal{O}_X$-modules and a quasi isomorphism:

$$\mathscr{G}^{\cdot} \to \mathscr{F}^{\cdot}|_U$$

We denote by Parf_X the full subcategory of $\mathrm{D}(\mathrm{Mod}\,X)$ whose objects are perfect complexes. We leave the proof of the following result to the reader.

PROPOSITION 4. a) *Let X be any affine scheme and $\mathscr{F}^{\cdot}$ a perfect complex on X. Then there exists a bounded complex of locally free, finitely generated $\mathcal{O}_X$-modules $\mathscr{G}^{\cdot}$ and a quasi-isomorphism:*

$$\mathscr{G}^{\cdot} \to \mathscr{F}^{\cdot}$$

(i.e., globally on X:)

Let $\alpha : \mathscr{F}^{\cdot} \to \mathscr{F}^{\cdot\prime}$ be a map in the category Parf_X, *and suppose we are given quasi-isomorphisms*:

$$\mathrm{p} : \mathscr{G}^{\cdot} \to \mathscr{F}^{\cdot} \quad \textit{and} \quad \mathrm{p}' : \mathscr{G}^{\cdot\prime} \to \mathscr{F}^{\cdot\prime}$$

where $\mathscr{G}^{\cdot}$ and $\mathscr{F}^{\cdot\prime}$ are bounded complexes of locally free $\mathcal{O}_X$-modules, then there exists up to homotopy a unique map

$$\beta : \mathscr{G}^{\cdot} \to \mathscr{G}^{\cdot\prime}$$

such that $\mathrm{p}'\beta = \alpha\mathrm{p}$ *in* Parf_X.

b) *If*

$$0 \to \mathscr{F}^{\cdot\prime} \to \mathscr{F}^{\cdot} \to \mathscr{F}^{\cdot\prime\prime} \to 0$$

is a true triangle of perfect complexes there exists a true triangle of bounded complexes of finite locally free $\mathcal{O}_X$-modules.

$$0 \to \mathscr{G}^{\cdot\prime} \to \mathscr{G}^{\cdot} \to \mathscr{G}^{\cdot\prime\prime} \to 0$$

and an isomorphism in the category $\mathrm{VT}(\mathrm{Parf}_X)$

$$\begin{array}{ccccccccc} 0 & \to & \mathscr{G}^{\cdot\prime} & \to & \mathscr{G}^{\cdot} & \to & \mathscr{G}^{\cdot\prime\prime} & \to & 0 \\ & & \downarrow{\scriptstyle \mathrm{p}'} & & \downarrow{\scriptstyle \mathrm{p}} & & \downarrow{\scriptstyle \mathrm{p}''} & & \\ 0 & \to & \mathscr{F}^{\cdot\prime} & \to & \mathscr{F}^{\cdot} & \to & \mathscr{F}^{\cdot\prime\prime} & \to & 0 \end{array}$$

Moreover if

$$\begin{array}{ccccccccc} 0 & \to & \mathscr{F}^{\cdot\prime} & \to & \mathscr{F}^{\cdot} & \to & \mathscr{F}^{\cdot\prime\prime} & \to & 0 \\ & & \downarrow{\scriptstyle \alpha'} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha''} & & \\ 0 & \to & \mathscr{H}^{\cdot\prime} & \to & \mathscr{H}^{\cdot} & \to & \mathscr{H}^{\cdot\prime\prime} & \to & 0 \end{array}$$

is any morphism in VT(Parf_X) *and*

$$\begin{array}{ccccccccc} 0 & \to & \mathscr{K}^{\cdot\prime} & \to & \mathscr{K}^{\cdot} & \to & \mathscr{K}^{\cdot\prime\prime} & \to & 0 \\ & & \downarrow{\scriptstyle q'} & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle q''} & & \\ 0 & \to & \mathscr{H}^{\cdot\prime} & \to & \mathscr{H}^{\cdot} & \to & \mathscr{H}^{\cdot\prime\prime} & \to & 0 \end{array}$$

is an isomorphism with $\mathscr{K}^{\cdot\prime}, \mathscr{K}^{\cdot}$ *and* $\mathscr{K}^{\cdot\prime\prime}$ *bounded complexes of locally free* $\mathcal{O}_X$*-modules. Then there exists up to homotopy a unique map:*

$$\begin{array}{ccccccccc} 0 & \to & \mathscr{G}^{\cdot\prime} & \to & \mathscr{G}^{\cdot} & \to & \mathscr{G}^{\cdot\prime\prime} & \to & 0 \\ & & \downarrow{\scriptstyle \beta'} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \beta''} & & \\ 0 & \to & \mathscr{K}^{\cdot\prime} & \to & \mathscr{K}^{\cdot} & \to & \mathscr{K}^{\cdot\prime\prime} & \to & 0 \end{array}$$

such that $\alpha'\mathrm{p}' = \mathrm{q}'\beta', \alpha\mathrm{p} = \mathrm{q}\beta$ *and* $\alpha''\mathrm{p}'' = \mathrm{q}''\beta''$ *in* Parf_X.

c) *Same for diagrams of the form*

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \mathscr{F}^{\cdot\prime} & \to & \mathscr{F}^{\cdot} & \to & \mathscr{F}^{\cdot\prime\prime} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \mathscr{G}^{\cdot\prime} & \to & \mathscr{G}^{\cdot} & \to & \mathscr{G}^{\cdot\prime\prime} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \mathscr{H}^{\cdot\prime} & \to & \mathscr{H}^{\cdot} & \to & \mathscr{H}^{\cdot\prime\prime} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

DEFINITION 4. An extended determinant functor (f, i) from Parf-is to $\mathscr{P}is$ consists of the following data:

I) For every scheme X a functor

$$\mathrm{f}_X : \text{Parf-is}_X \to \mathscr{P}is_X$$

such that $\mathrm{f}_X(0) = 1$.

II) For every true triangle in Parf-is$_X$

$$0 \to F \xrightarrow{\alpha} G \xrightarrow{\beta} H \to 0$$

we have an isomorphism:

$$\mathrm{i}_X(\alpha, \beta) : \mathrm{f}_X(F) \otimes \mathrm{f}_X(H) \xrightarrow{\sim} \mathrm{f}_X(G)$$

such that for the particular true triangles

$$0 \to H = H \to 0 \to 0$$

and

$$0 \to 0 \to H = H \to 0$$

we have:

$$\mathrm{i}_X(1,0) = \mathrm{i}_X(0,1) = 1_{\mathrm{f}_X(H)}\,.$$

We require that:

i) Given an isomorphism of true triangles*:

$$\begin{tikzcd} 0 \arrow[r] & F \arrow[r, "\alpha"] \arrow[d, "u"] & G \arrow[r, "\beta"] \arrow[d, "v"] & H \arrow[r] \arrow[d, "w"] & 0 \\ 0 \arrow[r] & F' \arrow[r, "\alpha'"] & G' \arrow[r, "\beta'"] & H' \arrow[r] & 0 \end{tikzcd}$$

the diagram

$$\begin{tikzcd} \mathrm{f}_X(F)\otimes\mathrm{f}_X(H) \arrow[r, "\mathrm{i}_X(\alpha,\beta)", "\sim"'] \arrow[d, "\mathrm{f}_X(u)\otimes\mathrm{f}_X(w)", "\wr"'] & \mathrm{f}_X(G) \arrow[d, "\mathrm{f}_X(v)", "\wr"'] \\ \mathrm{f}_X(F')\otimes\mathrm{f}_X(H') \arrow[r, "\sim", "\mathrm{f}_X(\alpha'\beta')"'] & \mathrm{f}_X(G') \end{tikzcd}$$

commutes.

ii) Given a true triangle of true triangles, i.e. a commutative diagram

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & F \arrow[r, "\alpha"] \arrow[d, "u"] & G \arrow[r, "\beta"] \arrow[d, "u'"] & H \arrow[r] \arrow[d, "u''"] & 0 \\ 0 \arrow[r] & F' \arrow[r, "\alpha'"] \arrow[d, "v"] & G' \arrow[r, "\beta'"] \arrow[d, "v'"] & H' \arrow[r] \arrow[d, "v''"] & 0 \\ 0 \arrow[r] & F'' \arrow[r, "\alpha''"] \arrow[d] & G'' \arrow[r, "\beta''"] \arrow[d] & H'' \arrow[r] \arrow[d] & 0 \\ & 0 & 0 & 0 & \end{tikzcd}$$

the diagram:

$$\begin{tikzcd} \mathrm{f}_X(F)\otimes\mathrm{f}_X(H)\otimes\mathrm{f}_X(F'')\otimes\mathrm{f}_X(H'') \arrow[r, "\mathrm{i}_X(\alpha,\beta)\otimes\mathrm{i}_X(\alpha'',\beta'')"] \arrow[d, "\mathrm{i}_X(u,v)\otimes\mathrm{i}_X(u'',v'')\otimes(1\otimes\psi\otimes 1)", "\wr"'] & \mathrm{f}_X(G)\otimes\mathrm{f}_X(G'') \arrow[d, "\mathrm{i}_X(u',v')", "\wr"'] \\ \mathrm{f}_X(F')\otimes\mathrm{f}_X(H') \arrow[r, "\sim", "\mathrm{i}_X(\alpha'\beta')"'] & \mathrm{f}_X(G') \end{tikzcd}$$

* This means this diagram commutes as $\mathcal{O}_X$-modules and not just $v\cdot\alpha=\alpha'\cdot u$ in $\mathrm{D}(\mathrm{Mod}\,X)$: in fact, even assuming $v\cdot\alpha$ and $w\cdot\beta$ homotopic to $\alpha'\cdot u$ and $\beta'\cdot v$ respectively and all sheaves locally free this property will *not* hold for det!

commutes.

iii) f and i commute with base change. Written out this means: For every morphism of schemes

$$g:\ X \to Y$$

we have an isomorphism

$$\eta(g):\ \mathrm{f}_X\cdot \mathrm{L}g^* \xrightarrow{\sim} g^*\mathrm{f}_X$$

such that for every true triangle

$$0 \to F^{\cdot} \xrightarrow{u} G^{\cdot} \xrightarrow{v} H^{\cdot} \to 0$$

the diagram:

$$\begin{tikzcd}
\mathrm{f}_X(\mathrm{L}g^*F^{\cdot})\otimes \mathrm{f}_X(\mathrm{L}g^*H^{\cdot}) \arrow[r, "\mathrm{i}_X(\mathrm{L}g^*(u,v))", "\sim"'] \arrow[d, "\eta\cdot\eta", "\wr"'] & \mathrm{f}_X(\mathrm{L}g^*G^{\cdot}) \arrow[d, "\eta", "\wr"'] \\
g^*\mathrm{f}_Y(F^{\cdot})\otimes g^*\mathrm{f}_Y(H^{\cdot}) \arrow[r, "\mathrm{i}_Y(u,v)", "\sim"'] & g^*\mathrm{f}_Y(G^{\cdot})
\end{tikzcd}$$

commutes. Moreover if

$$X \xrightarrow{g} Y \xrightarrow{h} Z$$

are two consecutive morphisms, the diagram:

$$\begin{tikzcd}
\mathrm{f}_X(\mathrm{L}g^*\mathrm{L}h^*) \arrow[r, "\eta(g)", "\sim"'] \arrow[d, "\mathrm{f}_X(\theta)", "\wr"'] & g^*\mathrm{f}_Y\mathrm{L}h^* \arrow[r, "g^*\eta(h)", "\sim"'] & g^*h^*\mathrm{f}_Z \arrow[d, "\wr"'] \\
\mathrm{f}_X(\mathrm{L}(g\cdot h)^*) \arrow[rr, "\sim"] & & (g\cdot h)^*\mathrm{f}_Z
\end{tikzcd}$$

commutes where θ is the canonical isomorphism

$$\theta:\ \mathrm{L}g^*\cdot \mathrm{L}h^* \xrightarrow{\sim} \mathrm{L}(g\cdot h)^*\ ,$$

iv) On finite complexes of locally free $\mathcal{O}_X$-modules,

$$\mathrm{f} = \det \quad \text{and} \quad \mathrm{i} = \mathrm{i}\,.$$

Then using Proposition 4, one proves easily:

Theorem 2. There is one, and, up to canonical isomorphism, only one extended determinant functor (f,i), which we will write (det,i) again.

Remark. If $\mathcal{F}$ is a perfect complex and you filter it with subcomplexes such that the successive quotients $\mathrm{gr}^n(\mathcal{F}^{\cdot})$ are all perfect, then there is a canonical isomorphism:

$$\det(\mathcal{F}^{\cdot}) \xrightarrow{\approx} \otimes \det(\mathrm{gr}^n\mathcal{F}^{\cdot})\,.$$

This is constructed easily by induction on the number of steps in the filtration, using the isomorphisms $i(\alpha,\beta)$ at each stage and it has the compatibility property described after Proposition 1 above for ordinary $\det^*$. In particular:

a) if each $\mathscr{F}^n$ is itself perfect, i.e., has locally a finite free resolution, then

$$\det(\mathscr{F}^\cdot) \cong \otimes_n \det{}^*(\mathscr{F}^n)^{(-1)^n}$$

b) if the cohomology sheaves $H^n(\mathscr{F}^\cdot)$ of the complex are perfect — we call these complexes the objects of the subcategory $\mathrm{Parf}^0 \subset \mathrm{Parf}$ — then

$$\det(\mathscr{F}^\cdot) \cong \otimes_n \det{}^*(H^n(\mathscr{F}^\cdot))^{(-1)^n} .$$

This has various easy consequences:

COROLLARY 1. *Let $\mathscr{F}^\cdot$ and $\mathscr{G}^\cdot$ be two objects of* $\mathrm{Parf}^0{}_X$ *and suppose α and β*

$$\alpha,\beta\colon \mathscr{F}^\cdot \rightrightarrows \mathscr{G}^\cdot$$

are two quasi-isomorphisms such that $H^i(\alpha) = H^i(\beta)$ for each i. Then $\det(\alpha) = \det(\beta)$.

COROLLARY 2. *Let*

$$\to \mathscr{F}_1^\cdot \xrightarrow{u} \mathscr{F}_2^\cdot \xrightarrow{v} \mathscr{F}_3^\cdot \xrightarrow{w} T\mathscr{F}_1^\cdot \to$$

be an ordinary triangle in Parf_X *such that the $\mathscr{F}_j$ are in* $\mathrm{Parf}^\circ{}_X$. *We then have an isomorphism*

$$\det(\mathscr{F}_1^\cdot) \otimes \det(\mathscr{F}_3^\cdot) \xrightarrow[\sim]{i_X(u,v,w)} \det(\mathscr{F}_2^\cdot)$$

which is functorial with respect to such triangles.

PROOF.

$$\det(\mathscr{F}_1^\cdot)\otimes\det(\mathscr{F}_3^\cdot) \cong [\otimes\det{}^*(H^n(\mathscr{F}_1^\cdot))^{(-1)^n}]\otimes[\otimes\det{}^*(H^n(\mathscr{F}_3^\cdot))^{(-1)^n}]$$

and

$$\det(\mathscr{F}_2^\cdot) \cong \otimes\det{}^*(H^n(\mathscr{F}_2^\cdot))^{(-1)^n} .$$

But the long exact cohomology sequence $H^\cdot(u,v,w)$ is an acyclic complex with perfect sheaves at each stage, so

$$\begin{aligned} 1_X &\xrightarrow{\sim} \det(H^\cdot(u,v,w)) \\ &\cong \otimes \det{}^*(n^{\text{th}} \text{ sheaf of } H^\cdot(u,v,w))^{(-1)^n} \\ &\cong [\otimes \det{}^*(H^n(\mathscr{F}_1^\cdot))^{(-1)^n}]\otimes[\otimes \det{}^*(H^n(\mathscr{F}_2^\cdot))^{(-1)^{n+1}}] \\ &\quad \otimes[\otimes \det{}^*(H^n(\mathscr{F}_3^\cdot))^{(-1)^n}] . \end{aligned}$$

We tried for some time to extend i to ordinary triangles, but in general this is not possible. It is true that for each ordinary triangle we can find an isomorphism but it is by no means functorial or unique (cf. footnote to Definition 4 above). We have seen that i extends when the complexes are good, we will now see that it also extends when the schemes are good (i.e., reduced).

PROPOSITION 6. *Let X be a reduced scheme $\mathscr{F}^\cdot$ and $\mathscr{G}^\cdot$ perfect complexes, α and β two quasi-isomorphism*

$$\alpha, \beta : \mathscr{F}^\cdot \rightrightarrows \mathscr{G}^\cdot$$

such that

a) *For each integer i there are finite filtrations*

$$F^\cdot(H^i(\mathscr{F}^\cdot)) \quad \text{and} \quad F^\cdot(H^i(\mathscr{G}^\cdot)) .$$

b) *For each generic point $x \in X$, the maps*

$$H^i(\alpha) \otimes 1_{\boldsymbol{k}(x)} \quad \text{and} \quad H^i(\beta) \otimes 1_{\boldsymbol{k}(x)}$$

are compatible with the induced filtrations on $H^i(\mathscr{F}^\cdot) \otimes \boldsymbol{k}(x)$ and $H^i(\mathscr{G}^\cdot) \otimes \boldsymbol{k}(x)$. (Note that $\boldsymbol{k}(x) = \mathcal{O}_{X,x}$) and we have

$$\mathrm{gr}(H^i(\alpha) \otimes 1_{\boldsymbol{k}(x)}) = \mathrm{gr}(H^i(\beta) \otimes 1_{\boldsymbol{k}(x)})$$

for each i.
Then

$$\det(\alpha) = \det(\beta) .$$

PROOF. Since X is reduced and det commutes with base change, we may as well assume $X = \mathrm{Spec}(k)$ where k is a field. However in this case we have

$$\mathrm{Parf}_X = \mathrm{Parf}^0{}_X$$

and so the proposition follows from the last one.

PROPOSITION 7. *Let X be a reduced scheme, then for each triangle of perfect complexes:*

$$\mathscr{F}^\cdot \xrightarrow{u} \mathscr{G}^\cdot \xrightarrow{v} \mathscr{H}^\cdot \xrightarrow{w} T\mathscr{F}^\cdot \to$$

we have a unique isomorphism.

$$\mathrm{i}_X(u, v, w) : \det(\mathscr{F}^\cdot) \otimes \det(\mathscr{H}^\cdot) \xrightarrow{\sim} \det(\mathscr{G}^\cdot)$$

which is functorial with respect to isomorphisms of triangles.

PROOF. First we represent the mapping w by a diagram of real maps

$$\begin{tikzcd} T^{-1}(\mathscr{H}^\cdot) \arrow[dr, "\mu"'] & & \mathscr{F}^\cdot \arrow[dl] \\ & I^\cdot & \end{tikzcd}$$

where $I^\cdot$ is injective.

The mapping cylinder of μ gives us a true triangle:

$$\begin{tikzcd} \ldots T^{-1}(\mathscr{H}^\cdot) \arrow[r] \arrow[d, equal] & I^\cdot \arrow[r] \arrow[d] & C_\mu^\cdot \arrow[r] & \mathscr{H}^\cdot \arrow[r] \arrow[d, equal] & TI^\cdot \arrow[r] \arrow[d] & \ldots \\ \ldots T^{-1}(\mathscr{H}^\cdot) \arrow[r] & \mathscr{F}^\cdot \arrow[r] & \mathscr{G}^\cdot \arrow[r] & \mathscr{H}^\cdot \arrow[r] & T\mathscr{F}^\cdot \arrow[r] & \ldots \end{tikzcd}$$

By the second axiom for triangles there exists a map (necessarily an isomorphism) $\lambda: C^\cdot \to \mathscr{G}^\cdot$ making the diagram above into an isomorphism of triangles. By Proposition 6 the map $\det(\lambda)$ does not depend on the choice of λ. If we represent w by a different diagram say:

$$\begin{tikzcd} T^{-1}(\mathscr{H}^\cdot) \arrow[dr, "\mu'"'] & & \mathscr{F}^\cdot \arrow[dl] \\ & I'^\cdot & \end{tikzcd}$$

we get a homotopy commutative diagram

$$\begin{tikzcd} & I^\cdot \arrow[dd, "\alpha"] \\ T^{-1}(\mathscr{H}^\cdot) \arrow[ur, "\mu"] \arrow[dr, "\mu'"'] & \\ & I'^\cdot \end{tikzcd}$$

If H is a homotopy we get a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & I^\cdot \arrow[r] \arrow[d, "\alpha"] & C_\mu^\cdot \arrow[r] \arrow[d, "\begin{pmatrix} \alpha & H \\ 0 & 1 \end{pmatrix}"] & \mathscr{H}^\cdot \arrow[r] \arrow[d, equal, "1"] & 0 \\ 0 \arrow[r] & I'^\cdot \arrow[r] & C_{\mu'}^\cdot \arrow[r] & \mathscr{H}^\cdot \arrow[r] & 0 \end{tikzcd}$$

i.e., a map of true triangles. It follows that if λ' is a map from $C_{\mu'}^\cdot$ to $\mathscr{G}^\cdot$ making

$$\begin{tikzcd} I'^\cdot \arrow[r] \arrow[d] & C_{\mu'}^\cdot \arrow[r] \arrow[d, "\lambda'"] & \mathscr{H}^\cdot \arrow[d, equal] \\ \mathscr{F}^\cdot \arrow[r] & \mathscr{G}^\cdot \arrow[r] & \mathscr{H}^\cdot \end{tikzcd}$$

into an isomorphism of triangles, then the diagram:

$$\begin{tikzcd}
\det(\mathscr{F}^\cdot)\otimes\det(\mathscr{H}^\cdot) \arrow[r] \arrow[d, equal] & \det(I^\cdot)\otimes\det(\mathscr{H}^\cdot) \arrow[r] \arrow[d, "\wr"] & \det(C^\cdot_\mu) \arrow[r, "\det(\lambda)"] \arrow[d, "\wr"] & \det(\mathscr{G}^\cdot) \arrow[d, equal] \\
\det(\mathscr{F}^\cdot)\otimes\det(\mathscr{H}^\cdot) \arrow[r] & \det(I''^\cdot)\otimes\det(\mathscr{H}^\cdot) \arrow[r] & \det(C^\cdot_{\mu'}) \arrow[r, "\det(\lambda')"] & \det(\mathscr{G}^\cdot)
\end{tikzcd}$$

commutes.

The composite map above we define to be $\mathrm{i}(u,v,w)$. It is clearly functorial.

Let $p\colon X \to Y$ be a proper morphism of finite Tor-dimension with Y noetherian. Recall that if $\mathscr{F}^\cdot$ is a perfect complex on X then $\mathrm{R}^\cdot p_*\mathscr{F}^\cdot$ is again perfect (cf. Proposition 4.8, SGA 6, expose 3 (Lecture Notes in Mathematics 225, p. 257, Springer-Verlag, Berlin-Heidelberg-New York). Hence to every perfect complex on X we can associate a graded invertible sheaf on Y

$$\det(\mathrm{R}^\cdot p_*(\mathscr{F}^\cdot)) .$$

True triangles on X have injective resolutions so $\mathrm{R}^\cdot p^*$ maps true triangles to true triangles. Hence for every true triangle

$$0 \to \mathscr{F}^\cdot \xrightarrow{\alpha} \mathscr{G}^\cdot \xrightarrow{\beta} \mathscr{H}^\cdot \to 0$$

on X we have an isomorphism:

$$\mathrm{i}_Y(\alpha,\beta) :\ \det(\mathrm{R}^\cdot p_*\mathscr{F}^\cdot)\otimes\det(\mathrm{R}p_*\mathscr{H}^\cdot) \xrightarrow{\sim} \det(\mathrm{R}p_*\mathscr{G}^\cdot)$$

which is functorial with respect to isomorphisms in $\mathrm{VT}(\mathrm{Parf}_X)$.

This operation commutes with base change too, i.e., given a morphism of noetherian schemes, $g\colon Y' \to Y$, let

$$\begin{aligned} X' &= X\times_Y Y' , \\ g' &= p_1 :\ X\times_Y Y' \to X , \\ p' &= p_2 :\ X\times_Y Y' \to Y' . \end{aligned}$$

Then there are canonical isomorphisms:

$$g^*\big(\det_Y(\mathrm{R}p_*(\mathscr{F}^\cdot))\big) \cong \det_{Y'}\big(\mathrm{L}g^*(\mathrm{R}p_*(\mathscr{F}^\cdot))\big) \cong \det_{Y'}(\mathrm{R}p'_*(\mathrm{L}g'^*\mathscr{F}^\cdot)) .$$

The last result of this chapter we state in the

Proposition 8. *Let $p\colon X \to Y$ be a proper morphism of noetherian schemes and suppose that Y is a regular scheme. We then have a functorial isomorphism:*

$$\det(\mathrm{R}^\cdot p_*\mathscr{F}^\cdot) \xrightarrow{\sim} \otimes_{p,q}\det(\mathrm{R}^q p_* H^p(\mathscr{F}^\cdot))^{(-1)^{p+q}} .$$

PROOF. The proof is easy by observing that on a noetherian regular scheme we have:

$$\mathrm{Parf}_X = \mathrm{Parf}^0{}_X ,$$

and using the spectral sequence

$$\mathrm{R}^q p^*(H^p(\mathscr{F}^\cdot)) \Rightarrow \mathrm{R}^{p+q} p_*(\mathscr{F}^\cdot) .$$

Chapter II: Div and Chow.

Let X be a notherian scheme, and

$$\lambda : \mathscr{F}^\cdot \to \mathscr{G}^\cdot$$

a map of perfect complexes in the derived categorical sense. We define the open set $U(\lambda)$ as follows:

$$U(\lambda) = \{x \in X \mid \begin{array}{l}\text{there exists a neighbourhood } V \text{ of } x\\ \text{in } X \text{ such that } \lambda \text{ restricted to } V\\ \text{is an isomorphism in } D(\mathrm{Mod}(V))\}\,.\end{array}$$

We define the support of λ to be the closed set:

$$\mathrm{Supp}(\lambda) = X - U(\lambda) .$$

Finally we say that λ is a *good* map if $\mathrm{Supp}(\lambda)$ contains no points of depth 0 or equivalently $U(\lambda)$ contains all points of depth 0.

Let again $\lambda : \mathscr{F}^\cdot \to \mathscr{G}^\cdot$ be a *good* map of perfect complexes, and let x be a point in X. By the very definition of a perfect complex, we can find a neighbourhood V containing x and two bounded complexes of coherent free $\mathcal{O}_X$-modules, say $\mathscr{E}_1^\cdot$ and $\mathscr{E}_2^\cdot$ plus, restricted to V, quasi-isomorphisms

$$\mathscr{E}_1^\cdot|_V \xrightarrow{\alpha} \mathscr{F}^\cdot|_V \quad \text{and} \quad \mathscr{E}_2^\cdot|_V \xrightarrow{\beta} \mathscr{G}^\cdot|_V .$$

By choosing basis for the various $\mathscr{E}_j^{i}$'s we get an isomorphism:

$$\mathcal{O}_X|_{V\cap U(\lambda)} \xrightarrow{\sim} \det(\mathscr{E}_1^\cdot)|_{V\cap U(\lambda)} \xrightarrow[\sim]{\det(\alpha)} \det(\mathscr{F}^\cdot)|_{V\cap U(\lambda)} \xrightarrow[\sim]{\det(\lambda)}$$
$$\det(\mathscr{G}^\cdot)|_{V\cap U(\lambda)} \xrightarrow[\sim]{\det(\beta)^{-1}} \det(\mathscr{E}_2^\cdot)|_{V\cap U(\lambda)} \xrightarrow{\sim} \mathcal{O}_X|_{V\cap U(\lambda)}$$

and this isomorphism determines a section $s \in \Gamma(V \cap U(\lambda), \mathcal{O}_X^*)$.

Since $V\cap U(\lambda)$ contains all points of depth 0 in V, $s=0$ defines a Cartier divisor $\delta(s)$ in V. Clearly $\delta(s)$ does not depend on the choice of $\mathscr{E}_1^\cdot$ and $\mathscr{E}_2^\cdot$, so we have defined a global divisor via the formula:

$$\mathrm{Div}(\lambda)|_V = \delta(s) .$$

It follows immediately from the definition that the canonical map on $U(\lambda)$

$$\det(\lambda):\ \det(\mathscr{F}^{\cdot})|_{U(\lambda)} \xrightarrow{\approx} \det(\mathscr{G}^{\cdot})|_{U(\lambda)}$$

extends to an isomorphism on the whole of X:

$$\det(\lambda):\ \det(\mathscr{F}^{\cdot})(\mathrm{Div}(\lambda)) \xrightarrow{\approx} \det(\mathscr{G}^{\cdot})\,.$$

In particular:

(i) $\mathrm{Supp}(\mathrm{Div}(\lambda)) \subset \mathrm{Supp}(\lambda)$

(ii) $\mathscr{O}(\mathrm{Div}(\lambda)) \approx \det(\mathscr{G}^{\cdot}) \otimes (\det(\mathscr{F}^{\cdot}))^{-1}$.

If $\mathscr{F}^{\cdot}$ is a perfect complex on X such that the zero map:

$$0^{\cdot} \to \mathscr{F}^{\cdot}$$

is a good map of complexes, we simply write

$$\mathrm{Div}(\mathscr{F}^{\cdot}) = \mathrm{Div}(0^{\cdot} \to \mathscr{F}^{\cdot})$$

and we have a canonical map:

$$\det(0):\ \mathscr{O}(\mathrm{Div}(\mathscr{F}^{\cdot})) \xrightarrow{\approx} \det(\mathscr{F}^{\cdot})\,.$$

This association of a divisor to every good map of perfect complexes satisfies some properties which we will summarize in the following:

THEOREM 3. (i) *Let $\lambda:\mathscr{F}^{\cdot} \to \mathscr{G}^{\cdot}$ and $\mu:\mathscr{G}^{\cdot} \to \mathscr{H}^{\cdot}$ be two good maps of perfect complexes, then the composition is good too and we have:*

$$\mathrm{Div}(\mu\cdot\lambda) = \mathrm{Div}(\mu) + \mathrm{Div}(\lambda)\,.$$

(ii) *Consider a strictly commutative diagram of short-exact sequences of perfect complexes:*

$$\begin{array}{ccccccccc}
0 \to & \mathscr{F}^{\cdot} & \longrightarrow & \mathscr{G}^{\cdot} & \longrightarrow & \mathscr{H}^{\cdot} & \to 0 \\
& \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & \\
0 \to & \mathscr{F}^{\cdot\prime} & \longrightarrow & \mathscr{G}^{\cdot\prime} & \longrightarrow & \mathscr{H}^{\cdot\prime} & \to 0\,.
\end{array}$$

Then of any two if the vertical maps are good, so is the third and we have:

$$\mathrm{Div}(\alpha) - \mathrm{Div}(\beta) + \mathrm{Div}(\gamma) = 0\,.$$

(iii) *Let*

$$0 \to \mathscr{F}^{\cdot} \xrightarrow{\lambda} \mathscr{G}^{\cdot} \xrightarrow{\mu} \mathscr{H}^{\cdot} \to 0$$

be a short exact sequence of perfect complexes such that λ is good, then $0^{\cdot} \to \mathscr{H}^{\cdot}$ is good and we have:

$$\mathrm{Div}(\lambda) = \mathrm{Div}(\mathscr{H}^{\cdot})\,.$$

(v) *Let $f:X \to Y$ be a morphism of noetherian schemes, $\lambda:\mathscr{F}^{\cdot} \to \mathscr{G}^{\cdot}$ a good map of perfect complexes on Y. Suppose that for each $x \in X$ of depth* 0, *$f(x) \in U(\lambda)$, then the map*:

$$\mathrm{L}f^*(\lambda) : \mathrm{L}f^*(\mathscr{F}^{\cdot}) \to \mathrm{L}f^{\cdot}(\mathscr{G}^{\cdot})$$

is good too, and we have:

$$\mathrm{Div}(\mathrm{L}f^*(\lambda)) = f^*(\mathrm{Div}(\lambda)) .$$

(vi) *Let X be a normal noetherian scheme and $\mathscr{F}^{\cdot}$ a good perfect complex on X. For every point x in X of depth* 1 *recall that $\mathcal{O}_X$ is a discrete rank* 1 *valuation ring, and since $\mathscr{F}^{\cdot}$ is good $H^i(\mathscr{F}^{\cdot})_X$ is a torsion $\mathcal{O}_X$-module of finite length, say*:

$$\mathrm{length}(H^i(\mathscr{F}^{\cdot})_x) = r_x{}^i(\mathscr{F}^{\cdot}) .$$

We define the number:

$$\mathrm{r}_x(\mathscr{F}^{\cdot}) = \textstyle\sum_{i=-\infty}^{\infty}(-1)^i \mathrm{r}_x{}^i(\mathscr{F}^{\cdot}) .$$

Since X is a normal noetherian scheme the group of Cartier-divisors injects into the group of Weil-divisors and we have

$$(*) \qquad \mathrm{Div}(\mathscr{F}^{\cdot}) = \sum_{\substack{x\in X\\ \mathrm{depth}(x)=1}} \mathrm{r}_x(\mathscr{F}^{\cdot})\cdot\overline{\{x\}} .$$

Proof. Everything is obvious except for v. Since a divisor is determined by its values at points of depth 1 we may assume that $X = \mathrm{Spec}(\mathcal{O})$ where $\mathcal{O}$ is a regular local ring of dimension 1.

For every good perfect complex $\mathscr{F}^{\cdot}$ on X we define:

$$\mathrm{Div}(\mathscr{F}^{\cdot}) = \mathrm{r}_x(\mathscr{F}^{\cdot})\cdot x$$

where x is the unique closed point of X. Clearly Div satisfies (i), (ii), and (iii). Since every coherent sheaf $\mathscr{F}$ on X with $\mathrm{Supp}(\mathscr{F}) \subset \{x\}$ can be considered as a perfect complex, it follows by induction that we can reduce the proof of the equality (*) to the case where $\mathscr{F}^{\cdot}$ is a complex of length 1, that is $\mathscr{F}^{\cdot} = \tilde{M}$ in degree 0 and 0 otherwise where M is a torsion $\mathcal{O}$-module. By the structure theorem for such modules we can find integers n_i, $1 \leq i \leq s$ such that

$$M \approx \textstyle\sum_{i=1}^{s} \mathcal{O}/\pi^{n_i}\mathcal{O} .$$

We then have a free resolution of M

$$0 \to \mathcal{O}_s \xrightarrow{\ d\ } \mathcal{O}_s \longrightarrow M \to 0$$

where d is given by the matrix

$$\begin{bmatrix} \pi^{n_1} & 0 & 0 & 0 \\ 0 & \pi^{n_2} & 0 & 0 \\ 0 & 0 & \pi^{n_3} & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 \dots & \pi^{n_s} \end{bmatrix}.$$

It follows that the local equation of $\mathrm{Div}(\tilde{M})$ is $\det(d) = \pi^{\Sigma n_i}$. Since length $M = \sum n_i$, the equality (*) follows.

Let $f: X \to Y$ be a morphism of noetherian schemes, and $\mathscr{F}^{\cdot}$ a perfect complex on X. We put:

$$\mathrm{Supp}(\mathscr{F}^{\cdot}) = \bigcup_i \mathrm{Supp}(H^i(\mathscr{F}^{\cdot}))\,.$$

For any point $y \in Y$ consider the fibre product

$$\begin{array}{ccc} \mathrm{Supp}(\mathscr{F}^{\cdot})_Y & \longrightarrow & \mathrm{Supp}(\mathscr{F}^{\cdot}) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\,k(y)) & \longrightarrow & Y \end{array}$$

Definition. Let f, X, Y and $\mathscr{F}^{\cdot}$ be as above. We will say that $\mathscr{F}^{\cdot}$ satisfies condition $Q_{(r)}$ if the following holds:

1) For each point $y \in Y$ of depth 0

$$\dim(\mathrm{Supp}(\mathscr{F}^{\cdot})_y) \leq r\,.$$

2) For each point $y \in Y$ of depth 1

$$\dim(\mathrm{Supp}(\mathscr{F}^{\cdot})_y) \leq r+1\,.$$

Proposition 9. *Let $f: X \to Y$ be a proper morphism of finite* Tor-*dimension. If $\mathscr{F}^{\cdot}$ is a perfect complex on X satisfying condition $Q_{(-1)}$ for the morphism f, then*

a) $\mathrm{Div}(\mathrm{R}f_*(\mathscr{F}^{\cdot}))$ *is defined,*
b) *for all line bundles $\mathscr{H}$ on X,*

$$\mathrm{Div}(\mathrm{R}f_*(\mathscr{F}^{\cdot})) = \mathrm{Div}(\mathrm{R}f_*(\mathscr{F}^{\cdot} \otimes \mathscr{H}))\,.$$

Proof. a) is clear and to prove (b), we may make a base change and replace Y by $\mathrm{Spec}\,\mathcal{O}_{y,Y}$, where $y \in Y$ has depth 0 or 1. Then $\mathrm{Supp}(\mathscr{F}^{\cdot})$ is finite over Y, hence there is an open neighborhood U

$$\mathrm{Supp}(\mathscr{F}^{\cdot}) \subset U \subset X$$

and an isomorphism of $\mathscr{H}|_U$ with $\mathcal{O}_U$. Therefore there is a sheaf of ideals $\mathscr{I} \subset \mathcal{O}_X$ such that $\mathrm{Supp}\,\mathcal{O}_X/\mathscr{I} \subset X - U$ and a homomorphism φ as follows:

$$0 \to \mathscr{I} \to \mathscr{H} \to \mathscr{K} \to 0$$

$$\mathrm{Supp}(\mathscr{K}) \subset X - U\,.$$

Then $\mathscr{F}^{\cdot} \otimes^L \mathcal{O}_X/\mathscr{I}$ and $\mathscr{F}^{\cdot} \otimes^L \mathscr{K}$ are acyclic, hence $\mathscr{F}^{\cdot} \otimes \mathscr{H}$ is quasi-isomorphic first to $\mathscr{F} \otimes^L \mathscr{I}$, and second to $\mathscr{F}^{\cdot}$. This proves (b).

Let $f: X \to Y$ be a morphism of noetherian schemes, $\mathscr{F}^{\cdot}$ a perfect complex on X and consider the function $Y \to \mathbf{Z}$ given by

$$y \to \dim(\mathrm{Supp}(\mathscr{F}^{\cdot})_y)\,.$$

It will be convenient to compute this function in a slightly different manner. Consider the fibre product:

$$\begin{array}{ccc} X_y & \xrightarrow{i} & X \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\,k(y)) & \longrightarrow & Y \end{array}$$

Lemma 1. *With the notations as above we have:*

$$\dim(\mathrm{Supp}(\mathscr{F}^{\cdot})_y) = \dim \mathrm{Supp}(\mathrm{L}i^*\mathscr{F}^{\cdot})\,.$$

Proof. We may assume X and Y affine, so let $X = \mathrm{Spec}(S)$, $Y = \mathrm{Spec}(R)$, and let $Y = [p]$. Let k be the field $k(y)$

$$k(y) = R_p/p \cdot R_p$$

we then have:

$$X_y = \mathrm{Spec}(S \otimes_R k)\,.$$

Also we may assume that $\mathscr{F}^{\cdot} = \tilde{M}^{\cdot}$ where $M^{\cdot}$ is a bounded complex of finite free S-modules, hence $\mathrm{L}i^*\mathscr{F}^{\cdot}$ is represented by $M^{\cdot} \otimes_S (S \otimes_R k) = M^{\cdot} \otimes_R k$. But there is a spectral sequence:

$$E_2^{-p,q} = \mathrm{Tor}^S{}_p(H^q(M^{\cdot}), S \otimes_R k) \Rightarrow H^n(M^{\cdot} \otimes_R k)\,.$$

If $x \in \bigcup \mathrm{Supp}(H^i(M^{\cdot}) \otimes_R k)$, let i_0 be the maximum of the indexes i such that:

$$x \in \mathrm{Supp}(H^i(M^{\cdot}) \otimes_R k)\,.$$

Then

$$x \in \mathrm{Supp}\big(\mathrm{Tor}^S{}_p(H^i(M), S \otimes_R k)\big)$$

for $i > i_0$. Consequently we have

$$x \in \bigcup_{p+q=i_0} \mathrm{Supp}(E_r^{p,q})$$

for all r, and hence

$$x \in \mathrm{Supp}(H^{i_0}(M^\cdot \otimes_R k)) .$$

Conversely, if $x \in \bigcup \mathrm{Supp}(H^i(M^\cdot \otimes_R k))$ we have $x \in \bigcup_{p,q} \mathrm{Supp}(E_r^{p,q})$ for all r, Since

$$\mathrm{Supp}\big(\mathrm{Tor}^S{}_r(H^q(M^\cdot), S\otimes_R k)\big) \subset \mathrm{Supp} H^q(M^\cdot)\otimes_R k$$

we are done.

Now we come to the main application of our techniques, namely to "*Chow points*". Let Y be a noetherian scheme, and E a locally free rank $n+1$ sheaf of $\mathcal{O}_Y$-modules. These define:

$P = \mathbf{P}(E)$, a $\mathbf{P}^n$-bundle over Y,
$\pi: P \to Y$ the projection,
$\mathcal{O}_p(1)$, the "tautological" line bundle (s.t. $\pi_*\mathcal{O}_p(1) = E$),
$\check{P} = \mathbf{P}(\check{E})$ the dual, $\mathcal{O}_{\check{P}}(1)$ its tautological line bundle,
$H \subset P \times_Y \check{P}$ the universal hyperplane, i.e.,

$$E\otimes\check{E} \cong \mathcal{O}_Y \oplus [\text{trace zero subsp. of } E\otimes\check{E}] \quad \text{canonically},$$

and if $1 \in \Gamma(\mathcal{O}_Y)$ corresponds to

$$\delta \in \Gamma(Y, E\otimes\check{E}) = \Gamma(P\times_Y\check{P}, p_1{}^*\mathcal{O}(1)\otimes p_2{}^*\mathcal{O}(1))$$

then $H = V(\delta)$.
$\mathcal{K}^\cdot_{(1)}$: the complex on $P\times_Y\check{P}$:

$$\begin{array}{ccccccc} 0 \to & p_1{}^*\mathcal{O}_P(-1)\otimes p_2{}^*\mathcal{O}_{\check{P}}(-1) & \xrightarrow{\otimes\delta} & \mathcal{O}_{P\times\check{P}} & \to 0 \\ & \| & & \| & \\ & \mathcal{K}^{-1}_{(1)} & & \mathcal{K}^{0}_{(1)} & \end{array}$$

which resolves $\mathcal{O}_H$.
$P\times_Y(\check{P})^k$ = the fibre product over Y,
$\mathcal{K}^\cdot_{(k)}$: the complex $\otimes_{i=2}^{k+1} p_{1,i}^*(\mathcal{K}^\cdot_{(1)})$ on $P\times_Y\check{P}^k$.
This complex is a resolution of $\mathcal{O}_{H_k}$, where

$$H_k = \bigcap_{i=2}^{k+1} p_{1,i}^{-1}(H) .$$

So much for the "universal" elements of our construction. Now say $\mathcal{F}^\cdot$ is a perfect complex on P and define:

$$\mathcal{F}^\cdot_{(k)}(n) = \mathrm{L}p_1{}^*(\mathcal{F}^\cdot(n)) \otimes^L \mathcal{K}^\cdot_{(k)}, \quad \text{on } P\times_Y\check{P}^k ,$$

$$\mathscr{L}_{(k)}(n) = \det(\mathrm{R}p_{2*}\mathscr{F}^{\cdot}{}_{(k)}(n)), \quad \text{on } \check{P}^k,$$
$$\mathscr{L}(n) = \det(\mathrm{R}\pi_*\mathscr{F}^{\cdot}(n)), \quad \text{on } Y.$$

LEMMA 2. *If $\mathscr{F}^{\cdot}$ satisfies condition $Q_{(r)}$ for the morphism $\pi: P \to Y$, and $r \geq k-1$, then $\mathscr{F}^{\cdot}{}_{(k)}$ satisfies condition $Q_{(r-k)}$ for the morphism $p_2: P \times_Y \check{P}^k \to \check{P}^k$.*

PROOF. By induction it is sufficient to prove the Lemma in case $k=1$ (with $\check{P}^{k-1}$ as the new Y and $\mathscr{F}^{\cdot}{}_{(k-1)}$ as the new $\mathscr{F}^{\cdot}$). If x is a point of $\hat{P}$, let $y=\pi(x) \in Y$ and let $k= \boldsymbol{k}(y)$.
Identifying the fibre of $\check{P}$ over y with $\check{\mathsf{P}}_k{}^n$, we get the diagram:

$$\begin{array}{ccccc} \mathrm{Spec}(\boldsymbol{k}(x)) & \longrightarrow & \check{\mathsf{P}}_k{}^n & \longrightarrow & \check{P} \\ & & \downarrow & & \downarrow{\scriptstyle p_2} \\ & & \mathrm{Spec}(k) & \longrightarrow & Y \end{array}$$

Since p_2 is flat, it follows from E.G.A., Chapitre IV, Proposition 6.3.1 that:

$$\mathrm{depth}(\mathcal{O}_{Y,y}) + \mathrm{depth}(\mathcal{O}_{\check{\mathsf{P}}_k{}^n,x}) = \mathrm{depth}(\mathcal{O}_{\check{P},x}).$$

From this and the previous lemma it follows that we may assume $Y= \mathrm{Spec}(k), P=\mathsf{P}_k{}^n, k$ a field, in which case the Lemma is straight-forward.

COROLLARY-DEFINITION. *If $\mathscr{F}^{\cdot}$ satisfies condition $Q_{(r)}$, then $\mathscr{F}^{\cdot}{}_{(r+1)}$ satisfies $Q_{(-1)}$, hence we can define the Chow divisor*

$$\mathrm{Chow}(\mathscr{F}^{\cdot}) = \mathrm{Div}(\mathrm{R}p_{2*}\mathscr{F}^{\cdot}{}_{(r+1)})$$

on $\check{P}^{r+1}$. Then $\mathrm{Chow}(\mathscr{F}^{\cdot}(n)) = \mathrm{Chow}(\mathscr{F}^{\cdot})$ *and there is a canonical isomorphism:*

$$\mathcal{O}_{\check{P}^{r+1}}(\mathrm{Chow}(\mathscr{F}^{\cdot})) \cong \mathscr{L}_{(r+1)}(n), \quad \textit{for every } n.$$

Next, we would like to compute $\mathscr{L}_{(k)}(n)$ in another way: since $\mathscr{K}^{\cdot}_{(k)}$ is locally free, each term $\mathscr{F}^{\cdot} \otimes \mathscr{K}_{(k)}{}^l$ is perfect, hence there is a canonical isomorphism

$$\mathscr{L}_{(k)}(n) = \det(\mathrm{R}p_{2*}\mathscr{F}^{\cdot}{}_{(k)}(n)) \cong \otimes_{l=0}^{k} \det(\mathrm{R}p_{2*}\mathrm{L}p_1{}^*\mathscr{F}^{\cdot}(n) \otimes^{L} \mathscr{K}_{(k)}^{-l})^{(-1)^l}.$$

On $\check{P}^k$, let $\mathscr{H}_i$ be the invertible sheaf $\mathcal{O}_{\check{P}}(1)$ pulled up from the i^{th} factor. Then by definition:

$$\mathscr{K}_{(k)}^{-l} = p_1{}^*(\mathcal{O}_P(-l)) \otimes p_2{}^* \textstyle\sum_{1<i_1<\ldots<i_l\leq k} \mathscr{H}_{i_1}^{-1} \otimes \cdots \otimes \mathscr{H}_{i_l}^{-1}$$

hence if $\check{\pi}\colon \check{P}^k \to Y$ denotes the projection:

$$\begin{aligned}\mathscr{L}_{(k)}(n) &\cong \otimes_{l=0}^{k}\otimes_{1<i_1<\ldots<i_l\leq k}\det\Big(\mathrm{R}p_{2*}(\mathrm{L}p_1{}^*\mathscr{F}^{\cdot}(n-l)\\ &\qquad \otimes p_2{}^*(\mathscr{H}_{i_1}^{-1}\otimes\ldots\otimes\mathscr{H}_{i_l}^{-1}))\Big)^{(-1)^l}\\ &\cong \otimes_{l=0}^{k}\otimes_{1<i_1<\ldots<i_l\leq k}\det\Big(\mathrm{L}\check{\pi}^*(\mathrm{R}\pi_*\mathscr{F}^{\cdot}(n-l))\\ &\qquad \otimes\mathscr{H}_{i_1}^{-1}\otimes\ldots\otimes\mathscr{H}_{i_l}^{-1}\Big)^{(-1)^l}\end{aligned}$$

On the other hand, it is easy to check that for any perfect complex and invertible sheaf:

$$\det(\mathscr{G}^{\cdot}\otimes\mathscr{L}) \cong \det(\mathscr{G}^{\cdot})\otimes\mathscr{L}\mathrm{rk}(\mathscr{G}^{\cdot})\,.$$

Note that

$$\mathrm{rk}(\mathscr{L}(n)) = \chi(\mathscr{F}^{\cdot}(n))$$

i.e. = the continuous function $Y \to \mathbf{Z}$ given by

$$y \to \Sigma(-1)^i\dim_{k(y)}H^i(\mathscr{F}^{\cdot}\otimes^L P_{k(y)})\,.$$

We abbreviate this to $\chi(n)$. Therefore we have canonical isomorphisms:

$$\mathscr{L}_{(k)}(n) \cong \otimes_{l=0}^{k}\otimes_{1\leq i_1<\ldots<i_l\leq k}\check{\pi}^*\mathscr{L}(n-l)^{(-1)^l}\otimes(\mathscr{H}_{i_1}\otimes\ldots\otimes\mathscr{H}_{i_l})^{(-1)^{l+1}\chi(n-l)}\,.$$

Now defined by induction:

a) "difference" sheaves:

$$\begin{aligned}\Delta\mathscr{L}(n) &= \mathscr{L}(n)\otimes\mathscr{L}(n-1)^{-1}\\ \Delta^k\mathscr{L}(n) &= \Delta^{k-1}\mathscr{L}(n)\otimes\Delta^{k-1}\mathscr{L}(n-1)^{-1}\\ &\cong \Delta^{k-2}\mathscr{L}(n)\otimes\Delta^{k-2}\mathscr{L}(n-1)^{-2}\otimes\Delta^{k-2}\mathscr{L}(n-2)\\ &\quad\ldots\\ &\cong \otimes_{l=0}^{k}\mathscr{L}(n-l)^{(-1)^l\binom{k}{l}}\end{aligned}$$

b) difference functions:

$$\begin{aligned}\chi_1(n) &= \chi(n)-\chi(n-1)\\ \chi_k(n) &= \chi_{k-1}(n)-\chi_{k-1}(n-1)\\ &\quad\ldots\\ &= \textstyle\sum_{l=0}^{k}(-1)^l\binom{k}{l}\chi(n-l)\end{aligned}$$

Then it follows easily that:

$$\mathscr{L}_{(k)}(n) \cong \check{\pi}^*(\Delta^k\mathscr{L}(n))\otimes(\mathscr{H}_1\otimes\ldots\otimes\mathscr{H}_k)^{\chi_{k-1}(n-1)}\,.$$

Combining this with above Corollary, if $\mathscr{F}^{\cdot}$ satisfies $Q_{(r)}$, then we find

$\chi_r(n)$ is independent of n

Up to canonical isomorphisms, $\pi^*(\Delta^{r+1}\mathscr{L}(n))$ is independent of n.

Since $\check{\pi}_*(\mathcal{O}_{\check{P}_k}) = \mathcal{O}_Y$, this implies that:

Up to canonical isomorphisms, $\Delta^{r+1}\mathcal{L}(n)$ independent of n. Going backwards, this implies that χ is a polynomial of degree at most r and that $\mathcal{L}(n)$ can be expanded as in the following final Theorem:

THEOREM 4. *Let Y be a noetherian scheme, E a locally free sheaf of rank $n+1$ on $Y, P = \mathsf{P}(E)$ and $\mathcal{F}^{\cdot}$ a perfect complex on P satisfying condition $Q_{(r)}$ for $\pi: P \to Y$. Then there are sheaves $\mathcal{M}_0, \ldots, \mathcal{M}_{r+1}$ on Y and canonical and functorial isomorphisms:*

$$\det(\mathrm{R}\pi_*\mathcal{F}^{\cdot}(n)) \cong \otimes_{k=0}^{r+1} \mathcal{M}_k^{\binom{n}{k}} .$$

Moreover the leading term $\mathcal{M}_{r+1}$ is related to the Chow divisor by a canonical isomorphism:

$$\check{\pi}^*\mathcal{M}_{r+1} \otimes (\mathcal{H}_1 \otimes \ldots \otimes \mathcal{H}_{r+1})^d \cong \mathcal{O}_{\hat{P}^{r+1}}(\mathrm{Chow}(\mathcal{F}^{\cdot}))$$

where $\check{\pi}: \check{P}^{r+1} \to Y$ is the projection,

$$\mathcal{H}_i = i^{\text{th}} \textit{ sheaf } \mathcal{O}_{\check{P}}(1) \textit{ on } \check{P}^{r+1},$$

$d\binom{n}{r}$ = leading term of the Hilbert polynomial $\chi(\mathcal{F}^{\cdot}(n))$.

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